Some Characteristic Properties of $AW(k)$-type Curves on Dual Unit Sphere

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Abstract: In this paper, we have considered some properties of $AW(k)$-type curves on the dual unit sphere. We have given some relationships between the curvatures of the base curves of the ruled surfaces in $\mathbb{R}^3$ which correspond to the dual spherical curves of these type. And also, we have examined the relations between the real integral invariants (the angle of pitch and the pitch) of the closed ruled surfaces in $\mathbb{R}^3$ which correspond to the dual closed spherical curves drawn by the dual unit vectors $\vec{\vec{N_1}}$, $\vec{\vec{N_2}}$ and $\vec{\vec{N_3}}$ on dual unit sphere.

Key words: Dual curve, Dual angle of pitch, Ruled surface, Pitch, Angle of pitch.

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1. Introduction

In a spatial motion, the trajectories of the oriented lines embedded in a moving rigid body are generally ruled surfaces. These lines may be the axes of the joints of spatial mechanisms or manipulators or the line of action of the end-of-arm tooling of a manipulator. Thus the geometry of ruled surfaces is important in the study of rational design problems in spatial mechanisms, especially for robot kinematics. The integral invariants of line trajectories seek to characterize the shape of the trajectory ruled surface and relate it to the motion of body carrying the line that generates it. An x-closed ruled surface generated by on x-oriented line fixed in the moving system is characterized by two real integral invariants; the pitch l_x and the angle of pitch λ_x. Recently, based on the dual spherical curves and invariants, the differential geometry of closed ruled surfaces has been studied in many papers, [1], [5], [8],..., [16].

In [3], Arslan and Özgür considered curves of the $AW(k)$-type , $1 \leq k \leq 3$. They gave curvature conditions of these kind of curves.

In this study, analogously to [3], we consider $AW(k)$-type curves on dual unit sphere. We investigate the curvature conditions of these kind of curves on dual unit sphere. According to E. Study transference principle, we give the curvature conditions of the base curve of the ruled surface in $\mathbb{R}^3$ which corresponds to the curve on the dual unit sphere.

Finally, we investigate the relationships among the geometric invariants of closed ruled surface in $\mathbb{R}^3$ which corresponds to the dual closed spherical curves drawn by the dual unit vectors $\vec{\widetilde{N}}_1$, $\vec{\widetilde{N}}_2$ and $\vec{\widetilde{N}}_3$ on dual unit sphere.

2. Basic concepts

DEFINITION 2.1. If a and a^* are real numbers and $\varepsilon^2 = 0$, the combination $A = a + \varepsilon a^*$ is called a dual number where $\varepsilon = (0, 1)$ is a dual unit.

W. K. Clifford defined the dual numbers and showed that they form an algebra, not a field. The pure dual numbers εa^* are zero divisors, $(\varepsilon a^*)(\varepsilon b^*) = 0$. No number εa^* has an inverse in the algebra. But the other laws of the algebra of dual numbers are the same as the laws of algebra of complex numbers. This means that dual numbers form a ring over the real number field. For example two dual numbers $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$ are added componentwise $A + B = (a + b) + \varepsilon(a^* + b^*)$ and they are multiplied by $A \cdot B = ab + \varepsilon(a^* b + ab^*)$. For the equality of A and B, we have $A = B \Leftrightarrow a = b$ and $a^* = b^*$ [4].

The set of dual numbers is denoted by D. The set $D^3 = D \times D \times D$ is a module over the ring D. It is clear that any dual vector $\vec{X}$ in D^3, consists of two real vectors $\vec{x}$ and $\vec{x^*}$ in $\mathbb{R}^3$. The elements of D^3 are called the dual vectors.

DEFINITION 2.2. Let $\vec{X}$ and $\vec{Y}$ be two dual vectors. Then the inner product of them is defined as follows

$$\langle \vec{X}, \vec{Y} \rangle = \langle \vec{x}, \vec{y} \rangle + \varepsilon \left(\langle \vec{x^*}, \vec{y} \rangle + \langle \vec{x}, \vec{y^*} \rangle \right).$$

DEFINITION 2.3. For a given dual vector $\vec{X}$, the norm is defined by

$$\|\vec{X}\| = \sqrt{\langle \vec{X}, \vec{X} \rangle} = \|\vec{x}\| + \varepsilon \frac{\langle \vec{x}, \vec{x^*} \rangle}{\|\vec{x}\|}, \quad \vec{x} \neq \vec{0}.$$

DEFINITION 2.4. The set

$$\left\{ \vec{X} = \vec{x} + \varepsilon \vec{x^*} \ : \ \|\vec{X}\| = (1,0), \ \vec{x}, \vec{x^*} \in \mathbb{R}^3 \right\}$$

is called dual unit sphere in D^3.

THEOREM 2.1. (E. Study) *A unit vector $\vec{X} = \vec{x} + \varepsilon \vec{x^*}$ on unit dual sphere corresponds to one and only oriented line in $\mathbb{R}^3$, where the real part $\vec{x}$ shows the direction of the line and the dual part $\vec{x^*}$ shows the vectorial moment of the unit vector $\vec{x}$ with respect to origin* [7].

According to E. Study Theorem a differentiable curve $\vec{X}$ on the dual unit sphere, depending on a real parameter s, represents a differentiable family of straight lines in $\mathbb{R}^3$ which is called as ruled surface.

If the dual unit spheres K and K' respectively correspond to a line space H and H' then K/K' corresponds to a spatial motion which will be denoted by H/H'. Then H is the moving space with respect to the fixed space H'.

During the one-parameter closed spatial motion H/H', each fixed line of the moving space H, generally, generates a closed ruled surface in the fixed space H' [11].

Let K be a moving dual unit sphere generated by a dual orthonormal system

$$\left\{ \vec{X_1}, \ \vec{X_2} = \frac{\vec{X_1}}{\|\vec{X_1}\|}, \ \vec{X_3} = \vec{X_1} \wedge \vec{X_2} \right\}, \quad \vec{X_i} = \vec{x_i} + \varepsilon \vec{x_i^*}; \quad i = 1,2,3$$

and K' be a fixed dual unit sphere with the same center. Then the derivative equations of the dual spherical closed motion with respect to K'-sphere of the K-sphere are

$$d\vec{X_i} = \sum_{j=1}^{3} \Omega_i^j \vec{X_j}, \ \Omega_i^j(t) = \omega_i^j(t) + \varepsilon {\omega^*}_i^{\,j}(t), \ \Omega_i^j = -\Omega_j^i, \ t \in I, \quad i = 1,2,3.$$

The dual Steiner vector of this closed motion is defined by

$$\vec{D} = \oint \vec{\Psi}, \ \vec{\Psi} = \vec{\psi} + \varepsilon \vec{\psi^*} \tag{2.1}$$

where $\vec{\Psi} = \Omega_2^3 \vec{X_1} + \Omega_3^1 \vec{X_2} + \Omega_1^2 \vec{X_3}$ is the instantaneous Darboux vector of the motion. According to E. Study transference principle the differentiable dual

closed curve $\vec{X}_1(t)$, $t \in I \subset \mathbb{R}$, is considered as a closed ruled surface in $\mathbb{R}^3$ and denoted by x_1-c.r.s..

A dual invariant which is called the dual angle of pitch of x_1-c.r.s. is given by

$$\Lambda_{X_1} = -\langle \vec{D}, \vec{X}_1 \rangle = \lambda_{x_1} - \varepsilon l_{x_1} \tag{2.2}$$

in [8], [9] in terms of the real integral invariants.

3. Main results

In this section we discuss the curves of dual $AW(k)$-type on dual unit sphere, similar to curves of $AW(k)$-type in $\mathbb{R}^3$ which are studied in [3]. We denote these type curves by $DAW(k)$-type.

(For more details for $AW(k)$-type curves see also [2]).

Let $\vec{\Gamma}(s) = \vec{\gamma}(s) + \varepsilon \vec{\gamma}^*(s)$ be a unit speed spherical curve on the dual unit sphere and $\left\{ \vec{V}_1, \vec{V}_2, \vec{V}_3 \right\}$ be the dual Frenet frame of $\vec{\Gamma}$. The Frenet formulas of $\vec{\Gamma}$ are defined:

$$\begin{bmatrix} d\vec{V}_1 \\ d\vec{V}_2 \\ d\vec{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & \widehat{\kappa}_1 & 0 \\ -\widehat{\kappa}_1 & 0 & \widehat{\kappa}_2 \\ 0 & -\widehat{\kappa}_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \end{bmatrix} \tag{3.1}$$

where $\vec{V}_1 = \vec{v}_1 + \varepsilon \vec{v}_1^*$, $\vec{V}_2 = \vec{v}_2 + \varepsilon \vec{v}_2^*$ and $\vec{V}_3 = \vec{v}_3 + \varepsilon \vec{v}_3^*$, $\widehat{\kappa}_1 = \kappa_1 + \varepsilon \kappa_1^*$ and $\widehat{\kappa}_2 = \kappa_2 + \varepsilon \kappa_2^*$ are the Frenet curvatures of Γ.

From (2.1) and (3.1) the dual Steiner vector of this closed motion is

$$\vec{D} = \oint \widehat{\kappa}_2 \vec{V}_1 + \widehat{\kappa}_1 \vec{V}_3 \tag{3.2}$$

PROPOSITION 3.1. *Let $\vec{\Gamma} : I \subset \mathbb{R} \to D^3$ be a unit speed spherical curve on the dual unit sphere with arclength parameter s. Similar with Proposition 3 in [3], we have*

$$\vec{\Gamma}' = \vec{V}_1, \tag{3.3}$$

$$\vec{\Gamma}'' = \widehat{\kappa}_1 \vec{V}_2, \tag{3.4}$$

$$\vec{\Gamma}''' = -\widehat{\kappa}_1^2 \vec{V}_1 + \widehat{\kappa}_1' \vec{V}_2 + \widehat{\kappa}_1 \widehat{\kappa}_2 \vec{V}_3, \tag{3.5}$$

$$\vec{\Gamma}^{(iv)} = -3\widehat{\kappa}_1 \widehat{\kappa}_1' \vec{V}_1 + (-\widehat{\kappa}_1^3 + \widehat{\kappa}_1'' - \widehat{\kappa}_1 \widehat{\kappa}_2^2) \vec{V}_2 + (2\widehat{\kappa}_1' \widehat{\kappa}_2 + \widehat{\kappa}_1 \widehat{\kappa}_2') \vec{V}_3 \tag{3.6}$$

where primes denote differentiation with respect to s.

Notation: Let us write

$$\vec{N}_1 = \widehat{\kappa}_1 \vec{V}_2 \tag{3.7}$$

$$\vec{N}_2 = \widehat{\kappa}_1' \vec{V}_2 + \widehat{\kappa}_1 \widehat{\kappa}_2 \vec{V}_3 \tag{3.8}$$

$$\vec{N}_3 = (-\widehat{\kappa}_1^3 + \widehat{\kappa}_1'' - \widehat{\kappa}_1 \widehat{\kappa}_2^2)\vec{V}_2 + (2\widehat{\kappa}_1' \kappa_2 + \widehat{\kappa}_1 \widehat{\kappa}_2')\vec{V}_3 \tag{3.9}$$

where $\vec{N}_1 = \vec{n_1} + \varepsilon \vec{n_1^*}$, $\vec{N}_2 = \vec{n_2} + \varepsilon \vec{n_2^*}$ and $\vec{N}_3 = \vec{n_3} + \varepsilon \vec{n_3^*}$.

DEFINITION 3.1. The unit speed spherical Frenet curves on the dual unit sphere are
i) of type $DAW(1)$ if they satisfy $\vec{N}_3 = 0$,
ii) of type $DAW(2)$ if they satisfy

$$\|\vec{N}_2\|^2 \vec{N}_3 = \langle \vec{N}_3, \vec{N}_2 \rangle \vec{N}_2, \tag{3.10}$$

iii) of type $DAW(3)$ if they satisfy

$$\|\vec{N}_1\|^2 \vec{N}_3 = \langle \vec{N}_3, \vec{N}_1 \rangle \vec{N}_1. \tag{3.11}$$

From Definition 3.1 we may give the following propositions.

PROPOSITION 3.2. *The unit speed spherical curve is of type $DAW(1)$ if and only if*

$$\widehat{\kappa}_1'' - \widehat{\kappa}_1^3 - \widehat{\kappa}_1 \widehat{\kappa}_2^2 = 0 \tag{3.12}$$

$$2\widehat{\kappa}_1' \widehat{\kappa}_2 + \widehat{\kappa}_1 \widehat{\kappa}_2' = 0. \tag{3.13}$$

Separating the relations (3.12)-(3.13) into real and dual parts, we have

$$\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2 = 0, \tag{3.14}$$

$$2\kappa_1' \kappa_2 + \kappa_1 \kappa_2' = 0 \tag{3.15}$$

and

$$\kappa_1^{*''} - 3\kappa_1^2 \kappa_1^* - \kappa_1^* \kappa_2^2 - 2\kappa_1 \kappa_2 \kappa_2^* = 0, \tag{3.16}$$

$$2\kappa_2 \kappa_1^{*'} + 2\kappa_2^* \kappa_1' + \kappa_2' \kappa_1^* + \kappa_1 \kappa_2^{*'} = 0. \tag{3.17}$$

PROPOSITION 3.3. *The unit speed spherical curve is of type $DAW(2)$ if and only if*

$$2\widehat{\kappa}_1'^2\widehat{\kappa}_2 + \widehat{\kappa}_1\widehat{\kappa}_1'\widehat{\kappa}_2' = \widehat{\kappa}_1''\widehat{\kappa}_1\widehat{\kappa}_2 - \widehat{\kappa}_1^4\widehat{\kappa}_2 - \widehat{\kappa}_1^2\widehat{\kappa}_2^3. \qquad (3.18)$$

Separating the relation (3.18) into real and dual parts, we have

$$2\kappa_1'^2\kappa_2 + \kappa_1\kappa_1'\kappa_2' = \kappa_1''\kappa_1\kappa_2 - \kappa_1^4\kappa_2 - \kappa_1^2\kappa_2^3 \qquad (3.19)$$

and

$$2\kappa_1'^2\kappa_2^* + 4\kappa_1'\kappa_1^{*'}\kappa_2 + \kappa_1\kappa_1'\kappa_2^{*'} + \kappa_1\kappa_2'\kappa_1^{*'} + \kappa_1'\kappa_1^*\kappa_2' \qquad (3.20)$$
$$= \kappa_1'\kappa_1''\kappa_2^* + \kappa_1''\kappa_1^*\kappa_2 + \kappa_1\kappa_1^{*''}\kappa_2 - \kappa_1^4\kappa_2^* + 4\kappa_1^3\kappa_2\kappa_1^* - 2\kappa_1\kappa_1^*\kappa_2^3 - 3\kappa_1^2\kappa_2^2\kappa_2^*.$$

PROPOSITION 3.4. *The unit speed spherical curve is of type $DAW(3)$ if and only if*

$$2\widehat{\kappa}_1'\widehat{\kappa}_2 + \widehat{\kappa}_1\widehat{\kappa}_2' = 0. \qquad (3.21)$$

Separating the relation (3.21) into real and dual parts, we have

$$2\kappa_1'\kappa_2 + \kappa_1\kappa_2' = 0 \qquad (3.22)$$

and

$$2\kappa_2\kappa_1^{*'} + 2\kappa_2^*\kappa_1' + \kappa_2'\kappa_1^* + \kappa_1\kappa_2^{*'} = 0. \qquad (3.23)$$

From (3.15) and (3.22) we may give the following result.

COROLLARY 3.1. *The right helicoids are of type $DAW(1)$ and $DAW(3)$ on the dual unit sphere.*

By solving differential equation (3.15), we have

$$\kappa_1^2\kappa_2 = c = const.$$

So we get the following result.

COROLLARY 3.2. *There is the relation $\kappa_1^2\kappa_2 = c = const.$ between the curvatures of the base curves of the ruled surfaces in $\mathbb{R}^3$ which correspond to the dual unit speed spherical curves of $DAW(1)$ and $DAW(3)$-types.*

Let us consider the dual unit vectors $\vec{\tilde{N}}_1$, $\vec{\tilde{N}}_2$ and $\vec{\tilde{N}}_3$ which are fixed in the moving Frenet frame $\left\{\vec{V}_1, \vec{V}_2, \vec{V}_3\right\}$ and expressed as follows:

$$
\begin{aligned}
\vec{\tilde{N}}_1 &= \frac{\vec{\tilde{N}}_1}{\|\vec{\tilde{N}}_1\|} = \vec{V}_2 \\
\vec{\tilde{N}}_2 &= \frac{\vec{\tilde{N}}_2}{\|\vec{\tilde{N}}_2\|} = \widehat{A}(\widehat{\kappa}_1' \vec{V}_2 + \widehat{\kappa}_1 \widehat{\kappa}_2 \vec{V}_3) \\
\vec{\tilde{N}}_3 &= \frac{\vec{\tilde{N}}_3}{\|\vec{\tilde{N}}_3\|} = \widehat{B}\left((\widehat{\kappa}_1'' - \widehat{\kappa}_1^3 - \widehat{\kappa}_1\widehat{\kappa}_2^2)\vec{V}_2 + (2\widehat{\kappa}_1'\widehat{\kappa}_2 + \widehat{\kappa}_1\widehat{\kappa}_2')\vec{V}_3\right)
\end{aligned}
\tag{3.24}
$$

where

$$
\widehat{A} = \frac{1}{a} - \varepsilon\left(\frac{b}{a^3}\right), \quad a = \left(\kappa_1'^2 + \kappa_1^2\kappa_2^2\right)^{\frac{1}{2}}, \quad b = \kappa_1'\kappa_1^{*'} + \kappa_1^2\kappa_2\kappa_2^* + \kappa_1\kappa_1^*\kappa_2^2
$$

and

$$
\begin{aligned}
\widehat{B} &= \frac{1}{c} - \varepsilon\left(\frac{d}{c^3}\right), \; c = \left(\left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2\right)^2 + \left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right)^2\right)^{\frac{1}{2}} \\
d &= (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2)(\kappa_1^{*''} - 3\kappa_1^2\kappa_1^* - \kappa_1^*\kappa_2^2 - 2\kappa_1\kappa_2\kappa_2^*) \\
&\quad + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')(2\kappa_2\kappa_1^{*'} + 2\kappa_2^*\kappa_1' + \kappa_2'\kappa_1^* + \kappa_1\kappa_2^{*'})
\end{aligned}
$$

Now, we consider dual spherical closed curves drawn by the dual unit vectors $\vec{\tilde{N}}_1$, $\vec{\tilde{N}}_2$ and $\vec{\tilde{N}}_3$, during the one-parameter dual spherical closed motion K/K'. According to E. Study transference principle these curves correspond to closed ruled surfaces in $\mathbb{R}^3$. Our aim is to investigate the relationships among the integral invariants (the pitch and angle of pitch) of these closed ruled surfaces which correspond to dual spherical closed curves.

According to the equations (2.2), (3.2) and (3.24) the dual angles of pitch of the closed ruled surfaces drawn on K' by the dual unit vectors $\vec{\tilde{N}}_1$, $\vec{\tilde{N}}_2$ and $\vec{\tilde{N}}_3$ may be written as follows,

$$
\begin{aligned}
\Lambda_{\widehat{N}_1} &= -\left\langle \vec{D}, \vec{\tilde{N}}_1 \right\rangle \\
&= -\left\langle \oint \widehat{\kappa}_2 \vec{V}_1 + \widehat{\kappa}_1 \vec{V}_3, \vec{V}_2 \right\rangle = 0
\end{aligned}
\tag{3.25}
$$

and similarly

$$\Lambda_{\widehat{N_2}} = \widehat{A}\widehat{\kappa}_1\widehat{\kappa}_2\Lambda_{V_3}, \tag{3.26}$$

$$\Lambda_{\widehat{N_3}} = \widehat{B}(2\widehat{\kappa}_1'\widehat{\kappa}_2 + \widehat{\kappa}_1\widehat{\kappa}_2')\Lambda_{V_3}. \tag{3.27}$$

Separating the relations (3.25)-(3.27) into real and dual parts, we have

$$\begin{aligned}\lambda_{n_1} &= 0, \\ l_{n_1} &= 0,\end{aligned} \tag{3.28}$$

$$\lambda_{n_2} = \frac{\kappa_1\kappa_2}{\left(\kappa_1'^2 + \kappa_1^2\kappa_2^2\right)^{\frac{1}{2}}}\lambda_{v_3},$$

$$l_{n_2} = \left(\frac{\kappa_1\kappa_2^* + \kappa_1^*\kappa_2}{\left(\kappa_1'^2 + \kappa_1^2\kappa_2^2\right)^{\frac{1}{2}}} - \frac{b\kappa_1\kappa_2}{\left(\kappa_1'^2 + \kappa_1^2\kappa_2^2\right)^{\frac{3}{2}}}\right)\lambda_{v_3} + \frac{\kappa_1\kappa_2}{\left(\kappa_1'^2 + \kappa_1^2\kappa_2^2\right)^{\frac{1}{2}}}l_{v_3}, \tag{3.29}$$

$$\lambda_{n_3} = \frac{2\kappa_1'\kappa_2 + \kappa_1\kappa_2'}{\left(\left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2\right)^2 + \left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right)^2\right)^{\frac{1}{2}}}\lambda_{v_3},$$

$$\begin{aligned}l_{n_3} = (&\frac{2\kappa_2\kappa_1^{*'} + 2\kappa_2^*\kappa_1' + \kappa_2'\kappa_1^* + \kappa_1\kappa_2^{*'}}{\left(\left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2\right)^2 + \left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right)^2\right)^{\frac{1}{2}}} + \\ &\frac{d(2\kappa_1'\kappa_2 + \kappa_1\kappa_2')}{\left(\left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2\right)^2 + \left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right)^2\right)^{\frac{3}{2}}})\lambda_{v_3} \\ &+ \frac{2\kappa_1'\kappa_2 + \kappa_1\kappa_2'}{\left(\left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2\right)^2 + \left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right)^2\right)^{\frac{1}{2}}}l_{v_3}.\end{aligned} \tag{3.30}$$

Thus, we may give the following propositions.

PROPOSITION 3.5. *The ruled surface which correspond to the dual spherical closed curves drawn by dual unit vector $\overrightarrow{\widehat{N_1}}$ is cylinder (developable surface).*

PROPOSITION 3.6. *The relationships among the real integral invariants and the curvatures of the base curves of the closed ruled surfaces in $\mathbb{R}^3$ which correspond to dual spherical closed curve drawn by dual unit vector $\overrightarrow{\widehat{N_2}}$ and $\overrightarrow{\widehat{N_3}}$ can be given by equations (3.29) and (3.30).*

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