

## Continuity of the Norm of Composition Operators

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*Abstract:* Let  $\phi$  be an analytic self-map of the open unit disk  $\mathbb{D}$  in the complex plane. Such a map induces a composition operator  $C_\phi$  on weighted Banach spaces of holomorphic functions. We study when the norm of composition operators acting on weighted Banach spaces of analytic functions is continuous at a symbol.

*Key words:* norm of a composition operator, continuity, weighted Banach spaces of holomorphic functions.

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### 1. INTRODUCTION

Let  $\mathbb{D}$  denote the open unit disk in the complex plane  $\mathbb{C}$  and  $H(\mathbb{D})$  the space of all analytic functions on  $\mathbb{D}$ . An analytic self-map  $\phi$  of  $\mathbb{D}$  induces through composition a linear composition operator

$$C_\phi : H(\mathbb{D}) \rightarrow H(\mathbb{D}), f \mapsto f \circ \phi.$$

Such operators link operator theoretical questions with classical complex analysis. Moreover, they occur in a variety of natural problems. Therefore, the investigation of composition operators has a long and rich history, see [9] and [17].

We are interested in composition operators acting in the following setting. Let  $v : \mathbb{D} \rightarrow (0, \infty)$  be a bounded and continuous function (*weight*). We consider the *weighted Banach spaces of holomorphic functions*

$$H_v^\infty := \left\{ f \in H(\mathbb{D}); \|f\|_v := \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty \right\}$$

endowed with the weighted sup-norm  $\|\cdot\|_v$ . Weighted spaces of this type arise naturally in the study of several problems related to e.g. spectral theory, Fourier analysis, partial differential equations and convolution equations. See e.g. [2], [3], [4] for further information.

Another important fact is that composition operators acting on the Bloch space are closely connected to weighted composition operators acting on weighted Banach spaces of holomorphic functions. This parallelism has been explained by Contreras and Hernandez-Díaz in [8] and Montes in [14] and [15]. Another important paper on this topic is [13].

Motivated by the work of Pokorný and Shapiro [16] in this article we consider the question when the norm of a composition operator is continuous with respect to the inducing symbol  $\phi$ . Moreover, this article improves the results we got in [18].

## 2. NOTATIONS, DEFINITIONS AND AUXILIARY RESULTS

**2.1. WEIGHTS AND WEIGHTED SPACES.** We will denote the closed unit ball of  $H_v^\infty$  by  $B_v^\infty$ . When dealing with weighted Bergman spaces of infinite order an important tool is the so called *associated weight* introduced by Anderson and Duncan in [1] and thoroughly studied by Bierstedt, Bonet and Taskinen in [4]. For a weight  $v$  the associated weight  $\tilde{v}$  is given by

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|; f \in B_v^\infty\}} = \frac{1}{\|\delta_z\|_{H_v^\infty}}, z \in \mathbb{D},$$

where  $\delta_z$  denotes the point evaluation of  $z$ . The associated weight  $\tilde{v}$  is continuous,  $\tilde{v} \geq v > 0$ , and for every  $z \in \mathbb{D}$  we can find  $f_z \in B_v^\infty$  such that  $f_z(z) = \frac{1}{\tilde{v}(z)}$ . Since quite often it is rather complicated to compute the associated weights, we are interested in giving easy conditions on the weight, to find out, when it is *essential*, i.e. when there is a constant  $C > 0$  such that

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \text{ for every } z \in \mathbb{D}.$$

Examples of essential weights as well as conditions when weights are essential may be found in [4], [5] and [6]. We are mainly interested in the class of *radial* weights, i.e. weights which satisfy  $v(z) = v(|z|)$  for every  $z \in \mathbb{D}$ . If a radial weight  $v$  satisfies the Lusky condition (L1) (due to Lusky [12])

$$(L1) \quad \inf_k \frac{v(1 - 2^{-k-1})}{v(1 - 2^{-k})} > 0,$$

then  $v$  is essential. The most famous examples of weights satisfying the Lusky condition (L1) are the *standard weights*  $v_\alpha(z) = (1 - |z|^2)^\alpha$ ,  $\alpha > 0$ , and the *logarithmic weights*  $w_\beta(z) = (1 - \log(1 - |z|^2))^\beta$ ,  $\beta < 0$ .

A radial non-increasing weight with  $\lim_{|z| \rightarrow 1^-} v(z) = 0$  is called *typical*. In the sequel every radial weight is assumed to be non-increasing.

2.2. GEOMETRY OF THE UNIT DISK. In order to treat differences of composition operators we need some geometric data of the unit disk. A very important tool in the investigation of differences of composition operators is the *pseudohyperbolic metric* which is given by

$$\rho(z, a) := |\varphi_a(z)|, \quad z, a \in \mathbb{D},$$

where  $\varphi_a(z) := \frac{a-z}{1-\bar{a}z}$  for every  $z, a \in \mathbb{D}$ , is the *Möbius transformation* which interchanges 0 and  $a$ . One of the most important properties of the pseudohyperbolic metric is that it is Möbius invariant, or more precisely that

$$\rho(\sigma(z), \sigma(a)) = \rho(z, a) \text{ for all automorphisms } \sigma \text{ of } \mathbb{D}, \quad z, a \in \mathbb{D}.$$

Moreover, the pseudohyperbolic metric is a true metric, it even satisfies a stronger version of the triangle inequality. For  $z, a, b \in \mathbb{D}$  we have

$$\rho(z, a) \leq \frac{\rho(z, b) + \rho(b, a)}{1 + \rho(z, b)\rho(b, a)}.$$

In [10] Domański and Lindström showed that the pseudohyperbolic metric is equivalent to

$$\rho_v(z, p) = \sup\{|f(z)|\tilde{v}(z); f \in B_v^\infty, f(p) = 0\}.$$

2.3. THE MAPPINGS. Let  $\mathcal{C}$  be the set of all bounded composition operators acting on  $H_v^\infty$  endowed with the operator norm. By  $ASM(\mathbb{D})$  we denote the set of all analytic self-maps of  $\mathbb{D}$  equipped with the sup-norm

$$\|\phi\|_\infty := \sup_{z \in \mathbb{D}} |\phi(z)|.$$

Now, we have a look at the following maps

$$\begin{aligned} N &: (ASM(\mathbb{D}), \|\cdot\|_\infty) \rightarrow \mathbb{R}, \quad \phi \mapsto \|C_\phi\| \\ K &: (ASM(\mathbb{D}), \|\cdot\|_\infty) \rightarrow \mathcal{C}, \quad \phi \mapsto C_\phi. \end{aligned}$$

In [16], Pokorný and Shapiro characterized the continuity of these maps in the setting of the classical Hardy space  $H^2$ . Now, we pose the following question:

*For which functions  $\phi \in ASM(\mathbb{D})$  is the map  $N$  resp.  $K$  continuous?*

## 3. CONTINUITY OF THE NORM

LEMMA 1. (Lindström, Wolf [11]) *Let  $v$  be a radial weight satisfying the Lusky condition (L1) such that  $v$  is continuously differentiable with respect to  $|z|$ . Then there is a constant  $C > 0$  such that*

$$\left| \frac{v(p)}{v(z)} - 1 \right| \leq C\rho(p, z)$$

for every  $z, p \in \mathbb{D}$ .

THEOREM 2. *Let  $v$  be a typical weight satisfying the condition (L1) such that  $v$  is continuously differentiable w.r.t.  $|z|$  and  $\phi$  be an analytic self-map of  $\mathbb{D}$ . Then  $K$  is continuous at  $\phi$  if and only if  $C_\phi$  is compact.*

*Proof.* First, we assume that  $K$  is continuous at  $\phi$  and prove that  $C_\phi$  is compact. We do this indirectly and suppose that  $C_\phi$  is not compact. Thus, w.l.o.g. by [6] Theorem 3.3 we may assume that there is a sequence  $(z_k)_k \subset \mathbb{D}$  with  $|\phi(z_k)| \rightarrow 1$  such that  $\lim_{k \rightarrow \infty} \frac{v(z_k)}{v(\phi(z_k))} = \alpha > 0$ . Now, we consider the sequence

$$\phi_n(z) = \left(1 - \frac{1}{n}\right) \phi(z) \text{ for every } z \in \mathbb{D}.$$

Now, by [7] the operator norm of the difference  $C_\phi - C_{\phi_n}$  is given by

$$\|C_\phi - C_{\phi_n}\| = \sup_{z \in \mathbb{D}} \max \left\{ \frac{v(z)}{v(\phi(z))}, \frac{v(z)}{v(\phi_n(z))} \right\} \rho(\phi(z), \phi_n(z)).$$

Thus, we obtain

$$\begin{aligned} \|C_\phi - C_{\phi_n}\| &= \sup_{z \in \mathbb{D}} \max \left\{ \frac{v(z)}{v(\phi(z))}, \frac{v(z)}{v(\phi_n(z))} \right\} \rho(\phi(z), \phi_n(z)) \\ &\geq \lim_{k \rightarrow \infty} \max \left\{ \frac{v(z_k)}{v(\phi(z_k))}, \frac{v(z_k)}{v(\phi_n(z_k))} \right\} \rho(\phi(z_k), \phi_n(z_k)) \\ &\geq \lim_{k \rightarrow \infty} \max \left\{ \frac{v(z_k)}{v(\phi(z_k))}, \frac{v(z_k)}{v(\phi_n(z_k))} \right\} \frac{1}{n} \frac{|\phi(z_k)|}{1 - (1 - 1/n)|\phi(z_k)|^2} \geq \alpha. \end{aligned}$$

Hence  $K$  cannot be continuous at  $\phi$  which is a contradiction.

Conversely, we assume that  $C_\phi$  is compact and have to show that  $K$  is continuous. Since  $C_\phi$  is bounded, by [6] Proposition 2.1 we can find a constant  $M > 0$  such that  $\sup_{z \in \mathbb{D}} \frac{v(z)}{v(\phi(z))} \leq M$ . Now, let  $(\phi_n)_n$  be an arbitrary sequence

of analytic self-maps of  $\mathbb{D}$  such that  $\|\phi - \phi_n\|_\infty \rightarrow 0$ . Fix  $\varepsilon > 0$ . Since  $C_\phi$  is compact, there is  $0 < r_0 < 1$  such that for every  $z \in \mathbb{D}$  with  $|\phi(z)| > r_0$

$$\frac{v(z)}{v(\phi(z))} < \frac{\varepsilon}{2C},$$

where  $C$  denotes the constant in Lemma 1. Moreover, there is  $n_0 \in \mathbb{N}$  with  $\rho(\phi(z), \phi_n(z)) < \frac{\varepsilon}{2MC}$  for every  $z \in \mathbb{D}$  with  $|\phi(z)| \leq r_0$  and every  $n \geq n_0$ . Next, for every  $z \in \mathbb{D}$  and every  $n \geq n_1$  we obtain

$$\begin{aligned} \left| \frac{v(z)}{v(\phi_n(z))} - \frac{v(z)}{v(\phi(z))} \right| &\leq \frac{v(z)}{v(\phi(z))} \left| 1 - \frac{v(\phi(z))}{v(\phi_n(z))} \right| \\ &\leq C \frac{v(z)}{v(\phi(z))} \rho(\phi(z), \phi_n(z)) < \frac{\varepsilon}{2}. \end{aligned}$$

Hence

$$\frac{v(z)}{v(\phi_n(z))} < \frac{v(z)}{v(\phi(z))} + \frac{\varepsilon}{2}$$

for every  $z \in \mathbb{D}$  and every  $n \geq n_0$ . Putting this together for a fixed  $\varepsilon > 0$  we can find  $0 < r_1 < 1$  and  $n_1 \in \mathbb{N}$  such that

1.  $\max \left\{ \frac{v(z)}{v(\phi(z))}, \frac{v(z)}{v(\phi_n(z))} \right\} < \frac{\varepsilon}{2}$  for every  $n \geq n_1$  and every  $z \in \mathbb{D}$  with  $|\phi(z)| > r_1$ ,
2.  $\max \left\{ \frac{v(z)}{v(\phi(z))}, \frac{v(z)}{v(\phi_n(z))} \right\} < R$  for every  $n \geq n_1$  and every  $z \in \mathbb{D}$  with  $|\phi(z)| \leq r_1$ ,
3.  $\rho(\phi(z), \phi_n(z)) < \frac{\varepsilon}{2R}$  for every  $n \geq n_1$  and every  $z \in \mathbb{D}$  with  $|\phi(z)| \leq r_1$ .

Finally, we obtain

$$\begin{aligned} \|C_\phi - C_{\phi_n}\| &= \sup_{z \in \mathbb{D}} \max \left\{ \frac{v(z)}{v(\phi(z))}, \frac{v(z)}{v(\phi_n(z))} \right\} \rho(\phi(z), \phi_n(z)) \\ &\leq \sup_{|\phi(z)| \leq r_1} \max \left\{ \frac{v(z)}{v(\phi(z))}, \frac{v(z)}{v(\phi_n(z))} \right\} \rho(\phi(z), \phi_n(z)) \\ &\quad + \sup_{|\phi(z)| > r_1} \max \left\{ \frac{v(z)}{v(\phi(z))}, \frac{v(z)}{v(\phi_n(z))} \right\} \rho(\phi(z), \phi_n(z)) \\ &\leq R \frac{\varepsilon}{2R} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for every  $n \geq n_1$ . Thus, the claim follows. ■

COROLLARY 3. *Let  $v$  be a typical weight satisfying the condition (L1) such that  $v$  is continuously differentiable with respect to  $|z|$ . Moreover, let  $\phi$  be an analytic self-map of  $\mathbb{D}$ . If  $C_\phi$  is compact, then the map  $N$  is continuous at  $\phi$ .*

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