



On $H_3(1)$ Hankel determinant for certain subclass of analytic functions

D. VAMSHEE KRISHNA^{1,®}, D. SHALINI²

¹ *Department of Mathematics, GIS, GITAM University
Visakhapatnam-530 045, A.P., India*

² *Department of Mathematics, Dr. B. R. Ambedkar University
Srikakulam-532 410, A.P., India*

vamsheekrishna1972@gmail.com, shaliniraj1005@gmail.com

Received February 21, 2019
Accepted September 3, 2019

Presented by Manuel Maestre

Abstract: The objective of this paper is to obtain an upper bound to Hankel determinant of third order for any function f , when it belongs to certain subclass of analytic functions, defined on the open unit disc in the complex plane.

Key words: Analytic function, upper bound, third Hankel determinant, positive real function.

AMS *Subject Class.* (2010): 30C45, 30C50.

1. INTRODUCTION

Let A denotes the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture also called as Coefficient conjecture, which states that for a univalent function its n^{th} - Taylor's coefficient is bounded by n (see [4]). The bounds for the coefficients of these functions give information about their geometric properties. For example, the n^{th} -coefficient gives information about the area where as the second coefficient of functions in the family S yields the growth and distortion properties of the function. A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. The Hankel determinant of f for

® Corresponding author



$q \geq 1$ and $n \geq 1$ was defined by Pommerenke [20], which has been investigated by many authors, as follows.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \quad (1.2)$$

It is worth of citing some of them. Ehrenborg [7] studied the Hankel determinant of exponential polynomials. Noor [18] determined the rate of growth of $H_q(n)$ as $n \rightarrow \infty$ for the functions in S with bounded boundary rotation. The Hankel transform of an integer sequence and some of its properties were discussed by Layman (see [13]). It is observed that $H_2(1)$, the Fekete-Szegő functional is the classical problem settled by Fekete-Szegő [8] is to find for each $\lambda \in [0, 1]$, the maximum value of the coefficient functional, defined by $\phi_\lambda(f) := |a_3 - \lambda a_2^2|$ over the class S and was proved by using Loewner method. Ali [1] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegő functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ when $f^{-1} \in \widetilde{ST}(\alpha)$, the class of strongly starlike functions of order α ($0 < \alpha \leq 1$). In recent years, the research on Hankel determinants has focused on the estimation of $|H_2(2)|$, where

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2,$$

known as the second Hankel determinant obtained for $q = 2$ and $n = 2$ in (1.2). Many authors obtained an upper bound to the functional $|a_2 a_4 - a_3^2|$ for various subclasses of univalent and multivalent analytic functions. It is worth citing a few of them. The exact (sharp) estimates of $|H_2(2)|$ for the subclasses of S namely, bounded turning, starlike and convex functions denoted by \mathcal{R} , S^* and \mathcal{K} respectively in the open unit disc E , that is, functions satisfying the conditions $\operatorname{Re} f'(z) > 0$, $\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > 0$ and $\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > 0$ were proved by Janteng et al. [11, 10] and determined the bounds as $4/9$, 1 and $1/8$ respectively. For the class $S^*(\psi)$ of Ma-Minda starlike functions, the exact bound of the second Hankel determinant was obtained by Lee et al. [15]. Choosing $q = 2$ and $n = p + 1$ in (1.2), we obtain the second Hankel determinant for the p -valent function (see [24]), as follows.

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1} a_{p+3} - a_{p+2}^2,$$

The case $q = 3$ appears to be much more difficult than the case $q = 2$. Very few papers have been devoted to the third order Hankel determinant denoted by $H_3(1)$, obtained for $q = 3$ and $n = 1$ in (1.2), also called as Hankel determinant of third kind, namely

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \quad (a_1 = 1).$$

Expanding the determinant, we have

$$H_3(1) = a_1(a_3a_5 - a_4^2) + a_2(a_3a_4 - a_2a_5) + a_3(a_2a_4 - a_3^2), \quad (1.3)$$

equivalently

$$H_3(1) = H_2(3) + a_2J_2 + a_3H_2(2),$$

where $J_2 = (a_3a_4 - a_2a_5)$ and $H_2(3) = (a_3a_5 - a_4^2)$.

Babalola [2] is the first one, who tried to estimate an upper bound for $|H_3(1)|$ for the classes \mathcal{R} , S^* and \mathcal{K} . As a result of this paper, Raza and Malik [22] obtained an upper bound to the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. Sudharsan et al. [23] derived an upper bound to the third kind Hankel determinant for a subclass of analytic functions. Bansal et al. [3] improved the upper bound for $|H_3(1)|$ for some of the classes estimated by Babalola [2] to some extent. Recently, Zaprawa [25] improved all the results obtained by Babalola [2]. Further, Orhan and Zaprawa [19] obtained an upper bound to the third kind Hankel determinant for the classes S^* and \mathcal{K} functions of order alpha. Very recently, Kowalczyk et al. [12] calculated sharp upper bound to $|H_3(1)|$ for the class of convex functions \mathcal{K} and showed as $|H_3(1)| \leq \frac{4}{135}$, which is far better than the bound obtained by Zaprawa [25]. Lecko et al. [14] determined sharp bound to the third order Hankel determinant for starlike functions of order 1/2. Motivated by the results obtained by different authors mentioned above and who are working in this direction (see [5]), in this paper, we are making an attempt to obtain an upper bound to the functional $|H_3(1)|$ for the function f belonging to the class, defined as follows.

DEFINITION 1.1. A function $f(z) \in A$ is said to be in the class $Q(\alpha, \beta, \gamma)$ with $\alpha, \beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$, if it satisfies the condition that

$$\operatorname{Re} \left\{ \alpha \frac{f(z)}{z} + \beta f'(z) \right\} \geq \gamma, \quad z \in E. \quad (1.4)$$

This class was considered and studied by Zhi- Gang Wang et al. [26].

In obtaining our results, we require a few sharp estimates in the form of lemmas valid for functions with positive real part.

Let \mathcal{P} denotes the class of functions consisting of g , such that

$$g(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (1.5)$$

which are analytic in E and $\operatorname{Re}g(z) > 0$ for $z \in E$. Here g is called the Caratheodory function [6].

LEMMA 1.2. ([9]) *If $g \in \mathcal{P}$, then the sharp estimate $|c_k - \mu c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N} = \{1, 2, 3, \dots\}$, with $n > k$ and $\mu \in [0, 1]$.*

LEMMA 1.3. ([17]) *If $g \in \mathcal{P}$, then the sharp estimate $|c_k - c_k c_{n-k}| \leq 2$, holds for $n, k \in \mathbb{N}$, with $n > k$.*

LEMMA 1.4. ([21]) *If $g \in \mathcal{P}$ then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $g(z) = \frac{1+z}{1-z}$, $z \in E$.*

In order to obtain our result, we refer to the classical method devised by Libera and Zlotkiewicz [16], used by several authors.

2. MAIN RESULT

THEOREM 2.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$, ($\alpha, \beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$) then*

$$|H_3(1)| \leq 4t_1^2 \left[\frac{k_1\alpha^6 + k_2\alpha^5 + k_3\alpha^4\beta + k_4\alpha^3\beta^2 + k_5\alpha^2\beta^3 + k_6\alpha\beta^4 + k_7\beta^5}{(\alpha + 2\beta)^2(\alpha + 3\beta)^3(\alpha + 4\beta)^2(\alpha + 5\beta)} \right],$$

where $k_1 = 2$, $k_2 = 2(18\beta + 1)$, $k_3 = 2(132\beta + 15)$, $k_4 = 2(511\beta + 87)$, $k_5 = (2179\beta + 490)$, $k_6 = 12(203\beta + 56)$, $k_7 = 12(93\beta + 30)$ and $t_1 = (\alpha + \beta - \gamma)$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$. By virtue of Definition 1.1, there exists an analytic function $g \in \mathcal{P}$ in the open unit disc E with $g(0) = 1$ and $\operatorname{Re}\{g(z)\} > 0$ such that

$$\frac{1}{\alpha + \beta - \gamma} \left\{ \alpha \frac{f(z)}{z} + \beta f'(z) - \gamma \right\} = g(z) \quad (2.1)$$

Using the series representation for f and g in (2.1), upon simplification, we obtain

$$\sum_{n=2}^{\infty} (\alpha + n\beta) a_n z^{n-2} = (\alpha + \beta - \gamma) \sum_{n=1}^{\infty} c_n z^{n-1}. \quad (2.2)$$

The coefficient of z^{t-2} , where t is an integer with $t \geq 2$ in (2.2) is given by

$$a_t = \frac{(\alpha + \beta - \gamma)c_{t-1}}{(\alpha + t\beta)}, \quad \text{with } t \geq 2. \quad (2.3)$$

Substituting the values of a_2 , a_3 , a_4 and a_5 from (2.3) in the functional given in (1.3), it simplifies to

$$\begin{aligned} H_3(1) = (\alpha + \beta - \gamma)^2 & \left[\frac{c_2 c_4}{(\alpha + 3\beta)(\alpha + 5\beta)} - \frac{(\alpha + \beta - \gamma)c_2^3}{(\alpha + 3\beta)^3} - \frac{c_3^2}{(\alpha + 4\beta)^2} \right. \\ & \left. - \frac{(\alpha + \beta - \gamma)c_1^2 c_4}{(\alpha + 2\beta)^2(\alpha + 5\beta)} + \frac{2(\alpha + \beta - \gamma)c_1 c_2 c_3}{(\alpha + 2\beta)(\alpha + 3\beta)(\alpha + 4\beta)} \right]. \end{aligned} \quad (2.4)$$

On grouping the terms in the expression (2.4), in order to apply the lemmas, we have

$$\begin{aligned} H_3(1) = t_1^2 & \left[\frac{c_4(c_2 - t_1 c_1^2)}{(\alpha + 2\beta)^2(\alpha + 5\beta)} - \frac{c_3}{(\alpha + 4\beta)^2} \left\{ c_3 - \frac{t_1(\alpha + 4\beta)c_1 c_2}{(\alpha + 2\beta)(\alpha + 3\beta)} \right\} \right. \\ & + \frac{c_2(c_4 - t_1 c_2^2)}{(\alpha + 3\beta)^3} - \frac{c_2}{(\alpha + 3\beta)(\alpha + 4\beta)^2} \left\{ c_4 - \frac{t_1(\alpha + 4\beta)c_1 c_3}{(\alpha + 2\beta)(\alpha + 4\beta)} \right\} \\ & \left. + \frac{(d_1 \alpha^6 + d_2 \alpha^5 + d_3 \alpha^4 \beta + d_4 \alpha^3 \beta^2 + d_5 \alpha^2 \beta^3 + d_6 \alpha \beta^4 + d_7 \beta^5) c_2 c_4}{(\alpha + 2\beta)^2(\alpha + 3\beta)^3(\alpha + 4\beta)^2(\alpha + 5\beta)} \right], \end{aligned} \quad (2.5)$$

with $d_1 = 1$, $d_2 = (18\beta - 1)$, $d_3 = (133\beta - 19)$, $d_4 = 4(129\beta - 35)$, $d_5 = 2(554\beta - 249)$, $d_6 = 8(156\beta - 107)$, $d_7 = 4(144\beta - 143)$ and $t_1 = (\alpha + \beta - \gamma)$.

On applying the triangle inequality in (2.5), we have

$$\begin{aligned} |H_3(1)| & \leq t_1^2 \left[\frac{|c_4|(c_2 - t_1 c_1^2)|}{(\alpha + 2\beta)^2(\alpha + 5\beta)} + \frac{|c_3|}{(\alpha + 4\beta)^2} \left| c_3 - \frac{t_1(\alpha + 4\beta)c_1 c_2}{(\alpha + 2\beta)(\alpha + 3\beta)} \right| \right. \\ & + \frac{|c_2|(c_4 - t_1 c_2^2)|}{(\alpha + 3\beta)^3} + \frac{|c_2|}{(\alpha + 3\beta)(\alpha + 4\beta)^2} \left| c_4 - \frac{t_1(\alpha + 4\beta)c_1 c_3}{(\alpha + 2\beta)(\alpha + 4\beta)} \right| \\ & \left. + \frac{|d_1 \alpha^6 + d_2 \alpha^5 + d_3 \alpha^4 \beta + d_4 \alpha^3 \beta^2 + d_5 \alpha^2 \beta^3 + d_6 \alpha \beta^4 + d_7 \beta^5| |c_2| |c_4|}{(\alpha + 2\beta)^2(\alpha + 3\beta)^3(\alpha + 4\beta)^2(\alpha + 5\beta)} \right]. \end{aligned} \quad (2.6)$$

Upon using the lemmas given in (1.2), (1.3) and (1.4) in the inequality (2.6), it simplifies to

$$|H_3(1)| \leq 4t_1^2 \left[\frac{k_1\alpha^6 + k_2\alpha^5 + k_3\alpha^4\beta + k_4\alpha^3\beta^2 + k_5\alpha^2\beta^3 + k_6\alpha\beta^4 + k_7\beta^5}{(\alpha + 2\beta)^2(\alpha + 3\beta)^3(\alpha + 4\beta)^2(\alpha + 5\beta)} \right], \quad (2.7)$$

with $k_1 = 2$, $k_2 = 2(18\beta + 1)$, $k_3 = 2(132\beta + 15)$, $k_4 = 2(511\beta + 87)$, $k_5 = (2179\beta + 490)$, $k_6 = 12(203\beta + 56)$, $k_7 = 12(93\beta + 30)$ and $t_1 = (\alpha + \beta - \gamma)$. This completes the proof of the theorem. ■

Remark 2.2. For the values $\alpha = 1 - \sigma$, $\beta = \sigma$, $\gamma = 0$, so that $(\alpha + \beta - \gamma) = 1$ in (2.7), we obtain

$$|H_3(1)| \leq 4 \left[\frac{63\sigma^6 + 312\sigma^5 + 411\sigma^4 + 414\sigma^3 + 188\sigma^2 + 44\sigma + 4}{(1 + \sigma)^2(1 + 2\sigma)^3(1 + 3\sigma)^2(1 + 4\sigma)} \right]. \quad (2.8)$$

Remark 2.3. Choosing $\sigma = 1$ in the expression (2.8), it coincides with the result obtained by Zaprawa [25].

ACKNOWLEDGEMENTS

The authors are extremely grateful to the esteemed reviewers for a careful reading of the manuscript and making valuable suggestions leading to a better presentation of the paper.

REFERENCES

- [1] R.M. ALI, Coefficients of the inverse of strongly starlike functions, *Bull. Malays. Math. Sci. Soc.*, (2) **26** (1) (2003), 63–71.
- [2] K.O. BABALOLA, On $H_3(1)$ Hankel determinant for some classes of univalent functions, in “Inequality Theory and Applications 6” (ed. Cho, Kim and Dragomir), Nova Science Publishers, New York, 2010, 1–7.
- [3] D. BANSAL, S. MAHARANA, J.K. PRAJAPAT, Third order Hankel determinant for certain univalent functions, *J. Korean Math. Soc.* **52** (6) (2015), 1139–1148.
- [4] L. DE BRANGES, A proof of Bieberbach conjecture, *Acta Math.* **154** (1-2) (1985), 137–152.
- [5] N.E. CHO, B. KOWALCZYK, O.S. KWON, A. LECKO, Y.J. SIM, The bounds of some determinants for starlike functions of order Alpha, *Bull. Malays. Math. Sci. Soc.* **41** (1) (2018), 523–535.
- [6] P.L. DUREN, “Univalent Functions”, Grundlehren der Mathematischen Wissenschaften, 259, Springer-Verlag, New York, 1983.

- [7] R. EHRENBORG, The Hankel determinant of exponential polynomials, *Amer. Math. Monthly* **107** (6) (2000), 557–560.
- [8] M. FEKETE, G. SZEGÖ, Eine bemerkung uber ungerade schlichte funktionen, *J. Lond. Math. Soc.* **8** (2) (1933), 85–89.
- [9] T. HAYAMI, S. OWA, Generalized Hankel determinant for certain classes, *Int. J. Math. Anal.* **4** (49-52) (2010), 2573–2585.
- [10] A. JANTENG, S.A. HALIM, M. DARUS, Coefficient inequality for a function whose derivative has a positive real part, *J. Inequal. Pure Appl. Math.* **7** (2) (2006), Article 50, 1–5.
- [11] A. JANTENG, S.A. HALIM, M. DARUS, Hankel Determinant for starlike and convex functions, *Int. J. Math. Anal.* **1** (13-16) (2007), 619–625.
- [12] B. KOWALCZYK, A. LECKO, Y.J. SIM, The sharp bound for the Hankel determinant of the Third kind for convex functions, *Bull. Aust. Math. Soc.* **97** (3) (2018), 435–445.
- [13] J.W. LAYMAN, The Hankel transform and some of its properties, *J. Integer Seq.* **4** (1) (2001), Article 01.1.5, 1–11.
- [14] A. LECKO, Y.J. SIM, B. ŚMIAROWSKA, The Sharp Bound of the Hankel Determinant of the Third kind for starlike functions of order $1/2$, *Complex Anal. Oper. Theory* **13** (5) (2019), 2231–2238.
- [15] S.K. LEE, V. RAVICHANDRAN, S. SUPRAMANIAM, Bounds for the second Hankel determinant of certain univalent functions, *J. Inequal. Appl.* (2013), 2013:281, 1–17.
- [16] R.J. LIBERA, E.J. ZLOTKIEWICZ, Coefficient bounds for the inverse of a function with derivative in \mathcal{P} , *Proc. Amer. Math. Soc.* **87** (2) (1983), 251–257.
- [17] A.E. LIVINGSTON, The coefficients of multivalent close-to-convex functions, *Proc. Amer. Math. Soc.* **21** (1969), 545–552.
- [18] K.I. NOOR, Hankel determinant problem for the class of functions with bounded boundary rotation, *Rev. Roumaine Math. Pures Appl.* **28** (8) (1983), 731–739.
- [19] H. ORHAN, P. ZAPRAWA, Third Hankel determinants for starlike and convex functions of order α , *Bull. Korean Math. Soc.* **55** (1) (2018), 165–173.
- [20] CH. POMMERENKE, On the coefficients and Hankel determinants of univalent functions, *J. Lond. Math. Soc.* **s1-41** (1) (1966), 111–122.
- [21] CH. POMMERENKE, G. JENSEN “Univalent Functions”, *Studia Mathematica/Mathematische Lehrbücher 25*, Vandenhoeck und Ruprecht, Gottingen, 1975.
- [22] M. RAZA, S.N. MALIK, Upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli, *J. Inequal. Appl.* (2013), 2013:412, 1–8.

- [23] T.V. SUDHARSAN, S.P. VIJAYALAKSHMI, B.A. STEPHEN, Third Hankel determinant for a subclass of analytic functions, *Malaya J. Math.* **2**(4) (2014), 438–444.
- [24] D. VAMSHEE KRISHNA, T. RAMREDDY, Coefficient inequality for certain p -valent analytic functions, *Rocky Mountain J. Math.* **44**(6) (2014), 1941–1959.
- [25] P. ZAPRAWA, Third Hankel determinants for subclasses of univalent functions, *Mediterr. J. Math.* **14**(1) (2017), Article 19, 1–10.
- [26] ZHI-GANG WANG, CHUN-YI GAO, SHAO-MOU YUAN, On the univalence of certain analytic functions, *J. Inequal. Pure Appl. Math.* **7**(1) (2006), Article 9, 1–4.