EXTRACTA MATHEMATICAE
Vol. **35**, Num. 1 (2020), 69 – 97
doi:10.17398/2605-5686.35.1.69
Available online April 29, 2020

Hom-Jordan and Hom-alternative bimodules

S. Attan, H. Hounnon, B. Kpamegan

Département de Mathématiques, Université d'Abomey-Calavi
01 BP 4521, Cotonou 01, Bénin

syltane2010@yahoo.fr , hi.hounnon@fast.uac.bj , kpamegan_bernadin@yahoo.fr

Received March 21, 2019 and, in revised form, April 8, 2020 Presented by Consuelo Martínez
Accepted April 15, 2020

Abstract: In this paper, Hom-Jordan and Hom-alternative bimodules are introduced. It is shown that Jordan and alternative bimodules are twisted via endomorphisms into Hom-Jordan and Hom-alternative bimodules respectively. Some relations between Hom-associative bimodules, Hom-Jordan and Hom-alternative bimodules are given.

Key words: Bimodules, alternative algebras, Jordan algebras, Hom-alternative algebras, Hom-Jordan algebras, Hom-associative algebras.

AMS *Subject Class.* (2010): 17A30, 17B10, 17C50, 17D05.

1. Introduction

Algebras where the identities defining the structure are twisted by a homomorphism are called Hom-algebras. They have been intensively investigated in the literature recently. Hom-algebra started from Hom-Lie algebras introduced and discussed in [6, 10, 11, 12], motivated by quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras. Hom-associative algebras were introduced in [15] while Hom-alternative and Hom-Jordan algebras are introduced in [14], [23] as twisted generalizations of alternative and Jordan algebra respectively. The reader is referred to [20] for applications of alternative algebras to projective geometry, buildings, and algebraic groups and to [4, 9, 16, 19] for discussions about the important roles of Jordan algebras in physics, especially quantum mechanics.

The anti-commutator of a Hom-alternative algebra gives rise to a Hom-Jordan algebra [23]. Starting with a Hom-alternative algebra $(A, \cdot, \alpha)$, it is known that the Jordan product

$$x * y = \frac{1}{2}\,(x \cdot y + y \cdot x)$$

gives a Hom-Jordan algebra $A^+ = (A, *, \alpha)$. In other words, Hom-alternative algebras are Hom-Jordan-admissible [23].

ISSN: 0213-8743 (print), 2605-5686 (online)

The notion of bimodule for a class of algebras defined by multilinear identities has been introduced by Eilenberg [3]. If $\mathcal{H}$ is in the class of associative algebras or in the one of Lie algebras then this notion is the familiar one for which we are in possession of well-worked theories. The study of bimodule (or representation) of Jordan algebras was initiated by N. Jacobson [7]. Subsequently the alternative case was considered by Schafer [17].

Modules over an ordinary algebra has been extended to the ones of Hom-algebras in many works [2, 18, 21, 22].

The aim of this paper is to introduce Hom-alternative bimodules and Hom-Jordan bimodules and to discuss about some findings. The paper is organized as follows. In section two, we recall basic notions related to Hom-algebras and modules over Hom-associative algebras. Section three is devoted to the introduction of Hom-alternative bimodules . Proposition 3.7 shows that from a given Hom-alternative bimodule, a sequence of this kind of bimodules can be obtained. Theorem 3.8 establishes that, an alternative bimodule gives rise to a bimodule over the corresponding twisted algebra. It is also proved that a direct sum of a Hom-alternative algebra and a module over this Hom-algebra is again a Hom-alternative algebra (Theorem 3.11). In section four, we introduce Hom-Jordan modules and attest similar results as in the previous section. Furthermore, it is proved that a Hom-Jordan special left and right module, with an additional condition, has a bimodule structure over this Hom-algebra (Theorem 4.10). Finally, Proposition 4.12 shows that a bimodule over a Hom-associative algebra has a bimodule structure over its plus Hom-algebra. All vector spaces are assumed to be over a fixed ground field $\mathbb{K}$ of characteristic 0.

2. Preliminaries

We recall some basic notions introduced in [6, 15, 21] related to Hom-algebras and while dealing of any binary operation we will use juxtaposition in order to reduce the number of braces, i.e., e.g., for "$\cdot$", $xy \cdot \alpha(z)$ means $(x \cdot y) \cdot \alpha(z)$. Also, for the map $\mu : A^{\otimes 2} \to A$, we will write sometimes $\mu(a \otimes b)$ as $\mu(a, b)$ or ab for $a, b \in A$ and if V is another vector space, $\tau_1 : A \otimes V \to V \otimes A$ (resp. $\tau_2 : V \otimes A \to A \otimes V$) denote the twist isomorphism $\tau_1(a \otimes v) = v \otimes a$ (resp. $\tau_2(v \otimes a) = a \otimes v$).

Definition 2.1. A Hom-module is a pair (M, α_M) consisting of a $\mathbb{K}$-module M and a linear self-map $\alpha_M : M \to M$. A morphism $f : (M, \alpha_M) \to (N, \alpha_N)$ of Hom-modules is a linear map $f : M \to N$ such that $f \circ \alpha_M = \alpha_N \circ f$.

Definition 2.2. ([15, 21]) A Hom-algebra is a triple (A, μ_A, α_A) in which (A, α_A) is a Hom-module, $\mu : A^{\otimes 2} \to A$ is a linear map. The Hom-algebra (A, μ, α) is said to be multiplicative if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ (multiplicativity). A morphism $f : (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B)$ of Hom-algebras is a morphism of the underlying Hom-modules such that $f \circ \mu_A = \mu_B \circ f^{\otimes 2}$.

An important class of Hom-algebras that is considered here is the one of Hom-alternative algebras. These algebras have been introduced in [14] and more studied in [23].

Definition 2.3. Let (A, μ, α) be a Hom-algebra.

(i) The Hom-associator of A is the linear map $as_A : A^{\otimes 3} \to A$ defined as $as_A = \mu \circ (\mu \otimes \alpha - \alpha \otimes \mu)$. A multiplicative Hom-algebra (A, μ, α) is said to be Hom-associative algebra if $as_A = 0$.

(ii) A Hom-alternative algebra [14] is a multiplicative Hom-algebra (A, μ, α) that satisfies

$$as_A(x, x, y) = 0 \qquad \text{(left Hom-alternativity)}\,, \tag{1}$$

$$as_A(x, y, y) = 0 \qquad \text{(right Hom-alternativity)} \tag{2}$$

for all $x, y \in A$.

(iii) Let (A, μ, α) be a Hom-alternative algebra. A Hom-subalgebra of (A, μ, α) is a linear subspace H of A, which is closed for the multiplication μ and invariant by α, that is, $\mu(x, y) \in H$ and $\alpha(x) \in H$ for all $x, y \in H$. If furthermore $\mu(a, b) \in H$ and $\mu(b, a) \in H$ for all $(a, b) \in A \times H$, then H is called a two-sided Hom-ideal of A.

Now, we prove:

Proposition 2.4. *Let (A, μ, α) be a Hom-alternative algebra and I be a two-sided Hom-ideal of (A, μ, α). Then $(A/I, \bar{\mu}, \bar{\alpha})$ is a Hom-alternative algebra where $\bar{\mu}(\bar{x}, \bar{y}) = \overline{\mu(x, y)}$ and $\bar{\alpha}(\bar{x}) = \overline{\alpha(x)}$ for all $\bar{x}, \bar{y} \in A/I$.*

Proof. First, note that the multiplicativity of $\bar{\mu}$ with respect to $\bar{\alpha}$ follows from the one of μ with respect to α. Next, pick $\bar{x}, \bar{y} \in A/I$. Then the left Hom-alternativity (1) in $(A/I, \bar{\mu}, \bar{\alpha})$ is proved as follows

$$\begin{aligned} as_{A/I}(\bar{x}, \bar{x}, \bar{y}) &= \bar{\mu}(\bar{\mu}(\bar{x}, \bar{x}), \bar{\alpha}(\bar{y})) - \bar{\mu}(\bar{\alpha}(\bar{x}), \bar{\mu}(\bar{x}, \bar{y}) \\ &= \overline{\mu(\mu(x, x)\alpha(y)) - \mu(\alpha(x), \mu(x, y))} = \overline{as_A(x, x, y))} = \bar{0}\,. \end{aligned}$$

Similarly, we get (2) and therefore $(A/I, \bar{\mu}, \bar{\alpha})$ is a Hom-alternative algebra. ∎

As Hom-alternative algebras, Hom-Jordan algebras are fundamental objects of this paper. They appear as cousins of Hom-alternative algebras and these two Hom-algebras are related as Jordan and alternative algebras.

DEFINITION 2.5. ([23]) (i) A Hom-Jordan algebra is a multiplicative Hom-algebra (A, μ, α) such that $\mu \circ \tau = \mu$ (commutativity of μ) and the so-called Hom-Jordan identity holds

$$as_A(\mu(x, x,), \alpha(y), \alpha(x)) = 0, \forall\ (x, y) \in A^2 \tag{3}$$

where, $\tau : A^{\otimes 2} \to A^{\otimes 2},\ \tau(a \otimes b) = b \otimes a$, is the twist isomorphism.

(ii) Let (A, μ, α) be a Hom-Jordan algebra. A Hom-subalgebra of (A, μ, α) is a linear subspace H of A, which is closed for the multiplication μ and invariant by α, that is, $\mu(x, y) \in H$ and $\alpha(x) \in H$ for all $x, y \in H$. If furthermore $\mu(a, b) \in H$ for all $(a, b) \in A \times H$, then H is called a two-sided Hom-ideal (or simply Hom-ideal) of A [5].

Similarly as a Hom-alternative algebra case, if H is a Hom-ideal of a Hom-Jordan algebra (A, μ, α), then $(A/H, \bar{\mu}, \bar{\alpha})$ is a Hom-Jordan algebra where $\bar{\mu}(\bar{x}, \bar{y}) = \overline{\mu(x, y)}$ for all $\bar{x}, \bar{y} \in A/H$ and $\bar{\alpha} : A/H \to A/H$ is naturally induced by α, inherits a Hom-Jordan algebra structure, which is named quotient Hom-Jordan algebra.

Remark 2.6. In [14] Makhlouf defined a Hom-Jordan algebra as a commutative multiplicative Hom-algebra satisfying $as_A(x^2, y, \alpha(x)) = 0$, which becomes the identity (3) if y is replaced by $\alpha(y)$.

The proof of the following result can be found in [23] where the product $*$, differs from the one given here by a factor of $\frac{1}{2}$.

PROPOSITION 2.7. *Let (A, μ, α) be a Hom-alternative algebra. Then $A^+ = (A, *, \alpha)$ is a Hom-Jordan algebra where $x * y = xy + yx$ for all $x, y \in A$.*

EXAMPLE 2.8. From the eight-dimensional Hom-alternative algebra $O_\alpha = (O, \mu_\alpha, \alpha)$ with basis $\{e_0, e_{1,2}, e_3, e_4, e_5, e_6, e_7\}$ [23, Example 3.19], constructed from the octonion algebra which is an eight-dimensional alternative algebra, we obtain, the Hom-Jordan algebra $O^+_\alpha = (O, * = \mu_\alpha + \mu_\alpha \circ \tau, \alpha)$ where the non zero products are: $e_0 * e_0 = 2e_0$, $e_0 * e_1 = e_1 * e_0 = 2e_5$, $e_0 * e_2 = e_2 * e_0 = 2e_6$, $e_0 * e_3 = e_3 * e_0 = 2e_7$, $e_0 * e_4 = e_4 * e_0 = 2e_1$, $e_0 * e_5 = e_5 * e_0 = 2e_2$, $e_0 * e_6 = e_6 * e_0 = 2e_3$, $e_0 * e_7 = e_7 * e_0 = 2e_4$, $e_1 * e_1 = e_2 * e_2 = e_3 * e_3 =$

$e_4 * e_4 = e_5 * e_5 = e_6 * e_6 = e_7 * e_7 = -2e_0$ and the twisting map α is given by $\alpha(e_0) = e_0$, $\alpha(e_1) = e_5$, $\alpha(e_2) = e_6$, $\alpha(e_3) = e_7$, $\alpha(e_4) = e_1$, $\alpha(e_5) = e_2$, $\alpha(e_6) = e_3$, $\alpha(e_7) = e_4$.

A. Makhlouf proved that the plus algebra of any Hom-associative algebra is a Hom-Jordan algebra as defined in [14]. Here, we prove the same result for the Hom-Jordan algebra as defined in [23] (see also Definition 2.5 above).

PROPOSITION 2.9. *Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. Then $A^+ = (A, *, \alpha)$ is a Hom-Jordan algebra where $x * y = xy + yx$ for all $x, y \in A$.*

Proof. The commutativity of $*$ is obvious. We compute the Hom-Jordan identity as follows:

$$\begin{aligned}
&as_{A^+}\big(x^2, \alpha(x), \alpha(y)\big)\\
&\quad = (x^2 * \alpha(y)) * \alpha^2(x) - \alpha(x^2) * (\alpha(y) * \alpha(x))\\
&\quad = (x^2 \cdot \alpha(y)) \cdot \alpha^2(x) + (\alpha(y) \cdot x^2) \cdot \alpha^2(x) + \alpha^2(x) \cdot (x^2 \cdot \alpha(y))\\
&\qquad + \alpha^2(x) \cdot (\alpha(y) \cdot x^2) - \alpha(x^2) \cdot (\alpha(y) \cdot \alpha(x)) - \alpha(x^2) \cdot (\alpha(x) \cdot \alpha(y))\\
&\qquad - (\alpha(y) \cdot \alpha(x)) \cdot \alpha(x^2) - (\alpha(x) \cdot \alpha(y)) \cdot \alpha(x^2) \ \text{(by a direct computation)}\\
&\quad = (\alpha(y) \cdot x^2) \cdot \alpha^2(x) + \alpha^2(x) \cdot (x^2 \cdot \alpha(y)) - \alpha(x^2) \cdot (\alpha(x) \cdot \alpha(y))\\
&\qquad - (\alpha(y) \cdot \alpha(x)) \cdot \alpha(x^2) \quad \text{(by the Hom-associativity)}\\
&\quad = (\alpha(y) \cdot x^2) \cdot \alpha^2(x) + \alpha^2(x) \cdot (x^2 \cdot \alpha(y)) - (\alpha(x) \cdot \alpha(x)) \cdot \alpha(x \cdot y)\\
&\qquad - \alpha(yx) \cdot (\alpha(x) \cdot \alpha(x)) \quad \text{(by the multiplicativity)}\\
&\quad = (\alpha(y) \cdot x^2) \cdot \alpha^2(x) + \alpha^2(x) \cdot (x^2 \cdot \alpha(y)) - \alpha^2(x) \cdot (\alpha(x) \cdot (x \cdot y))\\
&\qquad - ((yx) \cdot \alpha(x)) \cdot \alpha^2(x) \quad \text{(by the Hom-associativity)}\\
&\quad = 0 \quad \text{(by the Hom-associativity)}\,.
\end{aligned}$$

Then $A^+ = (A, *, \alpha)$ is a Hom-Jordan algebra. ∎

EXAMPLES 2.10. (i) Consider the three-dimensional Hom-associative algebra $\mathcal{A} = (A, \mu_A, \alpha_A)$ over $\mathbb{K}$ with basis (e_1, e_2, e_3) defined by $\mu_A(e_1, e_1) = e_1$, $\mu_A(e_2, e_2) = e_2$, $\mu_A(e_3, e_3) = e_1$, $\mu_A(e_1, e_3) = \mu_A(e_3, e_1) = -e_3$ and $\alpha_A(e_1) = e_1$, $\alpha_A(e_3) = -e_3$ (see [24, Theorem 3.12], Hom-algebra A'^3_3). Using the product $*$ in Proposition 2.9, the triple $\mathcal{A}^+ = (A, *, \alpha_A)$ is Hom-Jordan algebra where, $e_1 * e_1 = 2e_1$, $e_2 * e_2 = 2e_2$, $e_3 * e_3 = 2e_1$, $e_1 * e_3 = e_3 * e_1 = -2e_3$.

(ii) From the three-dimensional Hom-associative algebra $\mathcal{B} = (B, \mu_B, \alpha_B)$ over $\mathbb{K}$ with basis (e_1, e_2, e_3) defined by $\mu_B(e_1, e_1) = e_1$, $\mu_B(e_2, e_2) = e_1$, $\mu_B(e_3, e_3) = e_3$, $\mu_B(e_1, e_2) = \mu_B(e_2, e_1) = -e_2$ and $\alpha_B(e_1) = e_1$, $\alpha_B(e_2) = -e_2$ (see [24, Theorem 3.12], Hom-algebra A'^3_5). Then the triple $\mathcal{B}^+ = (B, *, \alpha_B)$ is a Hom-Jordan algebra, where "$*$" is the product in Proposition 2.9 and $e_1 * e_1 = 2e_1$, $e_2 * e_2 = 2e_1$, $e_3 * e_3 = 2e_3$, $e_1 * e_2 = e_2 * e_1 = -2e_2$.

Let us consider the following definitions which will be used in next sections.

Definition 2.11. Let (A, μ, α_A) be any Hom-algebra.

(i) A Hom-module (V, α_V) is called an A-bimodule if it comes equipped with a left and a right structure maps on V that is morphisms $\rho_l : A \otimes V \to V$, $a \otimes v \mapsto a \cdot v$ and $\rho_r : V \otimes A \to V$, $v \otimes a \mapsto v \cdot a$ of Hom-modules respectively.

(ii) A morphism $f : (V, \alpha_V, \rho_l, \rho_r) \to (W, \alpha_W, \rho'_l, \rho'_r)$ of A-bimodules is a morphism of the underlying Hom-modules such that

$$f \circ \rho_l = \rho'_l \circ (Id_A \otimes f) \qquad \text{and} \qquad f \circ \rho_r = \rho'_r \circ (f \otimes Id_A).$$

(iii) Let (V, α_V) be an A-bimodule with structure maps ρ_l and ρ_r. Then the module Hom-associator of V is a trilinear map $as_{A,V}$ defined as:

$$\begin{aligned} as_{A,V} \circ Id_{V \otimes A \otimes A} &= \rho_r \circ (\rho_r \otimes \alpha_A) - \rho_l \circ (\alpha_V \otimes \mu), \\ as_{A,V} \circ Id_{A \otimes V \otimes A} &= \rho_r \circ (\rho_l \otimes \alpha_A) - \rho_l \circ (\alpha_A \otimes \rho_r), \\ as_{A,V} \circ Id_{A \otimes A \otimes V} &= \rho_l \circ (\mu \otimes \alpha_V) - \rho_l \circ (\alpha_A \otimes \rho_l). \end{aligned}$$

Remark 2.12. The module Hom-associator given above is a generalization of the one given in [2].

Now, let consider the following notion for Hom-associative algebras.

Definition 2.13. Let (A, μ, α_A) be a Hom-associative algebra and (M, α_M) be a Hom-module.

(i) A left Hom-associative A-module structure on M consists of a morphism $\rho : A \otimes M \to M$ of Hom-modules, such that

$$\rho \circ (\alpha_A \otimes \rho) = \rho \circ (\mu \otimes \alpha_M) \tag{4}$$

(ii) A right Hom-associative A-module structure on M consists of a morphism $\rho : M \otimes A \to M$ of Hom-modules, such that

$$\rho \circ (\alpha_M \otimes \mu) = \rho \circ (\rho \otimes \alpha_A) \tag{5}$$

(iii) A Hom-associative A-bimodule structure on M consists of two structure maps $\rho_l : A \otimes M \to M$ and $\rho_r : M \otimes A \to M$ such that (M, α_M, ρ_l) is a left A-module, (M, α_M, ρ_r) is a right A-module and that the following Hom-associativity (or operator commutativity) condition holds:

$$\rho_l \circ (\alpha_A \otimes \rho_r) = \rho_r \circ (\rho_l \otimes \alpha_A) \tag{6}$$

Remark 2.14. Actually, left Hom-associative A-module, right Hom-associative A-module and Hom-associative A-bimodule have been already introduced in [21, 22] where they are called left A-module, right A-module and A-bimodule respectively. The expressions, used in Definition 2.13 for these notions, are motivated by the unification of our terminologies.

3. Hom-alternative bimodules

In this section, we give the definition of Hom-alternative (bi)modules. We prove that from a given Hom-alternative bimodule, a sequence of this kind of bimodules can be constructed. It is also proved that a direct sum of a Hom-alternative algebra and a bimodule over this Hom-algebra is a Hom-alternative algebra called a split null extension of the considered Hom-algebra.

First, we start by the following notion, due to [2], where it is called a module over a left (resp. right) Hom-alternative algebra. However, we call it a Hom-alternative left (resp. right) module in this paper.

Definition 3.1. Let (A, μ, α_A) be a Hom-alternative algebra.

(i) A left Hom-alternative A-module is a Hom-module (V, α_V) with a left structure map $\rho_l : A \otimes V \to V$, $a \otimes v \mapsto a \cdot v$ such that

$$as_{A,V}(x, y, v) = -as_{A,V}(y, x, v) \qquad \text{for all } x, y \in A \text{ and } v \in V.$$

(ii) A right Hom-alternative A-module is a Hom-module (V, α_V) with a right structure map $\rho_r : V \otimes A \to V$, $v \otimes a \mapsto v \cdot a$ such that

$$as_{A,V}(v, x, y) = -as_{A,V}(v, y, x) \qquad \text{for all } x, y \in A \text{ and } v \in V.$$

Now, as a generalization of alternative bimodules [8, 17], one has:

DEFINITION 3.2. Let (A, μ, α_A) be a Hom-alternative algebra. A Hom-alternative A-bimodule is a Hom-module (V, α_V) with a (left) structure map $\rho_l : A \otimes V \to V$, $a \otimes v \mapsto a \cdot v$ and a (right) structure map $\rho_r : V \otimes A \to V$, $v \otimes a \mapsto v \cdot a$ such that the following equalities hold:

$$as_{A,V}(a, v, b) = -as_{A,V}(v, a, b) = as_{A,V}(b, a, v) = -as_{A,V}(a, b, v) \qquad (7)$$

for all $(a, b, v) \in A^{\times 2} \times V$.

Remarks 3.3. (i) The relation (7) is equivalent to

$$as_{A,V}(a, v, b) = -as_{A,V}(v, a, b) = as_{A,V}(b, a, v) = -as_{A,V}(b, v, a)$$

or since the field's characteristic is 0 to

$$as_{A,V}(a, v, b) = -as_{A,V}(v, a, b) = as_V(b, a, v) \text{ and } as_{A,V}(a, a, v) = 0\,.$$

(ii) If $\alpha_A = Id_A$ and $\alpha_V = Id_V$ then V is the so-called alternative bimodule for the alternative algebra (A, μ) [8, 17].

EXAMPLES 3.4. Here are some examples of Hom-alternative A-bimodules.

(i) Let (A, μ, α_A) be a Hom-alternative algebra. Then (A, α_A) is a Hom-alternative A-bimodule where the structure maps are $\rho_l(a, b) = \mu(a, b)$ and $\rho_r(a, b) = \mu(b, a)$. More generally, if B is a two-sided Hom-ideal of (A, μ, α_A), then (B, α_A) is a Hom-alternative A-bimodule where the structure maps are $\rho_l(a, x) = \mu(a, x)$ and $\rho_r(x, b) = \mu(x, b)$ for all $x \in B$ and $(a, b) \in A^{\times 2}$.

(ii) If (A, μ) is an alternative algebra and M is an alternative A-bimodule [8] in the usual sense, then (M, Id_M) is a Hom-alternative $\mathbb{A}$-bimodule where $\mathbb{A} = (A, \mu, Id_A)$ is a Hom-alternative algebra.

(iii) If $f : (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B)$ is a surjective morphism of Hom-alternative algebras, then (B, α_B) becomes a Hom-alternative A-bimodule via f, i.e, the structure maps are defined as $\rho_l : (a, b) \mapsto \mu_B(f(a), b)$ and $\rho_r : (b, a) \mapsto \mu_B(b, f(a))$ for all $(a, b) \in A \times B$. Indeed one can remark that $as_{A,B} \circ (Id_A \otimes f \otimes Id_A) = f \circ as_A$.

In order to give another example of Hom-alternative bimodules , let us consider the following

DEFINITION 3.5. An abelian extension of Hom-alternative algebras is a short exact sequence of Hom-alternative algebras

$$0 \to (V, \alpha_V) \xrightarrow{i} (A, \mu_A, \alpha_A) \xrightarrow{\pi} (B, \mu_B, \alpha_B) \to 0$$

where (V, α_V) is a trivial Hom-alternative algebra, i and π are morphisms of Hom-algebras. Furthermore, if there exists a morphism $s : (B, \mu_B, \alpha_B) \to (A, \mu_A, \alpha_A)$ such that $\pi \circ s = id_B$ then the abelian extension is said to be split and s is called a section of π.

EXAMPLE 3.6. Given an abelian extension as in the previous definition, the Hom-module (V, α_V) inherits a structure of a Hom-alternative B-bimodule and the actions of the Hom-algebra (B, μ_B, α_B) on V are as follows. For any $x \in B$, there exist $\tilde{x} \in A$ such that $x = \pi(\tilde{x})$. Let x acts on $v \in V$ by $x \cdot v := \mu_A(\tilde{x}, i(v))$ and $v \cdot x := \mu_A(i(v), \tilde{x})$. These are well-defined, as another lift $\tilde{x'}$ of x is written $\tilde{x'} = \tilde{x} + v'$ for some $v' \in V$ and thus $x \cdot v = \mu_A(\tilde{x}, i(v)) = \mu_A(\tilde{x'}, i(v))$ and $v \cdot x = \mu_A(i(v), \tilde{x}) = \mu_A(i(v), \tilde{x'})$ because V is trivial. The actions property follow from the Hom-alternativity identity. In case these actions of B on V are trivial, one speaks of a central extension.

The following result describes a sequence of Hom-alternative bimodules by twisting the structure maps of a given bimodule over this Hom-algebra.

PROPOSITION 3.7. *Let (A, μ, α_A) be a Hom-alternative algebra and (V, α_V) be a Hom-alternative A-bimodule with the structure maps ρ_l and ρ_r. Then the maps*

$$\rho_l^{(n)} = \rho_l \circ (\alpha_A^n \otimes Id_V)$$
$$\rho_r^{(n)} = \rho_r \circ (Id_V \otimes \alpha_A^n)$$

give the Hom-module (V, α_V) the structure of a Hom-alternative A-bimodule that we denote by $V^{(n)}$

Proof. It is clear that $\rho_l^{(n)}$ and $\rho_r^{(n)}$ are structure maps on $V^{(n)}$. Next, observe that for all $x, y \in A$ and $v \in V$,

$$\begin{aligned} as_{A,V^{(n)}}(x, v, y) &= \rho_r^{(n)}(\rho_l^{(n)}(x, v), \alpha_A(y)) - \rho_l^{(n)}(\alpha_A(x), \rho_r^{(n)}(v, y)) \\ &= \rho_r(\rho_l(\alpha_A^n(x), v), \alpha_A^{n+1}(y)) - \rho_l(\alpha_A^{n+1}(x), \rho_r(v, \alpha_A^n(y)) \\ &= as_{A,V}(\alpha_A^n(x), v, \alpha_A^n(y)) \end{aligned}$$

and similarly

$$as_{A,V^{(n)}}(v,x,y) = as_{A,V}(v,\alpha_A^n(x),\alpha_A^n(y))\,,$$
$$as_{A,V^{(n)}}(y,x,v) = as_{A,V}(\alpha_A^n(y),\alpha_A^n(x),v)\,,$$
$$as_{A,V^{(n)}}(x,y,v) = as_{A,V}(\alpha_A^n(x),\alpha_A^n(y),v)\,.$$

Therefore, equalities of (7) in $V^{(n)}$ derive from the one in V. ∎

We know that alternative algebras can be deformed into Hom-alternative algebras via an endomorphism. The following result shows that alternative bimodules can be deformed into Hom-alternative bimodules via an endomorphism. This provides a large class of examples of Hom-alternative bimodules.

THEOREM 3.8. *Let (A,μ) be an alternative algebra, V be an alternative A-bimodule with the structure maps ρ_l and ρ_r, α_A be an endomorphism of the alternative algebra A and α_V be a linear self-map of V such that $\alpha_V \circ \rho_l = \rho_l \circ (\alpha_A \otimes \alpha_V)$ and $\alpha_V \circ \rho_r = \rho_r \circ (\alpha_V \otimes \alpha_A)$.*

Write A_{α_A} for the Hom-alternative algebra $(A,\mu_{\alpha_A},\alpha_A)$ and V_{α_V} for the Hom-module (V,α_V). Then the maps

$$\tilde{\rho}_l = \alpha_V \circ \rho_l \qquad \text{and} \qquad \tilde{\rho}_r = \alpha_V \circ \rho_r$$

give the Hom-module V_{α_V} the structure of a Hom-alternative A_{α_A}-bimodule.

Proof. Trivially, $\tilde{\rho}_l$ and $\tilde{\rho}_r$ are structure maps on V_{α_V}. The proof of (7) for V_{α_V} follows directly by the fact that $as_{A,V_{\alpha_V}} = \alpha_V^2 \circ as_{A,V}$ and the relation (7) in V. ∎

COROLLARY 3.9. *Let (A,μ) be an alternative algebra, V be an alternative A-bimodule with the structure maps ρ_l and ρ_r, α_A an endomorphism of the alternative algebra A and α_V be a linear self-map of V such that $\alpha_V \circ \rho_l = \rho_l \circ (\alpha_A \otimes \alpha_V)$ and $\alpha_V \circ \rho_r = \rho_r \circ (\alpha_V \otimes \alpha_A)$.*

Write A_{α_A} for the Hom-alternative algebra $(A,\mu_{\alpha_A},\alpha_A)$ and V_{α_V} for the Hom-module (V,α_V). Then the maps

$$\tilde{\rho_l}^{(n)} = \rho_l \circ \left(\alpha_A^{n+1} \otimes \alpha_V\right) \qquad \text{and} \qquad \tilde{\rho_r}^{(n)} = \rho_r \circ \left(\alpha_V \otimes \alpha_A^{n+1}\right)$$

give the Hom-module V_α the structure of a Hom-alternative A_{α_A}-bimodule for each $n \in \mathbb{N}$.

LEMMA 3.10. *Let (A, μ, α_A) be a Hom-alternative algebra and (V, α_V) be a Hom-alternative A-bimodule with the structure maps ρ_l and ρ_r. Then the following relation*

$$as_{A,V}(v, a, a) = 0 \tag{8}$$

holds for all $a \in A$ and $v \in V$.

Proof. Using (7), for all $(a, b) \in A^{\times 2}$ and $v \in V$ we have $-as_{A,V}(v, a, b) = as_{A,V}(a, v, b)$ and $as_{A,V}(v, b, a) = -as_{A,V}(a, b, v)$. Moreover again from (7), we get $as_{A,V}(a, v, b) = -as_{A,V}(a, b, v)$ and then $-as_{A,V}(v, a, b) = as_{A,V}(v, b, a)$. It follows that $as_{A,V}(v, a, a) = 0$ since the field $\mathbb{K}$ is of characteristic 0. ∎

The following result shows that a direct sum of a Hom-alternative algebra and a bimodule over this Hom-algebra, is still a Hom-alternative, called the split null extension determined by the given bimodule.

THEOREM 3.11. *Let (A, μ, α_A) be a Hom-alternative algebra and (V, α_V) be a Hom-alternative A-bimodule with the structure maps ρ_l and ρ_r. Defining on $A \oplus V$ the bilinear map $\tilde{\mu} : (A \oplus V)^{\otimes 2} \to A \oplus V$, $\tilde{\mu}(a+m, b+n) := ab + a \cdot n + m \cdot b$ and the linear map $\tilde{\alpha} : A \oplus V \to A \oplus V$, $\tilde{\alpha}(a+m) := \alpha_A(a) + \alpha_V(m)$, then $E = (A \oplus V, \tilde{\mu}, \tilde{\alpha})$ is a Hom-alternative algebra.*

Proof. The multiplicativity of $\tilde{\alpha}$ with respect to $\tilde{\mu}$ follows from the one of α with respect to μ and the fact that ρ_l and ρ_r are morphisms of Hom-modules. Next

$$\begin{aligned}
as_E(a + m,&a + m, b + n)\\
&= \tilde{\mu}(\tilde{\mu}(a + m, a + m), \tilde{\alpha}(b + n)) - \tilde{\mu}(\tilde{\alpha}(a + m), \tilde{\mu}(a + m, b + n))\\
&= \tilde{\mu}(a^2 + a \cdot m + m \cdot a, \alpha_A(b) + \alpha_V(n))\\
&\quad - \tilde{\mu}(\alpha_A(a) + \alpha_V(m), ab + a \cdot n + m \cdot b)\\
&= a^2\alpha_A(b) + a^2 \cdot \alpha_V(n) + (a \cdot m) \cdot \alpha_A(b) + (m \cdot a) \cdot \alpha_A(b)\\
&\quad - \alpha_A(a)(ab) - \alpha_A(a) \cdot (a \cdot n) - \alpha_A(a) \cdot (m \cdot b) - \alpha_V(m) \cdot (ab)\\
&= \underbrace{as_A(a, a, b)}_{0} + \underbrace{as_V(a, a, n)}_{0} + \underbrace{as_{A,V}(a, m, b) + as_{A,V}(m, a, b)}_{0}\\
&\quad\quad \text{(by (1), Remarks 3.3 and (7))}\\
&= 0\,.
\end{aligned}$$

Similarly, we compute

$$
\begin{aligned}
as_E(a+m, & b+n, b+n) \\
&= \tilde{\mu}(\tilde{\mu}(a+m, b+n), \tilde{\alpha}(b+n)) - \tilde{\mu}(\tilde{\alpha}(a+m), \tilde{\mu}(b+n, b+n)) \\
&= \tilde{\mu}(ab + a\cdot n + m\cdot b, \alpha_A(b) + \alpha_V(n)) \\
&\quad - \tilde{\mu}(\alpha_A(a) + \alpha_V(m), b^2 + b\cdot n + b\cdot b) \\
&= (ab)\alpha_A(b) + (ab)\cdot\alpha_V(m) + (a\cdot n)\cdot\alpha_A(b) + (m\cdot b)\cdot\alpha_A(b) \\
&\quad - \alpha_A(a)(b^2) - \alpha_A(a)\cdot(b\cdot n) - \alpha_A(a)\cdot(n\cdot b) - \alpha_V(m)\cdot b^2 \\
&= \underbrace{as_A(a,b,b)}_{0} + \underbrace{as_{A,V}(a,b,n) + as_{A,V}(a,n,b)}_{0} + \underbrace{as_{A,V}(m,b,b)}_{0} \\
&\quad \text{(by (2), (7) and (8))} \\
&= 0\,.
\end{aligned}
$$

We then conclude that $(A\oplus V, \tilde{\mu}, \tilde{\alpha})$ is a Hom-alternative algebra. ∎

Remark 3.12. Consider the split null extension $A\oplus V$ determined by the Hom-alternative bimodule (V, α_V) of the Hom-alternative algebra (A, μ, α_A) in the previous theorem. Write elements $a+v$ of $A\oplus V$ as (a,v). Then, there is an injective homomorphism of Hom-modules $i: V\to A\oplus V$ given by $i(v) = (0, v)$ and a surjective homomorphism of Hom-modules $\pi: A\oplus V\to A$ given by $\pi(a, v) = a$. Moreover $i(V)$ is a two-sided Hom-ideal of $A\oplus V$ such that $A\oplus V/i(V)\cong A$. On the other hand, there is a morphism of Hom-algebras $\sigma: A\to A\oplus V$ given by $\sigma(a) = (a, 0)$ which is clearly a section of π. Hence, we obtain the abelian split exact sequence of Hom-alternative algebras and (V, α_V) is a Hom-alternative A-bimodule via π.

4. Hom-Jordan bimodules

In this section, we study Hom-Jordan bimodules. It is observed that similar results for Hom-alternative bimodules hold for Hom-Jordan bimodules. Some of them require an additional condition. Furthermore, relations between Hom-associative bimodules and Hom-Jordan bimodules are given on the one hand, and on the other hand, relations between left (resp. right) Hom-alternative modules and left(resp. right) special Hom-Jordan modules are proved. First, we have:

Definition 4.1. Let (A, μ, α_A) be a Hom-Jordan algebra.

(i) A right Hom-Jordan A-module is a Hom-module (V, α_V) with a right structure map $\rho_r : V \otimes A \to V$, $v \otimes a \mapsto v \cdot a$ such that the following conditions hold:

$$\begin{aligned}\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab) \\ = (\alpha_V(v) \cdot bc) \cdot \alpha_A^2(a) + (\alpha_V(v) \cdot ca) \cdot \alpha_A^2(b) \\ + (\alpha_V(v) \cdot ab) \cdot \alpha_A^2(c)\,,\end{aligned} \tag{9}$$

$$\begin{aligned}\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab) \\ = ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \\ + \alpha_V^2(v) \cdot ((ac)\alpha_A(b))\end{aligned} \tag{10}$$

for all $a, b, c \in A$ and $v \in V$.

(ii) A left Hom-Jordan A-module is a Hom-module (V, α_V) with a left structure map $\rho_l : A \otimes V \to V$, $a \otimes v \mapsto a \cdot v$ such that the following conditions hold:

$$\begin{aligned}\alpha_A(bc) \cdot \alpha_V(a \cdot v) + \alpha_A(ca) \cdot \alpha_V(b \cdot v) + \alpha_A(ab) \cdot \alpha_V(c \cdot v) \\ = \alpha_A^2(a) \cdot (bc \cdot \alpha_V(v)) + \alpha_A^2(b) \cdot (ca \cdot \alpha_V(v)) \\ + \alpha_A^2(c) \cdot (ab \cdot \alpha_V(v))\,,\end{aligned} \tag{11}$$

$$\begin{aligned}\alpha_A(bc) \cdot \alpha_V(a \cdot v) + \alpha_A(ca) \cdot \alpha_V(b \cdot v) + \alpha_A(ab) \cdot \alpha_V(c \cdot v) \\ = \alpha_A^2(c) \cdot (\alpha_A(b) \cdot (a \cdot v)) + \alpha_A^2(a) \cdot (\alpha_A(b) \cdot (c \cdot v)) \\ + ((ac)\alpha_A(b)) \cdot \alpha_V^2(v)\end{aligned} \tag{12}$$

for all $a, b, c \in A$ and $v \in V$.

The following result allows to introduce the notion of right special Hom-Jordan modules.

THEOREM 4.2. *Let (A, μ, α_A) be a Hom-Jordan algebra, (V, α_V) be a Hom-module and $\rho_r : V \otimes A \to V$, $a \otimes v \mapsto v \cdot a$, be a bilinear map satisfying*

$$\alpha_V \circ \rho_r = \rho_r \circ (\alpha_V \otimes \alpha_A) \tag{13}$$

and

$$\alpha_V(v) \cdot (ab) = (v \cdot a) \cdot \alpha_A(b) + (v \cdot b) \cdot \alpha_A(a) \tag{14}$$

for all $(a, b) \in A^{\times 2}$ and $v \in V$. Then (V, α, ρ_r) is a right Hom-Jordan A-module called a right special Hom-Jordan A-module.

Proof. It suffices to prove (9) and (10). For all $(a, b) \in A^{\times 2}$ and $v \in V$, we have:

$$\begin{aligned}
&\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab) \\
&\quad = \alpha_V(v \cdot a) \cdot \alpha_A(b)\alpha_A(c) + \alpha_V(v \cdot b) \cdot \alpha_A(c)\alpha_A(a) \\
&\qquad + \alpha_V(v \cdot c) \cdot \alpha_A(a)\alpha_A(b) \quad \text{(multiplicativity)} \\
&\quad = ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha^2(c) + ((v \cdot a) \cdot \alpha_A(c)) \cdot \alpha^2(b) \\
&\qquad + ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha^2(a) + ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha^2(c) \\
&\qquad + ((v \cdot c) \cdot \alpha_A(a)) \cdot \alpha^2(b) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha^2(a) \quad \text{(by (14))} \\
&\quad = [\alpha_V(v) \cdot ab - (v \cdot b)\alpha_A(a)] \cdot \alpha^2(c) + ((v \cdot a) \cdot \alpha_A(c)) \cdot \alpha^2(b) \\
&\qquad + [\alpha_V(v) \cdot bc - (v \cdot c)\alpha_A(b)] \cdot \alpha^2(a) + ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha^2(c) \\
&\qquad + [\alpha_V(v) \cdot ca - (v \cdot a) \cdot \alpha_A(c)] \cdot \alpha^2(b) \\
&\qquad + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha^2(a) \quad \text{(again by (14))} \\
&\quad = (\alpha_V(v) \cdot bc) \cdot \alpha_A^2(a) + (\alpha_V(v) \cdot ca) \cdot \alpha_A^2(b) + (\alpha_V(v) \cdot ab) \cdot \alpha_A^2(c)
\end{aligned}$$

and thus, we get (9). Finally, (10) is proved as follows:

$$\begin{aligned}
&\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab) \\
&\quad = \alpha_V(v \cdot a) \cdot \alpha_A(b)\alpha_A(c) + \alpha_V(v \cdot b) \cdot \alpha_A(c)\alpha_A(a) \\
&\qquad + \alpha_V(v \cdot c) \cdot \alpha_A(a)\alpha_A(b) \quad \text{(multiplicativity)} \\
&\quad = ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \cdot a) \cdot \alpha_A(c)) \cdot \alpha_A^2(b) \\
&\qquad + ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha_A^2(a) + ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha_A^2(c) \\
&\qquad + ((v \cdot c) \cdot \alpha_A(a)) \cdot \alpha_A^2(b) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \quad \text{(by ((14))} \\
&\quad = ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + [\alpha_V(v) \cdot ac - ((v \cdot c) \cdot \alpha_A(a)] \cdot \alpha_A^2(b) \\
&\qquad + ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha_A^2(a) + ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha_A^2(c) \\
&\qquad + ((v \cdot c) \cdot \alpha_A(a)) \cdot \alpha_A^2(b) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \quad \text{(again by (14))} \\
&\quad = ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + \alpha_V^2(v) \cdot ((ac)\alpha_A(b)) \\
&\qquad - (\alpha_V(v) \cdot \alpha_A(b)) \cdot \alpha_A(ac) + ((v \cdot b) \cdot \alpha_A(c)) \cdot \alpha_A^2(a) \\
&\qquad + ((v \cdot b) \cdot \alpha_A(a)) \cdot \alpha_A^2(c) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \quad \text{(again by (14))}
\end{aligned}$$

$$
\begin{aligned}
&= ((v\cdot a)\cdot \alpha_A(b))\cdot \alpha_A^2(c) + \alpha_V^2(v)\cdot((ac)\alpha_A(b)) - (\alpha_V(v\cdot b)\cdot \alpha_A(ac)\\
&\quad + ((v\cdot b)\cdot \alpha_A(c))\cdot \alpha_A^2(a) + ((v\cdot b)\cdot \alpha_A(a))\cdot \alpha_A^2(c)\\
&\quad + ((v\cdot c)\cdot \alpha_A(b))\cdot \alpha_A^2(a) \quad \text{(by (13))}\\
&= ((v\cdot a)\cdot \alpha_A(b))\cdot \alpha_A^2(c) + \alpha_V^2(v)\cdot((ac)\alpha_A(b)) - ((v\cdot b)\cdot \alpha_A(a))\cdot \alpha_A^2(c)\\
&\quad - ((v\cdot b)\cdot \alpha_A(c))\cdot \alpha_A^2(a) + ((v\cdot b)\cdot \alpha_A(c))\cdot \alpha_A^2(a)\\
&\quad + ((v\cdot b)\cdot \alpha_A(a))\cdot \alpha_A^2(c) + ((v\cdot c)\cdot \alpha_A(b))\cdot \alpha_A^2(a) \quad \text{(by (14))}\\
&= ((v\cdot a)\cdot \alpha_A(b))\cdot \alpha_A^2(c) + ((v\cdot c)\cdot \alpha_A(b))\cdot \alpha_A^2(a) + \alpha_V^2(v)\cdot((ac)\alpha_A(b))
\end{aligned}
$$

which is (10). ∎

Similarly, the following result can be proved.

THEOREM 4.3. *Let (A,μ,α_A) be a Hom-Jordan algebra, (V,α_V) be a Hom-module and $\rho_l : A\otimes V \to V$, $v\otimes a \mapsto a\cdot v$, be a bilinear map satisfying*

$$\alpha_V \circ \rho_l = \rho_l \circ (\alpha_A \otimes \alpha_V)$$

and

$$(ab)\cdot \alpha_V(v) = \alpha_A(a)\cdot(b\cdot v) + \alpha_A(b)\cdot(a\cdot v) \tag{15}$$

for all $(a,b)\in A^{\times 2}$ and $v\in V$. Then (V,α,ρ_l) is a left Hom-Jordan A-module called a left special Hom-Jordan A-module.

It is well known that the plus algebra of any Hom-alternative algebra is a Hom-Jordan algebra. The next result shows that any left (resp. right) Hom-alternative module a is also a left (resp. right) module over its plus Hom-algebra.

PROPOSITION 4.4. *Let (A,μ,α_A) be a Hom-alternative algebra and (V,α_V) be a Hom-module.*

(i) *If (V,α_V) is a right Hom-alternative A-module with the structure map ρ_r then (V,α_V) is a right special Hom-Jordan A^+-module with the same structure map ρ_r.*

(ii) *If (V,α_V) is a left Hom-alternative A-module with the structure map ρ_l then (V,α_V) is a left special Hom-Jordan A^+-module with the same structure map ρ_l.*

Proof. It suffices to prove (14) and (15).

(i) If (V, α_V) is a right Hom-alternative A-module with the structure map ρ_r, then for all $(x, y, v) \in A \times A \times V$, $as_{A,V}(v, x, y) = -as_{A,V}(v, y, x)$ by (8), i.e., $\alpha_V(v) \cdot (xy) + \alpha_V(v) \cdot (yx) = (v \cdot x) \cdot \alpha_A(y) + (v \cdot y) \cdot \alpha_A(x)$. Thus $\alpha_V(v) \cdot (x * y) = \alpha_V(v) \cdot (xy) + \alpha_V(v) \cdot (yx) = (v \cdot x) \cdot \alpha_A(y) + (v \cdot y) \cdot \alpha_A(x)$. Therefore (V, α_V) is a right special Hom-Jordan A^+-module by Theorem 4.2.

(ii) If (V, α_V) is a left Hom-alternative A-module with the structure map ρ_l, then for all $(x, y, v) \in A \times A \times V$, $as_{A,V}(x, y, v) = -as_{A,V}(y, x, v)$ by Remarks 3.3 and then $(xy) \cdot \alpha_V(v) + (yx) \cdot \alpha_V(v) = \alpha_A(x) \cdot (y \cdot v) + \alpha_A(y) \cdot (x \cdot v)$. Thus $(x * y) \cdot \alpha_V(v) = (xy) \cdot \alpha_V(v) + (yx) \cdot \alpha_V(v) = \alpha_A(x) \cdot (y \cdot v) + \alpha_A(y) \cdot (x \cdot v)$. Therefore (V, α_V) is a left special Hom-Jordan A^+-module by Theorem 4.3. ∎

Now, we give the definition of a Hom-Jordan bimodule.

DEFINITION 4.5. Let (A, μ, α_A) be a Hom-Jordan algebra.
A Hom-Jordan A-bimodule is a Hom-module (V, α_V) with a left structure map $\rho_l : A \otimes V \to V$, $a \otimes v \mapsto a \cdot v$ and a right structure map $\rho_r : V \otimes A \to V$, $v \otimes a \mapsto v \cdot a$, such that the following conditions hold:

$$\rho_r \circ \tau_1 = \rho_l \,, \tag{16}$$

$$\begin{aligned} &\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab) \\ &\quad = (\alpha_V(v) \cdot bc) \cdot \alpha_A^2(a) + (\alpha_V(v) \cdot ca) \cdot \alpha_A^2(b) \\ &\qquad + (\alpha_V(v) \cdot ab) \cdot \alpha_A^2(c) \,, \end{aligned} \tag{17}$$

$$\begin{aligned} &\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab) \\ &\quad = ((v \cdot a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \cdot c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \\ &\qquad + ((ac)\alpha_A(b)) \cdot \alpha_V^2(v) \,, \end{aligned} \tag{18}$$

for all $a, b, c \in A$ and $v \in V$.

In term of the module Hom-associator, using the relation (16) and the fact that the structure maps are morphisms, the relations (17) and (18) are respectively

$$\circlearrowleft_{(a,b,c)} as_{A,V}(\alpha_A(a), \alpha_V(v), bc) = 0 \,, \tag{19}$$

$$\begin{aligned} &as_{A,V}(v \cdot a, \alpha_A(b), \alpha_A(c)) + as_{A,V}(v \cdot c, \alpha_A(b), \alpha_A(a)) \\ &\qquad + as_{A,V}(ac, \alpha_A(b), \alpha_V(v)) = 0 \,. \end{aligned} \tag{20}$$

Remarks 4.6. (i) One can note that (17) and (18) are the same identities as (9) and (10) respectively.

(ii) Since $\rho_r \circ \tau_1 = \rho_l$, nothing is lost in dropping one of the compositions. Thus the term Hom-Jordan module can be used for Hom-Jordan bimodule.

(iii) Since the field is of characteristic 0, the identity (19) implies

$$as_{A,V}(\alpha_A(a), \alpha_V(v), a^2) = 0 .$$

(iv) If $\alpha_A = Id_A$ and $\alpha_V = Id_V$ then V is reduced to the so-called Jordan module of the Jordan algebra (A, μ) [7, 8].

Examples 4.7. Here are some examples of Hom-Jordan bimodules.

(i) Let (A, μ, α_A) be a Hom-Jordan algebra. Then (A, α_A) is a Hom-Jordan A-bimodule where the structure maps are $\rho_l = \rho_r = \mu$. More generally, if B is a Hom-ideal of (A, μ, α_A), then (B, α_A) is a Hom-Jordan A-bimodule where the structure maps are $\rho_l(a, x) = \mu(a, x) = \mu(x, a) = \rho_r(x, a)$ for all $(a, x) \in A \times B$.

(ii) If (A, μ) is a Jordan algebra and M is a Jordan A-bimodule [8] in the usual sense then (M, Id_M) is a Hom-Jordan $\mathbb{A}$-bimodule where $\mathbb{A} = (A, \mu, Id_A)$ is a Hom-Jordan algebra.

(iii) If $f : (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B)$ is a surjective morphism of Hom-Jordan algebras, then (B, α_B) becomes a Hom-Jordan A-bimodule via f, i.e, the structure maps are defined by $\rho_l : (a, b) \mapsto \mu_B(b, f(a))$ and $\rho_r : (b, a) \mapsto \mu_B(f(a), b)$ for all $(a, b) \in A \times B$.

As in the case of Hom-alternative algebras, in order to give another example of Hom-Jordan bimodules, let us consider the following

Definition 4.8. An abelian extension of Hom-Jordan algebras is a short exact sequence of Hom-Jordan algebras

$$0 \to (V, \alpha_V) \xrightarrow{i} (A, \mu_A, \alpha_A) \xrightarrow{\pi} (B, \mu_B, \alpha_B) \to 0$$

where (V, α_V) is a trivial Hom-Jordan algebra, i and π are morphisms of Hom-algebras. Furthermore, if there exists a morphism $s : (B, \mu_B, \alpha_B) \to (A, \mu_A, \alpha_A)$ such that $\pi \circ s = id_B$ then the abelian extension is said to be split and s is called a section of π.

Example 4.9. Given an abelian extension as in the previous definition, the Hom-module (V, α_V) inherits a structure of a Hom-Jordan B-bimodule and the actions of the Hom-algebra (B, μ_B, α_B) on V are as follows. For any $x \in B$, there exist $\tilde{x} \in A$ such that $x = \pi(\tilde{x})$. Let x acts on $v \in V$ by $x \cdot v := \mu_A(\tilde{x}, i(v))$ and $v \cdot x := \mu_A(i(v), \tilde{x})$. These are well-defined, as another lift $\tilde{x'}$ of x is written $\tilde{x'} = \tilde{x} + v'$ for some $v' \in V$ and thus $x \cdot v = \mu_A(\tilde{x}, i(v)) = \mu_A(\tilde{x'}, i(v))$ and $v \cdot x = \mu_A(i(v), \tilde{x}) = \mu_A(i(v), \tilde{x'})$ because V is trivial. The actions property follow from the Hom-Jordan identity. In case these actions of B on V are trivial, one speaks of a central extension.

The next result shows that a special left and right Hom-Jordan module has a Hom-Jordan bimodule structure under a specific condition.

THEOREM 4.10. *Let (A, μ, α_A) be a Hom-Jordan algebra and (V, α_V) be both a left and a right special Hom-Jordan A-module with the structure maps ρ_1 and ρ_2 respectively such that the Hom-associativity (or operator commutativity) condition holds*

$$\rho_2 \circ (\rho_1 \otimes \alpha_A) = \rho_1 \circ (\alpha_A \otimes \rho_2)\,. \tag{21}$$

Define the bilinear maps $\rho_l : A \otimes V \to V$ and $\rho_r : V \otimes A \to V$ by

$$\rho_l = \rho_1 + \rho_2 \circ \tau_1 \quad \text{and} \quad \rho_r = \rho_1 \circ \tau_2 + \rho_2\,. \tag{22}$$

Then $(V, \alpha_V, \rho_l, \rho_r)$ is a Hom-Jordan A-bimodule.

Proof. It is clear that ρ_l and ρ_r are structure maps and (16) holds. To prove relations (17) and (18), let put $\rho_l(a \otimes v) := a \diamond v$, i.e., $a \diamond v = a \cdot v + v \cdot a$ for all $(a, v) \in A \times V$. We have then $\rho_r(v \otimes a) := v \diamond a = a \cdot v + v \cdot a$ for all $(a, v) \in A \times V$. Therefore for all $(a, b, v) \in A \times A \times V$, we have

$$\begin{aligned}
&\alpha_V(v \diamond a) \diamond \alpha_A(bc) + \alpha_V(v \diamond b) \diamond \alpha_A(ca) + \alpha_V(v \diamond c) \diamond \alpha_A(ab)\\
&\quad = \alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(a \cdot v) \cdot \alpha_A(bc) + \alpha_A(bc) \cdot \alpha_V(v \cdot a)\\
&\qquad + \alpha_A(bc) \cdot \alpha_V(a \cdot v) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(b \cdot v) \cdot \alpha_A(ca)\\
&\qquad + \alpha_A(ca) \cdot \alpha_V(v \cdot b) + \alpha_A(ca) \cdot \alpha_V(b \cdot v) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)\\
&\qquad + \alpha_V(c \cdot v) \cdot \alpha_A(ab) + \alpha_A(ab) \cdot \alpha_V(v \cdot c) + \alpha_A(ab) \cdot \alpha_V(c \cdot v)\\
&\qquad\quad \text{(by a straightforward computation)}\\
&\quad = \{\alpha_V(v \cdot a) \cdot \alpha_A(bc) + \alpha_V(v \cdot b) \cdot \alpha_A(ca) + \alpha_V(v \cdot c) \cdot \alpha_A(ab)\}\\
&\qquad + \{\alpha_A(bc) \cdot (\alpha_V(v) \cdot \alpha_A(a)) + \alpha_A(ca) \cdot (\alpha_V(v) \cdot \alpha_A(b))\\
&\qquad + \alpha_A(ab) \cdot (\alpha_V(v) \cdot \alpha_A(c))\} + \{\alpha_A(bc) \cdot \alpha_V(a \cdot v) + \alpha_A(ca) \cdot \alpha_V(b \cdot v)
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_A(ab)\cdot\alpha_V(c\cdot v)\} + \{(\alpha_A(a)\cdot\alpha_V(v))\cdot\alpha_A(bc) \\
&+ (\alpha_A(b)\cdot\alpha_V(v))\cdot\alpha_A(ca) + (\alpha_A(c)\cdot\alpha_V(v))\cdot\alpha_A(ab)\} \\
&\quad \text{(rearranging terms and noting that } \rho_1 \text{ and } \rho_2 \text{ are morphisms)} \\
= &\{(\alpha_V(v)\cdot bc)\cdot\alpha_A^2(a) + (\alpha_V(v)\cdot ca)\cdot\alpha_A^2(b) + (\alpha_V(v)\cdot ab)\cdot\alpha_A^2(c)\} \\
&+ \{(bc\cdot\alpha_V(v))\cdot\alpha_A^2(a) + (ca\cdot\alpha_V(v))\cdot\alpha_A^2(b) + (ab\cdot\alpha_V(v))\cdot\alpha_A^2(c)\} \\
&+ \{\alpha_A^2(a)\cdot(bc\cdot\alpha_V(v)) + \alpha_A^2(b)\cdot(ca\cdot\alpha_V(v)) + \alpha_A^2(c)\cdot(ab\cdot\alpha_V(v))\} \\
&+ \{\alpha_A^2(a)\cdot(\alpha_V(v)\cdot bc) + \alpha_A^2(b)\cdot(\alpha_V(v)\cdot ca) + \alpha_A^2(c)\cdot(\alpha_V(v)\cdot ab)\} \\
&\quad \text{(by (9), (11) and (21))} \\
= &\{(\alpha_V(v)\diamond bc)\cdot\alpha_A^2(a) + (\alpha_V(v)\diamond ca)\cdot\alpha_A^2(b) + (\alpha_V(v)\diamond ab)\cdot\alpha_A^2(c)\} \\
&+ \{\alpha_A^2(a)\cdot(\alpha_V(v)\diamond bc) + \alpha_A^2(b)\cdot(\alpha_V(v)\diamond ca) + \alpha_A^2(c)\cdot(\alpha_V(v)\diamond ab)\} \\
&\quad \text{(by the definition of } \diamond) \\
= &(\alpha_V(v)\diamond bc)\diamond\alpha_A^2(a) + (\alpha_V(v)\diamond ca)\diamond\alpha_A^2(b) + (\alpha_V(v)\diamond ab)\diamond\alpha_A^2(c) \\
&\quad \text{(again by the definition of } \diamond).
\end{aligned}
$$

Therefore, we get (17). Finally, we have:

$$
\begin{aligned}
&\alpha_V(v\diamond a)\diamond\alpha_A(bc) + \alpha_V(v\diamond b)\diamond\alpha_A(ca) + \alpha_V(v\diamond c)\diamond\alpha_A(ab) \\
&= \alpha_V(v\cdot a)\cdot\alpha_A(bc) + \alpha_V(a\cdot v)\cdot\alpha_A(bc) + \alpha_A(bc)\cdot\alpha_V(v\cdot a) \\
&\quad + \alpha_A(bc)\cdot\alpha_V(a\cdot v) + \alpha_V(v\cdot b)\cdot\alpha_A(ca) + \alpha_V(b\cdot v)\cdot\alpha_A(ca) \\
&\quad + \alpha_A(ca)\cdot\alpha_V(v\cdot b) + \alpha_A(ca)\cdot\alpha_V(b\cdot v) + \alpha_V(v\cdot c)\cdot\alpha_A(ab) \\
&\quad + \alpha_V(c\cdot v)\cdot\alpha_A(ab) + \alpha_A(ab)\cdot\alpha_V(v\cdot c) + \alpha_A(ab)\cdot\alpha_V(c\cdot v) \\
&\qquad \text{(by a straightforward computation)} \\
&= \{\alpha_V(v\cdot a)\cdot\alpha_A(bc) + \alpha_V(v\cdot b)\cdot\alpha_A(ca) + \alpha_V(v\cdot c)\cdot\alpha_A(ab)\} \\
&\quad + \{(\alpha_V(a\cdot v)\cdot\alpha_A(b)\alpha_A(c) + (\alpha_V(b\cdot v))\cdot\alpha_A(c)\alpha_A(a) \\
&\quad + (\alpha_V(c\cdot v)\cdot\alpha_A(a)\alpha_A(b)\} + \{\alpha_A(bc)\cdot\alpha_V(a\cdot v) + \alpha_A(ca)\cdot\alpha_V(b\cdot v) \\
&\quad + \alpha_A(ab)\cdot\alpha_V(c\cdot v)\} + \{\alpha_A(b)\alpha_A(c)\cdot\alpha_V(v\cdot a) \\
&\quad + \alpha_A(c)\alpha_A(a)\cdot\alpha_V(v\cdot b) + \alpha_A(a)\alpha_A(b)\cdot\alpha_V(v\cdot c)\} \\
&\qquad \text{(rearranging terms and using the multiplicativity of } \alpha_A) \\
&= \{\underbrace{((v\cdot a)\cdot\alpha_A(b))\cdot\alpha_A^2(c)}_{1} + \underbrace{((v\cdot c)\cdot\alpha_A(b))\cdot\alpha_A^2(a)}_{2} + \underbrace{\alpha_V^2(v)\cdot((ac)\alpha_A(b))}_{5}\}
\end{aligned}
$$

$$
\begin{aligned}
&+\{\underbrace{((a\cdot v)\cdot\alpha_A(b))\cdot\alpha_A^2(c)}_{1}+((a\cdot v)\cdot\alpha_A(c))\cdot\alpha_A^2(b)\\
&+((b\cdot v)\cdot\alpha_A(c))\cdot\alpha_A^2(a)+((b\cdot v)\cdot\alpha_A(a))\cdot\alpha_A^2(c)\\
&+((c\cdot v)\cdot\alpha_A(a))\cdot\alpha_A^2(b)+\underbrace{((c\cdot v)\cdot\alpha_A(b))\cdot\alpha_A^2(a)}_{2}\}\\
&+\{\underbrace{\alpha_A^2(c)\cdot(\alpha_A(b)\cdot(a\cdot v))}_{3}+\underbrace{\alpha_A^2(a)\cdot(\alpha_A(b)\cdot(c\cdot v))}_{4}\\
&+\underbrace{((ac)\alpha_A(b))\cdot\alpha_V^2(v)}_{5}\}+\{\alpha_A^2(b)\cdot(\alpha_A(c)\cdot(v\cdot a))\\
&+\underbrace{\alpha_A^2(c)\cdot(\alpha_A(b)\cdot(v\cdot a))}_{3}+\alpha_A^2(a)\cdot(\alpha_A(c)\cdot(v\cdot b))\\
&+\alpha_A^2(c)\cdot(\alpha_A(a)\cdot(v\cdot b))+\underbrace{\alpha_A^2(a)\cdot(\alpha_A(b)\cdot(v\cdot c))}_{4}\\
&+\alpha_A^2(b)\cdot(\alpha_A(a)\cdot(v\cdot c))\}\quad\text{(by (10), (12), (14) and (15))}\\
=&((v\diamond a)\cdot\alpha_A(b))\cdot\alpha_A^2(c)+((v\diamond c)\cdot\alpha_A(b))\cdot\alpha_A^2(a)\\
&+\alpha_A^2(c)\cdot(\alpha_A(b)\cdot(v\diamond a))+\alpha_A^2(a)\cdot(\alpha_A(b)\cdot(v\diamond c))\\
&+\alpha_V^2(v)\diamond((ac)\alpha_A(b))+((a\cdot v)\cdot\alpha_A(c))\cdot\alpha_A^2(b)\\
&+((b\cdot v)\cdot\alpha_A(c))\cdot\alpha_A^2(a)+((b\cdot v)\cdot\alpha_A(a))\cdot\alpha_A^2(c)\\
&+((c\cdot v)\cdot\alpha_A(a))\cdot\alpha_A^2(b)+\alpha_A^2(b)\cdot(\alpha_A(c)\cdot(v\cdot a))\\
&+\alpha_A^2(a)\cdot(\alpha_A(c)\cdot(v\cdot b))+\alpha_A^2(c)\cdot(\alpha_A(a)\cdot(v\cdot b))\\
&+\alpha_A^2(b)\cdot(\alpha_A(a)\cdot(v\cdot c))\\
=&((v\diamond a)\cdot\alpha_A(b))\cdot\alpha_A^2(c)+((v\diamond c)\cdot\alpha_A(b))\cdot\alpha_A^2(a)\\
&+\alpha_A^2(c)\cdot(\alpha_A(b)\cdot(v\diamond a))+\alpha_A^2(a)\cdot(\alpha_A(b)\cdot(v\diamond c))\\
&+\alpha_V^2(v)\diamond((ac)\alpha_A(b))+(\alpha_A(a)\cdot(v\cdot c))\cdot\alpha_A^2(b)\\
&+(\alpha_A(b)\cdot(v\cdot c))\cdot\alpha_A^2(a)+(\alpha_A(b)\cdot(v\cdot a))\cdot\alpha_A^2(c)\\
&+(\alpha_A(c)\cdot(v\cdot a))\cdot\alpha_A^2(b)+\alpha_A^2(b)\cdot((c\cdot v)\cdot\alpha_A(a))\\
&+\alpha_A^2(a)\cdot((c\cdot v)\cdot\alpha_A(b))+\alpha_A^2(c)\cdot((a\cdot v)\cdot\alpha_A(b))\\
&+\alpha_A^2(b)\cdot((a\cdot v)\cdot\alpha_A(c))\quad\text{(by (21))}
\end{aligned}
$$

$$
\begin{aligned}
&= ((v \diamond a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \diamond c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a) \\
&\quad + \alpha_A^2(c) \cdot (\alpha_A(b) \cdot (v \diamond a)) + \alpha_A^2(a) \cdot (\alpha_A(b) \cdot (v \diamond c)) \\
&\quad + \alpha_V^2(v) \diamond ((ac)\alpha_A(b)) + \underbrace{\alpha_A^2(a) \cdot ((v \cdot c) \cdot \alpha_A(b))}_{6} \\
&\quad + \underbrace{(\alpha_A(b) \cdot (v \cdot c)) \cdot \alpha_A^2(a)}_{7} + \underbrace{(\alpha_A(b) \cdot (v \cdot a)) \cdot \alpha_A^2(c)}_{8} \\
&\quad + \underbrace{\alpha_A^2(c) \cdot ((v \cdot a) \cdot \alpha_A(b))}_{9} + \underbrace{(\alpha_A(b) \cdot (c \cdot v)) \cdot \alpha_A^2(a)}_{7} \\
&\quad + \underbrace{\alpha_A^2(a) \cdot ((c \cdot v) \cdot \alpha_A(b))}_{6} + \underbrace{\alpha_A^2(c) \cdot ((a \cdot v) \cdot \alpha_A(b))}_{9} \\
&\quad + \underbrace{(\alpha_A(b) \cdot (a \cdot v)) \cdot \alpha_A^2(c))}_{8} \quad \text{(again by (21))} \\
&= \underbrace{((v \diamond a) \cdot \alpha_A(b)) \cdot \alpha_A^2(c)}_{10} + \underbrace{((v \diamond c) \cdot \alpha_A(b)) \cdot \alpha_A^2(a)}_{11} \\
&\quad + \underbrace{\alpha_A^2(c) \cdot (\alpha_A(b) \cdot (v \diamond a))}_{13} + \underbrace{\alpha_A^2(a) \cdot (\alpha_A(b) \cdot (v \diamond c))}_{12} \\
&\quad + \alpha_V^2(v) \diamond ((ac)\alpha_A(b)) + \underbrace{\alpha_A^2(a) \cdot ((v \diamond c) \cdot \alpha_A(b))}_{12} \\
&\quad + \underbrace{(\alpha_A(b) \cdot (v \diamond c)) \cdot \alpha_A^2(a)}_{11} + \underbrace{(\alpha_A(b) \cdot (v \diamond a)) \cdot \alpha_A^2(c)}_{10} \\
&\quad + \underbrace{\alpha_A^2(c) \cdot ((v \diamond a) \cdot \alpha_A(b))}_{13} \\
&= ((v \diamond a) \diamond \alpha_A(b)) \cdot \alpha_A^2(c) + ((v \diamond c) \diamond \alpha_A(b)) \cdot \alpha_A^2(a) \\
&\quad + \alpha_A^2(a) \cdot ((v \diamond c) \diamond \alpha_A(b)) + \alpha_A^2(c) \cdot ((v \diamond a) \diamond \alpha_A(b)) \\
&\quad + \alpha_V^2(v) \diamond ((ac)\alpha_A(b)) \\
&= ((v \diamond a) \diamond \alpha_A(b)) \diamond \alpha_A^2(c) + ((v \diamond c) \diamond \alpha_A(b)) \diamond \alpha_A^2(a) \\
&\quad + \alpha_V^2(v) \diamond ((ac)\alpha_A(b))
\end{aligned}
$$

which is (18). ∎

The following result will be used below. It gives a relation between Hom-associative modules and special Hom-Jordan modules.

LEMMA 4.11. *Let (A, μ, α_A) be a Hom-associative algebra and (V, α_V) be a Hom-module.*

(i) *If (V, α_V) is a right Hom-associative A-module with the structure maps ρ_r then (V, α_V) is a right special Hom-Jordan A^+-module with the same structure map ρ_r.*

(ii) *If (V, α_V) is a left Hom-associative A-module with the structure maps ρ_l then (V, α_V) is a left special Hom-Jordan A^+-module with the same structure map ρ_l.*

Proof. It also suffices to prove (14) and (15).

(i) If (V, α_V) is a right Hom-associative A-module with the structure map ρ_r then for all $(x, y, v) \in A \times A \times V$, $\alpha_V(v) \cdot (a * b) = \alpha_V(v) \cdot (ab) + \alpha_V(v) \cdot (ba) = (v \cdot a) \cdot \alpha_A(b) + (v \cdot b) \cdot \alpha_A(a)$ where the last equality holds by (5). Then (V, α_V) is a right special Hom-Jordan A^+-module.

(ii) If (V, α_V) is a left Hom-associative A-module with the structure map ρ_l then for all $(x, y, v) \in A \times A \times V$, $(a * b) \cdot \alpha_V(v) = (ab) \cdot \alpha_V(v) + (ba) \cdot \alpha_V(v) = \alpha_A(a) \cdot (b \cdot v) + \alpha_A(b) \cdot (a \cdot v)$ where the last equality holds by (4). Then (V, α_V) is a left special Hom-Jordan A^+-module. ∎

Now, we prove that a Hom-associative module gives rise to a Hom-Jordan module for its plus Hom-algebra.

PROPOSITION 4.12. *Let (A, μ, α_A) be a Hom-associative algebra and $(V, \rho_1, \rho_2, \alpha_V)$ be a Hom-associative A-bimodule. Then $(V, \rho_l, \rho_r, \alpha_V)$ is a Hom-Jordan A^+-bimodule where ρ_l and ρ_r are defined as in (22).*

Proof. The proof follows from Lemma 4.11 , the Hom-associativity condition (6) and Theorem 4.10. ∎

The following elementary result will be used below. It gives a property of a module Hom-associator.

LEMMA 4.13. *Let (A, μ, α_A) be a Hom-Jordan algebra and (V, α_V) be an Hom-Jordan A-bimodule with the structure maps ρ_l and ρ_r. Then*

$$\alpha_V^n \circ as_{A,V} \circ Id_{A \otimes V \otimes A} = as_{A,V} \circ (\alpha_A^{\otimes n} \otimes \alpha_V^{\otimes n} \otimes \alpha_A^{\otimes n}) \,. \tag{23}$$

Proof. Using twice the fact that ρ_l and ρ_r are morphisms of Hom-modules, we get

$$\begin{aligned}
&\alpha_V^n \circ as_{A,V} \circ Id_{A\otimes V\otimes A} \\
&\quad = \alpha_V^n \circ (\rho_r \circ (\rho_l \otimes \alpha_A) - \rho_l \circ (\alpha_A \otimes \rho_r)) \\
&\quad = \alpha_V^n \circ \rho_r \circ (\rho_l \otimes \alpha_A) - \alpha_V^n \circ \rho_l \circ (\alpha_A \otimes \rho_r) \quad \text{(linearity of } \alpha_V^n) \\
&\quad = \rho_r \circ (\alpha_V^n \circ \rho_l \otimes \alpha_A^{n+1}) - \rho_l \circ (\alpha_A^{n+1} \otimes \alpha_V^n \circ \rho_r) \\
&\quad = \rho_r \circ (\rho_l \circ (\alpha_A^n \otimes \alpha_V^n) \otimes \alpha_A^{n+1}) - \rho_l \circ (\alpha_A^{n+1} \otimes \rho_r \circ (\alpha_V^n \otimes \alpha_A^n)) \\
&\quad = (\rho_r \circ (\rho_l \otimes \alpha_A) - \rho_l \circ (\alpha_A \otimes \rho_r)) \circ (\alpha_A^{\otimes n} \otimes \alpha_V^{\otimes n} \otimes \alpha_A^{\otimes n}) \\
&\quad = as_{A,V} \circ (\alpha_A^{\otimes n} \otimes \alpha_V^{\otimes n} \otimes \alpha_A^{\otimes n}).
\end{aligned}$$

That ends the proof. ∎

The next result is similar to the one of Proposition 3.7, but an additional condition is needed.

PROPOSITION 4.14. *Let (A, μ, α_A) be a Hom-Jordan algebra and (V, α_V) be a Hom-Jordan A-bimodule with the structure maps ρ_l and ρ_r. Suppose that there exists $n \in \mathbb{N}$ such that $\alpha_V^n = Id_V$. Then the maps*

$$\rho_l^{(n)} = \rho_l \circ (\alpha_A^n \otimes Id_V), \tag{24}$$

$$\rho_r^{(n)} = \rho_r \circ (Id_V \otimes \alpha_A^n) \tag{25}$$

give the Hom-module (V, α_V) the structure of a Hom-Jordan A-bimodule that we denote by $V^{(n)}$.

Proof. Since the structure map ρ_l is a morphism of Hom-modules, we get:

$$\begin{aligned}
\alpha_V \circ \rho_l^{(n)} &= \alpha_V \circ \rho_l \circ (\alpha_A^n \otimes Id_V) \quad \text{(by (24))} \\
&= \rho_l \circ (\alpha_A^{n+1} \otimes \alpha_V) \\
&= \rho_l \circ (\alpha_A^n \otimes Id_V) \circ (\alpha_A \otimes \alpha_V) \\
&= \rho_l^{(n)} \circ (\alpha_A \otimes \alpha_V)
\end{aligned}$$

Then, $\rho_l^{(n)}$ is a morphism. Similarly, we get that $\rho_r^{(n)}$ is a morphism and that (16) holds for $V^{(n)}$. Next, we compute

$$
\begin{aligned}
&\circlearrowleft_{(a,b,c)} as_{A,V^{(n)}}(\alpha_A(a),\alpha_V(v),ab)\\
&\quad=\circlearrowleft_{(a,b,c)}\{\rho_r^{(n)}(\rho_l^{(n)}(\alpha_A(a),\alpha_V(v)),\alpha_A(bc))-\rho_l^{(n)}(\alpha_A^2(a),\rho_r^{(n)}(\alpha_V(v),bc))\}\\
&\quad=\circlearrowleft_{(a,b,c)}\{\rho_r(\rho_l^{(n)}(\alpha_A(a),\alpha_V(v)),\alpha_A^{n+1}(bc))-\rho_l(\alpha_A^{n+2}(a),\rho_r^{(n)}(\alpha_V(v),bc))\}\\
&\quad=\circlearrowleft_{(a,b,c)}\{\rho_r(\rho_l(\alpha_A^{n+1}(a),\alpha_V(v)),\alpha_A^{n+1}(bc))-\rho_l(\alpha_A^{n+2}(a),\rho_r(\alpha_V(v),\alpha_A^{n}(bc))\}\\
&\quad=\circlearrowleft_{(a,b,c)}\{\rho_r(\rho_l(\alpha_A^{n+1}(a),\alpha_V(v)),\alpha_A(\alpha_A^{n}(bc)))\\
&\qquad\quad-\rho_l(\alpha_A(\alpha_A^{n+1}(a)),\rho_r(\alpha_V(v),\alpha_A^{n}(bc))\}\\
&\quad=\circlearrowleft_{(a,b,c)} as_{A,V}(\alpha_A^{n+1}(a),\alpha_V(v),\alpha_A^{n}(bc))\\
&\quad=\circlearrowleft_{(a,b,c)} as_{A,V}(\alpha_A^{n+1}(a),\alpha_V^{n+1}(v),\alpha_A^{n}(bc))\quad\text{(by the hypothesis } \alpha_V=\alpha_V^{n+1})\\
&\quad=\alpha_V^{n}(\circlearrowleft_{(a,b,c)} as_{A,V}(\alpha_A(a),\alpha_V(v),bc))\quad\text{(by (23) and the linearity of } \alpha_V^{n})\\
&\quad=0\quad\text{(by (19) in } V).
\end{aligned}
$$

Then we get (19) for $V^{(n)}$. Finally remarking that

$$
\begin{aligned}
&as_{A,V^{(n)}}(\rho_r^{n}(v,a),\alpha_A(b),\alpha_A(c))\\
&\quad=as_{A,V^{(n)}}(v\cdot\alpha_A^{n}(a),\alpha_A(b),\alpha_A(c))\\
&\quad=\rho_r^{n}(\rho_r^{n}(v\cdot\alpha_A^{n}(a),\alpha_A(b),\alpha_A^{2}(c))-\rho_r^{n}(\alpha_V(v)\cdot\alpha_A^{n+1}(a),\mu(\alpha_A(b),\alpha(_A(c))\\
&\quad=\rho_r(\rho_r(v\cdot\alpha_A^{n}(a),\alpha_A^{n+1}(b),\alpha_A^{n+2}(c))\\
&\qquad\quad-\rho_r(\alpha_V(v)\cdot\alpha^{n+1}(a),\mu(\alpha_A^{n+1}(b),\alpha_A^{n+1}(c))\\
&\quad=\alpha_{A,V}(v\cdot\alpha_A^{n}(a),\alpha_A^{n+1}(b),\alpha_A^{n+1}(c))\,,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
as_{A,V^{(n)}}(\rho_r^{n}(v,c),\alpha_A(b),\alpha_A(a))&=as_{A,V}(v\cdot\alpha_A^{n}(c),\alpha_A^{n+1}(b),\alpha_A^{n+1}(a))\,,\\
as_{A,V^{(n)}}(ac,\alpha_A(b),\alpha_V(v))&=as_{A,V}(\alpha_A^{n}(a)\alpha_A^{n}(c),\alpha_A^{n+1}(b),\alpha_V(v))
\end{aligned}
$$

(20) is proved for $V^{(n)}$ as it follows:

$$
\begin{aligned}
&as_{A,V^{(n)}}(\rho_r^{n}(v,a),\alpha_A(b),\alpha_A(c))+as_{A,V^{(n)}}(\rho_r^{n}(v,c),\alpha_A(b),\alpha_A(a))\\
&\qquad\qquad\qquad\qquad+as_{A,V^{(n)}}(ac,\alpha_A(b),\alpha_V(v))\\
&\quad=\alpha_V(v\cdot\alpha_A^{n}(a),\alpha_A^{n+1}(b),\alpha_A^{n+1}(c))+as_{A,V}(v\cdot\alpha_A^{n}(c),\alpha_A^{n+1}(b),\alpha_A^{n+1}(a))\\
&\qquad+as_{A,V}(\alpha_A^{n}(a)\alpha_A^{n}(c),\alpha_A^{n+1}(b),\alpha_V(v))
\end{aligned}
$$

$$\begin{aligned}&= \alpha_V(v \cdot \alpha_A^n(a), \alpha_A(\alpha_A^n(b)), \alpha_A(\alpha_A^n(c)))\\&\quad + as_{A,V}(v \cdot \alpha_A^n(c), \alpha_A(\alpha_A^n(b)), \alpha_A(\alpha_A^n(a)))\\&\quad + as_{A,V}(\alpha_A^n(a)\alpha_A^n(c), \alpha_A(\alpha_A^n(b)), \alpha_V(v))\\&= 0 (\text{by (20) in } V).\end{aligned}$$

We conclude that $V^{(n)}$ is a Hom-Jordan A-bimodule. ∎

Example 4.15. Consider the Hom-Jordan algebra $\mathcal{A}^+$ of the Examples 2.10 and the subspace $V = \text{span}(e_1, e_3)$ of A. Then (V, μ_V, α_V) is a Hom-ideal of $\mathcal{A}^+$ where $\mu_V = \mu_{A_{|V}}$ and $\alpha_V = \alpha_{A_{|V}}$. It follows that $(V, \rho_l, \rho_r, \alpha_V)$ is a Hom-Jordan $\mathcal{A}^+$-bimodule where ρ_l and ρ_r are defined as in Examples 4.7. We have $\alpha_V^2 = Id_V$, then by Proposition 4.14, the structure maps $\rho_l^{(2)} = \rho_l \circ (\alpha_A^2 \otimes Id_V)$ and $\rho_r^{(2)} = \rho_r \circ (Id_V \otimes \alpha_A^2)$ give the Hom-module (V, α_V) the structure of a Hom-Jordan $\mathcal{A}^+$-bimodule that we denote by $V^{(2)}$.

Corollary 4.16. *Let (A, μ, α_A) be a Hom-Jordan algebra and (V, α_V) be a Hom-Jordan A-bimodule with the structure maps ρ_l and ρ_r such that α_V is an involution. Then (V, α_V) is a Hom-Jordan A-bimodule with the structure maps $\rho_l^{(2)} = \rho_l \circ (\alpha_A^2 \otimes Id_V)$ and $\rho_r^{(2)} = \rho_r \circ (Id_V \otimes \alpha_A^2)$.*

Example 4.17. Consider the Hom-Jordan algebra $\mathcal{B}^+$ of the Examples 2.10 and the subspace $V = \text{span}(e_1, e_2)$ of B. Then (V, μ_V, α_V) is a Hom-ideal of $\mathcal{B}^+$ where $\mu_V = \mu_{B_{|V}}$ and $\alpha_V = \alpha_{B_{|V}}$. Therefore $(V, \rho_l, \rho_r, \alpha_V)$ is a Hom-Jordan $\mathcal{B}^+$-bimodule where ρ_l and ρ_r are defined as in Examples 4.7. Note that α_V is involutive, i.e., $\alpha_V^2 = Id_V$, then by Corollary 4.16, the structure maps $\rho_l^{(2)} = \rho_l \circ (\alpha_B^2 \otimes Id_V)$ and $\rho_r^{(2)} = \rho_r \circ (Id_V \otimes \alpha_B^2)$ give the Hom-module (V, α_V) the structure of a Hom-Jordan $\mathcal{B}^+$-bimodule.

The following result is similar to theorem 3.8. It says that Jordan bimodules can be deformed into Hom-Jordan bimodules via an endomorphism.

Theorem 4.18. *Let (A, μ) be a Jordan algebra, V be a Jordan A-bimodule with the structure maps ρ_l and ρ_r, α_A be an endomorphism of the Jordan algebra A and α_V be a linear self-map of V such that $\alpha_V \circ \rho_l = \rho_l \circ (\alpha_A \otimes \alpha_V)$ and $\alpha_V \circ \rho_r = \rho_r \circ (\alpha_V \otimes \alpha_A)$. Write A_{α_A} for the Hom-Jordan algebra $(A, \mu_{\alpha_A}, \alpha_A)$ and V_{α_V} for the Hom-module (V, α_V). Then the maps:*

$$\tilde{\rho}_l = \alpha_V \circ \rho_l \qquad \text{and} \qquad \tilde{\rho}_r = \alpha_V \circ \rho_r$$

give the Hom-module V_{α_V} the structure of a Hom-Jordan A_{α_A}-bimodule.

Proof. It is easy to prove that the relation (16) for V_{α_V} holds and both maps $\tilde{\rho_l}$, $\tilde{\rho_r}$ are morphisms. Remarking that

$$as_{A,V_{\alpha_V}} = \alpha_V^2 \circ as_{A,V} \tag{26}$$

we first compute

$$\begin{aligned}
&\circlearrowleft_{(a,b,c)} as_{A,V_{\alpha_V}}(\alpha_A(a), \alpha_V(v), \mu_{\alpha_A}(b,c))\\
&\quad= \circlearrowleft_{(a,b,c)} \alpha_V^2(as_{A,V}(\alpha_A(a), \alpha_V(v), \alpha_A(bc))) \quad \text{(by (26))}\\
&\quad= \circlearrowleft_{(a,b,c)} \alpha_V^3((as_{A,V}(a, v, bc)) \quad \text{(by (23))}\\
&\quad= \alpha_V^3(\circlearrowleft_{(a,b,c)} (as_{A,V}(a, v, bc))\\
&\quad= 0 \quad \text{(by (19) in } V)
\end{aligned}$$

and then, we get (19) for V_{α_V}. Finally, we get

$$\begin{aligned}
&as_{A,V_{\alpha_V}}(\tilde{\rho_r}(v,a), \alpha_A(b), \alpha_A(c)) + as_{A,V_{\alpha_V}}(\tilde{\rho_r}(v,c), \alpha_A(b), \alpha_A(a))\\
&\qquad\qquad + as_{A,V_{\alpha_V}}(\mu_{\alpha_A}(a,c), \alpha_A(b), \alpha_V(v))\\
&= \alpha_V^2(as_{A,V}(\tilde{\rho_r}(v,a), \alpha_A(b), \alpha_A(c))) + \alpha_V^2(as_{A,V}(\tilde{\rho_r}(v,c), \alpha_A(b), \alpha_A(a)))\\
&\quad + \alpha_V^2(as_{A,V}(\mu_{\alpha_A}(a,c), \alpha_A(b), \alpha_V(v))) \quad \text{(by (26))}\\
&= \alpha_V^2(as_{A,V}(\alpha_V(v \cdot a), \alpha_A(b), \alpha_A(c)))\\
&\quad + \alpha_V^2(as_{A,V}(\alpha_V(v \cdot c), \alpha_A(b), \alpha_A(a)))\\
&\quad + \alpha_V^2(as_{A,V}(\alpha_A(ac), \alpha_A(b), \alpha_V(v)))\\
&= \alpha_V^3(as_{A,V}(v \cdot a, b, c)) + \alpha_V^3(as_{A,V}(v \cdot c, b, a))\\
&\quad + \alpha_V^3(as_{A,V}(ac, b, v)) \quad \text{(by 23)}\\
&= \alpha_V^3(as_{A,V}(v \cdot a, b, c) + as_{A,V}(v \cdot c, b, a) + as_{A,V}(ac, b, v))\\
&= 0 \quad \text{(by (20) in } V)
\end{aligned}$$

which is (20) for V_{α_V}. Therefore the Hom-module V_{α_V} has a Hom-Jordan A_{α_A}-bimodule structure. ∎

Corollary 4.19. *Let (A, μ) be a Jordan algebra, V be a Jordan A-bimodule with the structure maps ρ_l and ρ_r, α_A be an endomorphism of the Jordan algebra A and α_V be a linear self-map of V such that $\alpha_V \circ \rho_l = \rho_l \circ (\alpha_A \otimes \alpha_V)$ and $\alpha_V \circ \rho_r = \rho_r \circ (\alpha_V \otimes \alpha_A)$.*

Moreover, suppose that there exists $n \in \mathbb{N}$ such that $\alpha_V^n = Id_V$. Write A_{α_A} for the Hom-Jordan algebra $(A, \mu_{\alpha_A}, \alpha_A)$ and V_{α_V} for the Hom-module

(V, α_V). *Then the maps:*

$$\tilde{\rho}_l^{(n)} = \rho_l \circ (\alpha_A^{n+1} \otimes \alpha_V) \text{ and } \tilde{\rho}_r^{(n)} = \rho_r \circ (\alpha_V \otimes \alpha_A^{n+1}) \tag{27}$$

give the Hom-module V_α the structure of a Hom-Jordan A_{α_A}-bimodule for each $n \in \mathbb{N}$.

Proof. The proof follows from Proposition 4.14 and Theorem 4.18. ∎

Similarly to Hom-alternative algebras, the split null extension, determined by the given bimodule over a Hom-Jordan algebra, is constructed as follows:

THEOREM 4.20. *Let (A, μ, α_A) be a Hom-Jordan algebra and (V, α_V) be a Hom-Jordan A-bimodule with the structure maps ρ_l and ρ_r. Then $(A \oplus V, \tilde{\mu}, \tilde{\alpha})$ is a Hom-Jordan algebra where*
$\tilde{\mu} : (A \oplus V)^{\otimes 2} \to A \oplus V$, $\tilde{\mu}(a+m, b+n) := ab + a \cdot n + m \cdot b$ *and* $\tilde{\alpha} : A \oplus V \to A \oplus V$, $\tilde{\alpha}(a+m) := \alpha_A(a) + \alpha_V(m)$

Proof. First, the commutativity of $\tilde{\mu}$ follows from the one of μ. Next, the multiplicativity of $\tilde{\alpha}$ with respect to $\tilde{\mu}$ follows from the one of α with respect to μ and the fact that ρ_l and ρ_r are morphisms of Hom-modules. Finally, we prove the Hom-Jordan identity (3) for $E = A \oplus V$ as it follows

$$\begin{aligned}
&as_E(\tilde{\mu}(x+m, x+m), \tilde{\alpha}(y+n), \tilde{\alpha}(x+m)) \\
&\quad = \tilde{\mu}(\tilde{\mu}(\tilde{\mu}(x+m, x+m), \tilde{\alpha}(y+n)), \tilde{\alpha}^2(x+m)) - \tilde{\mu}(\tilde{\alpha}(\tilde{\mu}(x+m, x+m)), \\
&\qquad\qquad \tilde{\mu}(\tilde{\alpha}(y+n), \tilde{\alpha}(x+m))) \\
&\quad = \tilde{\mu}(\tilde{\mu}(x^2 + x \cdot m + m \cdot x, \alpha_A(y) - \tilde{\mu}(\alpha_A(x^2) + \alpha_V(n)), \alpha_A^2(x) + \alpha_V^2(m)) \\
&\qquad + \alpha_V(x \cdot m) + \alpha_V(m \cdot x), \tilde{\mu}(\alpha_A(y) + \alpha_V(n), \alpha_A(x) + \alpha_V(m))) \\
&\quad = \tilde{\mu}(x^2\alpha_A(y) + x^2 \cdot \alpha_V(n) + (x \cdot m) \cdot \alpha_A(y) + (m \cdot x) \cdot \alpha_A(y), \alpha_A^2(x) \\
&\qquad + \alpha_V^2(m)) - \tilde{\mu}(\alpha_A^2(x^2) + \alpha_V(x \cdot m) + \alpha_V(m \cdot x), \alpha_A(y)\alpha_A(x) \\
&\qquad + \alpha_A(y) \cdot \alpha_V(m) + \alpha_V(n) \cdot \alpha_A(x)) \\
&\quad = (x^2\alpha_A(y))\alpha_A^2(x) + (x^2\alpha_A(y)) \cdot \alpha_V^2(m) + (x^2 \cdot \alpha_V(n)) \cdot \alpha_A^2(x) \\
&\qquad + ((x \cdot m) \cdot \alpha_A(y)) \cdot \alpha_A^2(x) + ((m \cdot x) \cdot \alpha_A(y)) \cdot \alpha_A^2(x)) \\
&\qquad - \alpha_A(x^2)(\alpha_A(y)\alpha_A(x)) - \alpha_A(x^2) \cdot (\alpha_A(y) \cdot \alpha_V(m)) \\
&\qquad - \alpha_A(x^2) \cdot (\alpha_V(n) \cdot \alpha_A(x)) - \alpha_V(x \cdot m) \cdot (\alpha_A(y)\alpha_A(x)) \\
&\qquad - \alpha_V(m \cdot x) \cdot (\alpha_A(y)\alpha_A(x))
\end{aligned}$$

$$
\begin{aligned}
&= as_A(x^2, \alpha_A(y), \alpha_A(x)) + as_{A,V}(x^2, \alpha_A(y), \alpha_V(m)) \\
&\quad + as_{A,V}(x^2, \alpha_V(n), \alpha_A(x)) + as_{A,V}(x \cdot m, \alpha_A(y), \alpha_A(x)) \\
&\quad + as_{A,V}(m \cdot x, \alpha_A(y), \alpha_A(x)) \\
&= \underbrace{as_{A,V}(m \cdot x, \alpha_A(y), \alpha_A(x)) + as_{A,V}(m \cdot x, \alpha_A(y), \alpha_A(x))}_{0} \\
&\quad + \underbrace{as_{A,V}(x^2, \alpha_V(n), \alpha_A(x)) + as_{A,V}(x^2, \alpha_A(y), \alpha_V(m))}_{0} \\
&\quad + \underbrace{as_A(x^2, \alpha_A(y), \alpha_A(x))}_{0} \;=\; 0\,,
\end{aligned}
$$

where the first 0 follows from (20), the second from (19) (see Remarks 4.6) and the last from the Hom-Jordan identity (3) in A. We conclude then that $(A \oplus V, \tilde{\mu}, \tilde{\alpha})$ is a Hom-Jordan algebra. ∎

Similarly as Hom-alternative algebra case, let give the following:

Remark 4.21. Consider the split null extension $A \oplus V$ determined by the Hom-Jordan bimodule (V, α_V) for the Hom-Jordan algebra (A, μ, α_A) in the previous theorem. Write elements $a + v$ of $A \oplus V$ as (a, v). Then there is an injective homomorphism of Hom-modules $i : V \to A \oplus V$ given by $i(v) = (0, v)$ and a surjective homomorphism of Hom-modules $\pi : A \oplus V \to A$ given by $\pi(a, v) = a$. Moreover, $i(V)$ is a Hom-ideal of $A \oplus V$ such that $A \oplus V/i(V) \cong A$. On the other hand, there is a morphism of Hom-algebras $\sigma : A \to A \oplus V$ given by $\sigma(a) = (a, 0)$ which is clearly a section of π. Hence, we obtain the abelian split exact sequence of Hom-Jordan algebras and (V, α_V) is a Hom-Jordan bimodule for A via π.

References

[1] H. Ataguema, A. Makhlouf, S.D. Silvestrov, Generalization of n-ary Nambu algebras and beyond, *J. Math. Phys.* **50** (8) (2009), 083501, 15 pp.

[2] I. Bakayoko, B. Manga, Hom-alternative modules and Hom-Poisson comodules. `arXiv:1411.7957v1`

[3] S. Eilenberg, Extensions of general algebras, *Ann. Soc. Polon. Math.* **21** (1948), 125 – 34.

[4] F. Gürsey, C.-H. Tze, On The Role of Division, Jordan and Related Algebras in Particle Physics, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[5] N. Huang, L.Y. Chen, Y. Wang, Hom-Jordan algebras and their α^k-(a,b,c)-derivations, *Comm. Algebra* **46** (6) (2018), 2600 – 2614.

[6] J.T. Hartwig, D. Larsson, S.D. Silvestrov, Deformations of Lie algebras using σ-derivations, *J. Algebra* **292** (2006), 314 – 361.

[7] N. Jacobson, General representation theory of Jordan algebras, *Trans. Amer. Math. Soc.* **70** (1951), 509 – 530.

[8] N. Jacobson, Structure of alternative and Jordan bimodules, *Osaka Math. J.* **6** (1954), 1 – 71.

[9] P. Jordan, J. von Neumann, E. Wigner, On an algebraic generalization of the quantum mechanical formalism, *Ann. of Math. (2)* **35** (1934), 29 – 64.

[10] D. Larsson, S.D. Silvestrov, Quasi-Hom-Lie algebras, central extensions and 2-cocycle-like identities, *J. Algebra* **288** (2005), 321 – 344.

[11] D. Larsson, S.D. Silvestrov, Quasi-Lie algebras, in "Noncommutative Geometry and Representation Theory in Mathematical Physics", Contemp. Math., 391, Amer. Math. Soc., Providence, RI, 2005, 241 – 248.

[12] D. Larsson, S.D. Silvestrov, Quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$ using twisted derivations, *Comm. Algebra* **35** (12) (2007), 4303 – 4318.

[13] A. Makhlouf, S. Silvestrov, Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras, *Forum Math.* **22** (4) (2010), 715 – 739.

[14] A. Makhlouf, Hom-Alternative algebras and Hom-Jordan algebras, *Int. Electron. J. Algebra* **8** (2010), 177 – 190.

[15] A. Makhlouf, S.D. Silvestrov, Hom-algebra structures, *J. Gen. Lie Theory Appl.* **2** (2008), 51 – 64.

[16] S. Okubo, "Introduction to Octonion and other Non-associative Algebras in Physics", Cambridge University Press, Cambridge, 1995.

[17] R.D. Schafer, Representations of alternative algebras, *Trans. Amer. Math. Soc.* **72** (1952), 1 – 17.

[18] Y. Sheng, Representation of hom-Lie algebras, *Algebr. Represent. Theory* **15** (6) (2012), 1081 – 1098.

[19] T.A. Springer, F.D. Veldkamp, "Octonions, Jordan Algebras, and Exceptional Groups", Springer-Verlag, Berlin, 2000.

[20] J. Tits, R.M. Weiss, "Moufang Polygons", Springer-Verlag, Berlin, 2002.

[21] D. Yau, Hom-algebras and homology. `arXiv:0712.3515v1`

[22] D. Yau, Module Hom-algebras. `arXiv:0812.4695v1`

[23] D. Yau, Hom-Maltsev, Hom-alternative and Hom-Jordan algebras, *Int. Electron. J. Algebra* **11** (2012), 177 – 217.

[24] A. Zahary, A. Makhlouf, Structure and classification of Hom-associative algebras. `arXiv:1906.04969V1[math.RA]`