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Multifractal formalism of an inhomogeneous multinomial measure with various parameters

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Abstract: In this paper, we study the refined multifractal formalism in a product symbolic space and we estimate the spectrum of a class of inhomogeneous multinomial measures constructed on the product symbolic space.

Key words: Hausdorff dimension, packing dimension, fractal, multifractal.

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1. Introduction

The multifractal formalism of a measure μ aims to establish a relationship between the dimension of level set of the local Hölder exponent of μ to the Legendre transform of what is called the "free energy" function. A problem initially raised and studied for physical motivations [8, 9, 11, 12, 10]. It will be convenient to give a brief description of the multifractal formalism. Let $\mathbb{X}$ be a metric space. The local Hölder exponent $\alpha_\mu(x)$ at the point $x \in \mathbb{X}$ is defined to be

$$\alpha_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

where $B(x,r)$ stands for the ball of radius r centered at x. The measure μ is said to satisfy the multifractal formalism at $\alpha \geq 0$, if the Hausdorff dimension (dim) and the packing dimension (Dim) of the level set $E(\alpha)$ which is defined by

$$E(\alpha) = \{x \in \operatorname{supp}(\mu) \,:\, \alpha_\mu(x) = \alpha\},$$

are equal respectively to the value of the Legendre transform at α of a scale function τ_μ associated to the measure μ, i.e.,

$$\dim E(\alpha) = \operatorname{Dim} E(\alpha) = \tau_\mu^*(\alpha),$$

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where $f^*(x) = \inf_y (xy + f(y))$ is the Legendre transform of a function f and $\mathrm{supp}(\mu)$ is the topological support of μ.

The upper bound for $\dim E(\alpha)$ (respectively $\mathrm{Dim}\, E(\alpha)$) is obtained by a standard covering argument as Besicovisch's covering Theorem and Vitali's Lemma [13]. However, the lower bound is usually much harder to prove, it is related to the existence of an auxiliary measure such as a Gibbs measure [13] or a Frostman measure [3] which is supported by the set to be analyzed.

For this reason, F. Ben Nasr et al. [4] improved the Olsen's result in describing a class of measures satisfying the multifractal formalism and proposed a new sufficient condition that gives the lower bound. In such a situation, they concluded that $B_\mu(q) = b_\mu(q)$, where b_μ and B_μ are Olsen's functions. Besides, they constructed inhomogeneous Bernoulli products, such measures whose both multifractal dimension functions b_μ and B_μ agree at one or two points only. Which implies a valid refined multifractal formalism no more than two points. In [5], Ben Nasr and Peyrière constructed an example of a "bad" measure on the interval $\{0,1\}^{\mathbb{N}}$ for which the Olsen's functions b_μ and B_μ differ and the Hausdorff dimensions of the sets $E(\alpha)$ are given by the Legendre transform of b_μ, and their packing dimensions by the Legendre transform of B_μ, i.e., $b_\mu(q) < B_\mu(q)$ for all $q \in \{0,1\}$ and

$$\dim E(\alpha) = b_\mu^*(\alpha) \quad \text{and} \quad \mathrm{Dim}\, E(\alpha) = B_\mu^*(\alpha), \qquad \text{for some } \ \alpha \geq 0.$$

Shen [14] and Wu et al. [17, 18, 19] revisited this example such that the functions B_μ and b_μ can be real analytic. Motivated by these examples N. Attia and B. Selmi [1, 2] introduced and studied a new multifractal formalism based on the Hewitt-Stromberg measures and showed that this formalism is completely parallel to Olsen's multifractal formalism based on the Hausdorff and packing measures.

In the present work, let $2 \leq r_1 < r_2$ be two integers, we consider a class of measures defined on a product symbolic space $\mathbb{A}_1 \times \mathbb{A}_2$ endowed with the distance product where $\mathbb{A}_i = \{0, \ldots, r_i - 1\}$ for $i = 1, 2$, and constructed on the rectangles that flatten as their diameters tend to zero. However, these rectangles do not allow the calculation of the Hausdorff dimension, hence the difficulty of the problem. The aim of this paper is to study the validity of the refined multifractal formalism of this class of measures.

The paper is organized as follows. In Section 2, we give some notations and definitions which will be useful. In the third section we consider a sequence of finite partitions of a product symbolic space made of rectangles and we show through an example that the almost squares allow the calculation of the

Hausdorff and packing dimensions. In Section 4, we consider a variant of the refined multifractal formalism as already introduced by Ben Nasr and Peyrière [5] which we adapt it to almost squares and estimate the dimensions of the level sets $E(\alpha)$. Finally, we apply our results to a class of inhomogeneous measures defined on the product symbolic space.

2. Notations and definitions

In this section, we will recall the Hausdorff and packing measures and their dimensions. Let $(\mathbb{X}, d)$ be a separable metric space. The diameter of a non-empty set $E \subseteq \mathbb{X}$ is given by

$$\operatorname{diam} E = \sup \{d(x, y) \,:\, x, y \in E\},$$

with the convention that $\operatorname{diam}(\emptyset) = 0$.

We define the closed ball with center $x \in \mathbb{X}$ and radius $r > 0$ as

$$B(x, r) = \{y \in \mathbb{X} \,:\, d(x, y) \leq r\}.$$

A finite or countable collection of subsets $\{U_i\}_i$ of $\mathbb{X}$ is called a δ-cover of $E \subseteq \mathbb{X}$, if for each i we have $\operatorname{diam} U_i \leq \delta$ and $E \subset \bigcup_i U_i$.

Suppose that E is a subset of $\mathbb{X}$ and s is a non-negative number. For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_i \operatorname{diam}(U_i)^s \,:\, \{U_i\}_i \text{ is a } \delta\text{-cover of } E \right\}.$$

As δ decreases, the class of δ-covers of E is reduced. Therefore, this infimum increases and approaches a limit as $\delta \searrow 0$. Thus we define

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

We term $\mathcal{H}^s(E)$ the s-dimensional Hausdorff measure of E. Then we define the Hausdorff dimension of E as

$$\dim(E) = \sup \{s \geq 0 \,:\, \mathcal{H}^s(E) = \infty\} = \inf \{s \geq 0 \,:\, \mathcal{H}^s(E) = 0\}.$$

Remark 1. Notice that the covering of E with centered balls in E allow the calculation of the Hausdorff dimension of E, for more details see [7].

We will now define the packing measure. First, let define a δ-packing of $E \subset \mathbb{X}$ to be a finite or countable collection of disjoint balls $\{B(x_i, r_i)\}_i$ of diameter at most δ and with centers in E. For $s \geq 0$ and $\delta > 0$, let

$$\overline{\mathcal{P}}^s_\delta(E) = \sup \left\{ \sum_i (2r_i)^s \,:\, \{B(x_i, r_i)\}_i \text{ is a } \delta\text{-packing of } E \right\}.$$

From this the s-dimensional pre-packing measure $\overline{\mathcal{P}}^s$ of E is defined by

$$\overline{\mathcal{P}}^s(E) = \lim_{\delta \to 0} \overline{\mathcal{P}}^s_\delta(E).$$

Finally, we define the s-dimensional packing measure $\mathcal{P}^s(E)$ of E by

$$\mathcal{P}^s(E) = \inf \left\{ \sum_i \overline{\mathcal{P}}^s(E_i) \,:\, E \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

The packing dimension of E, denoted by $\mathrm{Dim}(E)$, is defined in the same way as Hausdorff dimension, that means

$$\mathrm{Dim}(E) = \sup \{ s \geq 0 \,:\, \mathcal{P}^s(E) = \infty \} = \inf \{ s \geq 0 \,:\, \mathcal{P}^s(E) = 0 \}.$$

For more details about the Hausdorff, packing measures and their dimensions see [15, 16, 7].

3. Calculation of the Hausdorff and packing dimensions on the product symbolic space on different basis

For practical reasons, we shall need basic notions about the set of words on an alphabet. Let $2 \leq r_1 < r_2$ be two integers. For $i \in \{1, 2\}$, given $A_i = \{0, \dots, r_i - 1\}$ a finite alphabet. For all $n \in \mathbb{N}^*$, each element in A_i^n is denoted by a string of n letters or digits in A_i that we call a word; by convention A_i^0 is reduced to the empty word $\emptyset$. Let $A_i^* = \bigcup_{n \geq 0} A_i^n$ be the set of finite words built over A_i and $\mathbb{A}_i = A_i^{\mathbb{N}^*}$ the symbolic space over A_i.

The set $A_i^* \cup \mathbb{A}_i$ is endowed with the concatenation operation: If $\omega \in A_i^*$ and $\omega' \in A_i^* \cup \mathbb{A}_i$, we denote by $\omega.\omega'$ the word obtained by juxtaposition of the two words ω and ω'.

For each finite word $\omega \in A_i^*$, $[\omega]$ is the cylinder $\omega \cdot \mathbb{A}_i = \{\omega \cdot \omega' \,:\, \omega' \in \mathbb{A}_i\}$. Furthermore, if $\omega = \omega_1 \cdots \omega_k \cdots \in \mathbb{A}_i$ and $n \in \mathbb{N}$ then $\omega_{|n}$ stands for the prefix $\omega_1 \cdots \omega_n$ of ω for $n \geq 1$ and the empty word otherwise. Each set $\mathbb{A}_i$

is endowed with the ultrametric distance $d_i : (z, z') \in \mathbb{A}_i^2 \longmapsto r_i^{-|z\wedge z'|}$, where $z \wedge z'$ is defined to be the longest prefix common to both z and z' and $|z|$ the length of a word $z \in A_i^* \cup \mathbb{A}_i$. Then the product symbolic space $\mathbb{A}_1 \times \mathbb{A}_2$ is endowed with the ultrametric distance.

$$d((x, y), (x', y')) = \max(d_1(x, x'), d_2(y, y')).$$

In the next, if $\omega \in A_1^k$ and $\omega' \in A_2^{k'}$, we call $R(\omega, \omega')$ the rectangle obtained as the product of the cylinders $[\omega]$ and $[\omega']$. We denote by

$$\left|R(\omega, \omega')\right|_M = \sup\left(\frac{1}{r_1^k}, \frac{1}{r_2^{k'}}\right),$$

and

$$\left|R(\omega, \omega')\right|_m = \inf\left(\frac{1}{r_1^k}, \frac{1}{r_2^{k'}}\right).$$

We say that a sequence $\{\xi_n\}_{n\geq 1}$ of finite partitions of $\mathbb{A}_1 \times \mathbb{A}_2$ made of rectangles satisfies condition (1) if

$$\lim_{n\to\infty} \sup_{R\in\xi_n} \mathrm{diam}(R) = 0 \quad \text{and} \quad \xi_{n+1} \quad \text{is a refinement of } \xi_n. \tag{1}$$

In all over this work, we will consider a sequence $\{\xi_n\}_{n\geq 1}$ of finite partitions of $\mathbb{A}_1 \times \mathbb{A}_2$ made of rectangles verifying (1) and we put $\xi = \bigcup_{n\geq 1} \xi_n$. If R belongs to ξ_{n+1}, we define by $p(R)$ the element of ξ_n that contains it.

Let E be a nonempty subset of $\mathbb{A}_1 \times \mathbb{A}_2$ and s a strictly positive real number. For all $\varepsilon > 0$, a finite or countable collection of rectangles $\{R_j\}_j$ is called an ε-covering of E if $\mathrm{diam}(R_j) \leq \varepsilon$ for all j and $E \subset \bigcup_j R_j$.

Let

$$\mathcal{H}^s_{\xi,\varepsilon}(E) = \inf\left\{\sum_j \mathrm{diam}(R_j)^s \;:\; R_j \in \xi, \{R_j\}_j \text{ is an } \varepsilon\text{-covering of } E\right\}$$

and

$$\mathcal{H}^s_\xi(E) = \lim_{\varepsilon\to 0} \mathcal{H}^s_{\xi,\varepsilon}(E).$$

Finally, the dimension $\dim_\xi(E)$ is given by

$$\dim_\xi(E) = \inf\left\{s > 0 \;:\; \mathcal{H}^s_\xi(E) = 0\right\} = \sup\left\{s > 0 \;:\; \mathcal{H}^s_\xi(E) = \infty\right\}.$$

Here, we define an ε-packing of $E \subset \mathbb{A}_1 \times \mathbb{A}_2$ to be a finite or countable collection of disjoint rectangles $\{R_j\}_j$ of diameter not exceeding ε and with $R_j \cap E \neq \emptyset$. For $\varepsilon > 0$, we define

$$\overline{\mathcal{P}}^s_{\xi,\varepsilon}(E) = \sup \left\{ \sum_j \operatorname{diam}(R_j)^s \; : \; R_j \in \xi, \{R_j\}_j \text{ is an } \varepsilon\text{-packing of } E \right\}.$$

Then $\overline{\mathcal{P}}^s_{\xi,\varepsilon}(E)$ decreases as ε increases, so we may take the limit

$$\overline{\mathcal{P}}^s_{\xi}(E) = \lim_{\varepsilon \to 0} \overline{\mathcal{P}}^s_{\xi,\varepsilon}(E).$$

Unfortunately, $\overline{\mathcal{P}}^s_{\xi}(E)$ is not an outer measure, to overcome this difficulty we define

$$\mathcal{P}^s_{\xi}(E) = \inf \left\{ \sum_j \overline{\mathcal{P}}^s_{\xi}(E_j) \; : \; E \subseteq \bigcup_j E_j \right\}.$$

The definition of packing dimension parallels that of Hausdorff dimension. So, let $\operatorname{Dim}_\xi(E)$ defined such that

$$\begin{aligned} \operatorname{Dim}_\xi(E) &= \inf \left\{ s > 0 \; : \; \mathcal{P}^s_\xi(E) = 0 \right\} \\ &= \sup \left\{ s > 0 \; : \; \mathcal{P}^s_\xi(E) = \infty \right\}. \end{aligned}$$

In the following proposition we will give some conditions on a family ξ of rectangles of the symbolic space $\mathbb{A}_1 \times \mathbb{A}_2$ such that for every part E of $\mathbb{A}_1 \times \mathbb{A}_2$, we have

$$\dim(E) = \dim_\xi(E) \quad \text{and} \quad \operatorname{Dim}(E) = \operatorname{Dim}_\xi(E).$$

Proposition 3.1. *Suppose that*

(i) $\lim_{n\to\infty} \sup_{R \in \xi_n} \log |R|_m / \log |R|_M = 1,$

(ii) $\lim_{n\to\infty} \sup_{R \in \xi_n} \log |R|_M / \log |p(R)|_M = 1.$

Then for any part E of $\mathbb{A}_1 \times \mathbb{A}_2$, we have

$$\operatorname{Dim}_\xi(E) = \operatorname{Dim}(E), \tag{2}$$

$$\dim_\xi(E) = \dim(E). \tag{3}$$

Proof. In order to prove the equality (2), we start by proving that $\mathrm{Dim}_\xi(E) \le \mathrm{Dim}(E)$.

Let $t > \mathrm{Dim}(E)$ and $\eta > 0$ such that $\frac{t}{1+\eta} > \mathrm{Dim}(E)$. It follows from assumption (i) that there exists an integer n_0 such that for all $n \ge n_0$ and for all $R \in \xi_n$, we have

$$|R|_M^{1+\eta} \le |R|_m \,.$$

Take $\{E_j\}_j$ a cover of E and choose $\{R_k\}_k$ an ε-packing of E_j with $\varepsilon \le \inf\limits_{R\in\xi_{n_0}} \mathrm{diam}(R)$. For $j \in \mathbb{N}$, fix $x_k \in R_k \cap E_j$, we denote by $B_k = B(x_k, |R_k|_m)$. It is clear that $\{B_k\}_k$ is an ε-packing of E_j.

As $\varepsilon \le \inf\limits_{R\in\xi_{n_0}} \mathrm{diam}(R)$ we get for all integer k,

$$|R_k|_M^{1+\eta} \le |R_k|_m \tag{4}$$

and

$$\sum_k \mathrm{diam}(R_k)^t \le \sum_k \mathrm{diam}(B_k)^{\frac{t}{1+\eta}}.$$

Then,

$$\overline{\mathcal{P}}^t_{\xi,\varepsilon}(E_j) \le \overline{\mathcal{P}}^{\frac{t}{1+\eta}}_{\varepsilon}(E_j)$$

and as ε goes to 0, yields

$$\overline{\mathcal{P}}^t_{\xi}(E_j) \le \overline{\mathcal{P}}^{\frac{t}{1+\eta}}(E_j).$$

Therefore,

$$\mathcal{P}^t_{\xi}(E) \le \mathcal{P}^{\frac{t}{1+\eta}}(E) < +\infty$$

consequently,

$$\mathrm{Dim}_\xi(E) < t\,, \text{ for all } t > \mathrm{Dim}(E),$$

which implies that

$$\mathrm{Dim}_\xi(E) \le \mathrm{Dim}(E).$$

In order to obtain the other inequality, fix $t > \mathrm{Dim}_\xi(E)$ and $\eta > 0$ such that $\frac{t}{1+\eta} > \mathrm{Dim}_\xi(E)$. Using assumption (ii) there exists an integer n_0 such that for all $n \ge n_0$ and for all $R \in \xi_n$, we have

$$|P(R)|_M^{1+\eta} \le |R\,|_M \,. \tag{5}$$

Let $\{E_j\}_j$ be a cover of E and $\{B_k = B(x_k, r_k)\}_k$ an ε-packing of E_j with $\varepsilon \leq \inf\limits_{R\in\xi_{n_0}} \operatorname{diam}(R)$. If R_k is a rectangle such that

$$R_k \subset B(x_k, r_k) \qquad \text{and} \qquad P(R_k) \nsubseteq B(x_k, r_k), \tag{6}$$

then $\{R_k\}_k$ is an ε-packing of E_j. Since $\varepsilon \leq \inf\limits_{R\in\xi_{n_0}} \operatorname{diam}(R)$, we have for all $k \in \mathbb{N}$,

$$|P(R_k)|_M^{1+\eta} \leq |R_k|_M \,. \tag{7}$$

Taking into account relations (6) and (7), we have

$$\sum_k \operatorname{diam}(B_k)^t \leq \sum_k \operatorname{diam}(P(R_k))^t \leq \sum_k \operatorname{diam}(R_k)^{\frac{t}{1+\eta}}.$$

So,

$$\overline{\mathcal{P}}_{\varepsilon}^{t}(E_j) \leq \overline{\mathcal{P}}_{\xi,\varepsilon}^{\frac{t}{1+\eta}}(E_j).$$

As ε goes to zero,

$$\overline{\mathcal{P}}^{t}(E_j) \leq \overline{\mathcal{P}}_{\xi}^{\frac{t}{1+\eta}}(E_j).$$

Then, we obtain

$$\mathcal{P}^{t}(E) \leq \mathcal{P}_{\xi}^{\frac{t}{1+\eta}}(E) < +\infty.$$

Hence,

$$\operatorname{Dim}(E) \leq \operatorname{Dim}_{\xi}(E)$$

which achieves the proof of equality (2).

Now, we will be interested in proving the equality (3).

It is easy to see that $\mathcal{H}^t(E) \leq \mathcal{H}^t_{\xi}(E)$ then $\dim(E) \leq \dim_{\xi}(E)$. Let's prove that

$$\dim_{\xi}(E) \leq \dim(E).$$

Fix $t > \dim(E)$ and $\eta > 0$ such that $\frac{t}{1+\eta} + (2 - 2(1+\eta)^3) > \dim(E)$. Let ε be a positive number such that $\varepsilon \leq \inf\limits_{R\in\xi_{n_0}} \operatorname{diam}(R)$. Pick an ε-covering $\{R_j\}_j$ of E and set $B_j = B(x_j, |R_j|_M)$ such that $R_j \subseteq B_j$.

For all $j \in \mathbb{N}$, there exists a family of disjoint rectangles $\{R_{jk}\}_{k\in L_j}$ such that

$$\bigcup_{k\in L_j} R_{jk} \subset B_j\,, \quad P(R_{jk}) \nsubseteq B_j \quad \text{and} \quad B_j \subseteq \bigcup_{k\in L_j} P(R_{jk}).$$

In a first step, we will calculate the number of $P(R_{jk})$ that cover B_j. We denote by λ the Lebesgue measure on $\mathbb{A}_1 \times \mathbb{A}_2$. Using relations (4) and (5), we have

$$\lambda(P(R_{jk}))^{(1+\eta)^2} \leq \lambda(R_{jk})$$

and

$$\operatorname{diam}(B_j)^{2(1+\eta)^3} \leq \lambda(R_{jk}). \tag{8}$$

Let s and s' be two positive integers such that

$$r_1^{-(s+1)} < |R_{jk}|_M \leq r_1^{-s} \qquad \text{and} \qquad r_2^{-(s'+1)} < |R_{jk}|_M \leq r_2^{-s'}.$$

We have

$$\sum_{k \in L_j} \lambda(R_{jk}) \leq \lambda(B_j) \leq r_1^{-s} r_2^{-s'} \leq (r_1 r_2) \operatorname{diam}(B_j)^2. \tag{9}$$

It follows from inequalities (8) and (9) that

$$\sum_{k \in L_j} \operatorname{diam}(B_j)^{2(1+\eta)^3} \leq r_1 r_2 \operatorname{diam}(B_j)^2.$$

Hence,

$$\operatorname{card}(L_j) \leq r_1 r_2 \operatorname{diam}(B_j)^{2-2(1+\eta)^3}.$$

In a second step, we have

$$|P(R_{jk})|_M^{1+\eta} \leq |R_{jk}|_M \leq \operatorname{diam}(B_j)$$

and

$$\sum_{k \in L_j} |P(R_{jk})|_M^t \leq \sum_{k \in L_j} \operatorname{diam}(B_j)^{\frac{t}{1+\eta}}.$$

So,

$$\begin{aligned}\sum_j \operatorname{diam}(R_{jk})^t &\leq \sum_j \sum_{k \in L_j} |P(R_{jk})|_M^t \\ &\leq \sum_j (r_1 r_2) \operatorname{diam}(B_j)^{2-2(1+\eta)^3} \operatorname{diam}(B_j)^{\frac{t}{1+\eta}}\end{aligned}$$

and

$$\mathcal{H}^t_{\xi,\varepsilon}(E) \leq (r_1 r_2) \mathcal{H}_\varepsilon^{\frac{t}{1+\eta} + (2-2(1+\eta)^3)}(E).$$

Letting ε tend to 0 implies

$$\mathcal{H}_\xi^t(E) \le (r_1 r_2)\mathcal{H}^{\frac{t}{1+\eta}+2-(1+\eta)^3}(E).$$

Finally, we obtain

$$\dim_\xi(E) \le t.$$

And the result yields. ∎

Next, we set a generalization of the Billingsley Theorem [6] in our case. For this purpose, we introduce the following notations. If E is a non empty subset of $\mathbb{A}_1 \times \mathbb{A}_2$ and $x = (x_1, x_2) \in E$, let $\xi = \bigcup_{n\ge 1} \xi_n$ be a family of rectangles satisfying assumptions (i) and (ii) of Proposition 3.1 and $R_n(x)$ be the rectangle of ξ_n containing x.

In the sequel, we define by $\mathcal{P}(\mathbb{A}_1 \times \mathbb{A}_2)$ the set of Borel probability measures on $\mathbb{A}_1 \times \mathbb{A}_2$. For all $\mu \in \mathcal{P}(\mathbb{A}_1 \times \mathbb{A}_2)$ and $\varepsilon > 0$, and $E \in \mathbb{A}_1 \times \mathbb{A}_2$, we define

$$\mu_\varepsilon^\sharp(E) = \inf\left\{\sum_j \mu(R_j) \;:\; R_j \in \xi, \{R_j\}_j \text{ an } \varepsilon\text{-covering of } E\right\},$$

$$\mu^\sharp(E) = \lim_{\varepsilon\to 0} \mu_\varepsilon^\sharp(E)$$

and

$$\operatorname*{ess\,sup}_{x\in E,\mu^\sharp} \mathcal{A}(x) = \inf\left\{t \in \mathbb{R} \;:\; \mu^\sharp(\{x \in E \;:\; \mathcal{A}(x) > t\}) = 0\right\}.$$

Proposition 3.2. *Let E be a subset of $\mathbb{A}_1 \times \mathbb{A}_2$ and $\mu \in \mathcal{P}(\mathbb{A}_1 \times \mathbb{A}_2)$, we have*

(a) $\dim_\xi(E) \le \sup_{x\in E} \liminf_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))}$;

(b) $\operatorname{Dim}_\xi(E) \le \sup_{x\in E} \limsup_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))}$.

If $\mu^\sharp(E) > 0$, then we have

(c) $\dim_\xi(E) \ge \operatorname*{ess\,sup}_{x\in E,\mu^\sharp} \liminf_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))}$;

(d) $\operatorname{Dim}_\xi(E) \ge \operatorname*{ess\,sup}_{x\in E,\mu^\sharp} \limsup_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))}$.

Proof. Let us prove assumption (a). Take $\delta > \sup_{x\in E} \liminf_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))}$, then for all $x \in E$, there exists $k \geq n$ such that

$$\mu(R_k(x)) \geq \operatorname{diam}(R_k(x))^{\delta}.$$

Let ε be a positive number, there exists $\{R_j\}_j$ a family of pairwise disjoint rectangles such that $E \subset \bigcup_j R_j$ with

$$\mu(R_j) \geq \operatorname{diam}(R_j)^{\delta} \qquad \text{and} \qquad \operatorname{diam}(R_j) \leq \varepsilon.$$

We have

$$\sum_j \operatorname{diam}(R_j)^{\delta} \leq \sum_j \mu(R_j) < \infty.$$

Therefore, $\mathcal{H}^{\delta}_{\xi,\varepsilon}(E) < \infty$. Finally, when $\varepsilon \to 0$, we get $\dim_{\xi}(E) \leq \delta$ and the result easily follows.

To prove the assumption (b), take $\delta > \sup_{x\in E} \limsup_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))}$. For all $x \in E$, there exists $n \in \mathbb{N}$ such that, for all $k \geq n$ one has

$$\mu(R_k(x)) \geq \operatorname{diam}(R_k(x))^{\delta}.$$

Consider the set

$$E(n) = \left\{ x \in E \,:\, \text{for each } k \geq n,\, \mu(R_k(x)) \geq \operatorname{diam}(R_k(x))^{\delta} \right\}.$$

Let $\{E_k\}_k$ be a cover of E and $\{R_j\}_j$ be an ε-packing of $E(n) \cap E_k$ with $\varepsilon < \inf_{R\in\xi_{n_0}} \operatorname{diam}(R)$. One has

$$\sum_j \operatorname{diam}(R_j)^{\delta} \leq \sum_j \mu(R_j) < \infty.$$

From which $\overline{\mathcal{P}}^{\delta}_{\xi,\varepsilon}(E(n) \cap E_k) < \infty$. Then we get $\overline{\mathcal{P}}^{\delta}_{\xi}(E(n) \cap E_k) < \infty$ when $\varepsilon \to 0$. Since $E = \bigcup_n E(n)$, we obtain

$$\operatorname{Dim}_{\xi}(E) \leq \delta.$$

Hence (b).

Let us prove assumption (c). Take $\delta < \operatorname*{ess\,sup}_{x\in E,\mu^{\sharp}} \liminf_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))}$ and set

$$E_{\delta} = \left\{ x \in E \,:\, \liminf_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))} > \delta \right\}.$$

Let

$$E_n = \left\{ x \in E_\delta \ : \ \text{ for each } k \geq n, \ \mu(R_k(x)) \leq \operatorname{diam}(R_k(x))^\delta \right\}.$$

It is clear that $E_\delta = \bigcup_n E_n$. As we have $\mu^\sharp(E_\delta) > 0$, there exists $n \in \mathbb{N}$ such that $\mu^\sharp(E_n) > 0$. Then, for any ε-covering $\{R_j\}_j$ of E_n, one has

$$\mu_\varepsilon^\sharp(E_n) \leq \sum_j \mu(R_j) \leq \sum_j \operatorname{diam}(R_j)^\delta.$$

Therefore,

$$\mu_\varepsilon^\sharp(E_n) \leq \mathcal{H}^\delta_{\xi,\varepsilon}(E_n).$$

So,

$$0 < \mu^\sharp(E_n) \leq \mathcal{H}^\delta_\xi(E_n),$$

which implies

$$\dim_\xi(E) \geq \dim_\xi(E_\delta) \geq \dim_\xi(E_n) \geq \delta$$

and assumption (c) yields.

In order to prove assumption (d), let $\delta < \operatorname*{ess\,sup}_{x \in E, \mu^\sharp} \limsup_{n \to \infty} \dfrac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))}$, and put

$$E_\delta = \left\{ x \in E \ : \ \limsup_{n \to \infty} \frac{\log(\mu(R_n(x)))}{\log(\operatorname{diam}(R_n(x)))} > \delta \right\}.$$

We have $\mu^\sharp(E_\delta) > 0$, so there exists a subset F of E_δ such that $\mu^\sharp(F) > 0$. If $x \in F$, then for all $n \in \mathbb{N}$ there exists $k \geq n$ such that

$$\mu(R_k(x)) \leq \operatorname{diam}(R_k(x))^\delta \tag{10}$$

Let $\varepsilon > 0$ and $\{R_j\}_j$ an ε-packing of F satisfying (10). So,

$$\mu_\varepsilon^\sharp(F) \leq \sum_j \mu(R_j) \leq \sum_j \operatorname{diam}(R_j)^\delta.$$

Then

$$\mu_\varepsilon^\sharp(F) \leq \overline{\mathcal{P}}^\delta_{\xi,\varepsilon}(F).$$

This implies

$$0 < \mu^\sharp(F) \leq \overline{\mathcal{P}}^\delta_\xi(F).$$

Hence, if $F = \bigcup_j F_j$, one has

$$0 < \mu^\sharp(F) < \sum_j \mu^\sharp(F_j) \leq \sum_j \overline{\mathcal{P}}_\xi^\delta(F_j).$$

Thus,

$$\mathcal{P}_\xi^\delta(F) > 0.$$

Therefore,

$$\mathrm{Dim}_\xi(E_\delta) \geq \delta,$$

from which the result follows and we achieve the proof of Proposition 3.2. ∎

As a consequence of Proposition 3.2, we obtain the following corollary. We adopt the following convention

$$\frac{\log 0}{\log \rho} = +\infty, \qquad \text{for each } \rho > 0.$$

Corollary 1. *Let $\gamma \in \mathbb{R}$. If μ is a probability Borel measure on $\mathbb{A}_1 \times \mathbb{A}_2$ such that $\mu(E) > 0$, we consider a family ξ of rectangles verifying the assumptions of Proposition* 3.1 *and*

$$E \subset \left\{ x \in \mathbb{A}_1 \times \mathbb{A}_2 \, : \, \lim_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\mathrm{diam}(R_n(x)))} = \gamma \right\},$$

we have

$$\dim_\xi(E) = \mathrm{Dim}_\xi(E) = \gamma.$$

Next, we will be interested in adding an example of application of Corollary 1.

Example. Let $\{\xi_n\}_{n\geq 1}$ be a sequence of finite partitions of $\mathbb{A}_1 \times \mathbb{A}_2$ made of rectangles in the form $[\omega]\times[\omega']$, for all $(\omega,\omega') \in A_1^{q(n)} \times A_2^n$ and $\xi = \bigcup_{n\geq 1} \xi_n$, where the integer $q(n)$ is defined such that, for $n \in \mathbb{N}^*$

$$n\frac{\log(r_2)}{\log(r_1)} \leq q(n) < n\frac{\log(r_2)}{\log(r_1)} + 1.$$

It is clear that the family ξ satisfies the assumptions of Proposition 3.1.

For $\alpha \geq 0$, we consider the set

$$E_\alpha = \left\{ x \in \mathbb{A}_1 \times \mathbb{A}_2 \, : \, \lim_{n\to\infty} \frac{N_n^{\omega,\omega'}}{n}(x) = \alpha_{\omega,\omega'} \text{ for all } (\omega,\omega') \in A_1 \times A_2 \right\}$$

where for $(\omega,\omega')\in A_1\times A_2$, $N_n^{\omega,\omega'}(x)$ stands for the number of appearances of the couple (ω,ω') in the product word $x_{|n}\times y_{|n}$ and $\alpha=(\alpha_{\omega,\omega'})_{(\omega,\omega')\in A_1\times A_2}$ is a family of positive numbers such that

$$\sum_{(\omega,\omega')\in A_1\times A_2}\alpha_{\omega,\omega'}=1.$$

We propose to calculate the Hausdorff dimension of the set E_α. For this purpose, we consider the Bernoulli measure μ in $\mathbb{A}_1\times\mathbb{A}_2$ defined by

$$\mu([\omega_1\cdots\omega_n]\times[\omega'_1\cdots\omega'_n])=\prod_{k=1}^{n}\alpha_{\omega_k,\omega'_k}\qquad\text{for each } n\in\mathbb{N}^*.$$

We have

$$\mu([\omega_1\cdots\omega_{q(n)}]\times[\omega'_1\cdots\omega'_n])=\prod_{k=1}^{n}\alpha_{\omega_k,\omega'_k}\prod_{k=n+1}^{q(n)}\lambda_{\omega_k}$$

with $\lambda_{\omega_k}=\sum_{\omega'_k}\alpha_{\omega_k,\omega'_k}$.

It is clear that

$$E_\alpha\subset\left\{x\in\mathbb{A}_1\times\mathbb{A}_2\ :\ \lim_{n\to\infty}\frac{\log\mu(R_n(x))}{\log(\mathrm{diam}(R_n(x)))}=\gamma\right\},$$

where

$$\gamma=-\sum_{\omega,\omega'}\frac{\alpha_{\omega,\omega'}\log\alpha_{\omega,\omega'}}{\log r_2}+\left(\frac{1}{\log r_2}-\frac{1}{\log r_1}\right)\sum_{\omega}\lambda_\omega\log\lambda_\omega.$$

So, according to the strong law of large numbers we have $\mu(E_\alpha)=1$. By using Corollary 1 we have,

$$\dim_\xi(E_\alpha)=\mathrm{Dim}_\xi(E_\alpha)=\gamma,$$

which implies from Proposition 3.1 that

$$\dim(E_\alpha)=\mathrm{Dim}(E_\alpha)=\gamma.$$

Thus, any Borel set of E_α with dimension inferior to γ is of measure μ-zero.

4. A variant of the refined multifractal formalism in the product space $\mathbb{A}_1 \times \mathbb{A}_2$

4.1. Problematic. In this section, we will consider a sequence $\{\xi_n\}_{n\geq 1}$ of finite partitions of $\mathbb{A}_1 \times \mathbb{A}_2$ made of rectangles satisfying condition (1) and we put $\xi = \bigcup_{n\geq 1} \xi_n$.

In the following, we consider a Borel probability measure μ on $\mathbb{A}_1 \times \mathbb{A}_2$ and one defines its support $\operatorname{supp}(\mu)$ to be the complement of the set

$$\bigcup \{R \in \xi \,:\, \mu(R) = 0\}.$$

Then, we intend to underestimate the dimensions of the fractal sets $E_\mu(\gamma)$ for some values of γ, where

$$E_\mu(\gamma) = \left\{ x \in \operatorname{supp}(\mu) \,:\, \lim_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))} = \gamma \right\}.$$

Notice that the natural coverings of these iso-Hölder sets are made of rectangles which become thinner and thinner as their diameter tends to zero which doesn't allow the calculation of the Hausdorff and packing dimensions. For this purpose, we will consider a variant of the refined multifractal formalism as already introduced by F. Ben Nasr and J. Peyrière [5], adapted to rectangles.

Let us consider an auxiliary Borel probability measure ν on $\mathbb{A}_1 \times \mathbb{A}_2$. If E is a nonempty subset of $\mathbb{A}_1 \times \mathbb{A}_2$ then for $q, t \in \mathbb{R}$ and $\varepsilon > 0$, we introduce the following quantities:

$$\mathbf{H}^{q,t}_{\mu,\nu,\varepsilon}(E) = \inf \left\{ \sum_j \mu(R_j)^q \operatorname{diam}(R_j)^t \nu(R_j) \,:\, R_j \in \xi, \{R_j\}_j \text{ an } \varepsilon\text{-covering of } E \right\},$$

$$\mathbf{H}^{q,t}_{\mu,\nu}(E) = \lim_{\varepsilon\to 0} \mathbf{H}^{q,t}_{\mu,\nu,\varepsilon}(E),$$

and

$$\overline{\mathbf{P}}^{q,t}_{\mu,\nu,\varepsilon}(E) = \sup \left\{ \sum_j \mu(R_j)^q \operatorname{diam}(R_j)^t \nu(R_j) \,:\, R_j \in \xi,\ \{R_j\}_j \text{ an } \varepsilon\text{-packing of } E \right\},$$

$$\overline{\mathbf{P}}^{q,t}_{\mu,\nu}(E) = \lim_{\varepsilon\to 0} \overline{\mathbf{P}}^{q,t}_{\mu,\nu,\varepsilon}(E).$$

The function $\overline{\mathbf{P}}^{q,t}_{\mu,\nu}$ is called the packing pre-measure. In order to deal with an outer measure, one defines

$$\mathbf{P}^{q,t}_{\mu,\nu}(E) = \inf\left\{\sum_j \overline{\mathbf{P}}^{q,t}_{\mu,\nu}(E_j) \ :\ E \subset \bigcup_j E_j\right\}.$$

Let φ be the following function

$$\varphi(q) = \inf\left\{t \in \mathbb{R} \ :\ \overline{\mathbf{P}}^{q,t}_{\mu,\nu}(\operatorname{supp}(\mu)) = 0\right\}. \tag{11}$$

4.2. Main results. Let μ be a Borel probability measure on $\mathbb{A}_1 \times \mathbb{A}_2$. For $\alpha, \beta \in \mathbb{R}$, one sets

$$E_\mu(\alpha,\beta) = \underline{E}_\mu(\alpha) \cap \overline{E}_\mu(\beta),$$

where

$$\underline{E}_\mu(\alpha) = \left\{x \in \operatorname{supp}(\mu) \ :\ \liminf_{n\to\infty} \frac{\log\mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))} \geq \alpha\right\}$$

and

$$\overline{E}_\mu(\beta) = \left\{x \in \operatorname{supp}(\mu) \ :\ \limsup_{n\to\infty} \frac{\log\mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))} \leq \beta\right\}.$$

Theorem 4.1. *Assume that $\varphi(0) = 0$ and $\nu^\sharp(\operatorname{supp}(\mu)) > 0$. Then one has*

$$\begin{aligned}\dim_\xi E_\mu\big(-\varphi'_r(0), -\varphi'_l(0)\big) \\ \geq \inf\left\{\liminf_{n\to\infty} \frac{\log\nu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))} \ :\ x \in E_\mu(-\varphi'_r(0), -\varphi'_l(0))\right\}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Dim}_\xi E_\mu\big(-\varphi'_r(0), -\varphi'_l(0)\big) \\ \geq \inf\left\{\limsup_{n\to\infty} \frac{\log\nu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))} \ :\ x \in E_\mu(-\varphi'_r(0), -\varphi'_l(0))\right\},\end{aligned}$$

where φ'_r, φ'_l are respectively the left-hand and right-hand derivatives of φ.

Remark 2. The same result holds with

$$\psi(q) = \inf\left\{t \in \mathbb{R} \,:\, \mathbf{P}^{q,t}_{\mu,\nu}(\mathrm{supp}(\mu)) = 0\right\}.$$

The proof of Theorem 4.1 is an immediate consequence of the following proposition.

PROPOSITION 4.1. *Assume that* $\varphi(0) = 0$ *and* $\nu^\sharp(\mathrm{supp}(\mu)) > 0$. *Then one has*

$$\nu^\sharp(E_\mu(-\varphi'_r(0), -\varphi'_l(0))^c) = 0.$$

Proof. Take $\delta > -\varphi'_l(0)$, there exist two positive reals t and δ' such that $\delta > \delta' > -\varphi'_l(0)$ and $\delta' t > \varphi(-t)$ which implies $\mathbf{P}^{-t,\delta' t}_{\mu,\nu}(\mathrm{supp}(\mu)) = 0$. So, there exists a partition $\{E_j\}_j$ of $\mathrm{supp}(\mu)$ such that

$$\sum_j \overline{\mathbf{P}}^{-t,\delta' t}_{\mu,\nu}(E_j) \leq 1.$$

It results that $\overline{\mathbf{P}}^{-t,\delta t}_{\mu,\nu}(E_j) = 0$ for all j.

Now, consider the set

$$E_\delta = \left\{x \in \mathrm{supp}(\mu) \,:\, \limsup_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\mathrm{diam}(R_n(x)))} > \delta\right\}.$$

If $x \in E_\delta$, for all $n \in \mathbb{N}$ there exists $k \geq n$ such that

$$\mu(R_k(x)) \leq \mathrm{diam}(R_k(x))^\delta.$$

Let E be a subset of E_δ and set $F_j = E \cap E_j$. For $0 < \varepsilon \leq \inf_{R\in\xi_n} \mathrm{diam}(R)$ and for all j, one can find an ε-packing $\{R_{j_k}\}_k$ of F_j such that

$$\mu(R_{j_k}) \leq \mathrm{diam}(R_{j_k})^\delta.$$

So, we have

$$\begin{aligned}\nu^\sharp_\varepsilon(F_j) &\leq \sum_j \nu(R_j) \leq \sum_j \sum_k \nu(R_{j_k}) \\ &\leq \sum_j \sum_k \mu(R_{j_k})^{-t} \,\mathrm{diam}(R_{j_k})^{\delta t} \nu(R_{j_k}) \leq \sum_j \overline{\mathbf{P}}^{-t,\delta t}_{\mu,\nu,\varepsilon}(F_j) \;=\; 0.\end{aligned}$$

Then

$$\nu^\sharp(E_\delta) = 0.$$

We conclude that

$$\nu^\sharp\left(\left\{x \in \operatorname{supp}(\mu) \,:\, \limsup_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))} > -\varphi_l'(0)\right\}\right) = 0.$$

In the same way, one proves that

$$\nu^\sharp\left(\left\{x \in \operatorname{supp}(\mu) \,:\, \liminf_{n\to\infty} \frac{\log \mu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))} < -\varphi_r'(0)\right\}\right) = 0.$$

∎

Proof of Theorem 4.1. Assume that $\varphi(0) = 0$ and $\nu^\sharp(\operatorname{supp}(\mu)) > 0$. Then we have according to Proposition 4.1

$$\nu^\sharp(E_\mu(-\varphi_r'(0), -\varphi_l'(0))) > 0.$$

So, it is easy to see from Proposition 3.2 that

$$\dim_\xi E_\mu(-\varphi_r'(0), -\varphi_l'(0)) \geq \operatorname*{ess\,sup}_{x\in E_\mu(-\varphi_r'(0),-\varphi_l'(0)),\nu^\sharp} \liminf_{n\to\infty} \frac{\log \nu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))},$$

and

$$\operatorname{Dim}_\xi E_\mu(-\varphi_r'(0), -\varphi_l'(0)) \geq \operatorname*{ess\,sup}_{x\in E_\mu(-\varphi_r'(0),-\varphi_l'(0)),\nu^\sharp} \limsup_{n\to\infty} \frac{\log \nu(R_n(x))}{\log(\operatorname{diam}(R_n(x)))}.$$

However, as a property of ess sup, we know that if $\nu^\sharp(E_\mu(-\varphi_r'(0), -\varphi_l'(0))) > 0$, then

$$\inf_{x\in E_\mu(-\varphi_r'(0),-\varphi_l'(0))} \left\{\liminf_{n\to 0} \frac{\log \nu(R(x))}{\log(\operatorname{diam}(R_n(x)))}\right\}$$
$$\leq \operatorname*{ess\,sup}_{x\in E_\mu(-\varphi_r'(0),-\varphi_l'(0)),\nu^\sharp} \liminf_{n\to\infty} \frac{\log \nu(R(x))}{\log(\operatorname{diam}(R_n(x)))}$$

and the proof of the theorem follows. ∎

5. An example

In this section we give a large class of measures satisfying the result of Theorem 4.1. Let $\{\xi_n\}_{n\geq 1}$ be the sequence of finite partitions of $\mathbb{A}_1 \times \mathbb{A}_2$ made of rectangles of the form $[\omega] \times [\omega']$, for all $(\omega, \omega') \in A_1^{q(n)} \times A_2^n$ and $\xi = \bigcup_{n\geq 1} \xi_n$, where the integer $q(n)$ is defined such that, for $n \in \mathbb{N}^*$

$$n\frac{\log(r_2)}{\log(r_1)} \leq q(n) < n\frac{\log(r_2)}{\log(r_1)} + 1.$$

For $(i,j) \in A_1 \times A_2$, take $(p_{i,j})_{i,j}$ and $(q_{i,j})_{i,j}$ two sequences of non negative numbers such that

$$\sum_{i,j} p_{i,j} = \sum_{i,j} q_{i,j} = 1 \qquad \text{and} \qquad \lambda_i = \sum_j p_{i,j} = \sum_j q_{i,j}.$$

Let $(T_n)_{n\geq 1}$ be a sequence of integers defined by

$$T_1 = 1\ ,\ T_n < T_{n+1} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{T_n}{T_{n+1}} = 0.$$

Consider the family of parameters α_{i_k,j_k}

$$\alpha_{i_k,j_k} = \begin{cases} p_{i_k,j_k} & \text{if } T_{2n-1} \leq k < T_{2n}, \\ q_{i_k,j_k} & \text{if } T_{2n} \leq k < T_{2n+1}. \end{cases}$$

We define the measure μ on $\mathbb{A}_1 \times \mathbb{A}_2$ as follows

$$\mu\big([i_1 \cdots i_n] \times [j_1 \cdots j_n]\big) = \prod_{k=1}^{n} \alpha_{i_k,j_k}.$$

It is easy to see that

$$\mu\big([i_1 \cdots i_{q(n)}] \times [j_1 \cdots j_n]\big) = \mu\big([i_1 \cdots i_n] \times [j_1 \cdots j_n]\big) \cdot \lambda_{i_{n+1}} \cdots \lambda_{i_{q(n)}}.$$

In the sequel we will impose those monotony hypotheses

$$p_{0,0} < p_{0,1} < \cdots < p_{0,r_2-1} < p_{1,0} < \cdots < p_{1,r_2-1} < \cdots$$
$$\cdots < p_{r_1-1,0} < \cdots < p_{r_1-1,r_2-1},$$
$$q_{0,0} < q_{0,1} < \cdots < q_{0,r_2-1} < q_{1,0} < \cdots < q_{1,r_2-1} < \cdots$$
$$\cdots < q_{r_1-1,0} < \cdots < q_{r_1-1,r_2-1},$$
$$p_{0,0} < q_{0,0} \qquad \text{and} \qquad p_{r_1-1,r_2-1} > q_{r_1-1,r_2-1},$$

which prove the existence of a real x_0 such that $T(x_0) = W(x_0)$, where

$$T(x) = \sum_{i,j} \frac{p_{i,j}^x}{\sum\limits_{i,j} p_{i,j}^x} \log_{r_2} p_{i,j} + \left(\frac{1}{\log r_1} - \frac{1}{\log r_2}\right) \sum_{i,j} \frac{p_{i,j}^x}{\sum\limits_{i,j} p_{i,j}^x} \log \lambda_i$$

and

$$W(x) = \sum_{i,j} \frac{q_{i,j}^x}{\sum\limits_{i,j} q_{i,j}^x} \log_{r_2} q_{i,j} + \left(\frac{1}{\log r_1} - \frac{1}{\log r_2}\right) \sum_{i,j} \frac{q_{i,j}^x}{\sum\limits_{i,j} q_{i,j}^x} \log \lambda_i.$$

For this real x_0, we denote by

$$\tilde{p}_{i,j} = \frac{p_{i,j}^{x_0}}{\sum\limits_{i,j} p_{i,j}^{x_0}} \qquad \text{and} \qquad \tilde{q}_{i,j} = \frac{q_{i,j}^{x_0}}{\sum\limits_{i,j} q_{i,j}^{x_0}}.$$

Our aim is to estimate the dimensions of the sets $E_\mu(\gamma)$ for certain values of γ. To be done, we consider an auxiliary measure ν on $\mathbb{A}_1 \times \mathbb{A}_2$ defined as μ with the parameters $\tilde{p}_{i,j}$ and $\tilde{q}_{i,j}$ instead of $p_{i,j}$ and $q_{i,j}$ by

$$\nu([i_1 \cdots i_n] \times [j_1 \cdots j_n]) = \prod_{k=1}^{n} \tilde{\alpha}_{i_k,j_k}$$

where

$$\tilde{\alpha}_{i_k,j_k} = \begin{cases} \tilde{p}_{i_k,j_k} & \text{if } T_{2n-1} \leq k < T_{2n}, \\ \tilde{q}_{i_k,j_k} & \text{if } T_{2n} \leq k < T_{2n+1}. \end{cases}$$

Let $\tilde{\lambda}_i = \sum\limits_j \tilde{p}_{i,j} = \sum\limits_j \tilde{q}_{i,j}$. Then, we have the following result.

THEOREM 5.1. *For every*

$$\gamma \in \left(-\log_{r_2} \left(q_{r_1-1,r_2-1} \lambda_{r_1-1}^{\frac{\log(r_2)}{\log(r_1)}-1} \right), -\log_{r_2} \left(q_{0,0} \lambda_0^{\frac{\log(r_2)}{\log(r_1)}-1} \right) \right)$$

we have

$$\dim E_\mu(\gamma) \geq \min \{h(\tilde{p}), h(\tilde{q})\}$$

and

$$\operatorname{Dim} E_\mu(\gamma) \geq \max \{h(\tilde{p}), h(\tilde{q})\},$$

where

$$h(\tilde{p}) = -\sum_{i,j} \tilde{p}_{i,j} \log_{r_2} \tilde{p}_{i,j} + \left(\frac{1}{\log r_2} - \frac{1}{\log r_1}\right) \sum_i \tilde{\lambda}_i \log \tilde{\lambda}_i$$

and

$$h(\tilde{q}) = -\sum_{i,j} \tilde{q}_{i,j} \log_{r_2} \tilde{q}_{i,j} + \left(\frac{1}{\log r_2} - \frac{1}{\log r_1}\right) \sum_i \tilde{\lambda}_i \log \tilde{\lambda}_i.$$

In order to prove this theorem we will calculate the function φ defined in equation (11). For that, we need to use the following lemma.

LEMMA 5.1. *For $t \in \mathbb{R}$, one has*

$$\varphi(t) = \limsup_{n\to\infty} \frac{1}{n \log r_2} \log \sum_{R_n \cap \operatorname{supp}(\mu) \neq \emptyset} \mu(R_n)^t \nu(R_n).$$

Proof. For $t \in \mathbb{R}$, we denote by

$$\Phi(t) = \limsup_{n\to\infty} \frac{1}{n \log r_2} \log \sum_{R_n \cap \operatorname{supp}(\mu) \neq \emptyset} \mu(R_n)^t \nu(R_n).$$

We will prove that

$$\varphi(t) = \Phi(t).$$

Let's begin by proving that $\varphi(t) \leq \Phi(t)$.

For $\alpha > 0$ satisfying $\Phi(t) \leq \alpha$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\frac{1}{n \log r_2} \log \sum_{R_n \cap \operatorname{supp}(\mu) \neq \emptyset} \mu(R_n)^t \nu(R_n) \leq \alpha.$$

So,

$$\sum_{R_n \cap \operatorname{supp}(\mu) \neq \emptyset} \mu(R_n)^t \nu(R_n) r_2^{-n\alpha} \leq 1, \qquad \text{for each} \qquad n \geq n_0.$$

Then

$$\overline{\mathbf{P}}^{t,\alpha}_{\mu,\nu}(\operatorname{supp}(\mu)) \leq 1,$$

and

$$\alpha \geq \varphi(t),$$

which gives that

$$\Phi(t) \geq \varphi(t).$$

Next, we prove that $\varphi(t) \geq \Phi(t)$. Let $\alpha > \varphi(t)$, then

$$\overline{\mathbf{P}}^{t,\alpha}_{\mu,\nu}(\mathrm{supp}(\mu)) = 0.$$

For $\varepsilon > 0$, there exists an ε-packing $\{R_n\}_n$ of $\mathrm{supp}(\mu)$ such that

$$\sum_{R_n \cap \,\mathrm{supp}(\mu) \neq \emptyset} \mu(R_n)^t \nu(R_n) r_2^{-n\alpha} \leq 1.$$

Thus

$$\frac{1}{n \log r_2} \log \sum_{R_n \cap \,\mathrm{supp}(\mu) \neq \emptyset} \mu(R_n)^t \nu(R_n) \leq \alpha.$$

So,

$$\limsup_{n\to\infty} \frac{1}{n \log r_2} \log \sum_{R_n \cap \,\mathrm{supp}(\mu) \neq \emptyset} \mu(R_n)^t \nu(R_n) \leq \alpha$$

and

$$\Phi(t) \leq \alpha,$$

which prove Lemma 5.1. ∎

Now, we are able to prove Theorem 5.1. It is easy to see that

$$\varphi(t) = \sup\left(\log_{r_2} \sum_{i,j} p_{i,j}^t \tilde{p}_{i,j}, \log_{r_2} \sum_{i,j} q_{i,j}^t \tilde{q}_{i,j}\right) + \left(\frac{1}{\log r_1} - \frac{1}{\log r_2}\right) \log \sum_i \lambda_i^t \tilde{\lambda}_i$$

and

$$\varphi(0) = 0.$$

By the way, using the definitions of the sequences $(\tilde{p}_{i,j})$ and $(\tilde{q}_{i,j})$ and a simple computation of the derivative of φ at 0 we obtain

$$\varphi'(0) = \sum_{i,j} \tilde{p}_{i,j} \log_{r_2} p_{i,j} + \left(\frac{1}{\log r_1} - \frac{1}{\log r_2}\right) \sum_i \tilde{\lambda}_i \log \lambda_i.$$

Let $\gamma = -\varphi'(0)$, it is clear that

$$\gamma \in \left(-\log_{r_2}\left(q_{r_1-1,r_2-1} \lambda_{r_1-1}^{\frac{\log(r_2)}{\log(r_1)}-1}\right), -\log_{r_2}\left(q_{0,0} \lambda_0^{\frac{\log(r_2)}{\log(r_1)}-1}\right)\right).$$

Besides, using the strong law of large numbers we can see that

$$\liminf_{n\to\infty} \frac{\log_{r_2} \nu(R_n(x))}{-n} = \min\{h(\tilde{p}), h(\tilde{q})\}$$

and

$$\limsup_{n\to\infty} \frac{\log_{r_2} \nu(R_n(x))}{-n} = \max\{h(\tilde{p}), h(\tilde{q})\},$$

for ν-almost every x.

Then, it follows from Theorem 4.1 and Proposition 3.2 that

$$\dim E_\mu(\gamma) \geq \min\{h(\tilde{p}), h(\tilde{q})\}$$

and

$$\operatorname{Dim} E_\mu(\gamma) \geq \max\{h(\tilde{p}), h(\tilde{q})\}$$

which achieve the proof of Theorem 5.1.

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