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Stability of some essential B-spectra of pencil operators and application

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Abstract: In this paper, we give some results on the essential B-spectra of a linear operator pencil, which are used to determine the essential B-spectra of an integro-differential operator with abstract boundary conditions in the Banach space $L_p([-a,a]\times[-1,1])$, $p \geq 1$ and $a > 0$.

Key words: Operator pencil, finite-rank and power finite-rank perturbations, essential B-spectra, transport operator.

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1. Introduction

Let X be a Banach space. We will denote by $\mathcal{C}(X)$ (resp. $L(X)$) the set of all closed linear (resp. the algebra of all bounded) linear operators from X into X. For $T \in \mathcal{C}(X)$, we write $D(T) \subset X$ for the domain, $N(T) \subset X$ for the null space and $R(T) \subset X$ for the range of T. We denote by $\alpha(T)$ the dimension of $N(T)$ and $\beta(T)$ the codimension of $R(T)$ in X. For $T \in \mathcal{C}(X)$ and $M \in L(X)$, we define the resolvent of the linear operator pencil $\lambda M - T$, where $\lambda \in \mathbb{C}$, or the M-resolvent of T by

$$\rho_M(T) := \{\lambda \in \mathbb{C} : \lambda M - T \text{ has a bounded inverse}\},$$

and its spectrum by

$$\sigma(M,T) = \mathbb{C}\backslash\rho_M(T).$$

For $T \in \mathcal{C}(X)$, we define the set

$$\triangle(T) = \left\{n \in \mathbb{N} : \forall m \in \mathbb{N},\, m \geq n \Rightarrow R(T^n)\cap N(T) \subset R(T^m)\cap N(T)\right\}.$$

The degree of stable iteration of T is defined as $\text{dis}(T) = \inf \triangle(T)$, where $\text{dis}(T) = \infty$ if $\triangle(T) = \emptyset$.

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We define the set of upper semi-Fredholm operators by

$$\Phi_+(X) = \left\{T \in \mathcal{C}(X) \,:\, \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } X\right\},$$

and the set of lower semi-Fredholm operators by

$$\Phi_-(X) = \left\{T \in \mathcal{C}(X) \,:\, \beta(T) < \infty \text{ and } R(T) \text{ is closed in } X\right\}.$$

$\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ will denote the set of Fredholm operators from X into X. The index of a Fredholm operator T is defined by $i(T) = \alpha(T) - \beta(T)$.

According to [17], an operator $T \in \mathcal{C}(X)$ is called quasi-Fredholm of degree $d \in \mathbb{N}$ if the following three conditions are satisfied:

(i) $\mathrm{dis}(T) = d$;

(ii) $R(T^d) \cap N(T)$ is a closed and complemented subspace of X;

(iii) $R(T) + N(T^d)$ is a closed and complemented subspace of X.

This set of operators will be denoted by $QF(d)$.

Following [10, Definition 2.4], an operator $T \in \mathcal{C}(X)$ is called upper semi B-Fredholm (resp. lower semi B-Fredholm) if there exists an integer $d \in \mathbb{N}$ such that $T \in QF(d)$ and such that $N(T) \cap R(T^d)$ is of finite dimension (resp. $R(T) + N(T^d)$ is of finite codimension). These sets are denoted respectively by $\Phi_B^+(X)$ and $\Phi_B^-(X)$. We denote by $\Phi_B(X) := \Phi_B^+(X) \cap \Phi_B^-(X)$, the set of B-Fredholm operators from X into X. In this case, the index of T is defined as the integer: $\mathrm{ind}(T) = \dim(N(T) \cap R(T^d)) - \mathrm{codim}(R(T) + N(T^d))$. An operator $T \in \mathcal{C}(X)$ is called B-Weyl if it is a B-Fredholm operator of index zero. We will denote this set by $BW(X)$.

For $T \in \mathcal{C}(X)$, we define the ascent $a(T)$ of T by

$$a(T) = \inf\left\{n \in \mathbb{N} : N\big(T^n\big) = N\big(T^{n+1}\big)\right\},$$

and the descent $d(T)$ of T by

$$d(T) = \inf\left\{n \in \mathbb{N} : R\big(T^n\big) = R\big(T^{n+1}\big)\right\}.$$

We define respectively the set of Drazin invertible operators, left Drazin invertible operators and right Drazin invertible operators as follows:

$$DR(X) := \{T \in \mathcal{C}(X) \,:\, a(T) \text{ and } d(T) \text{ are both finite}\},$$
$$LD(X) := \left\{T \in \mathcal{C}(X) \,:\, a(T) \text{ is finite and } R\big(T^{a(T)+1}\big) \text{ is closed}\right\},$$
$$RD(X) := \left\{T \in \mathcal{C}(X) \,:\, d(T) \text{ is finite and } R\big(T^{d(T)}\big) \text{ is closed}\right\}.$$

An operator $T \in \mathcal{C}(X)$ is of Kato type if there exist an integer $d \in \mathbb{N}$ and a pair of two closed subspaces (N_1, N_2) of X such that:

(i) $X = N_1 \oplus N_2$;

(ii) $T(N_1) \subset N_1$ and $T_{/N_1}$ is semi-regular;

(iii) $T(N_2) \subset N_2$ and $(T_{/N_2})^d = 0$, i.e., $T_{/N_2}$ is nilpotent.

Note that, in the case of Hilbert spaces, Labrousse in [17, Theorem 3.2.2] has shown that, the set of Quasi-Fredholm operators coincides with the Kato type operators for the class of closed operators. According to [17, p. 206, Remark], this equivalence is also true in the case of Banach spaces.

For $T \in \mathcal{C}(X)$ and $M \in L(X)$, we define the B-Fredholm spectrum, Drazin spectrum, the upper semi B-Fredholm spectrum, the lower semi B-Fredholm spectrum, the left Drazin spectrum, the right Drazin spectrum, the B-Weyl spectrum, the closed-range spectrum and the Kato spectrum of the linear operator pencil $\lambda M - T$, where $\lambda \in \mathbb{C}$, or the pair (M,T) as follows:

$$\begin{aligned}
\sigma_{BF}(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, \lambda M - T \notin \Phi_B(X)\right\},\\
\sigma_{D}(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, \lambda M - T \notin DR(X)\right\},\\
\sigma_{BF^+}(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, \lambda M - T \notin \Phi_B^+(X)\right\},\\
\sigma_{BF^-}(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, \lambda M - T \notin \Phi_B^-(X)\right\},\\
\sigma_{LD}(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, \lambda M - T \notin LD(X)\right\},\\
\sigma_{RD}(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, \lambda M - T \notin RD(X)\right\},\\
\sigma_{BW}(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, \lambda M - T \notin BW(X)\right\},\\
\sigma_{ec}(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, R(\lambda M - T) \text{ is not closed}\right\},\\
\sigma_{ek}(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, \lambda M - T \text{ is not Kato}\right\}.
\end{aligned}$$

The point spectrum, the residual spectrum and the continuous spectrum of the pair (M,T), when $T \in \mathcal{C}(X)$ and $M \in L(X)$, are defined respectively by:

$$\begin{aligned}
\sigma_p(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, \lambda M - T \text{ is not injective}\right\},\\
\sigma_r(M,T) &= \left\{\lambda \in \mathbb{C} \,:\, N(\lambda M - T) = \{0\} \text{ and } \overline{R(\lambda M - T)} \subsetneq X\right\},\\
\sigma_c(M,T) &= \Big\{\lambda \in \mathbb{C} \,:\, N(\lambda M - T) = \{0\},\ \ \overline{R(\lambda M - T)} = X\\
&\qquad\qquad \text{and } R(\lambda M - T) \neq X\Big\},
\end{aligned}$$

where, $\overline{R(\lambda M - T)}$ is the closure of $R(\lambda M - T)$. The collection $\{\sigma_p(M,T), \sigma_r(M,T), \sigma_c(M,T)\}$ forms a partition of the spectrum $\sigma(M,T)$, which means that they are pairwise disjoint and $\sigma(M,T) = \sigma_p(M,T) \cup \sigma_r(M,T) \cup \sigma_c(M,T)$.

For $T \in \mathcal{C}(X)$ and $M \in L(X)$, the upper semi B-Fredholm, the lower semi B-Fredholm and the B-Fredholm resolvent of the linear operator pencil $\lambda M - T$, where $\lambda \in \mathbb{C}$, are defined respectively by

$$\begin{aligned}\rho_{BF^+}(M,T) &= \mathbb{C} \backslash \sigma_{BF^+}(M,T)\,,\\ \rho_{BF^-}(M,T) &= \mathbb{C} \backslash \sigma_{BF^-}(M,T)\,,\\ \rho_{BF}(M,T) &= \rho_{BF^+}(M,T) \cap \rho_{BF^-}(M,T)\,.\end{aligned}$$

The present paper is a generalization of the results obtained by A. Jeribi et al. in [15] and some of stability results obtained by M. Berkani et al. in [6] for the usual essential B-spectra. It generalizes also the works obtained by A. Jeribi in [13, 14] about the invariance of the S-essential spectra under weakly compact or strictly singular perturbations, which are not applied in the B-Fredholm theory. So that, we can use, by adding some hypothesis, the perturbations of the B-Fredholm spectra under finite-rank and power finite-rank commuting operators. More precisely, let $T_1, T_2 \in \mathcal{C}(X)$ be two commuting closed linear operators such that the bounded linear operator M commutes in the resolvent sense with T_1 and T_2 (see Definition 2.2) and satisfying $(\lambda M - T_1)^{-1} - (\lambda M - T_2)^{-1} \in \mathcal{F}(X)$ (resp. $(\lambda M - T_1)^{-1} - (\lambda M - T_2)^{-1} \in \mathcal{F}_p(X)$ or nilpotent) for some $\lambda \in \rho_M(T_1) \cap \rho_M(T_2)$. Then, we prove that $\sigma_*(M, T_1) = \sigma_*(M, T_2)$, where

$$\begin{aligned}\sigma_*(M,.) \in \big\{&\sigma_{BF}(M,.), \sigma_{BF^+}(M,.), \sigma_{BF^-}(M,.),\\ &\sigma_{BW}(M,.), \sigma_{LD}(M,.), \sigma_{RD}(M,.), \sigma_D(M,.)\big\}.\end{aligned}$$

These perturbation results are needed to extend the results obtained in [15], on the S-essential spectra of closed densely defined linear operators to essential B-spectra of operator pencil $\lambda M - T$, when $T \in \mathcal{C}(X)$, $M \in L(X)$ and $\lambda \in \mathbb{C}$. Moreover, under the additional hypothesis $M(C(T)) = C(T)$ (see Definition 2.1), we give the relationship between the closed-range spectrum and the Kato spectrum of the linear operator pencil $\lambda M - T$, when $T \in \mathcal{C}(X)$, $M \in L(X)$ and $\lambda \in \mathbb{C}$ (see Proposition 2.4), which generalizes a result obtained in [9, Proposition 3.2] in the case of bounded operators and [8, Corollary 3] for closed densely defined linear operators for the usual spectrum. We establish also, the equality between the closed-range spectrum and some essential B-spectra of operator pencil acting on a Banach space (Theorem 2.5). The obtained results, are finally used to describe the essential B-spectra of the operator pencil of the following integro-differential operator with abstract boundary conditions in the Banach space $X_p := L_p\big([-a,a] \times [-1,1], dxdy\big)$, $a > 0$, $1 \leq p < \infty$,

$$A_H = T_H + K\,,$$

where T_H, K and M are defined by

$$\begin{cases} T_H : D(T_H) \subseteq X_p & \longrightarrow & X_p \\ \qquad\qquad\qquad \psi & \longmapsto & -\xi \dfrac{\partial \psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi)\,, \\ D(T_H) = \left\{ \psi \in \mathcal{W}_p \,:\, \psi^i = H\psi^o \right\}, \end{cases}$$

where $\mathcal{W}_p := \left\{ \psi \in X_p \,:\, \xi \frac{\partial \psi}{\partial x} \in X_p \right\}$,

$$\begin{cases} K : X_p & \longrightarrow & X_p \\ \qquad u & \longmapsto & \displaystyle\int_{-1}^{1} k(x,\xi,\nu)u(x,\nu)d\nu\,, \end{cases}$$

and

$$\begin{cases} M : X_p & \longrightarrow & X_p \\ \qquad \varphi & \longmapsto & M(\varphi)(x,\xi) = \eta(\xi)\varphi(x,\xi)\,, \end{cases}$$

where $\sigma(.)$ and $\eta(.)$ are in $L^\infty(-1,1)$, $k(.,.,.)$ is a measurable function, and H is the boundary operator connecting the outgoing and the incoming fluxes.

The outline of this work is organized in the following way: in Section 2, we give some stability results of some essential B-spectra of linear operator pencil. The main results of this section are Theorem 2.3 and Theorem 2.5. In Section 3, we apply the results developed in Section 2 to characterize the B-essential spectra of a transport operator with abstract boundary conditions on L_p-spaces, $1 \le p < \infty$.

2. Stability of some essential B-spectra of pencil operators

We are interested, in this section, in some of stability results of the essential B-spectra of an operator pencil $\lambda M - T$, where $M \in L(X)$, $T \in \mathcal{C}(X)$ and $\lambda \in \mathbb{C}$.

Since, the Kato decomposition theorem, remains true in the case of Banach spaces as shown in [17, p. 206], we can directly use the following proposition inspired from [10], when necessary, in the case of Banach spaces without proof.

Proposition 2.1. *Let $T \in \mathcal{C}(X)$. If T is a semi B-Fredholm operator, then there exist two closed subspaces X_0 and X_1 of X such that*

(i) $X = X_0 \oplus X_1$,

(ii) $T(X_0) \subset X_0$ *and* $T_0 = T_{/X_0}$ *is a semi-Fredholm operator,*

(iii) $T(X_1) \subset X_1$ *and* $T_1 = T_{/X_1}$ *is a nilpotent operator.*

First we recall the following subspace, introduced by P. Saphar in [20], and it was defined by P. Aiena in [1] in purely algebraic terms.

Definition 2.1. The algebraic core $C(T)$ of a linear operator T is defined to be the greatest subspace $N \subset D(T)$ for which $T(N) = N$. For more details for the algebraic core $C(T)$, we can refer to [1].

Theorem 2.1. *Let $T \in \mathcal{C}(X)$ and $M \in L(X)$ such that $M(C(T)) = C(T)$ and $\rho_M(T) \neq \emptyset$. If T is a semi B-Fredholm operator, then there exists $\varepsilon > 0$ such that $T - \mu M$ is a semi-Fredholm operator, for each $\mu \in D(0,\varepsilon)\backslash\{0\}$. Moreover, we have $\alpha(T - \mu M)$ and $\beta(T - \mu M)$ are constants on $D(0,\varepsilon)\backslash\{0\}$.*

Proof. If $M = I$, then we obtain the result established in [7].

If $M \neq I$, then the fact that T is a semi B-Fredholm operator, this implies from Proposition 2.1, the existence of two T-invariant closed subspaces X_0 and X_1 such that

- $X = X_0 \oplus X_1$,
- $T(X_0) \subset X_0$ and $T_0 = T_{/X_0}$ is a semi-Fredholm operator,
- $T(X_1) \subset X_1$ and $T_1 = T_{/X_1}$ is nilpotent.

Since $M(C(T)) = C(T)$ and $T(C(T)) = C(T)$, then we can conclude, by using the definition of $C(T)$, that X_0 and X_1 are invariants under the operator M. So, we can consider $M_0 = M_{/X_0}$ and $M_1 = M_{/X_1}$ such that $M = M_0 \oplus M_1$.

Case 1: If $X_0 = \{0\}$, then we get $M_0 = 0$, $T_0 = 0$, $M = M_1$ and $T = T_1$ is nilpotent. Since, the operator T_1 is nilpotent then we have $T_1 - \mu I_1$ is invertible, for $\mu \neq 0$, where $I_1 = I_{/X_1}$. We have

$$T_1 - \mu M_1 = (T_1 - \mu I_1)[I_1 - \mu(T_1 - \mu I_1)^{-1}(M_1 - I_1)]$$

is invertible for μ such that

$$0 < |\mu| < \frac{1}{\|(T_1 - \mu I_1)^{-1}(M_1 - I_1)\|} = \gamma\,.$$

Hence, $T - \mu M = T_1 - \mu M_1$ is a semi-Fredholm operator, for μ such that $0 < |\mu| < \gamma$. Moreover, we have $\alpha(T - \mu M) = \alpha(T_1 - \mu M_1) = 0$ and $\beta(T - \mu M) = \beta(T_1 - \mu M_1) = 0$ on $D(0,\gamma)\backslash\{0\}$.

Case 2: Suppose $X_0 \neq \{0\}$.

- If $M_0 = 0$, then by using Case 1, we obtain that $T - \mu M = T_0 \oplus T_1 - \mu M_1$ is a semi-Fredholm operator for μ such that $0 < |\mu| < \gamma$.
- If $M_0 \neq 0$, then $T - \mu M = T_0 - \mu M_0 \oplus T_1 - \mu M_1$. It follows from [19, Theorem 7.9], that $T_0 - \mu M_0$ is a semi-Fredholm for μ such that

$\mu \in D\big(0, \frac{\varepsilon'}{\|M_0\|}\big)\backslash\{0\}$, for some $\varepsilon' > 0$. Set $\varepsilon = \min\big(\frac{\varepsilon'}{\|M_0\|}, \gamma\big)$. Therefore, the operator $T - \mu M$ is semi-Fredholm for μ such that $\mu \in D(0, \varepsilon)\backslash\{0\}$. Again, from [19, Theorem 7.9], we get $\alpha(T - \mu M) = \alpha(T_0 - \mu M_0)$ and $\beta(T - \mu M) = \beta(T_0 - \mu M_0)$ are constants on $D(0, \varepsilon)\backslash\{0\}$. ∎

Using Theorem 2.1, we can deduce the following result which is a generalization of [15, Proposition 2.1]

Corollary 2.1. *Let $T \in \mathcal{C}(X)$ and $M \in L(X)$ such that $M(C(T)) = C(T)$ and $\rho_M(T) \neq \emptyset$. Then,*

(i) *$\rho_{BF^+}(M,T)$, $\rho_{BF^-}(M,T)$ and $\rho_{BF}(M,T)$ are open subsets of $\mathbb{C}$;*

(ii) *$\operatorname{ind}(\lambda M - T)$ is constant on any component of $\rho_{BF^+}(M,T)$, $\rho_{BF^-}(M,T)$ and $\rho_{BF}(M,T)$.*

Proof. (i) Let $\lambda_0 \in \rho_{BF^+}(M,T)$, then from Theorem 2.1 there exists an $\varepsilon > 0$ such that $T - \mu M$ is an upper semi-Fredholm operator, for each $\mu \in D(\lambda_0, \varepsilon)\backslash\{0\}$. This implies that $\rho_{BF^+}(M,T)$ is an open subset of $\mathbb{C}$. The same proof is used to show that, $\rho_{BF^-}(M,T)$ and $\rho_{BF}(M,T)$ are open subsets of $\mathbb{C}$.

(ii) Let Ω be a component of $\rho_{BF^+}(M,T)$ (resp. $\rho_{BF^-}(M,T)$), $\lambda_0 \in \Omega$ be a fixed point and $\lambda_1 \in \Omega$ be an arbitrary point that are connected by a polygonal line Γ contained in Ω. It follows from the assertion (i) of this corollary that, for each $\mu \in \Gamma$, there exists an open disc $D(\mu, \varepsilon)$, such that $\operatorname{ind}(\mu M - T) = \operatorname{ind}(\lambda M - T)$, for each $\lambda \in D(\mu, \varepsilon)$. By the Heine-Borel theorem, there exist a finite number of open discs that cover Γ. This allows us to deduce that $\operatorname{ind}(\lambda_0 M - T) = \operatorname{ind}(\lambda_1 M - T)$. ∎

Now, we recall the following definition considered in [12] for bounded linear operators and it remains also true in the general case of closed linear operators:

Definition 2.2. Let $M \in L(X)$ and $T \in \mathcal{C}(X)$ such that $\rho_M(T) \neq \emptyset$. We say that M and T commute in the sense of resolvent if for all $\lambda \in \rho_M(T)$,

$$M(T - \lambda M)^{-1} = (T - \lambda M)^{-1} M\,.$$

Remarks 2.1. (a) If M and T commute in the sense of the resolvent, the assumption $M(C(T)) \subset C(T)$ is verified. Indeed, let $x \in C(T)$. Then, from [1, Theorem 1.8], there exists a sequence $(u_n) \subset D(T)$ such that $x = u_0$ and $Tu_{n+1} = u_n$, for every $n \in \mathbb{Z}_+$. Set $y_n = Mu_n$. The commutativity of the resolvent of the operators M and T, permits us to deduce that $M = (\lambda M - T)^{-1}M(\lambda M - T)$ on $D(T)$ and $TM = MT$ on $D(T)$, which entails that $Mu_n \in D(T)$. Thus, we get $y_0 = Mu_0 = Mx$ and $Ty_{n+1} = TMu_{n+1} = MTu_{n+1} = Mu_n = y_n$, for every $n \in \mathbb{Z}_+$. This, implies that $Mx \in M(C(T))$ and finally we obtain that $M(C(T)) \subset C(T)$.

(b) If M and T commute in the sense of the resolvent and M is invertible, then we get $M^{-1}(C(T)) \subset C(T)$ and finally we can conclude that $C(T) \subset M(C(T))$.

Proposition 2.2. *Let $M \in L(X)$ be an invertible operator and $T \in \mathcal{C}(X)$ such that $0 \in \rho_M(T)$. If $MT^{-1} = T^{-1}M$, then*

(i) *for $\lambda \neq 0$ and $n \geq 1$: $(T - \lambda M)^n(T^{-1})^n = (-\lambda)^n M^n \big(T^{-1} - \lambda^{-1}M^{-1}\big)^n$;*

(ii) *for all $n \geq 1$: $R\big((T - \lambda M)^n\big) = R\big((T^{-1} - \lambda^{-1}M^{-1})^n\big)$.*

Proof. (i) For $n = 1$, the equality is obvious. Let $n \geq 1$ and assume that $(T - \lambda M)^n(T^{-1})^n = (-\lambda)^n M^n(T^{-1} - \lambda^{-1}M^{-1})^n$, then

$$\begin{aligned}(T - \lambda M)^{n+1}(T^{-1})^{n+1} &= (T - \lambda M)\big[(T - \lambda M)^n \big(T^{-1}\big)^n\big]T^{-1} \\ &= (T - \lambda M)\big[(-\lambda)^n M^n \big(T^{-1} - \lambda^{-1}M^{-1}\big)^n\big]T^{-1} \\ &= (T - \lambda M)T^{-1}\big[(-\lambda)^n M^n \big(T^{-1} - \lambda^{-1}M^{-1}\big)^n\big] \\ &= (-\lambda)^{n+1}M^{n+1}\big(T^{-1} - \lambda^{-1}M^{-1}\big)^{n+1}.\end{aligned}$$

(ii) It follows from (i), that

$$R\big[M^n(T^{-1} - \lambda^{-1}M^{-1})^n\big] = R\big[(T^{-1} - \lambda^{-1}M^{-1})^n\big] \subseteq R\big[(T - \lambda M)^n\big].$$

Conversely, if $y \in R((T - \lambda M)^n)$, then there exists $x \in D(T^n)$ such that $y = (T - \lambda M)^n x$. The fact that $(T^{-1})^n(X) = D(T^n)$, this enable us the existence of $t \in X$ such that $x = (T^{-1})^n(t)$. Hence, $y = (T - \lambda M)^n \big(T^{-1}\big)^n(t)$ and finally $y = (-\lambda)^n M^n \big(T^{-1} - \lambda^{-1}M^{-1}\big)^n(t) \in R\big[(T^{-1} - \lambda^{-1}M^{-1})^n\big]$. ∎

Proposition 2.3. *Let $M \in L(X)$ be an invertible operator and $T \in \mathcal{C}(X)$ such that $0 \in \rho_M(T)$. If $MT^{-1} = T^{-1}M$, then*

(i) *for $\lambda \neq 0$ and $n \geq 1$: $(T^{-1})^n(T - \lambda M)^n = (-\lambda)^n M^n \big(T^{-1} - \lambda^{-1}M^{-1}\big)^n$ and $T^n(T^{-1} - \lambda^{-1}M^{-1})^n = (-\lambda^{-1})^n(M^{-1})^n(T - \lambda M)^n$ on $D(T^n)$;*

(ii) *for all $n \geq 1$: $N((T - \lambda M)^n) = N\big((T^{-1} - \lambda^{-1}M^{-1})^n\big)$.*

Proof. (i) Since T is invertible and $D(T-\lambda M)=D(T)$, then $R\big((T^{-1})^n\big) = D(T^n)$ and $R(T^n) = X$. Therefore, the operators $(T^{-1})^n(T-\lambda M)^n$ and $T^n\big(T^{-1}-\lambda^{-1}M^{-1}\big)^n$ are well defined. Then, we verify directly that $(T^{-1})^n(T-\lambda M)^n = (-\lambda)^n M^n\big(T^{-1}-\lambda^{-1}M^{-1}\big)^n$ and $T^n\big(T^{-1}-\lambda^{-1}M^{-1}\big)^n = (-\lambda^{-1})^n(M^{-1})^n(T-\lambda M)^n$ on $D(T^n)$.

(ii) It is a direct consequence of (i). ∎

Remark 2.2. If $M=I$, we recover the results obtained in [10].

Now, we state the main result of this section.

THEOREM 2.2. *Let $M\in L(X)$ be an invertible operator and $T\in\mathcal{C}(X)$ such that $0\in\rho_M(T)$. If $MT^{-1}=T^{-1}M$, then*

$$\sigma_*(M,T)=\left\{\lambda^{-1} : \lambda\in\sigma_*\big(M^{-1},T^{-1}\big)\setminus\{0\}\right\},$$

where

$$\sigma_*(M,T)\in\big\{\sigma_{BF^+}(M,T),\sigma_{BF^-}(M,T),\sigma_{BF}(M,T),\\ \sigma_D(M,T),\sigma_{BW}(M,T),\sigma_{LD}(M,T),\sigma_{RD}(M,T)\big\}.$$

Proof. Let $\lambda\neq 0$. By using Proposition 2.2 and Proposition 2.3, we get that $\lambda M^{-1}-T^{-1}$ is a B-Fredholm operator if and only if $\lambda^{-1}M-T$ is a B-Fredholm one. The same arguments are used to prove the other B-spectra. ∎

In order to give, the stability result of some essential B-spectra of operator pencil by means of some class of commuting perturbations which generalizes some results established in [6], we shall define the following class of operators: We say that a linear operator is of finite-rank if its range is of finite dimension. If there exists an integer $p\in\mathbb{N}^*$ such that $\dim R(T^p)<\infty$, then it is called a power finite-rank operator. We will denote by $\mathcal{F}(X)$ (resp. $\mathcal{F}_p(X)$) the set of all finite-rank linear bounded (resp. power finite-rank) operators.

In many applications (see Section 3) the perturbed operator is not of finite rank but we have some information about the difference of the resolvent, so the usual result.

THEOREM 2.3. *Let $M\in L(X)$ be an invertible operator and $T_1,T_2\in\mathcal{C}(X)$ such that M commutes with T_1 and T_2 in the sense of resolvent. If for some $\lambda\in\rho_M(T_1)\cap\rho_M(T_2)$, the operator $(T_1-\lambda M)^{-1}-(T_2-\lambda M)^{-1}\in\mathcal{F}(X)$,* *then*

$$\sigma_*(M,T_1) = \sigma_*(M,T_2)\,,$$

where, $\sigma_*(M,.) \in \{\sigma_{BF}(M,.), \sigma_{BF^+}(M,.), \sigma_{BF^-}(M,.), \sigma_{BW}(M,.)\}$.

Proof. Without loss of generality, we can assume that $\lambda = 0$, then $T_1^{-1} - T_2^{-1} \in \mathcal{F}(X)$. Let $\mu \neq 0$. The use of Theorem 2.2 shows that, $\mu M - T_1$ is a B-Fredholm operator if and only if $\mu^{-1}M^{-1} - T_1^{-1}$ is a B-Fredholm one. Since $T_1^{-1} - T_2^{-1} \in \mathcal{F}(X)$, then by using [3, Corollary 3.10], we get $\mu^{-1}M^{-1} - T_2^{-1}$ is a B-Fredholm operator. Again by Theorem 2.2, this is equivalent to $\mu M - T_2$ is also a B-Fredholm operator, which finish the proof of the B-Fredholm spectrum equality. For the upper semi B-Fredholm spectrum, the lower semi B-Fredholm spectrum and the B-Weyl spectrum, we use the same technique as above and [11, Proposition 2.7]. ∎

Definition 2.3. ([18]) Let X be a Banach space, $A : D(A) \subset X \to X$ and $T : D(T) \subset X \to X$ two linear operators. We say that A commutes with T, and we denote $AT = TA$, if

(i) $D(A) \subset D(T)$;

(ii) $Tx \in D(A)$ whenever $x \in D(A)$;

(iii) $AT = TA$ on $\{x \in D(A) \,:\, Ax \in D(T)\}$.

Under the additional hypothesis of commutativity of operators, we get a stronger version of Theorem 2.3:

Theorem 2.4. *Let $M \in L(X)$ be an invertible operator and $T_1, T_2 \in \mathcal{C}(X)$ such that M commutes with T_1 and T_2 in the resolvent sense and $T_1T_2 = T_2T_1$. If for some $\lambda \in \rho_M(T_1) \cap \rho_M(T_2)$, the operator $(T_1 - \lambda M)^{-1} - (T_2 - \lambda M)^{-1} \in \mathcal{F}_p(X)$, then*

$$\sigma_*(M,T_1) = \sigma_*(M,T_2)\,,$$

where

$$\sigma_*(M,T) \in \big\{\sigma_{BF}(M,.), \sigma_{BF^+}(M,.), \sigma_{BF^-}(M,.), \sigma_{BW}(M,.), \sigma_{LD}(M,.), \sigma_{RD}(M,.), \sigma_D(M,.)\big\}.$$

Proof. Without loss of generality, we can assume that $\lambda = 0$, then $T_1^{-1} - T_2^{-1} \in \mathcal{F}_p(X)$. Let $\mu \neq 0$. It follows from Theorem 2.2, that $\mu M - T_1$ is a B-Fredholm operator if and only if $\mu^{-1}M^{-1} - T_1^{-1}$ is a B-Fredholm one. Since, $T_1^{-1} - T_2^{-1} \in \mathcal{F}_p(X)$, then from [16], we obtain that $\mu^{-1}M^{-1} - T_2^{-1}$ is also a B-Fredholm operator, which is equivalent to $\mu M - T_1$ is a B-Fredholm operator by Theorem 2.2. This, shows that $\sigma_{BF}(M, T_1) = \sigma_{BF}(M, T_2)$. For the other equalities, we use the same technique as above. ∎

Corollary 2.2. *Let $M \in L(X)$ be an invertible operator and $T_1, T_2 \in \mathcal{C}(X)$ such that M commutes with T_1 and T_2 in the resolvent sense and $T_1T_2 = T_2T_1$. If for some $\lambda \in \rho_M(T_1) \cap \rho_M(T_2)$, the operator $(T_1 - \lambda M)^{-1} - (T_2 - \lambda M)^{-1}$ is nilpotent, then*

$$\sigma_*(M, T_1) = \sigma_*(M, T_2)\,,$$

where

$$\sigma_*() \in \big\{\sigma_{BF}(M,T), \sigma_{BF^+}(M,T), \sigma_{BF^-}(M,T), \sigma_{BW}(M,T), \sigma_{LD}(M,T), \sigma_{RD}(M,T), \sigma_D(M,T)\big\}.$$

Corollary 2.3. *Let $T \in \mathcal{C}(X)$, $M \in L(X)$ be an invertible operator and $Q \in L(X)$ a nilpotent operator such that $\rho_M(T) \neq \emptyset$. If $TQ = QT$ on $D(T)$, $MQ = QM$ and M, T commute in the resolvent sense, then*

$$\sigma_*(M, T) = \sigma_*(M, T + Q)\,,$$

where

$$\sigma_*() \in \big\{\sigma_{BF}(M,T), \sigma_{BF^+}(M,T), \sigma_{BF^-}(M,T), \sigma_{BW}(M,T), \sigma_{LD}(M,T), \sigma_{RD}(M,T), \sigma_D(M,T)\big\}.$$

Proof. Since, $TQ = QT$ and Q is a nilpotent operator, then $(\lambda M - T)^{-1}Q$ is nilpotent, for all $\lambda \in \rho_M(T)$. Thus, its spectral radius is equal to zero, which implies that $\lambda \in \rho_M(T + Q)$ and that

$$\begin{aligned}(\lambda M - T - Q)^{-1} &= (\lambda M - T)^{-1}(M - (\lambda I - T)^{-1}Q)^{-1} \\ &= (\lambda M - T)^{-1}\sum_{k=0}^{n}((\lambda M - T)^{-1}Q)^k \\ &= (\lambda M - T)^{-1} + (\lambda M - T)^{-1}Q\sum_{k=1}^{n-1}((\lambda M - T)^{-1})^k Q^{k-1}\end{aligned}$$

with n is the nilpotent-index of $(\lambda M - T)^{-1}Q$. Hence, $(\lambda M - T - Q)^{-1} - (\lambda M - T)^{-1}$ is nilpotent. So, we deduce from Corollary 2.2, that $\sigma_*(M, T + Q) = \sigma_*(M, T)$. ∎

The following proposition is proved in [12] for bounded linear operators and it holds also true in the general case of closed densely-defined linear operators.

PROPOSITION 2.4. *Let $T \in \mathcal{C}(X)$ be densely-defined linear operator and $M \in L(X)$ such that $M(C(T)) = C(T)$. If $\lambda \in \sigma_{ec}(M, T)$ is non-isolated, then $\lambda \in \sigma_{ek}(M, T)$.*

Remark 2.3. Proposition 2.4 is also true if we replace $\sigma_{ek}(M, T)$ by $\sigma_{qf}(M, T)$, where $\sigma_{qf}(M, T) = \{\lambda \in \mathbb{C} : \lambda M - T$ s not a Quazi-Fredholm operator$\}$.

The following theorem, shows the equality between the closed-range spectrum and some essential B-spectra of operator pencil acting on the Banach space.

THEOREM 2.5. *Let $T \in \mathcal{C}(X)$ be densely-defined linear operator and $M \in L(X)$ such that $M(C(T)) = C(T)$. If $\sigma_{ec}(M, T) = \sigma(M, T)$ and every $\lambda \in \sigma_{ec}(M, T)$ is non-isolated. Then*

$$\begin{aligned}\sigma(M, T) &= \sigma_{BF}(M, T) = \sigma_{BW}(M, T)\\ &= \sigma_{BF^+}(M, T) = \sigma_{BF^-}(M, T) = \sigma_D(M, T).\end{aligned}$$

Proof. Since $\sigma_{BF}(M, T) \subset \sigma(M, T)$, it suffices to show that $\sigma(M, T) \subset \sigma_{BF}(M, T)$. Let $\lambda \in \sigma(M, T)$, then from Proposition 2.4, we have $\lambda \in \sigma_{ek}(M, T)$. Since, a B-Fredholm operator is a Quasi-Fredholm one, this shows that $\sigma_{ek}(M, T) \subset \sigma_{BF}(M, T)$. The same arguments are used for the upper semi B-Fredholm, the lower semi B-Fredholm and the B-Weyl spectrum. The fact that, a Drazin invertible operator is a B-Fredholm one, then by using the same arguments as above we can prove that, $\sigma(M, T) = \sigma_D(M, T)$. Finally, we conclude that $\sigma(M, T) = \sigma_{ec}(M, T) = \sigma_{BF}(M, T) = \sigma_{BW}(M, T) = \sigma_{BF^+}(M, T) = \sigma_{BF^-}(M, T) = \sigma_D(M, T)$. ∎

3. APPLICATION

In this section, we will use the previous results to treat the essential B-spectra of a transport operator with abstract boundary conditions. Let

$$X_p := L_p\big((-a, a) \times (-1, 1), dx d\xi\big), \quad a > 0, \quad 1 \leq p < \infty.$$

We consider the following integro-differential operator with abstract boundary conditions:

$$A_H = T_H + K\,,$$

where T_H is defined by

$$\begin{cases} T_H : D(T_H) \subseteq X_p & \longrightarrow & X_p \\ \qquad\qquad\qquad \psi & \longmapsto & T_H\psi(x,\xi) = -\xi\dfrac{\partial\psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi)\,, \\ D(T_H) = \big\{\psi \in \mathcal{W}_p \,:\, \psi^i = H\psi^o\big\}\,, \end{cases}$$

where $\mathcal{W}_p := \big\{\varphi \in X_p \,:\, \xi\frac{\partial\varphi}{\partial x} \in X_p\big\}$ and $\sigma(.) \in L^\infty(-1,1)$; ψ^o, ψ^i are, respectively, the outgoing and the incoming fluxes related by the boundary operator H ("o" for the outgoing and "i" for the incoming), and given by

$$\begin{cases} \psi^i(\xi) \;=\; \psi(-a,\xi)\,, & \xi \in (0,1)\,, \\ \psi^i(\xi) \;=\; \psi(a,\xi)\,, & \xi \in (-1,0)\,, \\ \psi^o(\xi) \;=\; \psi(-a,\xi)\,, & \xi \in (-1,0)\,, \\ \psi^o(\xi) \;=\; \psi(a,\xi)\,, & \xi \in (0,1)\,. \end{cases}$$

The bounded collision operator K is defined by

$$\begin{cases} K : X_p & \longrightarrow & X_p \\ \qquad u & \longmapsto & K(u)(x,\xi) = \displaystyle\int_{-1}^{1} k(x,\xi,\nu)u(x,\nu)\,d\nu\,, \end{cases}$$

where the kernel $k : (-a,a) \times (-1,1) \times (-1,1) \longrightarrow \mathbb{R}$ is assumed to be measurable.

The following boundary spaces denoted by X_p^o and X_p^i are defined as follows:

$$X_p^o := L_p\big[\{-a\} \times (-1,0); |\xi| d\xi\big] \times L_p\big[\{a\} \times (0,1); |\xi| d\xi\big] := X_{1,p}^o \times X_{2,p}^o$$

equipped with the norm

$$\begin{aligned} \big\|u^\circ, X_p^\circ\big\| &:= \Big(\big\|u_1^\circ, X_{1,p}^\circ\big\|^p + \big\|u_2^\circ, X_{2,p}^\circ\big\|^p\Big)^{\frac{1}{p}} \\ &= \Big[\int_{-1}^{0} |u(-a,v)|^p\,|v|\,dv + \int_{0}^{1} |u(a,v)|^p\,|v|\,dv\Big]^{\frac{1}{p}}, \end{aligned}$$

and

$$X_p^i := L_p\big[\{-a\}\times(0,1);|\xi|d\xi\big]\times L_p\big[\{a\}\times(-1,0);|\xi|d\xi\big] := X_{1,p}^i\times X_{2,p}^i$$

equipped with the norm

$$\begin{aligned}\|u^i,X_p^i\| &:= \Big(\|u_1^i,X_{1,p}^i\|^p+\|u_2^i,X_{2,p}^i\|^p\Big)^{\frac{1}{p}}\\ &= \Big[\int_0^1 |u(-a,v)|^p\,|v|\,dv+\int_{-1}^0 |u(a,v)|^p\,|v|\,dv\Big]^{\frac{1}{p}}.\end{aligned}$$

In this part, we will determine the essential B-spectra of the pair (M,A_H), where M is the operator defined by

$$\begin{cases} M: X_p \longrightarrow X_p\\ \qquad \varphi \longmapsto M(\varphi)(x,\xi)=\eta(\xi)\varphi(x,\xi)\,,\end{cases}$$

where $\eta(.)\in L^\infty(-1,1)$.

To clarify our subsequent analysis, we define the following bounded operators introduced in [15]:

$$\begin{cases} M_\lambda: X_p^i\longrightarrow X_p^o\,, \qquad M_\lambda u:=\big(M_\lambda^+u,M_\lambda^-u\big)\,, \qquad \text{with}\\ (M_\lambda^+u)(-a,\xi):=u(-a,\xi)\,e^{\frac{-2a}{|\xi|}(\lambda\eta(\xi)+\sigma(\xi))}\,, \qquad 0<\xi<1\,,\\ (M_\lambda^-u)(a,\xi):=u(a,\xi)\,e^{\frac{-2a}{|\xi|}(\lambda\eta(\xi)+\sigma(\xi))}\,, \qquad -1<\xi<0\,,\end{cases}$$

$$\begin{cases} B_\lambda: X_p^i\longrightarrow X_p\,, \qquad B_\lambda u:=\chi_{(-1,0)}(\xi)B_\lambda^-u+\chi_{(0,1)}(\xi)B_\lambda^+u\,, \qquad \text{with}\\ (B_\lambda^-u)(x,\xi):=u(a,\xi)\,e^{-\frac{(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|a-x|}\,, \qquad -1<\xi<0\,,\\ (B_\lambda^+u)(x,\xi):=u(-a,\xi)\,e^{-\frac{(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|a+x|}\,, \qquad 0<\xi<1\,,\end{cases}$$

$$\begin{cases} G_\lambda: X_p\longrightarrow X_p^o\,, \qquad G_\lambda\varphi:=\big(G_\lambda^+\varphi,G_\lambda^-\varphi\big)\,, \qquad \text{with}\\ G_\lambda^-\varphi(a,\xi):=\frac{1}{|\xi|}\int_{-a}^a e^{\frac{-(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|a+x|}\varphi(x,\xi)\,dx\,, \qquad -1<\xi<0\,,\\ G_\lambda^+\varphi(-a,\xi):=\frac{1}{|\xi|}\int_{-a}^a e^{\frac{-(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|a-x|}\varphi(x,\xi)\,dx\,, \qquad 0<\xi<1\,,\end{cases}$$

and finally

$$\begin{cases} C_\lambda : X_p \longrightarrow X_p\,, \qquad C_\lambda\varphi = \chi_{(-1,0)}C_\lambda^-\varphi + \chi_{(0,1)}C_\lambda^+\varphi\,, \qquad \text{with} \\ \\ C_\lambda^-\varphi(x,\xi) := \dfrac{1}{|\xi|}\displaystyle\int_x^a e^{\frac{-(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|x-x'|}\varphi(x',\xi)\,dx'\,, \qquad -1<\xi<0\,, \\ C_\lambda^+\varphi(x,\xi) := \dfrac{1}{|\xi|}\displaystyle\int_{-a}^x e^{\frac{-(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|x-x'|}\varphi(x',\xi)\,dx'\,, \qquad 0<\xi<1\,, \end{cases}$$

where $\chi_{(0,1)}(.)$ and $\chi_{(-1,0)}(.)$ denote, respectively, the characteristic functions of the intervals $(0,1)$ and $(-1,0)$.

Note that, from [15, Proposition 3.1], the operators M_λ, B_λ, G_λ and C_λ are bounded respectively by $\exp(-2a\mu^*\operatorname{Re}\lambda)$, $(p\mu^*\operatorname{Re}\lambda)^{-1/p}$, $(\mu^*\operatorname{Re}\lambda)^{-1/q}$ and $(\mu^*\operatorname{Re}\lambda)^{-1}$ where q is the conjugate of p.

In what follows, we will show that the M-spectrum of T_0 (i.e., T_H with $H=0$) is the continuous spectrum $\sigma_c(M,T_0)$ of the pair (M,T_0).

Lemma 3.1. (i) *The point spectrum of the pair* (M,T_0) *is empty.*
(ii) *The residual spectrum of the pair* (M,T_0) *is empty.*

Proof. (i) We consider for $\lambda\in\mathbb{C}$ such that $\operatorname{Re}(\lambda)\le 0$, the eigenvalue problem $(\lambda M - T_0)\psi = 0$, where the unknown ψ must be in $D(T_0)$. His solution is formally given by

$$\psi(x,\xi) = K(\xi)e^{\frac{-1}{|\xi|}[\lambda\eta(\xi)+\sigma(\xi)]x}.$$

Moreover, since $\psi\in D(T_0)$, then we get $\psi^i = 0$. So we obtain $K(\xi)=0$ on $(-1,1)$. Consequently, $\psi=0$.

(ii) To prove that the residual spectrum $\sigma_r(M,T_0)$ is also empty, we shall determine the point spectrum of the adjoint operator pencil densely defined $\lambda M - T_0$, where $\lambda\in\mathbb{C}$.

The adjoint operators T_0^* and M^* are, respectively, given by:

$$\begin{cases} T_0^* : D(T_0^*)\subseteq X_q \quad \longrightarrow \quad X_q \\ \qquad\qquad\qquad \psi \quad \longmapsto \quad T_0^*\psi(x,\xi) = \xi\dfrac{\partial\psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi)\,, \\ D(T_0^*) = \big\{\psi\in\mathcal{W}_q \,:\, \psi^o = 0\big\}, \end{cases}$$

where q is the conjugate of p $\left(\frac{1}{p}+\frac{1}{q}=1\right)$,

$$\begin{cases} M^* : X_q \quad\longrightarrow\quad X_q \\ \qquad\quad \varphi \quad\longrightarrow\quad M^*(\varphi)(x,v) = \eta(v)\varphi(x,v)\,. \end{cases}$$

Let us consider the eigenvalue problem

$$(\lambda M^* - T_0^*)\psi = 0. \tag{3.1}$$

In view of the boundary conditions, a straightforward estimation shows that, the problem (3.1) admits only the trivial solution. Then, we obtain $\sigma_p(M^*, T_0^*) = \emptyset$. Since, $\sigma_r(M, T_0) \subseteq \sigma_p(M^*, T_0^*)$, then we can easily obtain the desired result. ∎

Now, by using the previous results, we can deduce the following theorem:

THEOREM 3.1. *Let $M \in L(X)$. Then,*

$$\sigma(M, T_0) = \sigma_c(M, T_0) = \sigma_{ec}(M, T_0) = \big\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\big\}.$$

Proof. Since, $\sigma(M, T_0) = \sigma_p(M, T_0) \cup \sigma_r(M, T_0) \cup \sigma_c(M, T_0)$, then by using Lemma 3.1, we deduce $\sigma(M, T_0) = \sigma_c(M, T_0)$. On the other hand, it follows from [15, Theorem 3.1], that $\sigma(M, T_0) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\}$. Combining these two results, we can obtain the assertion of the present theorem. ∎

Now, we are able to express the B-spectra of the pair (M, T_H):

THEOREM 3.2. *If the boundary operator H is of finite rank and commutes with M, and every point λ in $\sigma_{ec}(M, T_0)$ is non-isolated, then*

$$\sigma_*(M, T_H) = \sigma_*(M, T_0) = \big\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\big\},$$

where $\sigma_*(M, .) \in \big\{\sigma_{BF}(M, .), \sigma_{BF^+}(M, .), \sigma_{BF^-}(M, .), \sigma_{BW}(M, .)\big\}$.

Proof. According to [15], we have

$$(\lambda M - T_H)^{-1} = H \sum_{n \geq 0} B_\lambda (M_\lambda H)^n G_\lambda + C_\lambda, \tag{3.2}$$

where $C_\lambda = (\lambda M - T_0)^{-1}$. Since $(\lambda M - T_H)^{-1} - (\lambda M - T_0)^{-1} \in \mathcal{F}(X)$, this implies by Theorem 2.3 that $\sigma_*(M, T_H) = \sigma_*(M, T_0)$. The fact that M and T_H commute in the sense of the resolvent and M is invertible this allows us, by the use of Remarks 2.1. Theorem 2.5 and Theorem 3.1 to conclude that

$$\sigma_*(M, T_0) = \big\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\big\},$$

where $\sigma_*(M, .) \in \big\{\sigma_{BF}(M, .), \sigma_{BF^+}(M, .), \sigma_{BF^-}(M, .), \sigma_{BW}(M, .)\big\}$ ∎

Theorem 3.3. *Suppose that the boundary operator H and the collision operator K are of finite rank. If $MH = HM$ and every point λ in $\sigma_{ec}(M,T_0)$ is non-isolated, then we get*

$$\sigma_*(M,A_H) = \big\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq 0\big\},$$

where $\sigma_*(M,.) \in \big\{\sigma_{BF}(M,.), \sigma_{BF^+}(M,.), \sigma_{BF^-}(M,.), \sigma_{BW}(M,.)\big\}$.

Proof. Since, the collision operator K is finite rank, then it follows from Theorem 2.3 and Theorem 3.2, that $\sigma_*(M,A_H) = \sigma_*(M,T_H{+}K) = \sigma_*(M,T_H)$ $= \sigma_*(M,T_0)$, and finally we obtain that

$$\sigma_*(M,A_H) = \big\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq 0\big\},$$

where $\sigma_*(M,.) \in \big\{\sigma_{BF}(M,.), \sigma_{BF^+}(M,.), \sigma_{BF^-}(M,.), \sigma_{BW}(M,.)\big\}$. ∎

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