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Approximation by Polynomials in a Weighted Space of Infinitely Differentiable Functions with an Application to Hypercyclicity

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Abstract: A space of infinitely differentiable functions defined on an open cone of $\mathbb{R}^n$ and of prescribed growth near the boundary of the cone and at infinity is considered. The problem of polynomial approximation in this space is studied. It is shown that every linear continuous operator on this space that commutes with each partial derivative operator and is not a scalar multiple of the identity is hypercyclic.

Key words: Hypercyclic operators, polynomial approximation.

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1. Introduction

Let Ω be an open connected cone in $\mathbb{R}^n$ with apex at the origin, $\overline{\Omega}$ be the closure of Ω in $\mathbb{R}^n$.

Let $(h_m)_{m=1}^{\infty}$ be a sequence of positive functions $h_m \in C(\Omega)$ such that for all $m \in \mathbb{N}$ there exists $a_m \geq 0$ such that for all $x \in \Omega$

$$h_m(x) - h_{m+1}(x) \geq \left(\ln \frac{1}{\mathrm{d}(x)}\right)^+ - a_m ,$$

where $\mathrm{d}(x)$ is the distance from $x \in \Omega$ to the boundary $\partial\Omega$ of Ω, $t^+ = t$ for $t \geq 0$, and $t^+ = 0$ for $t < 0$.

Let $(\psi_m)_{m=1}^{\infty}$ be a sequence of positive functions $\psi_m \in C(\overline{\Omega})$ such that for each $m \in \mathbb{N}$:

(a) $\displaystyle\lim_{x\in\Omega,\, x\to\infty} \frac{\psi_m(x)}{\|x\|} = +\infty$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$,

(b) there exists $b_m \geq 0$ such that for all $x \in \Omega$

$$\psi_m(x) - \psi_{m+1}(x) \geq \ln(1 + \|x\|) - b_m .$$

Let $\varphi_m(x) = h_m(x) + \psi_m(x)$, $x \in \Omega$, $m \in \mathbb{N}$.

For each $m \in \mathbb{N}$ let

$$\mathcal{E}_m = \left\{ f \in C^m(\Omega) \, : \, p_m(f) = \sup_{x \in \Omega, \, |\alpha| \le m} \frac{|(D^\alpha f)(x)|}{\exp(\varphi_m(x))} < \infty \right\}.$$

Obviously, $\mathcal{E}_{m+1} \subset \mathcal{E}_m$ $(m \in \mathbb{N})$.

Let $\varphi = \{\varphi_m\}_{m=1}^\infty$ and $\mathcal{E}_\varphi(\Omega) = \bigcap_{m=1}^\infty \mathcal{E}_m$. Under usual operations of addition and multiplication by complex numbers $\mathcal{E}_\varphi(\Omega)$ is a linear space. Endow $\mathcal{E}_\varphi(\Omega)$ with the topology of projective limit of the spaces $\mathcal{E}_m$. Obviously, $\mathcal{E}_\varphi(\Omega)$ is a Fréchet space. In view of condition (b) on the family $(\psi_m)_{m=1}^\infty$ the space $\mathcal{E}_\varphi(\Omega)$ is invariant under multiplication by polynomials. In Section 3 it is shown that linear differential operators with constant coefficients are continuous on $\mathcal{E}_\varphi(\Omega)$.

In this paper the following two problems are considered:

1. approximation by polynomials in $\mathcal{E}_\varphi(\Omega)$;
2. hypercyclicity of linear continuous operators on $\mathcal{E}_\varphi(\Omega)$ commuting with each partial derivative operator.

The motivation to study the first problem is the following. In [10] M.M. Mannanov has considered the problem of description of a dual space to a weighted space of holomorphic functions on an unbounded convex domain of $\mathbb{C}^n$ with given majorants of growth near the boundary and at infinity in terms of the Laplace transform of linear continuous functionals on this space. To solve the problem he used the known scheme of the proof of the Polya-Martineau-Ehrenpreis theorem (see for details [7, Theorem 4.5.3]) and developed methods of the article [12]. Therefore he had to consider some space of infinitely differentiable functions on an unbounded convex domain of $\mathbb{R}^{2n}$ with given majorants of growth near the boundary of a domain and at infinity. But for this space approximation problems (for example, approximation by polynomials or by a system of exponentials) have not been studied. There was no need to study them to obtain the main result of [10].

Note that problems of approximation by polynomials in $\mathcal{E}_\varphi(\Omega)$ are studied here under minimal conditions on weight functions h_m and ψ_m. In Section 2 the following theorem is proved.

Theorem 1. *The polynomials are dense in* $\mathcal{E}_\varphi(\Omega)$.

From this theorem it easily follows that $\mathcal{E}_\varphi(\Omega)$ is a separable space.

Theorem 1 helps us to study the second problem. Recall that a linear continuous operator T on a separable locally convex space X is called *hypercyclic* if there is an element $x \in X$ such that its orbit $\text{Orb}\{x, T\} = \{x, Tx, T^2x, \ldots\}$ is dense in X. Hypercyclic operators have been extensively studied since the 1980's. For background, development, the most known results of theory of hypercyclic operators we refer the reader to the surveys of K.-G. Grosse-Erdmann [5] (where, in particular, results on hypercyclicity of operators in the real analysis setting are described), [6] and to the articles by R. Gethner and J.H. Shapiro [2], G. Godefroy and J.H. Shapiro [3], A. Montes-Rodríguez and N. Salas [11]. Note that C. Kitai [8] and R. Gethner and J.H. Shapiro [2] provided a useful sufficient condition (the so-called Hypercyclicity Criterion) for an operator to be hypercyclic. It was refined afterwards by many authors (see [1], [5], [6]). In some cases it is convenient to use the following result established by G. Godefroy and J.H. Shapiro [3] (see also [4, Corollary 1.10]) with the help of the Hypercyclicity Criterion.

Theorem A. (Godefroy-Shapiro Criterion) *Let X be a separable Fréchet space and $T : X \to X$ be a linear continuous operator. Suppose that $\bigcup_{|\lambda|<1} \ker(T - \lambda)$ and $\bigcup_{|\lambda|>1} \ker(T - \lambda)$ both span a dense subspace of X. Then T is hypercyclic.*

In Section 3, Theorem 1 and Theorem A are used to prove the following statement. For notation, see below.

Theorem 2. *Every linear continuous operator on $\mathcal{E}_\varphi(\Omega)$ that commutes with each partial derivative operator and is not a scalar multiple of the identity is hypercyclic.*

Let us fix some notation. For $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ $(\mathbb{C}^n)$, $\langle u, v \rangle = u_1v_1 + \cdots + u_nv_n$ and $\|u\|$ denotes the Euclidean norm in $\mathbb{R}^n(\mathbb{C}^n)$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$,

$$|\alpha| = \alpha_1 + \ldots + \alpha_n\,, \qquad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}\,, \qquad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}\,,$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\,, \qquad D_z^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}\,.$$

For multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$ the notation $\beta \le \alpha$ indicates that $\beta_j \le \alpha_j$ for $j = 1, 2, \ldots, n$.

For a positive function $\Phi \in C(\Omega)$ such that $\lim\limits_{x\in\Omega,\, x\to\infty} \dfrac{\Phi(x)}{\|x\|} = +\infty$ we set

$$\widetilde{\Phi}(x) := -\inf_{y\in\Omega} \bigl(\langle x, y\rangle + \Phi(y)\bigr)\,, \qquad x \in \mathbb{R}^n.$$

For the sake of simplicity, we put

$$\theta_m(x) := \exp(\varphi_m(x))\,, \qquad x \in \mathbb{R}^n.$$

If $S^{n-1} = \{x \in \mathbb{R}^n \,:\, \|x\| = 1\}$ we set

$$\mathrm{pr}(\Omega) := \Omega \cap S^{n-1}\,.$$

For each $\sigma \in \mathrm{pr}(\Omega)$, let

$$\psi_{m,\sigma}(t) := \psi_m(\sigma t)\,, \qquad t > 0\,.$$

For a function $u : (0,\infty) \to \mathbb{R}$, let

$$u[e](x) := u(e^x)\,, \qquad x \geq 0\,.$$

For $\Omega \subset \mathbb{R}^n$ and $\varepsilon > 0$, let $\Omega^{(\varepsilon)}$ be the ε-enlargement of Ω.

2. On approximation by polynomials in $\mathcal{E}_\varphi(\Omega)$

For a lower semi-continuous function $u : [0,\infty) \to \mathbb{R}$ such that

$$\lim_{x\to+\infty} \frac{u(x)}{x} = +\infty \tag{1}$$

let

$$u^*(x) = \sup_{y\geq 0} \bigl(xy - u(y)\bigr)\,, \qquad x \geq 0\,.$$

Note that $u^*(x) < \infty$ on $[0,\infty)$ and $\lim\limits_{x\to+\infty} \dfrac{u^*(x)}{x} = +\infty$.

Lemma 1. *Let a lower semi-continuous function $u : [0,\infty) \to \mathbb{R}$ satisfies the condition* (1). *Then*

$$(u[e])^*(x) + (u^*[e])^*(x) \leq x \ln x - x\,, \qquad x > 0\,.$$

Proof. Let $x > 0$. There exist numbers $t \geq 0$ and $\xi \geq 0$ such that

$$(u[e])^*(x) = xt - u(e^t)\,,$$
$$(u^*[e])^*(x) = x\xi - u^*(e^\xi)\,.$$

Thus,

$$(u[e])^*(x) + (u^*[e])^*(x) = xt - u(e^t) + x\xi - \sup_{\eta \geq 0} \left(e^\xi \eta - u(\eta)\right).$$

Hence, for each $\eta \geq 0$

$$(u[e])^*(x) + (u^*[e])^*(x) \leq xt - u(e^t) + x\xi - e^\xi \eta + u(\eta)\,.$$

Putting here $\eta = e^t$ we have

$$(u[e])^*(x) + (u^*[e])^*(x) \leq xt + x\xi - e^{\xi + t}.$$

Consequently,

$$\begin{aligned}(u[e])^*(x) + (u^*[e])^*(x) &\leq \sup_{y \geq 0} \left(xy - e^y\right) \\ &\leq \sup_{y \in \mathbb{R}} \left(xy - e^y\right) = x \ln x - x\,.\end{aligned}$$

∎

Proof of Theorem 1. Let ω be the function on $\mathbb{R}^n$ defined as follows:

$$\begin{aligned}\omega(t) &= c_\omega \exp\left(-\tfrac{1}{1 - \|t\|^2}\right) && \text{for } \|t\| < 1\,,\\ \omega(t) &= 0 && \text{for } \|t\| \geq 1\,,\end{aligned}$$

where $c_\omega > 0$ is chosen so that $\displaystyle\int_{\mathbb{R}^n} \omega(t)\,\mathrm{d}t = 1$.

For $\varepsilon > 0$ let $\omega_\varepsilon(t) = \varepsilon^{-n}\omega(\frac{t}{\varepsilon})$, $t \in \mathbb{R}^n$. For each $\nu \in \mathbb{N}$ let

$$K_\nu = \left\{x \in \Omega\,:\, \|x\| \leq \nu,\ \operatorname{dist}(x, \partial\Omega) \geq \frac{1}{\nu}\right\}.$$

Obviously, the closed sets K_ν are non-empty for $\nu \geq \nu_0$ (ν_0 is some positive integer) and $K_\nu \subset \operatorname{int} K_{\nu+1}$, $\bigcup_{\nu=\nu_0}^{\infty} K_\nu = \Omega$.

For $\nu \geq \nu_0$ let $r_\nu = \frac{1}{4}\left(\frac{1}{\nu} - \frac{1}{\nu+1}\right)$ and

$$\eta_\nu(x) = \int_{K_\nu^{(2r_\nu)}} \omega_{r_\nu}(x - y)\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^n.$$

Obviously, $\eta_\nu \in C_0^\infty(\mathbb{R}^n)$, $\eta_\nu(x) = 1$ for $x \in K_\nu^{(r_\nu)}$, $\eta_\nu(x) = 0$ for $x \notin K_\nu^{(3r_\nu)}$, $0 \le \eta_\nu(x) \le 1$ for $x \in \mathbb{R}^n$.

Since for each $\alpha \in \mathbb{Z}_+^n$

$$(D^\alpha \eta_\nu)(x) = \frac{1}{r_\nu^{n+|\alpha|}} \int_{K_\nu^{(2r_\nu)}} (D^\alpha \omega)\left(\frac{x-y}{r_\nu}\right) \mathrm{d}y\,, \qquad x \in \mathbb{R}^n,$$

we have for all $x \in \mathbb{R}^n$

$$\begin{aligned} |(D^\alpha \eta_\nu)(x)| &\le \frac{1}{r_\nu^{n+|\alpha|}} \cdot \sup_{t \in \mathbb{R}^n} |(D^\alpha \omega)(t)| \cdot \mu\Big(K_\nu^{(2r_\nu)}\Big) \\ &\le \frac{m_\alpha}{r_\nu^{n+|\alpha|}} \cdot \mu\Big(K_\nu^{(2r_\nu)}\Big) \le \frac{M_\alpha (1+\nu)^n}{r_\nu^{n+|\alpha|}}\,, \end{aligned}$$

where $m_\alpha = \sup\limits_{t \in \mathbb{R}^n} |(D^\alpha \omega)(t)|$, $M_\alpha = \frac{m_\alpha \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$, Γ is the Gamma-function and μ denotes n-dimensional Lebesgue measure. Thus, for each $\alpha \in \mathbb{Z}_+^n$

$$|(D^\alpha \eta_\nu)(x)| \le M_\alpha 4^{n+|\alpha|} (\nu+1)^{3n+2|\alpha|}, \qquad x \in \mathbb{R}^n, \tag{2}$$

where $M_\alpha > 0$ does not depend on $\nu \ge \nu_0$.

Now let $f \in \mathcal{E}_\varphi(\Omega)$. This means that $f \in C^\infty(\Omega)$ and for each $m \in \mathbb{N}$ there exists $c_m > 0$ such that

$$|(D^\alpha f)(x)| \le c_m \theta_m(x)\,, \qquad x \in \Omega\,,\ |\alpha| \le m\,. \tag{3}$$

Let us approximate f by polynomials in $\mathcal{E}_\varphi(\Omega)$. There are three steps in the proof.

1. For every positive integer $\nu \ge \nu_0$ let $f_\nu(x) = f(x)\eta_\nu(x)$, $x \in \Omega$. Obviously, $f_\nu \in \mathcal{E}_\varphi(\Omega)$. Note that $\operatorname{supp}(f_\nu) \subset K_{\nu+1}$.

Let us show that $f_\nu \to f$ in $\mathcal{E}_\varphi(\Omega)$ as $\nu \to \infty$.

First note that for each $m \in \mathbb{N}$

$$\begin{aligned} \sup_{x\in\Omega} \frac{|f_\nu(x) - f(x)|}{\theta_m(x)} &= \sup_{x\in\Omega} \frac{|f(x)|(1-\eta_\nu(x))}{\theta_m(x)} \le \sup_{x \in \Omega\setminus K_\nu} \frac{|f(x)|}{\theta_m(x)} \\ &\le p_{m+1}(f) \cdot \exp\left(\sup_{x\in\Omega\setminus K_\nu} \big(\varphi_{m+1}(x) - \varphi_m(x)\big)\right). \end{aligned}$$

Let $T_\nu = \Omega \cap \{x \in \mathbb{R}^n : \|x\| > \nu\}$, $S_\nu = \Omega \setminus (K_\nu \cup T_\nu)$, $\nu \ge \nu_0$. Since for each $m \in \mathbb{N}$ one has (see the properties of the families $(h_m)_{m=1}^\infty$ and $(\psi_m)_{m=1}^\infty$) that

$$\begin{aligned} &\varphi_{m+1}(x) - \varphi_m(x) \le -\ln(1+\nu) + a_m + b_m\,, \qquad x \in T_\nu\,, \\ &\varphi_{m+1}(x) - \varphi_m(x) \le -\ln\nu + a_m + b_m\,, \qquad x \in S_\nu\,, \end{aligned}$$

then

$$\exp\left(\sup_{x\in\Omega\setminus K_\nu}\big(\varphi_{m+1}(x)-\varphi_m(x)\big)\right)\to 0\,,\quad \nu\to+\infty\,. \tag{4}$$

Hence, for each $m\in\mathbb{N}$

$$\sup_{x\in\Omega}\frac{|f_\nu(x)-f(x)|}{\theta_m(x)}\to 0\quad\text{as }\nu\to\infty\,. \tag{5}$$

Further, for $x\in\Omega$ and $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|>0$ we have

$$\begin{aligned} D^\alpha(f_\nu(x)-f(x)) = &\sum_{\beta\le\alpha,\,|\beta|<|\alpha|} C_\alpha^\beta (D^\beta f)(x)(D^{\alpha-\beta}\eta_\nu)(x) \\ &+ \sum_{\beta\le\alpha,\,|\beta|<|\alpha|} (D^\alpha f)(x)(\eta_\nu(x)-1)\,, \end{aligned} \tag{6}$$

where $C_\alpha^\beta=\prod_{j=1}^n C_{\alpha_j}^{\beta_j}$ and $C_{\alpha_j}^{\beta_j}$ are the combinatorial numbers.

Denote by $F_\nu(x)$ the first term of the right-hand side in (6). We have for each $m\in\mathbb{N}$

$$\sup_{\substack{x\,\in\,\Omega\\ 1\le|\alpha|\le m}}\frac{|F_\nu(x)|}{\theta_m(x)} \le \sup_{\substack{x\,\in\,K_{\nu+1}\setminus K_\nu\\ 1\le|\alpha|\le m}}\frac{\sum_{\beta\le\alpha,\,|\beta|<|\alpha|} C_\alpha^\beta|(D^\beta f)(x)(D^{\alpha-\beta}\eta_\nu)(x)|}{\theta_m(x)}\,.$$

With the help of the inequalities (2) and (3) we have for each $s\in\mathbb{N}$

$$\sup_{\substack{x\,\in\,\Omega\\ 1\le|\alpha|\le m}}\frac{|F_\nu(x)|}{\theta_m(x)} \le \sup_{\substack{x\,\in\,K_{\nu+1}\setminus K_\nu\\ 1\le|\alpha|\le m}}\frac{2^{mn}4^{n+m}p_{m+s}(f)(\nu+1)^{3n+2m}}{e^{\varphi_m(x)-\varphi_{m+s}(x)}}\,. \tag{7}$$

Let $R_\nu=(K_{\nu+1}\setminus K_\nu)\cap\{\nu\le\|x\|\le\nu+1\}$, $P_\nu=(K_{\nu+1}\setminus K_\nu)\cap\{\|x\|\le\nu\}$. Note that for each $k\in\mathbb{N}$

$$\begin{aligned} \varphi_k(x)-\varphi_{k+1}(x) &\ge \ln(1+\nu)-a_k-b_k\,, &&x\in R_\nu\,,\\ \varphi_k(x)-\varphi_{k+1}(x) &\ge \ln\nu-a_k-b_k\,, &&x\in P_\nu\,. \end{aligned}$$

Thus, for each $s\in\mathbb{N}$ one obtains

$$\sup_{x\in R_\nu,\,1\le|\alpha|\le m}\frac{|F_\nu(x)|}{\theta_m(x)} \le \frac{2^{mn}4^{n+m}p_{m+s}(f)(\nu+1)^{3n+2m}}{(1+\nu)^s e^{-a_m-\cdots-a_{m+s}-b_m-\cdots-b_{m+s}}}\,,$$

$$\sup_{x\in P_\nu,\,1\le|\alpha|\le m}\frac{|F_\nu(x)|}{\theta_m(x)}\le\frac{2^{mn}4^{n+m}p_{m+s}(f)(\nu+1)^{3n+2m}}{\nu^s e^{-a_m-\cdots-a_{m+s}-b_m-\cdots-b_{m+s}}}\,.$$

Letting $s=3n+2m+1$ we get from these two estimates and (7) that for each $m\in\mathbb{N}$

$$\sup_{x\in\Omega,\,1\le|\alpha|\le m}\frac{|F_\nu(x)|}{\theta_m(x)}\to 0\quad\text{as }\nu\to\infty\,. \tag{8}$$

Now let us consider the second term in (6). For arbitrary $m\in\mathbb{N}$ we have

$$\sup_{\substack{x\in\Omega\\1\le|\alpha|\le m}}\frac{|(D^\alpha f)(x)(\eta_\nu(x)-1)|}{\theta_m(x)}\le\sup_{\substack{x\in\Omega\setminus K_\nu\\1\le|\alpha|\le m}}\frac{|(D^\alpha f)(x)|}{\theta_m(x)}$$

$$\le\ p_{m+1}(f)\exp\left(\sup_{x\in\Omega\setminus K_\nu}\big(\varphi_{m+1}(x)-\varphi_m(x)\big)\right).$$

Using (4) we get

$$\sup_{x\in\Omega,\,1\le|\alpha|\le m}\frac{|(D^\alpha f)(x)(\eta_\nu(x)-1)|}{\theta_m(x)}\to 0\,,\quad \nu\to\infty\,.$$

From this, (8) and (5) it follows that $p_m(f_\nu-f)\to 0$ as $\nu\to\infty$ for each $m\in\mathbb{N}$. Thus, the sequence $(f_\nu)_{\nu=1}^\infty$ converges to f in $\mathcal{E}_\varphi(\Omega)$ as $\nu\to\infty$.

2. Fix a positive integer $\nu\ge\nu_0$. Let $h\not\equiv 0$ be an entire function of exponential type at most 1 such that $h\in L_1(\mathbb{R})$ and $h(x)\ge 0$ for $x\in\mathbb{R}$. For example, we can take $h(z)=\dfrac{\sin^2\frac{z}{2}}{z^2}$, $z\in\mathbb{C}$. By the Paley-Wiener theorem [9] there exists a function $g\in C(\mathbb{R})$ with support in $[-1,1]$ such that

$$h(z)=\int_{-1}^{1}g(t)e^{izt}\,\mathrm{d}t\,,\qquad z\in\mathbb{C}\,.$$

From this representation it follows that for each $k\in\mathbb{Z}_+$

$$|h^{(k)}(x)|\le 2\max_{|t|\le 1}|g(t)|\,,\qquad x\in\mathbb{R}\,. \tag{9}$$

Let $H(z_1,z_2,\dots,z_n)=h(z_1)h(z_2)\cdots h(z_n)$. It is an entire function in $\mathbb{C}^n$. From (9) it follows that there exists a positive constant $C_H>0$ such that for each $\alpha\in\mathbb{Z}_+^n$

$$|(D^\alpha H)(x)|\le C_H\,,\qquad x\in\mathbb{R}^n\,. \tag{10}$$

Let $\int\limits_{\mathbb{R}^n} H(x)\,\mathrm{d}x = A$. Define a function $\tilde{f}_\nu$ on $\mathbb{R}^n$ as follows: $\tilde{f}_\nu(x) = f_\nu(x)$, $x \in \Omega$; $\tilde{f}_\nu(x) = 0$, $x \in \mathbb{R}^n \setminus \Omega$. Obviously, $\tilde{f}_\nu \in C^\infty(\mathbb{R}^n)$.

For $\lambda > 1$ let

$$\tilde{f}_{\nu,\lambda}(x) = \frac{\lambda^n}{A}\int_{\mathbb{R}^n} \tilde{f}_\nu(y) H(\lambda(x-y))\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^n\,.$$

It is clear that $\tilde{f}_{\nu,\lambda} \in C^\infty(\mathbb{R}^n)$. Moreover, $\tilde{f}_{\nu,\lambda}$ admits holomorphic continuation in $\mathbb{C}^n$. Note that for all $\alpha \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n$

$$\begin{aligned}\left|(D^\alpha \tilde{f}_{\nu,\lambda})(x)\right| &\le \frac{\lambda^n}{A}\int_{\mathbb{R}^n} \left|(D^\alpha \tilde{f}_\nu)(y)\right| H(\lambda(x-y))\,\mathrm{d}y \\ &\le \max_{y\in K_{\nu+1}} \left|(D^\alpha f_\nu)(y)\right| \cdot \frac{\lambda^n}{A}\int_{\mathbb{R}^n} H(\lambda(x-y))\,\mathrm{d}y \\ &= \max_{y\in K_{\nu+1}} \left|(D^\alpha f_\nu)(y)\right|.\end{aligned}$$

Let $f_{\nu,\lambda}$ be the restriction of $\tilde{f}_{\nu,\lambda}$ on Ω. Obviously, $f_{\nu,\lambda} \in \mathcal{E}_\varphi(\Omega)$.

Let us show that $f_{\nu,\lambda} \to f_\nu$ in $\mathcal{E}_\varphi(\Omega)$ as $\lambda \to +\infty$.

Take an arbitrary $m \in \mathbb{N}$ and let $r(\lambda) = \lambda^{-\frac{2n}{2n+1}}$ $(\lambda > 1)$. Note that for all $\alpha \in \mathbb{Z}_+^n, x \in \Omega$

$$\begin{aligned}&(D^\alpha f_{\nu,\lambda})(x) - (D^\alpha f_\nu)(x) \\ &\quad = \frac{\lambda^n}{A}\int_{\mathbb{R}^n} \big((D^\alpha \tilde{f}_\nu)(y) - (D^\alpha \tilde{f}_\nu)(x)\big) H(\lambda(x-y))\,\mathrm{d}y \\ &\quad = \frac{\lambda^n}{A}\int_{\{y\in\mathbb{R}^n:\|y-x\|\le r(\lambda)\}} \big((D^\alpha \tilde{f}_\nu)(y) - (D^\alpha \tilde{f}_\nu)(x)\big) H(\lambda(x-y))\,\mathrm{d}y \\ &\qquad + \frac{\lambda^n}{A}\int_{\{y\in\mathbb{R}^n:\|y-x\|> r(\lambda)\}} \big((D^\alpha \tilde{f}_\nu)(y) - (D^\alpha \tilde{f}_\nu)(x)\big) H(\lambda(x-y))\,\mathrm{d}y\,.\end{aligned}$$

Denote the terms on the right-hand side of this equality by $I_{1,\alpha}(x)$ and $I_{2,\alpha}(x)$, respectively. Let $C_{\nu,m} = \sup\limits_{t\in\mathbb{R}^n,\,|\beta|\le m+1} \left|(D^\beta \tilde{f}_\nu)(t)\right|$. Then

$$\begin{aligned}\sup_{x\in\Omega,\,|\alpha|\le m} |I_{1,\alpha}(x)| &\le \frac{\pi^{\frac{n}{2}}\sqrt{n}C_H C_{\nu,m}}{A\Gamma(\frac{n}{2}+1)}\lambda^{-\frac{n}{2n+1}}\,, \\ \sup_{x\in\Omega,\,|\alpha|\le m} |I_{2,\alpha}(x)| &\le \frac{2C_{\nu,m}}{A}\int\limits_{\|u\|>\lambda^{\frac{1}{2n+1}}} H(u)\,\mathrm{d}u\,.\end{aligned}$$

From these two estimates it follows that

$$\sup_{x\in\Omega,\,|\alpha|\le m} |(D^\alpha f_{\nu,\lambda})(x) - (D^\alpha f_\nu)(x)| \to 0$$

as $\lambda \to +\infty$. Hence, $p_m(f_{\nu,\lambda} - f_\nu) \to 0$ as $\lambda \to +\infty$. Since $m \in \mathbb{N}$ was arbitrary then $f_{\nu,\lambda} \to f_\nu$ in $\mathcal{E}_\varphi(\Omega)$ as $\lambda \to +\infty$.

3. *For fixed $\lambda > 0$ and positive integer $\nu \ge \nu_0$ let us approximate $f_{\nu,\lambda}$ by polynomials in $\mathcal{E}_\varphi(\Omega)$.*

For $N \in \mathbb{N}$ let

$$U_N(x) = H(0) + \sum_{k=1}^{N} \frac{\displaystyle\sum_{1\le i_1\le n} \cdots \sum_{1\le i_k\le n} \frac{\partial^k H}{\partial x_{i_1}\cdots\partial x_{i_k}}(0)x_{i_1}\cdots x_{i_k}}{k!} .$$

For $x \in \mathbb{R}^n$ we have

$$|H(x) - U_N(x)| \le \frac{\displaystyle\sum_{1\le i_1\le n} \cdots \sum_{1\le i_{N+1}\le n} \sup_{\xi\in[0,x]} \left|\frac{\partial^{N+1} H}{\partial x_{i_1}\cdots\partial x_{i_{N+1}}}(\xi)x_{i_1}\cdots x_{i_{N+1}}\right|}{(N+1)!} .$$

Using the inequality (10) we get

$$|H(x) - U_N(x)| \le \frac{C_H n^{N+1}\|x\|^{N+1}}{(N+1)!}, \qquad x \in \mathbb{R}^n . \tag{11}$$

Let

$$V_N(x) = \frac{\lambda^n}{A}\int_{\mathbb{R}^n} \tilde{f}_\nu(y) U_N(\lambda(x-y))\,\mathrm{d}y, \qquad x \in \mathbb{R}^n .$$

It is clear that V_N is a polynomial of degree at most N. We claim that the sequence $(V_N)_{N=1}^\infty$ converges to $f_{\nu,\lambda}$ in $\mathcal{E}_\varphi(\Omega)$ as $N \to \infty$. Let $m \in \mathbb{N}$ be arbitrary. For $\alpha \in \mathbb{Z}_+^n$ and $x \in \Omega$ we have

$$(D^\alpha f_{\nu,\lambda})(x) - (D^\alpha V_N)(x) = \frac{\lambda^n}{A}\int_{\mathbb{R}^n} (D^\alpha \tilde{f}_\nu)(y)\big(H(\lambda(x-y)) - U_N(\lambda(x-y))\big)\,\mathrm{d}y .$$

Using the inequality (11) and taking into account that $\tilde{f}_\nu$ has a bounded support we can find positive constants C_1 and C_2 (depending on n, λ, ν and m) such that for all $N \in \mathbb{N}$, $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le m$, $x \in \Omega$

$$\big|(D^\alpha f_{\nu,\lambda})(x) - (D^\alpha V_N)(x))\big| \le \frac{C_1 C_2^N (1+\|x\|)^{N+1}}{(N+1)!} .$$

Thus, for each $N\in \mathbb{N}$

$$p_m(f_{\nu,\lambda} - V_N) \le \frac{C_1 C_2^N}{(N+1)!} \sup_{x\in\Omega} \frac{(1+\|x\|)^{N+1}}{\theta_m(x)} .$$

Furthermore,

$$\begin{aligned}
\sup_{x\in\Omega} \frac{(1+\|x\|)^{N+1}}{\theta_m(x)} &\le \sup_{x\in\Omega} \frac{(1+\|x\|)^{N+1}}{e^{\psi_m(x)}} \\
&= \exp\left(\sup_{r>0,\, \sigma\in \mathrm{pr}(\Omega)} \big((N+1)\ln(r+1) - \psi_m(r\sigma)\big)\right) \\
&\le 2^{N+1} \exp\left(\left(\sup_{r>1,\, \sigma\in \mathrm{pr}(\Omega)} \big((N+1)\ln r - \psi_m(r\sigma)\big)\right)^+\right) \\
&= 2^{N+1} \exp\left(\left(\sup_{t>0,\, \sigma\in \mathrm{pr}(\Omega)} \big((N+1)t - \psi_m(e^t\sigma)\big)\right)^+\right) \\
&= 2^{N+1} \exp\left(\left(\sup_{\sigma\in (\mathrm{pr}\,\Omega)} \left(\sup_{t>0} \big((N+1)t - \psi_{m,\sigma}(e^t)\big)\right)\right)^+\right) \\
&= 2^{N+1} \exp\left(\left(\sup_{\sigma\in \mathrm{pr}(\Omega)} (\psi_{m,\sigma}[e])^*(N+1)\right)^+\right).
\end{aligned}$$

Now applying Lemma 1 we obtain

$$\sup_{x\in\Omega} \frac{(1+\|x\|)^{N+1}}{\theta_m(x)} \le 2^{N+1} \max\left(1, \frac{(N+1)^{N+1}}{e^{N+1} e^{\inf\limits_{\sigma\in \mathrm{pr}(\Omega)} (\psi^*_{m,\sigma}[e])^*(N+1)}}\right) .$$

Thus, for each $N\in \mathbb{N}$

$$p_m(f_{\nu,\lambda} - V_N) \le \frac{C_1 C_2^N 2^{N+1}}{(N+1)!} \max\left(1, \frac{(N+1)^{N+1}}{e^{N+1} e^{\inf\limits_{\sigma\in \mathrm{pr}(\Omega)} (\psi^*_{m,\sigma}[e])^*(N+1)}}\right) . \quad (12)$$

Since $N! \ge \frac{N^N}{e^N}$ for all $N \in \mathbb{N}$, we get

$$\frac{C_1 C_2^N 2^{N+1}}{(N+1)!} \cdot \frac{(N+1)^{N+1}}{e^{N+1} e^{\inf\limits_{\sigma\in \mathrm{pr}(\Omega)} (\psi^*_{m,\sigma}[e])^*(N+1)}} \le \frac{C_1 C_2^N 2^{N+1}}{e^{\inf\limits_{\sigma\in \mathrm{pr}(\Omega)} (\psi^*_{m,\sigma}[e])^*(N+1)}} . \quad (13)$$

Note that uniformly for $\sigma \in \mathrm{pr}(\Omega)$ one has

$$\lim_{\xi \to +\infty} \frac{(\psi^*_{m,\sigma}[e])^*(\xi)}{\xi} = +\infty\,. \tag{14}$$

This is so because for each $\sigma \in \mathrm{pr}(\Omega)$

$$(\psi^*_{m,\sigma}[e])^*(\xi) \ge \xi t - \psi^*_{m,\sigma}(e^t)\,, \qquad \xi > 0\,,\ t > 0\,,$$

and

$$\begin{aligned}\psi^*_{m,\sigma}(e^t) &= \sup_{r \ge 0} \left(e^t r - \psi_{m,\sigma}(r)\right)\\ &\le \sup_{r\ge 0,\, \sigma \in \mathrm{pr}(\Omega)} \left(e^t r - \psi_m(r\sigma)\right) = \sup_{x \in \Omega} \left(e^t \|x\| - \psi_m(x)\right).\end{aligned}$$

Using (13) and (14) we have from (12) that $p_m(f_{\nu,\lambda} - V_N) \to 0$ as $N \to \infty$.

From the conclusions of all three above steps, one derives that each function $f \in \mathcal{E}_\varphi(\Omega)$ can be approximated by polynomials in $\mathcal{E}_\varphi(\Omega)$. ∎

3. Application of Theorem 1 to hypercyclicity

The following auxiliary results will be used in the proof of Theorem 2.

Lemma 2. *Partial derivative operators are continuous on $\mathcal{E}_\varphi(\Omega)$.*

Proof. Let $m \in \mathbb{N}$ be arbitrary. Since $\varphi_m(x) - \varphi_{m+1}(x) \ge -a_m - b_m$ for all $x \in \Omega$, we get for each $f \in \mathcal{E}_\varphi(\Omega)$, all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le m$ and $j = 1, \dots, n$ that

$$\begin{aligned}\left|\left(D^\alpha\left(\frac{\partial}{\partial x_j} f\right)\right)(x)\right| &\le p_{m+1}(f) e^{\varphi_{m+1}(x)}\\ &\le p_{m+1}(f) e^{\varphi_m(x) + a_m + b_m}\,, \qquad x \in \Omega\,.\end{aligned}$$

Thus,

$$p_m\left(\frac{\partial}{\partial x_j} f\right) \le e^{a_m + b_m} p_{m+1}(f)\,, \qquad f \in \mathcal{E}_\varphi(\Omega)\,.$$

This means that the operators $\frac{\partial}{\partial x_j}$ are continuous on $\mathcal{E}_\varphi(\Omega)$. ∎

Corollary 1. *Let $P(x) = \sum\limits_{|\alpha| \le N} a_\alpha x^\alpha$ be a polynomial in $\mathbb{R}^n$ ($N \in \mathbb{N}$). Then the operator $\sum\limits_{|\alpha| \le N} a_\alpha D^\alpha$ is continuous on $\mathcal{E}_\varphi(\Omega)$.*

Note that for each $z \in \mathbb{C}^n$ the function $f_z(\xi) := \exp(i\langle \xi, z\rangle)$ belongs to $\mathcal{E}_\varphi(\Omega)$ since for each $m \in \mathbb{N}$

$$p_m(f_z) \le (1 + \|z\|)^m \exp\left(\widetilde{\varphi}_m(\operatorname{Im} z)\right).$$

So for each functional $\Phi \in (\mathcal{E}_\varphi(\Omega))'$ its Fourier-Laplace transform $\hat{\Phi}(z) = \Phi(e^{i\langle \xi, z\rangle})$ is well defined everywhere on $\mathbb{C}^n$.

LEMMA 3. *Let $\Phi \in (\mathcal{E}_\varphi(\Omega))'$. Then $\hat{\Phi}$ is an entire function on $\mathbb{C}^n$. Moreover, for each $\alpha \in \mathbb{Z}_+^n$*

$$(D_z^\alpha \hat{\Phi})(z) = \Phi\Big((i\xi)^\alpha e^{i\langle \xi, z\rangle}\Big), \qquad z \in \mathbb{C}^n. \tag{15}$$

Proof. Fix Φ as in the hypothesis as well as an arbitrary point $\zeta \in \mathbb{C}^n$. For each $z \in \mathbb{C}^n$ such that $\|z - \zeta\| < 1$ let

$$g_{z,\zeta}(\xi) := e^{i\langle \xi, z\rangle} - e^{i\langle \xi, \zeta\rangle} - i\langle \xi, z - \zeta\rangle e^{i\langle \xi, \zeta\rangle}, \qquad \xi \in \mathbb{R}^n. \tag{16}$$

Using the inequality

$$|(D^\alpha g_{z,\zeta})(\xi)| \le (1 + \|\zeta\|)^{|\alpha|}(1 + \|\xi\| + \|\xi\|^2)e^{\|\xi\|}\|z - \zeta\|^2 e^{\langle \xi, -\operatorname{Im}\zeta\rangle}, \quad \alpha \in \mathbb{Z}_+^n,$$

positivity of functions h_m and condition (a) on the system $(\psi_m)_{m=1}^\infty$ for each $m \in \mathbb{N}$ we can find a constant $C > 0$ depending on ζ and m such that

$$p_m(g_{z,\zeta}) \le C\|z - \zeta\|^2. \tag{17}$$

Since Φ is a continuous functional then $\Phi(g_{z,\zeta}) = o(\|z - \zeta\|)$, $z \to \zeta$. And now since Φ is linear we get

$$\hat{\Phi}(z) - \hat{\Phi}(\zeta) = \sum_{j=1}^n \Phi\Big(i\xi_j e^{i\langle \xi, \zeta\rangle}\Big)(z_j - \zeta_j) + o(\|z - \zeta\|), \qquad z \to \zeta.$$

Therefore, $\hat{\Phi}$ is holomorphic at the point ζ. Since $\zeta \in \mathbb{C}^n$ was arbitrary, then $\hat{\Phi}$ is an entire function.

The second part of the statement is evident. The lemma is proved. ∎

LEMMA 4. *Let $\mathcal{O}$ be a non-empty open set in $\mathbb{C}^n$. Then the system $\{\exp(i\langle \xi, z\rangle)\}_{z \in \mathcal{O}}$ is complete in $\mathcal{E}_\varphi(\Omega)$.*

Proof. Let S be an arbitrary linear continuous functional on $\mathcal{E}_\varphi(\Omega)$ such that $S\big(e^{i\langle\xi,z\rangle}\big) = 0$ for each $z \in \mathcal{O}$. By using the Hahn-Banach theorem, our task is to show that S is a zero functional. By Lemma 3 the Fourier-Laplace transform $\hat{S}$ of S is an entire function on $\mathbb{C}^n$. So by the uniqueness theorem $\hat{S}(z) = 0$ for each $z \in \mathbb{C}^n$. Now using (15) we get $S(\xi^\alpha) = 0$ for all $\alpha \in \mathbb{Z}_+^n$. Thus, $S(p) = 0$ for each polynomial p. By Theorem 1 polynomials are dense in $\mathcal{E}_\varphi(\Omega)$. Therefore, $S = 0$ and the proof is complete. ∎

Denote by $L(\mathcal{E}_\varphi(\Omega))$ the set of linear continuous operators on $\mathcal{E}_\varphi(\Omega)$. Let $T \in L(\mathcal{E}_\varphi(\Omega))$. Define the function F_T on $\Omega \times \mathbb{C}^n$ by the rule $F_T(\xi, z) = T(f_z)(\xi)$. For each fixed $z \in \mathbb{C}^n$ let $f_{j,z}(\xi) := i\xi_j \exp(i\langle\xi, z\rangle)$.

Lemma 5. *Let $T \in L(\mathcal{E}_\varphi(\Omega))$. Then the function F_T is an entire function in the second variable.*

Proof. Let T be a linear continuous operator on $\mathcal{E}_\varphi(\Omega)$. Then for each $k \in \mathbb{N}$ there exist numbers $c_k > 0$ and $m \in \mathbb{N}$ such that for all $g \in \mathcal{E}_\varphi(\Omega)$

$$p_k(T(g)) \le c_k p_m(g)\,. \tag{18}$$

Let $\xi \in \Omega$, $\zeta \in \mathbb{C}^n$ be arbitrary points. For each $z \in \mathbb{C}^n$ such that $\|z - \zeta\| < 1$ consider the function $g_{z,\zeta}$ (see (16)). From (18) it follows that

$$|T(g_{z,\zeta})(\xi)| \le c_k p_m(g_{z,\zeta}) e^{\varphi_k(\xi)}\,, \qquad \xi \in \Omega\,.$$

From this, the inequality (17) and linearity of T we get for each $\xi \in \Omega$

$$F_T(\xi, z) - F_T(\xi, \zeta) = \sum_{j=1}^{n} T(f_{j,\zeta})(\xi)(z_j - \zeta_j) + o(\|z - \zeta\|)\,, \qquad z \to \zeta\,.$$

Therefore, for each fixed $\xi \in \Omega$, $F_T(\xi, z)$ is holomorphic at the point ζ as a function of z. Since $\zeta \in \mathbb{C}^n$ was arbitrary, then the assertion of lemma is proved. ∎

Proof of Theorem 2. Since T commutes with each partial derivative operator, then for each $z \in \mathbb{C}^n$ and $j = 1, \dots, n$

$$D_j T(f_z) = T D_j(f_z) = T(iz_j f_z) = iz_j T(f_z)\,.$$

From this it follows that for each $z \in \mathbb{C}^n$ there is a complex number $a_T(z)$ such that

$$T(f_z) = a_T(z) f_z\,. \tag{19}$$

Thus, for all $z \in \mathbb{C}^n$, $\xi \in \Omega$ we have $F_T(\xi, z) = a_T(z)e^{i\langle \xi, z\rangle}$. Using Lemma 5 we get that a_T is an entire function on $\mathbb{C}^n$. Taking into account that T is not a scalar multiple of the identity and Lemma 4 we conclude that a_T is not a constant function. Note that, if Ω is a non-empty open set in $\mathbb{C}^n$, then the system $\{T(f_z)\}_{z\in\Omega}$ is complete in $\mathcal{E}_\varphi(\Omega)$. It is easy to show using the representation (19) and Lemma 4. Consider the sets $W_1 = \{z \in \mathbb{C}^n : |a_T(z)| < 1\}$ and $W_2 = \{z \in \mathbb{C}^n : |a_T(z)| > 1\}$. They are open in $\mathbb{C}^n$. Let X_0 be the linear span of the system $\{T(f_z)\}_{z\in W_1}$, Y_0 be the linear span of the system $\{T(f_z)\}_{z\in W_2}$. The sets X_0 and Y_0 are dense in $\mathcal{E}_\varphi(\Omega)$. Therefore, linear spans of the sets $\bigcup\limits_{|\lambda|<1} \ker(T-\lambda)$ and $\bigcup\limits_{|\lambda|>1} \ker(T-\lambda)$ are dense in $\mathcal{E}_\varphi(\Omega)$. Thus, all the conditions of Theorem A are fulfilled. Hence, the operator T is hypercyclic. ∎

Corollary 2. *Let* $P(x) = \sum\limits_{|\alpha|\le m} a_\alpha x^\alpha$ *be a non-constant polynomial in* $\mathbb{R}^n$ *($N \in \mathbb{N}$). Then the operator* $\sum\limits_{|\alpha|\le N} a_\alpha D^\alpha$ *is hypercyclic in* $\mathcal{E}_\varphi(\Omega)$.

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