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On the Berezin Symbols and Toeplitz Operators*

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Abstract: The present paper mainly gives some new applications of Berezin symbols. In particular, the Berezin symbol is used in approximation problem for H^∞-functions. We study also asymptotic multiplicativity of the Berezin symbols. Moreover, we study the solvability of some Riccati operator equations of the form $XAX + XB - CX = D$ on the Toeplitz algebra $\mathcal{T}$, which is the C^*-subalgebra of the operator algebra $\mathcal{B}(L_a^2)$ generated by the Toeplitz operators $\{T_g : g \in H^\infty\}$ on the Bergman space $L_a^2(\mathbb{D})$. We characterize compactness of truncated Toeplitz operators $A_\varphi = P_{K_\theta} T_\varphi \mid K_\theta$, $\varphi \in L^\infty(\mathbb{T})$, in terms of Berezin symbols. The spectrum of model operators $\varphi(M_\theta)$, $\varphi \in H^\infty$, is localized in terms of the so-called Berezin set by proving that $\sigma(\varphi(M_\theta)) \subset closBer(\varphi(M_\theta))$. Reducing subspaces of n-tuple of invertible operators on the Hilbert space H are described.

Key words: Berezin symbol, Hardy space, Bergman space, Toeplitz operator, truncated Toeplitz operator, inner function.

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1. Introduction and notations

The Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$\|f\|_2^2 := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{it})\right|^2 dt < +\infty.$$

For $\lambda \in \mathbb{D}$, the reproducing kernel (Szegö kernel) of H^2 is the function $k_\lambda \in H^2$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for every $f \in H^2$. The normalized reproducing kernel $\widehat{k}_\lambda$ is the function $\frac{k_\lambda}{\|k_\lambda\|_2}$. It is well known that $k_\lambda(z) = \left(1 - \overline{\lambda} z\right)^{-1}$. The set of bounded linear operators on the Hilbert space H is denoted by $\mathcal{B}(H)$.

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For $\varphi \in L^\infty(\mathbb{T})$, where $\mathbb{T} = \partial\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ is the unit circle, the Toeplitz operator T_φ with symbol φ is the operator on H^2 defined by $T_\varphi f = P_+(\varphi f)$; here P_+ is the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. Note that if $\varphi \in H^\infty$ (the set of all bounded analytic functions on $\mathbb{D}$), then T_φ is just the operator of multiplication by φ on H^2.

For $A \in \mathcal{B}(H^2)$, the Berezin symbol (transform) of A is the complex valued function $\widetilde{A}$ on $\mathbb{D}$ defined by

$$\widetilde{A}(\lambda) = \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle.$$

It is well known (see Zhu [24], Engliś [8] and Karaev [14]) that $\widetilde{T_\varphi} = \widetilde{\varphi}$, where $\widetilde{\varphi}$ denotes the harmonic extension of φ into the unit disc $\mathbb{D}$.

Often the behavior of the Berezin transform of an operator provides important information about the operator itself. For example, it is well known (see, for instance, Zhu [24], Fricain [9] and Yang [23]) that in most functional Hilbert spaces of analytic functions, including Hardy and Bergman spaces, the Berezin symbol uniquely determines the operator, that is $A = B$ if and only if $\widetilde{A} = \widetilde{B}$. It is also known (see Nordgren and Rosenthal [19]) that compact operators on the so-called standard functional reproducing kernel Hilbert spaces are completely characterized by the boundary behavior of Berezin symbols of their unitary orbits.

In the present paper we mainly study some new applications of Berezin symbols. In particular, the Berezin transform is used in approximation problem for H^∞-functions (see Theorem 1 in Section 2). In Section 3 of this article, we study the asymptotical multiplicatively of the Berezin symbols. In Section 4, we consider the Riccati equations of the form

$$XAX + XB - CX = D$$

with bounded operators A, B, C, D on the Bergman space $L_a^2(\mathbb{D})$ and study the solvability of such operator equations on the Toeplitz algebra $\mathcal{T}$ which is the C^*-subalgebra of $\mathcal{B}(L_a^2)$ generated by $\{T_g : g \in H^\infty\}$. Section 5 characterizes the compact truncated Toeplitz operators (which are defined by $A_\varphi = P_{K_\theta} T_\varphi \mid K_\theta$ for any symbol $\varphi \in L^\infty(\mathbb{T})$ and inner function θ) in terms of Berezin symbols. One important subclass of truncated Toeplitz operators is the class of model operators $\varphi(M_\theta)$, $\varphi \in H^\infty$, of Sz. -Nagy and Foias. In Section 6, we give in terms of the so-called Berezin set some localizations of spectrum of model operators $\varphi(M_\theta)$ by proving that $\sigma\big(\varphi(M_\theta)\big) \subset \overline{Ber\big(\varphi(M_\theta)\big)}$. Section 7 describes the reducing subspaces of n-tuple of invertible operators on a Hilbert space H, which is important in the invariant subspace problem for H.

Before giving the results of this paper, let us introduce some necessary notations. The symbol $H^\infty = H^\infty(\mathbb{D})$ denotes the Banach algebra of functions bounded and analytic on the unit disc $\mathbb{D}$ equipped with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$. A function $\theta \in H^\infty$ such that $|\theta(\zeta)| = 1$ almost everywhere in $\mathbb{T}$ is called an inner function. It is convenient to establish a natural embedding of the space H^2 in the space $L^2 = L^2(\mathbb{T})$ by associating to each function $f \in H^2$ its radial boundary values $(bf)(\zeta) := \lim_{r\to 1^-} f(r\zeta)$, which (by the Fatou Theorem [12]) exist for m-almost all $\zeta \in \mathbb{T}$; where m is the normalized Lebesgue measure on $\mathbb{T}$. Then we have

$$H^2 = \left\{f \in L^2 : \widehat{f}(n) = 0,\ n < 0\right\}$$

where $\widehat{f}(n) := \int_{\mathbb{T}} \overline{\zeta}^n f(\zeta)dm(\zeta)$ is the Fourier coefficient of the function f. We denote

$$H^2_- = \left\{f \in L^2 : \widehat{f}(n) = 0,\ n \geq 0\right\}$$

If $\varphi \in L^\infty = L^\infty(\mathbb{T})$, then the Hankel operator H_φ is defined by $H_\varphi f = P_-\varphi f$, $f \in H^2$, where $P_- = I - P_+$. The harmonic extension of function $\varphi \in L^\infty$ is denoted, as before, by the symbol $\widetilde{\varphi}$, and the space of all bounded harmonic functions on $\mathbb{D}$ is denoted by $h^\infty(\mathbb{D})$. For any inner function θ the model space is defined by $K_\theta = H^2 \Theta \theta H^2$.

The Berezin set and Berezin number of the operator $A \in \mathcal{B}(H^2)$ is defined respectively by

$$Ber(A) := Range\big(\widetilde{A}\big) = \left\{\widetilde{A}(\lambda) : \lambda \in \mathbb{D}\right\}$$

and

$$ber(A) := \sup\left\{\big|\widetilde{A}(\lambda)\big| : \lambda \in \mathbb{D}\right\}.$$

The symbol $W(A)$ denotes the numerical range of the operator $A \in \mathcal{B}(H^2)$, and $w(A)$ its numerical radius (see Halmos [11]):

$$W(A) = \left\{ \langle Ax, x\rangle : \|x\|_2 = 1\right\}$$
$$w(A) = \sup\left\{ |\langle Ax, x\rangle| : \|x\|_2 = 1\right\}.$$

2. Approximation of H^∞-functions

In this section we consider an approximation problem for H^∞-functions φ on level sets of inner functions θ using Berezin symbols of some operators associated to φ and θ.

Theorem 1. *Let $\varphi \in H^\infty, \|\varphi\|_\infty \leq 1$, be any function, and θ be any nonconstant inner function. For any nonzero $A \in \{T_\theta\}' := \{B \in \mathcal{B}(H^2) : BT_\theta = T_\theta B\}$ (the commutant of the Toeplitz operator T_θ) we denote*

$$N_{\varphi,\theta,A} := T_\varphi(I - AT_\theta T_\theta^*)$$

For any arbitrary but fixed $\varepsilon \in (0,1)$, let $L_{\varepsilon,\theta} := \{z \in \mathbb{D} : |\theta(z)| \leq \varepsilon\}$ be an ε-level set of θ. Then

$$\left\|\varphi - \widetilde{N}_{\varphi,\theta,A}\right\|_{L^\infty(L_{\varepsilon,\theta})} \leq \varepsilon^2 ber(A).$$

In other words,

$$\left\|\varphi - \widetilde{N}_{\varphi,\theta,A}\right\|_{L^\infty(L_{\varepsilon,\theta})} = O\left(\varepsilon^2\right) \text{ as } \varepsilon \to 0$$

Proof. Let $\widehat{k}_\lambda(z) = \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\bar{\lambda}z}$ be the normalized reproducing kernel of the Hardy space H^2. Then we obtain:

$$\begin{aligned}
\widetilde{N}_{\varphi,\theta,A}(\lambda) &= \left\langle N_{\varphi,\theta,A}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle = \left\langle T_\varphi(I - AT_\theta T_\theta^*)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&= \left\langle T_\varphi\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle - \left\langle T_\varphi AT_\theta T_\theta^*\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&= \left\langle \widehat{k}_\lambda, T_\varphi^*\widehat{k}_\lambda\right\rangle - \left\langle T_\varphi T_\theta A(\overline{\theta(\lambda)}\widehat{k}_\lambda), \widehat{k}_\lambda\right\rangle \\
&= \left\langle \widehat{k}_\lambda, \overline{\varphi(\lambda)}\widehat{k}_\lambda\right\rangle - \overline{\theta(\lambda)}\left\langle A\widehat{k}_\lambda, T_{\varphi\theta}^*\widehat{k}_\lambda\right\rangle \\
&= \varphi(\lambda) - \varphi(\lambda)\left|\theta(\lambda)\right|^2\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\
&= \varphi(\lambda)\left(1 - |\theta(\lambda)|^2\widetilde{A}(\lambda)\right)
\end{aligned}$$

thus

$$\widetilde{N}_{\varphi,\theta,A}(\lambda) = \varphi(\lambda)\left(1 - |\theta(\lambda)|^2\widetilde{A}(\lambda)\right)$$

for all $\lambda \in \mathbb{D}$. From the last formula we obtain that

$$\begin{aligned}
\left|\varphi(\lambda) - \widetilde{N}_{\varphi,\theta,A}(\lambda)\right| &= \left|\widetilde{A}(\lambda)\right| |\varphi(\lambda)| \, |\theta(\lambda)|^2 \leq \|\varphi\|_\infty ber(A) \, |\theta(\lambda)|^2 \\
&\leq ber(A) \, |\theta(\lambda)|^2
\end{aligned}$$

for all $\lambda \in \mathbb{D}$. In particular,

$$\left|\varphi(\lambda) - \widetilde{N}_{\varphi,\theta,A}(\lambda)\right| \leq ber(A)\varepsilon^2$$

for all $\lambda \in L_{\varepsilon,\theta}$, and hence

$$\left\|\varphi - \widetilde{N}_{\varphi,\theta,A}\right\|_{L^\infty(L_{\varepsilon,\theta})} \leq ber(A)\varepsilon^2,\ 0 < \varepsilon < 1,$$

which proves the theorem. ∎

In the next section, we give another application of the operator $N_{\varphi,\theta,A}$.

3. On the asymptotic multiplicative property of the Berezin symbol on the Hardy and Bergman spaces

This section focuses on the asymptotic multiplicative property of the Berezin symbol $\widetilde{A}$ of a given operator A on the Hardy and Bergman spaces.

Recall that the Berezin symbol is called asymptotically multiplicative on $\mathcal{B}(\mathcal{H}(\Omega))$ if $\lim_{\lambda\to\partial\Omega}\left(\widetilde{AB}(\lambda) - \widetilde{A}(\lambda)\widetilde{B}(\lambda)\right) = 0$.

Note that a complete investigation of a multiplicative property of the Berezin symbol $\widetilde{A}$ of a given linear operator $A : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$, where $\mathcal{H}(\Omega)$ is a functional Hilbert space of analytic functions on a region Ω in $\mathbb{C}^n$, apparently, was started by Kiliç in [17]. Namely, Kiliç showed that (see [17, Theorem 1]) $\widetilde{AB} = \widetilde{A}\widetilde{B}$ for all B in $\mathcal{B}(\mathcal{H}(\Omega))$ if and only if A is a multiplication operator M_φ, where φ is a multiplier; moreover, $\varphi = \widetilde{A}$. In particular, Kiliç proved [17, Corollary 2] that if A is a bounded operator on the Hardy space H^2, then $\widetilde{AB}(\lambda) = \widetilde{A}(\lambda)\widetilde{B}(\lambda)$ for all B in $\mathcal{B}\left(H^2\right)$ if and only if A is an analytic Toeplitz operator T_φ, $\varphi \in H^\infty$. Moreover, $\varphi = \widetilde{A}$.

Here we will prove that if T_φ ($\varphi \in L^\infty(\mathbb{T})$) is any Toeplitz operator on the Hardy space H^2 and B is an arbitrary operator on H^2, then $\lim_{\lambda\to\mathbb{T}}\left(\widetilde{T_\varphi B}(\lambda) - \widetilde{\varphi}(\lambda)\widetilde{B}(\lambda)\right) = 0$. A similar result is proved also for the Bergman space operators. (For practical results on when the Berezin symbol is asymptotic multiplicative on $\mathcal{B}\left(L_a^2\right)$, see Axler and Zheng's paper [4]). Our argument mainly is based on the following lemma (see Engliś [7, Theorem 6] and Karaev [14, Lemma 1.1]) which shows that the normalized reproducing kernels $\widehat{k}_\lambda$ of H^2 are, loosely speaking, asymptotic eigenfunctions for any Toeplitz operator T_φ, $\varphi \in L^\infty(\mathbb{T})$.

Lemma 1. *Let $\varphi \in L^\infty(\mathbb{T})$ and let $\widetilde{\varphi}$ be its harmonic extension (by the poisson formula) into $\mathbb{D}$. Then $T_\varphi\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda \to 0$ radially, i.e.,*

$$\lim_{r\to 1}\left\|T_\varphi\widehat{k}_{re^{it}} - \widetilde{\varphi}(re^{it})\widehat{k}_{re^{it}}\right\|_2 = 0$$

for almost all $t \in [0, 2\pi)$.

THEOREM 2. *Let $\varphi \in L^\infty(\mathbb{T})$, and $A \in \mathcal{B}(H^2)$ be an arbitrary operator. Then*

$$\left(\widetilde{T_\varphi A}(\lambda) - \widetilde{T}_\varphi(\lambda)\widetilde{A}(\lambda)\right) \to 0$$

as $\lambda \to \mathbb{T}$ radially.

Proof. Since $\widetilde{T}_\varphi = \widetilde{\varphi}$, by applying Lemma 1, we have:

$$\begin{aligned}\left|\widetilde{T_\varphi A}(\lambda) - \widetilde{T}_\varphi(\lambda)\widetilde{A}(\lambda)\right| &= \left|\left\langle T_\varphi A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle - \widetilde{\varphi}(\lambda)\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \\ &= \left|\left\langle A\widehat{k}_\lambda, T_\varphi^*\widehat{k}_\lambda\right\rangle - \left\langle A\widehat{k}_\lambda, \overline{\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\right\rangle\right| \\ &= \left|\left\langle A\widehat{k}_\lambda, T_{\overline{\varphi}}\widehat{k}_\lambda - \widetilde{\overline{\varphi}}(\lambda)\widehat{k}_\lambda\right\rangle\right| \\ &\le \left\|A\widehat{k}_\lambda\right\| \left\|T_{\overline{\varphi}}\widehat{k}_\lambda - \widetilde{\overline{\varphi}}(\lambda)\widehat{k}_\lambda\right\|_2 \\ &\le \|A\| \left\|T_{\overline{\varphi}}\widehat{k}_\lambda - \widetilde{\overline{\varphi}}(\lambda)\widehat{k}_\lambda\right\|_2 \to 0\end{aligned}$$

as $\lambda \to \mathbb{T}$ radially, which proves the theorem. ∎

It is not difficult to see that in case $A = P_E$, where $E \subset H^2$ is an arbitrary closed subspace, Theorem 2 essentially improves Proposition 2.3 of [15]. Before stating our results in the Bergman space, we will begin by recalling some notations and well known facts.

Let dA denote Lebesgue area measure of $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equal 1. Recall that the Bergman space $L_a^2 = L_a^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$. For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $k_\lambda(z) = (1 - \overline{\lambda}z)^{-2}$. The normalized reproducing kernel for the Bergman space L_a^2 is denoted by $\widehat{k}_\lambda(z) = \frac{k_\lambda(z)}{\|k_\lambda\|_{L_a^2}}$. For $A \in \mathcal{B}(L_a^2)$, its Berezin symbol (transform) is the function $\widetilde{A}$ on $\mathbb{D}$ defined by $\widetilde{A}(\lambda) = \langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle$. For $u \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator T_u with symbol u is the operator on L_a^2 defined by $T_u f = P(uf)$, where P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L_a^2.

The Berezin transform $\widetilde{u}$ of a function $u \in L^\infty(\mathbb{D}, dA)$ is defined to be the Berezin transform of the Toeplitz operator T_u, i.e., $\widetilde{u} = \widetilde{T}_u$, and hence

$$\widetilde{u}(\lambda) = \int_{\mathbb{D}} u(z)\left|\widehat{k}_\lambda(z)\right|^2 dA(z).$$

The Berezin transform of a function in $L^\infty = L^\infty(\mathbb{D}, dA)$ often plays the same important role in the theory of Bergman spaces as the harmonic extension of a function in $L^\infty(\mathbb{T})$ does in the theory of Hardy spaces.

Following by Axler and Zheng [4], note that the Toeplitz algebra $\mathcal{T}$ is the C^*-subalgebra of $\mathcal{B}(L_a^2)$ generated by $\{T_g : g \in H^\infty\}$. We let $\mathcal{U}$ denote the C^*-subalgebra of $L^\infty(\mathbb{D}, dA)$ generated H^∞. It is well known (see [4, Proposition 4.5]) that $\mathcal{U}$ equals the closed subalgebra of L^∞ generated by the set of bounded harmonic functions on $\mathbb{D}$. Although the map $u \to T_u$ is not multiplicative on $L^\infty(\mathbb{D}, dA)$, the identities $T_u^* = T_{\overline{u}}$, $T_{\overline{g}} T_u T_f = T_{\overline{g}uf}$ hold for all $u \in L^\infty$ and $g, f \in H^\infty$. This implies that $\mathcal{T}$ equals the closed subalgebra of $\mathcal{B}(L_a^2)$ generated by Toeplitz operators with bounded harmonic symbols, and that $\mathcal{T}$ also equals the closed subalgebra of $\mathcal{B}(L_a^2)$ generated by $\{T_u : u \in \mathcal{U}\}$.

It is well known that (see Ahern, Floers and Rudin [1] and Engliś [7]) a function in $L^\infty(\mathbb{D}, dA)$ equals its Berezin transform if and only if it is harmonic.

The following two results are due to Axler and Zheng (see [3, Corollary 3.4 and Corollary 3.7]).

Lemma 2. *If $u \in \mathcal{U}$, then $\widetilde{u} - u$ has nontangential limit 0 at almost every point of $\partial\mathbb{D}$.*

Lemma 3. *If $u \in \mathcal{U}$, then the function $\lambda \longmapsto \left\| T_{u-u(\lambda)} \widehat{k}_\lambda \right\|_{L_a^2}$ has nontangential limit 0 at almost every point of $\partial\mathbb{D}$.*

Now we are ready to state our results concerning Bergman space operators.

Theorem 3. *Let u be a bounded harmonic function on $\mathbb{D}$, and $A : L_a^2 \to L_a^2$ be an arbitrary bounded operator. Then the function*

$$\widetilde{T_u A}(\lambda) - \widetilde{T_u}(\lambda)\widetilde{A}(\lambda)$$

has nontangential limit 0 at almost every point of $\partial\mathbb{D}$.

Proof. As mentioned above, $\widetilde{T_u} = \widetilde{u} = u$. Then, by using Lemma 3, we obtain:

$$\begin{aligned}\left|\widetilde{T_u A}(\lambda) - \widetilde{T_u}(\lambda)\widetilde{A}(\lambda)\right| &= \left|\left\langle T_u A \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle - u(\lambda)\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right| \\ &= \left|\left\langle A\widehat{k}_\lambda, T_{\overline{u} - \overline{u(\lambda)}} \widehat{k}_\lambda \right\rangle\right| \le \|A\| \left\| T_{\overline{u - u(\lambda)}} \widehat{k}_\lambda \right\|_{L_a^2} \to 0\end{aligned}$$

as λ approaches to $\partial\mathbb{D}$ nontangentially for almost all points of $\partial\mathbb{D}$, which proves the theorem. ∎

THEOREM 4. *Let $u \in \mathcal{U}$, and $A \in \mathcal{B}(L_a^2)$ be an arbitrary operator. Then the function*

$$\widetilde{T_u A}(\lambda) - \widetilde{T_u}(\lambda)\widetilde{A}(\lambda)$$

has nontagential limit 0 *at almost every point of* $\partial\mathbb{D}$.

Proof. Since $\widetilde{T_u}(\lambda) = \widetilde{u}(\lambda)$, using Lemmas 2 and 3, we have:

$$\begin{aligned}
\left|\widetilde{T_u A}(\lambda) - \widetilde{T_u}(\lambda)\widetilde{A}(\lambda)\right| &= \left|\left\langle T_u A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle - \widetilde{u}(\lambda)\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right| \\
&= \left|\left\langle A\widehat{k}_\lambda, T_{\overline{u}-\overline{\widetilde{u}}(\lambda)}\widehat{k}_\lambda \right\rangle\right| \leq \|A\| \left\|T_{\overline{u}-\overline{\widetilde{u}}(\lambda)}\widehat{k}_\lambda\right\|_{L_a^2} \\
&= \|A\| \left\|T_{\overline{u}-\overline{u(\lambda)}+(\overline{u(\lambda)}-\overline{\widetilde{u}}(\lambda))}\widehat{k}_\lambda\right\|_{L_a^2} \\
&\leq \|A\| \left(\left\|T_{\overline{u-u(\lambda)}}\widehat{k}_\lambda\right\|_{L_a^2} + \left\|\left(\overline{u(\lambda) - \widetilde{u}(\lambda)}\right)\widehat{k}_\lambda\right\|_{L_a^2}\right) \\
&= \|A\| \left(\left\|T_{\overline{u-u(\lambda)}}\widehat{k}_\lambda\right\|_{L_a^2} + |u(\lambda) - \widetilde{u}(\lambda)|\right) \to 0
\end{aligned}$$

as λ tends to $\partial\mathbb{D}$ at almost every point of $\partial\mathbb{D}$, which proves the theorem. ∎

4. ON SOLVABILITY OF THE RICCATI EQUATIONS

The present section is devoted to the solvability of Riccati equations on the Toeplitz algebra $\mathcal{T}$ defined previously in Section 3.

We will consider Riccati equations of the form

$$XAX + XB - CX - D = 0 \tag{1}$$

with bounded operators A, B, C, D on the Bergman space $L_a^2(\mathbb{D})$.

We recall that the solvability of Riccati equations in concrete operator classes is one of the important problems in operator theory. Namely, if $\mathcal{P}_H$ denotes the set of all orthogonal projections from a Hilbert space H onto its closed subspaces and $A \in \mathcal{B}(H)$ is an arbitrary operator, then the existence of a nontrivial solution of the Riccati equation

$$XAX = AX$$

in $\mathcal{P}_H$ is equivalent to the solution of the well-known invariant subspace problem in the Hilbert space H.

The main result of this section is the following theorem.

THEOREM 5. *Let $B = T_u^*$, $C = T_v$ be Toeplitz operators on L_a^2, where $u, v \in H^\infty$ are nonconstant functions, and let A and D be linear bounded operators on the Bergman space L_a^2. Let $T_\varphi \in \mathcal{B}(L_a^2)$ be a Toeplitz operator with $\varphi \in \mathcal{U}$.*

(a) *If T_φ is a solution of the Riccati equation* (1), *then the function*

$$\widetilde{A}(\lambda)\big(\widetilde{\varphi}(\lambda)\big)^2 + \big(\overline{u(\lambda)} - v(\lambda)\big)\widetilde{\varphi}(\lambda) - \widetilde{D}(\lambda)$$

has nontangential limit 0 *at almost every point of $\mathbb{T} = \partial\mathbb{D}$.*

(b) *Assume that the following two nonzero nontangential limits $\widetilde{A}_{nt}(\zeta) := \lim_{\lambda\to\zeta\in\mathbb{T}} \widetilde{A}(\lambda)$ and $\widetilde{D}_{nt}(\zeta) := \lim_{\lambda\to\zeta\in\mathbb{T}} \widetilde{D}(\lambda)$ for almost all $\zeta \in \mathbb{T}$ exist and verify*

$$\big(\overline{u}(\zeta) - v(\zeta)\big)^2 + 4\widetilde{A}_{nt}(\zeta)\widetilde{D}_{nt}(\zeta) = 0 \tag{2}$$

for almost all $\zeta \in \mathbb{T}$. If T_φ satisfies the Riccati equation (1), *then*

$$\varphi(\zeta) = \pm i \left(\frac{\widetilde{D}_{nt}(\zeta)}{\widetilde{A}_{nt}(\zeta)}\right)^{\frac{1}{2}}$$

for almost all $\zeta \in \mathbb{T}$.

Proof. (a) If $T_\varphi \in \mathcal{B}(L_a^2)$ is a solution of equation (1), then considering that the Berezin symbol uniquely determines the operator on the Bergman space L_a^2, for every $\lambda \in \mathbb{D}$ we have:

$$\begin{aligned}
0 &= \Big\langle \big(T_\varphi A T_\varphi + T_\varphi T_u^* - T_v T_\varphi - D\big)\widehat{k}_\lambda, \widehat{k}_\lambda \Big\rangle \\
&= \Big\langle T_\varphi A T_\varphi \widehat{k}_\lambda, \widehat{k}_\lambda \Big\rangle + \Big\langle T_\varphi T_u^* \widehat{k}_\lambda, \widehat{k}_\lambda \Big\rangle - \Big\langle T_v T_\varphi \widehat{k}_\lambda, \widehat{k}_\lambda \Big\rangle - \Big\langle D\widehat{k}_\lambda, \widehat{k}_\lambda \Big\rangle \\
&= \Big\langle A T_\varphi \widehat{k}_\lambda, T_{\overline{\varphi}} \widehat{k}_\lambda \Big\rangle + \overline{u(\lambda)} \Big\langle T_\varphi \widehat{k}_\lambda, \widehat{k}_\lambda \Big\rangle - v(\lambda) \Big\langle T_\varphi \widehat{k}_\lambda, \widehat{k}_\lambda \Big\rangle - \Big\langle D\widehat{k}_\lambda, \widehat{k}_\lambda \Big\rangle \\
&= \Big\langle A\big((T_\varphi - \widetilde{\varphi}(\lambda))\widehat{k}_\lambda + \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\big), T_{\overline{\varphi}}\widehat{k}_\lambda \Big\rangle + \big(\overline{u(\lambda)} - v(\lambda)\big)\widetilde{T}_\varphi(\lambda) - \widetilde{D}(\lambda) \\
&= \Big\langle A\big(T_{\varphi - \widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\big), T_{\overline{\varphi}}\widehat{k}_\lambda \Big\rangle + \widetilde{\varphi}(\lambda) \Big\langle A\widehat{k}_\lambda, T_{\overline{\varphi}}\widehat{k}_\lambda - \widetilde{\overline{\varphi}}(\lambda)\widehat{k}_\lambda + \widetilde{\overline{\varphi}}(\lambda)\widehat{k}_\lambda \Big\rangle \\
&\quad + \big(\overline{u(\lambda)} - v(\lambda)\big)\widetilde{\varphi}(\lambda) - \widetilde{D}(\lambda) \\
&= \Big\langle A\big(T_{\varphi - \widetilde{\varphi}(\lambda)}\big)\widehat{k}_\lambda, T_{\overline{\varphi}}\widehat{k}_\lambda \Big\rangle + \widetilde{\varphi}(\lambda) \Big\langle A\widehat{k}_\lambda, T_{\overline{\varphi} - \widetilde{\overline{\varphi}}(\lambda)}\widehat{k}_\lambda \Big\rangle \\
&\quad + \big(\widetilde{\varphi}(\lambda)\big)^2 \widetilde{A}(\lambda) + \big(\overline{u(\lambda)} - v(\lambda)\big)\widetilde{\varphi}(\lambda) - \widetilde{D}(\lambda)
\end{aligned}$$

Thus,

$$\widetilde{A}(\lambda)\big(\widetilde{\varphi}(\lambda)\big)^2 + \big(\overline{u(\lambda)} - v(\lambda)\big)\widetilde{\varphi}(\lambda) - \widetilde{D}(\lambda) = \Big\langle A\big(T_{\widetilde{\varphi}(\lambda)-\varphi}\widehat{k}_\lambda\big), T_{\overline{\varphi}}\widehat{k}_\lambda\Big\rangle + \widetilde{\varphi}(\lambda)\big\langle A\widehat{k}_\lambda, T_{\overline{\widetilde{\varphi}(\lambda)}-\overline{\varphi}}\widehat{k}_\lambda\big\rangle \tag{4.1}$$

for all $\lambda \in \mathbb{D}$. It is easy to see that Lemma 2 and Lemma 3 imply that the function $\lambda \mapsto \big\|T_{\varphi-\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\big\|_{L_a^2}$ has nontangential limit 0 at almost every point of $\mathbb{T}$. Then we have from equality (3) that

$$\Big|\widetilde{A}(\lambda)\big(\widetilde{\varphi}(\lambda)\big)^2 + \big(\overline{u(\lambda)} - v(\lambda)\big)\widetilde{\varphi}(\lambda) - \widetilde{D}(\lambda)\Big| \le \|A\|\|T_{\overline{\varphi}}\|\big\|T_{\varphi-\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\big\|_{L_a^2} + \|A\|ber(T_\varphi)\big\|T_{\overline{\varphi}-\overline{\widetilde{\varphi}(\lambda)}}\widehat{k}_\lambda\big\|_{L_a^2} \to 0$$

at almost every point of $\mathbb{T}$, which proves (a).

(b) If T_φ is a solution of the Riccati equation (1), then by Lemma 2 and item (a), we obtain that $\widetilde{\varphi}(\zeta) = \varphi(\zeta)$ exists at almost every point $\zeta \in \mathbb{T}$ and satisfies

$$\widetilde{A}_{nt}(\zeta)\varphi^2(\zeta) + \big(\overline{u}(\zeta) - v(\zeta)\big)\varphi(\zeta) - \widetilde{D}_{nt}(\zeta) = 0$$

for almost all $\zeta \in \mathbb{T}$. This equality can be written as

$$\widetilde{A}_{nt}(\zeta)\left(\varphi(\zeta) + \frac{\overline{u}(\zeta) - v(\zeta)}{2\widetilde{A}_{nt}(\zeta)}\right)^2 = \frac{\big(\overline{u}(\zeta) - v(\zeta)\big)^2 + 4\widetilde{A}_{nt}(\zeta)\widetilde{D}_{nt}(\zeta)}{4\widetilde{A}_{nt}(\zeta)},$$

which gives by virtue of condition (2) that

$$\varphi(\zeta) = \frac{v(\zeta) - \overline{u}(\zeta)}{2\widetilde{A}_{nt}(\zeta)}$$

or

$$\varphi^2(\zeta) = -\frac{\widetilde{D}_{nt}(\zeta)}{\widetilde{A}_{nt}(\zeta)}$$

which implies that $\varphi(\zeta) = \pm i\left(\frac{\widetilde{D}_{nt}(\zeta)}{\widetilde{A}_{nt}(\zeta)}\right)^{\frac{1}{2}}$ for almost all $\zeta \in \mathbb{T}$. This proves (b). The theorem is proved. ∎

Now we will consider the solvability of the following equation in the set of Toeplitz operators on the Bergman space L_a^2:

$$X_1T_{\varphi_1} + X_2T_{\varphi_2} + \cdots + X_nT_{\varphi_n} = I,$$

where $T_{\varphi_i}, i = 1, 2, \ldots, n$, are given Toeplitz operators on the Bergman space L_a^2. Again, by applying Lemmas 2 and 3, we will prove here the following theorem.

THEOREM 6. *Let $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathcal{U}$. If there exists functions $\psi_1, \psi_2, \ldots, \psi_n \in \mathcal{U}$ such that*

$$T_{\psi_1} T_{\varphi_1} + \cdots + T_{\psi_n} T_{\varphi_n} = I,$$

where $T_{\psi_i}, T_{\varphi_i} (i = 1, 2, \ldots, n)$ are Toeplitz operators on L_a^2, then

$$\underset{\mathbb{T}}{ess} \inf \left(|\varphi_1| + \cdots + |\varphi_n| \right) > 0$$

Proof. Since

$$T_{\psi_1} T_{\varphi_1} + \cdots + T_{\psi_n} T_{\varphi_n} = I,$$

we have:

$$\begin{aligned}
1 &= \widetilde{T_{\psi_1} T_{\varphi_1}}(\lambda) + \cdots + \widetilde{T_{\psi_n} T_{\varphi_n}}(\lambda) = \left\langle T_{\psi_1} T_{\varphi_1} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \cdots + \left\langle T_{\psi_n} T_{\varphi_n} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&= \left\langle T_{\varphi_1} \widehat{k}_\lambda, T_{\overline{\psi}_1} \widehat{k}_\lambda \right\rangle + \cdots + \left\langle T_{\varphi_n} \widehat{k}_\lambda, T_{\overline{\psi}_n} \widehat{k}_\lambda \right\rangle \\
&= \left\langle T_{\varphi_1} \widehat{k}_\lambda, \widetilde{\overline{\psi}}_1(\lambda) \widehat{k}_\lambda \right\rangle + \cdots + \left\langle T_{\varphi_n} \widehat{k}_\lambda, \widetilde{\overline{\psi}}_n(\lambda) \widehat{k}_\lambda \right\rangle + \left\langle T_{\varphi_1} \widehat{k}_\lambda, T_{\overline{\psi}_1} \widehat{k}_\lambda - \widetilde{\overline{\psi}}_1(\lambda) \widehat{k}_\lambda \right\rangle \\
&\quad + \cdots + \left\langle T_{\varphi_n} \widehat{k}_\lambda, T_{\overline{\psi}_n} \widehat{k}_\lambda - \widetilde{\overline{\psi}}_n(\lambda) \widehat{k}_\lambda \right\rangle \\
&= \widetilde{\psi}_1(\lambda) \widetilde{T}_{\varphi_1}(\lambda) + \cdots + \widetilde{\psi}_n(\lambda) \widetilde{T}_{\varphi_n}(\lambda) + \left\langle T_{\varphi_1} \widehat{k}_\lambda, T_{\overline{\psi}_1 - \widetilde{\overline{\psi}}_1(\lambda)} \widehat{k}_\lambda \right\rangle \\
&\quad + \cdots + \left\langle T_{\varphi_n} \widehat{k}_\lambda, T_{\overline{\psi}_n - \widetilde{\overline{\psi}}_n(\lambda)} \widehat{k}_\lambda \right\rangle \\
&= \widetilde{\psi}_1(\lambda) \widetilde{\varphi}_1(\lambda) + \cdots + \widetilde{\psi}_n(\lambda) \widetilde{\varphi}_n(\lambda) + \left\langle T_{\varphi_1} \widehat{k}_\lambda, T_{\overline{\psi_1 - \widetilde{\psi}_1(\lambda)}} \widehat{k}_\lambda \right\rangle \\
&\quad + \cdots + \left\langle T_{\varphi_n} \widehat{k}_\lambda, T_{\overline{\psi_n - \widetilde{\psi}_n(\lambda)}} \widehat{k}_\lambda \right\rangle
\end{aligned}$$

Therefore

$$\begin{aligned}
1 &\le \left| \widetilde{\psi}_1(\lambda) \right| \left| \widetilde{\varphi}_1(\lambda) \right| + \cdots + \left| \widetilde{\psi}_n(\lambda) \right| \left| \widetilde{\varphi}_n(\lambda) \right| + \left\| T_{\varphi_1} \widehat{k}_\lambda \right\|_{L_a^2} \left\| T_{\overline{\psi_1 - \widetilde{\psi}_1(\lambda)}} \widehat{k}_\lambda \right\|_{L_a^2} + \cdots \\
&\quad + \left\| T_{\varphi_n} \widehat{k}_\lambda \right\|_{L_a^2} \left\| T_{\overline{\psi_n - \widetilde{\psi}_n(\lambda)}} \widehat{k}_\lambda \right\|_{L_a^2} \\
&\le \left| \widetilde{\psi}_1(\lambda) \right| \left| \widetilde{\varphi}_1(\lambda) \right| + \cdots + \left| \widetilde{\psi}_n(\lambda) \right| \left| \widetilde{\varphi}_n(\lambda) \right| + \|\varphi_1\|_{L^\infty} \left\| T_{\overline{\psi_1 - \widetilde{\psi}_1(\lambda)}} \widehat{k}_\lambda \right\|_{L_a^2} + \cdots \\
&\quad + \|\varphi_n\|_{L^\infty} \left\| T_{\overline{\psi_n - \widetilde{\psi}_n(\lambda)}} \widehat{k}_\lambda \right\|_{L_a^2}.
\end{aligned}$$

By applying Lemmas 2 and 3, we have that the functions $\lambda \mapsto \left\| T_{\overline{\psi_i - \widetilde{\psi}_i(\lambda)}} \widehat{k}_\lambda \right\|_{L_a^2}$ $(i = 1, \ldots, n)$ have nontangential limits 0 at almost every point of $\mathbb{T}$ (this claim can be considered as the Bergman space analog of Engliś's result [7, Theorem 6]). Then we have from the last inequality that

$$\begin{aligned} 1 &\le \left|\psi_1(\zeta)\right| \left|\varphi_1(\zeta)\right| + \cdots + \left|\psi_n(\zeta)\right| \left|\varphi_n(\zeta)\right| \\ &\le \|\psi_1\|_{L^\infty(\mathbb{T})} \left|\varphi_1(\zeta)\right| + \cdots + \|\psi_n\|_{L^\infty(\mathbb{T})} \left|\varphi_n(\zeta)\right| \end{aligned}$$

for almost all $\zeta \in \mathbb{T}$, which shows that

$$\underset{\mathbb{T}}{ess\,\inf} \Big(\left|\varphi_1(\zeta)\right| + \cdots + \left|\varphi_n(\zeta)\right| \Big) \ge \frac{1}{\max\left\{ \|\psi_i\|_{L^\infty(\mathbb{T})} : 1 \le i \le n \right\}} > 0$$

which proves the theorem. ∎

5. A characterization of compact truncated Toeplitz operators

In this section we characterize compact truncated Toeplitz operators in terms of Berezin symbols. Namely, we will prove the following theorem.

Theorem 7. *For $\varphi \in L^\infty(\mathbb{T})$, let $T_\varphi : H^2 \to H^2$ be a Toeplitz operator and $A_\varphi := P_\theta T_\varphi \mid K_\theta$ be a truncated Toeplitz operator, where $P_\theta := P_{K_\theta} = I - T_\theta T_{\overline{\theta}}$ is the orthogonal projection of H^2 onto K_θ. Then A_φ is a compact operator if and only if*

$$\lim_{\lambda \to \mathbb{T}} \left(\widetilde{P}_{U^{-1}K_\theta} \widetilde{U^{-1} A_\varphi U}^{\,U^{-1}K_\theta} \right)(\lambda) = 0$$

for all unitary operators $U \in \mathcal{B}(H^2)$.

Proof. Let us set $B := A_\varphi P_\theta$. Clearly $B \in \mathcal{B}(H^2)$. For every unitary operator $U \in \mathcal{B}(H^2)$, we have that

$$U^{-1}BU = U^{-1}A_\varphi P_\theta U = \left(U^{-1}A_\varphi U\right)\left(U^{-1}P_\theta U\right) = U^{-1}A_\varphi U P_{U^{-1}K_\theta}$$

By considering that $P_{U^{-1}K_\theta} k_\lambda = k_{U^{-1}K_\theta,\lambda}$ for every $\lambda \in \mathbb{D}$, where $k_\lambda(z) = (1 - \overline{\lambda} z)^{-1}$ is the reproducing kernel of the Hardy space H^2 and $k_{U^{-1}K_\theta,\lambda}$ is the reproducing kernel of $U^{-1}K_\theta$, we have:

$$\begin{aligned}\widetilde{U^{-1}BU}(\lambda) &= \left\langle U^{-1}BU\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle = \left\langle U^{-1}A_\varphi UP_{U^{-1}K_\theta}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\ &= \frac{1}{\|k_\lambda\|_2^2}\left\langle U^{-1}A_\varphi UP_{U^{-1}K_\theta}k_\lambda, k_\lambda \right\rangle \\ &= \left(1-|\lambda|^2\right)\left\langle U^{-1}A_\varphi Uk_{U^{-1}K_\theta,\lambda}, P_{U^{-1}K_\theta}k_\lambda + (I-P_{U^{-1}K_\theta})k_\lambda \right\rangle \\ &= \left(1-|\lambda|^2\right)\left[\left\langle U^{-1}A_\varphi Uk_{U^{-1}K_\theta,\lambda}, k_{U^{-1}K_\theta,\lambda}\right\rangle \right. \\ &\quad \left. + \left\langle U^{-1}A_\varphi Uk_{U^{-1}K_\theta,\lambda}, (I-P_{U^{-1}K_\theta})k_\lambda \right\rangle\right].\end{aligned}$$

It is not difficult to see that

$$\left\langle U^{-1}A_\varphi Uk_{U^{-1}K_\theta,\lambda}, (I-P_{U^{-1}K_\theta})k_\lambda \right\rangle = 0.$$

Indeed,

$$\begin{aligned}&\left\langle U^{-1}A_\varphi Uk_{U^{-1}K_\theta,\lambda}, (I-P_{U^{-1}K_\theta})k_\lambda \right\rangle \\ &\qquad = \left\langle A_\varphi U\left(U^{-1}P_\theta U\right)k_\lambda, U - U\left(U^{-1}P_\theta U\right)k_\lambda \right\rangle \\ &\qquad = \left\langle A_\varphi P_\theta Uk_\lambda, (I-P_\theta)Uk_\lambda \right\rangle = 0\end{aligned}$$

because $A_\varphi P_\theta Uk_\lambda \in K_\theta$ and $(I-P_\theta)Uk_\lambda \in \theta H^2$. Thus

$$\begin{aligned}\widetilde{U^{-1}BU}(\lambda) &= \left(1-|\lambda|^2\right)\left\langle U^{-1}A_\varphi Uk_{U^{-1}K_\theta,\lambda}, k_{U^{-1}K_\theta,\lambda}\right\rangle \\ &= \left(1-|\lambda|^2\right)\left\|k_{U^{-1}K_\theta,\lambda}\right\|_2^2 \left\langle U^{-1}A_\varphi U\widehat{k}_{U^{-1}K_\theta,\lambda}, \widehat{k}_{U^{-1}K_\theta,\lambda}\right\rangle \\ &= \left(1-|\lambda|^2\right)\left\|k_{U^{-1}K_\theta,\lambda}\right\|_2^2 \widetilde{U^{-1}A_\varphi U}^{U^{-1}K_\theta}(\lambda),\end{aligned}$$

or

$$\widetilde{U^{-1}BU}(\lambda) = \left(1-|\lambda|^2\right)\left\|k_{U^{-1}K_\theta,\lambda}\right\|_2^2 \widetilde{U^{-1}A_\varphi U}^{U^{-1}K_\theta}(\lambda)$$

for all $\lambda \in \mathbb{D}$. On the other hand,

$$\begin{aligned}\left\|k_{U^{-1}K_\theta,\lambda}\right\|_2^2 &= \left\|P_{U^{-1}K_\theta}k_\lambda\right\|_2^2 = \left\langle P_{U^{-1}K_\theta}k_\lambda, k_\lambda\right\rangle \\ &= \|k_\lambda\|_2^2\left\langle P_{U^{-1}K_\theta}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\ &= \frac{1}{1-|\lambda|^2}\widetilde{P}_{U^{-1}K_\theta}(\lambda)\end{aligned}$$

and hence

$$(1-|\lambda|^2)\left\|k_{U^{-1}K_\theta,\lambda}\right\|_2^2 = \widetilde{P}_{U^{-1}K_\theta}(\lambda), \quad \lambda \in \mathbb{D}$$

Therefore

$$\widetilde{U^{-1}BU}\,(\lambda)=\widetilde{P}_{U^{-1}K_\theta}(\lambda)\widetilde{U^{-1}A_\varphi U}^{U^{-1}K_\theta}(\lambda)\,,\quad (\lambda\in\mathbb{D}) \tag{4}$$

for all unitary operators $U\in\mathcal{B}(H^2)$. It is clear that $B\in\sigma_\infty(H^2)$ (the Schatten-Neumann ideal of compact operators on H^2) if and only if $A_\varphi\in\sigma_\infty(K_\theta)$. Then, by virtue of a well known theorem due to Nordgren and Rosenthal [19, Theorem 2.7] and equality (4), we assert that A_φ is compact in K_θ if and only if

$$\lim_{\lambda\to\mathbb{T}}\big(\widetilde{P}_{U^{-1}K_\theta}(\lambda).\widetilde{U^{-1}A_\varphi U}\,(\lambda)\,\big)=0$$

for every unitary operator $U\in\mathcal{B}(H^2)$, which proves the theorem. ∎

6. Some results for the spectra of model operators $\varphi(M_\theta)$ and Toeplitz operators

One of important subclass of truncated Toeplitz operators is the class of model operators $\varphi(M_\theta)$ of Sz-Nagy and Foias [22] defined by

$$\varphi(M_\theta)f=P_\theta(\varphi f),\quad f\in K_\theta,$$

for every function $\varphi\in H^\infty$.

In this section, we give in terms of the Berezin set some localizations of spectrums of model operators $\varphi(M_\theta)$. Namely, we will prove that $\sigma\big(\varphi(M_\theta)\big)\subset\overline{Ber\big(\varphi(M_\theta)\big)}$. We also localize the numerical range of Toeplitz operators on the Bergman space L_a^2.

Before giving our results, let us give some necessary definitions, notations and auxiliary results, which can be found, for example, in the book [18, Chapter 3] of Nikolski.

Let θ be an inner function. By the spectrum $\sigma(\theta)$ of θ we mean the complement (with respect to the whole closed disc $\overline{\mathbb{D}}$) of the set of all points λ, $\lambda\in\overline{\mathbb{D}}$, such that the function $\frac{1}{\theta}$ can be continued analytically into a (full) neighborhood of λ.

Furthermore,

$$\mathbb{Z}(\theta):=\{\lambda\in\mathbb{D}:\theta(\lambda)=0\}$$

is the set of all zeros of θ and *supp* μ denotes the support of the measure (any measure) μ.

If $\theta = B.\exp\left(-\int \frac{\zeta+z}{\zeta-z}\, d\mu^s(\zeta)\right)$ is the canonical factorization of θ, then

$$\sigma(\theta) = \overline{\mathbb{Z}(\theta)} \cup supp\ \mu^s = \left\{\lambda \in \overline{\mathbb{D}} : \varliminf_{\zeta\to\lambda,\ \zeta\in\mathbb{D}} |\theta(\zeta)| = 0\right\}$$

and

$$\sigma(M_\theta) = \sigma(\theta), \quad \sigma_p(M_\theta) = \sigma(\theta) \cap \mathbb{D} = \mathbb{Z}(\theta),$$

where $M_\theta = P_\theta z \mid K_\theta$ is the model operator.

Spectral mapping theorem for $\varphi(M_\theta)$ says that if θ is an inner function and $\varphi \in H^\infty$, then

$$\sigma\big(\varphi(M_\theta)\big) = \left\{\zeta \in \mathbb{C} : \inf_{\mathbb{D}}\left(|\theta(z)| + |\varphi(z) - \zeta\mathbf{1}|\right) = 0\right\} \tag{5}$$

THEOREM 8. *For every $\varphi \in H^\infty$ and inner function θ we have $\sigma\big(\varphi(M_\theta)\big) \subset \overline{Ber\big(\varphi(M_\theta)\big)}$ and $\sigma_p\big(\varphi(M_\theta)\big) \subset Ber\big(\varphi(M_\theta)\big)$.*

Proof. Since the reproducing kernel of the subspace $K_\theta = H^2 \ominus \theta H^2$ is the function

$$k_{\theta,\lambda}(z) := \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\overline{\lambda}z}$$

and $\widetilde{T_\psi} = \widetilde{\psi}$ for every function $\psi \in L^\infty(\mathbb{T})$, where T_ψ is the Toeplitz operators on the Hardy space H^2, it is not difficult to prove that (see, Karaev [14])

$$\widetilde{\varphi(M_\theta)}(\lambda) := \left\langle \varphi(M_\theta)\frac{k_{\theta,\lambda}}{\|k_{\theta,\lambda}\|_2}, \frac{k_{\theta,\lambda}}{\|k_{\theta,\lambda}\|_2}\right\rangle = \frac{1}{1-|\theta(\lambda)|^2}\big(\varphi(\lambda) - \theta(\lambda)\widetilde{\varphi\overline{\theta}}(\lambda)\big) \tag{6}$$

for all $\lambda \in \mathbb{D}$, which shows that

$$Ber\big(\varphi(M_\theta)\big) = Range\left(\frac{1}{1-|\theta|^2}\big(\varphi - \theta\widetilde{\varphi\overline{\theta}}\big)\right).$$

It follows from formula (6) that

$$\varphi(z) = \big(1 - |\theta(z)|^2\big)\widetilde{\varphi(M_\theta)}(z) + \theta(z)\widetilde{\varphi\overline{\theta}}(z). \tag{7}$$

Formula (5) means that

$$\sigma\big(\varphi(M_\theta)\big) = \left\{\zeta \in \mathbb{C} : \lim_{z\to\lambda} \varphi(z) = \zeta,\ \lambda \in \sigma(\theta)\right\}.$$

Then it follows from (7) that

$$\underline{\lim}_{z\to\lambda,\ \lambda\in\sigma(\theta)} \varphi(z)=\zeta$$

if and only if

$$\underline{\lim}_{z\to\lambda,\ \lambda\in\sigma(\theta)} \widetilde{\varphi(M_\theta)}(\lambda)=\zeta,$$

which clearly means that for every $\zeta\in\sigma\big(\varphi(M_\theta)\big)$, $\zeta\in\overline{Ber\big(\varphi(M_\theta)\big)}$, that is $\sigma\big(\varphi(M_\theta)\big)\subset\overline{Ber\big(\varphi(M_\theta)\big)}$. Since $\varphi(\lambda)=\big(\widetilde{\varphi(M_\theta)}\big)(\lambda)$ for every $\lambda\in\mathbb{Z}(\theta)$ (see formula (7)) and $\sigma_p\big(\varphi(M_\theta)\big)=\varphi\big(\mathbb{Z}(\theta)\big)$, we obtain that $\sigma_p\big(\varphi(M_\theta)\big)=\widetilde{\varphi(M_\theta)}\big(\mathbb{Z}(\theta)\big)$, and hence $\sigma_p\big(\varphi(M_\theta)\big)\subset Ber(\varphi(M_\theta)\big)$, which completes the proof of theorem. ∎

THEOREM 9. *Let T_φ, $\varphi\in L^\infty(\mathbb{D})$, be any Toeplitz operator acting on the Bergman space L_a^2, and let $W(T_\varphi)$ denote the numerical range of T_φ. Then*

$$W(T_\varphi)\subset clos \bigcup_{g\in(L_a^2)_1\cap H^\infty} Range\big(\widetilde{|g|^2\varphi}\big), \tag{8}$$

where $\left(L_a^2\right)_1:=\left\{f\in L_a^2:\|f\|_{L_a^2}=1\right\}$ is the unit sphere of the space L_a^2 and $\widetilde{|g|^2\varphi}$ is the Berezin transform of the function $|g|^2\varphi\in L^\infty(\mathbb{D})$.

Proof. Let $\widehat{k}_\lambda(z)=\left(1-|\lambda|^2\right)(1-\overline{\lambda}z)^{-2}$ be the normalized reproducing kernel of the Bergman space $L_a^2(\mathbb{D})$. Then for every $f\in(L_a^2)_1\cap H^\infty$ we have:

$$\begin{aligned}\langle T_\varphi f,f\rangle&=\big\langle P(\varphi f),f\big\rangle=\langle\varphi f,f\rangle=\int_{\mathbb{D}}\varphi(z)\,|f(z)|^2\,dA(z)\\&=\int_{\mathbb{D}}\varphi(z)\,|f(z)|^2\left|\widehat{k}_0(z)\right|^2dA(z)=\Big\langle T_{\varphi|f|^2}\widehat{k}_0,\widehat{k}_0\Big\rangle\\&=\widetilde{\varphi\,|f|^2}(0)\in Range\big(\widetilde{\varphi\,|f|^2}\big),\end{aligned}$$

and therefore, $\langle T_\varphi f,f\rangle\in\bigcup_{g\in(L_a^2)_1\cap H^\infty}Range\big(\widetilde{\varphi\,|g|^2}\big)$, and thus $\langle T_\varphi h,h\rangle\in clos\bigcup_{g\in(L_a^2)_1\cap H^\infty}Range\big(\widetilde{\varphi\,|g|^2}\big)$ for every $h\in(L_a^2)_1$, because H^∞ is dense in L_a^2. This gives inclusion (8). The theorem is proved. ∎

Since $\sigma(T_\varphi)\subset\overline{W(T_\varphi)}$, the following is an immediate corollary of inclusion (8).

Corollary 1. *$\sigma(T_\varphi) \subset clos \bigcup_{g\in(L_a^2)_1\cap H^\infty} Range\big(\widetilde{\varphi\,|g|^2}\big)$ for every Toeplitz operator T_φ, with $\varphi \in L^\infty(\mathbb{D})$, on the Bergman space L_a^2.*

7. On the reducing subspaces for n-tuple of invertible operators

Let H be a separable complex Hilbert space and A be a bounded linear operator on H. We recall that a reducing subspace of A is a common invariant subspace $E \subset H$ for A and A^*, that is $AE \subset E$ and $A^*E \subset E$, or equivalently, $AE \subset E$ and $AE^\perp \subset E^\perp$, where $E^\perp := H \ominus E$.

In the present section, we will consider the n-tuples of invertible operators $(A_1, A_2, \ldots, A_n)$ and describe in terms of normal operators their reducing subspaces.

Note that the study of common invariant subspace of pairs (N_1, N_2) of nilpotent operators N_1, N_2 of index 2 on H plays a key role in the problem of the existence of nontrivial invariant subspace for Hilbert space operators. Namely, one of the equivalent forms of the invariant subspace problem for Hilbert spaces is the following (see [21, Theorem 1]): any operator on a Hilbert space H has a nontrivial closed invariant subspace if and if only every pair of nilpotent operators of index two on H has a common nontrivial invariant subspace. In [2], the reader can find many results concerning the existence of common invariant subspace of some quadratic operators (see [2, Theorem 3 and Corollary 9]. For more related results see also [16]).

Before giving our results, we recall that every invertible operator $A \in \mathcal{B}(H)$ induces an inner automorphism α_A of $\mathcal{B}(H)$ defined by

$$\alpha_A(X) := AXA^{-1}, \quad X \in \mathcal{B}(H).$$

Theorem 10. *Let $(A_1, A_2, \ldots, A_n)$ be a n-tuple of invertible operators $A_i \in \mathcal{B}(H), i = 1, 2, \ldots, n$ and $E \subset H$ be a closed nontrivial subspace (i.e., $\{0\} \neq E \neq H$). Then E is a reducing subspace for the n-tuple $(A_1, A_2, \ldots, A_n)$ if and only if there exist an integer $k \geq 2$ and a nonzero operator $T \in \mathcal{B}(H)$ such that:*

(a) $\alpha_{A_i}(T) = T^k$ $(i = 1, 2, \ldots, n)$;

(b) *T is a normal operator with $\overline{R(T)} = E$ and $\sigma(T) \subseteq \{z \in \mathbb{C} : z^k = z\}$.*

Proof. If $E \subset H$ reduces $A_1, A_2, \ldots, A_n$, then $P_E A_i = A_i P_E$ $(i = 1, 2, \ldots, n)$, where P_E is an orthogonal projection onto E, that is, $P_E^2 A_i = A_i P_E$, or $\alpha_{A_i}(P_E) = P_E^2$ $(i = 1, 2, \dots, n)$. From this, by setting $k = 2$ and $T = P_E$, immediately follows the necessity of the theorem.

Conversely, from the equalities $\alpha_{A_i}(T) = T^k$ $(i = 1, 2, \dots, n)$ (see condition (a)) it is easy to obtain that $\sigma(T^k) = \sigma(T)$. Then, by using condition (b), we have that $\sigma(T^k) = \left\{z^k : z \in \sigma(T)\right\}$, and hence $\sigma(T^k - T) = \left\{z^k - z : z \in \sigma(T)\right\} = \{0\}$. Since $T^k - T$ is a normal operator, this means that $T^k - T = 0$ (because for the normal operator its norm and spectral radius coincide), that is $T^k = T$. By considering this, from (a) we have

$$TA_i = A_iT, \ (i = 1, 2, \dots, n) \tag{9}$$

Then, by Fuglede-Putnam theorem we have

$$TA_i^* = A_i^*T, \ (i = 1, 2, \dots, n) \tag{10}$$

The equalities (9) and (10) imply that $\overline{R(T)}$ is a reducing subspace for the set $(A_1, A_2, \dots, A_n)$, and thus by condition (b) E is a reducing subspace for the n-tuple $(A_1, A_2, \dots, A_n)$. The theorem is proved. ∎

Corollary 2. *Let θ be an inner function other than a linear fractional transformation, $T_{\varphi_i \circ \theta}, i = 1, 2, \dots, n$, be the invertible Toeplitz operators on H^2 with $\varphi_i \in L^\infty(\mathbb{T})$, $i = 1, 2, \dots, n$, and $E \subset H^2$ be a nontrivial closed subspace. Then E is a reducing subspace for the n-tuple $(T_{\varphi_1 \circ \theta}, \dots, T_{\varphi_n \circ \theta})$ if and only if there exist an integer $k \geq 2$ and a nonzero operator $T \in \mathcal{B}(H^2)$ satisfying*

(a) $\alpha_{T_{\varphi_i \circ \theta}}(T) = T^k$ $(i = 1, 2, \dots, n)$;

(b) *T is a normal operator with $\overline{R(T)} = E$ and $\sigma(T) \subseteq \left\{z \in \mathbb{C} : z^k = z\right\}$.*

We recall that the existence of nontrivial reducing subspaces of Toeplitz operators $T_{\varphi \circ \theta}$ is proved by Nordgren for $\varphi \in H^\infty$ [20, Theorem 2] and by Karaev for $\varphi \in L^\infty(\mathbb{T})$ [15, Theorem 1].

Corollary 3. *Let $N_1, N_2, \dots, N_n \in \mathcal{B}(H)$ be nilpotent operators and $E \subset H$ be a nontrivial closed subspace. Then E is a reducing subspace for the n-tuple $(N_1, N_2, \dots, N_n)$ if and only if there exist an integer $k \geq 2$ and a nonzero operator $T \in \mathcal{B}(H)$ such that:*

(a) $\alpha_{I+N_i}(T) = T^k \quad (i = 1, 2, \dots, n)$;

(b) *T is a normal operator with $\overline{R(T)} = E$ and $\sigma(T) \subseteq \left\{z \in \mathbb{C} : z^k = z\right\}$.*

In conclusion, we recall that in [10] Halmos has shown that if H is finite-dimensional, then every invariant subspace of nilpotent operator $\mathcal{N}$ on H is the range of an operator from the commutant of $\mathcal{N}$, i.e., $E \in Lat(\mathcal{N})$ if and only if $E = XH$ for some $X \in \{\mathcal{N}\}^{'}$. For infinite dimensional H the finite dimensional invariant subspace of nilpotent operator $\mathcal{N}$ is described by Barraa and Charles in [6]. Namely, they have shown that every finite-dimensional $E \in Lat(\mathcal{N})$ is the range of an operator from the commutant $\{\mathcal{N}\}^{'}$ of $\mathcal{N}$. The hyperinvariant subspaces of nilpotent operators in Banach space are described in [5].

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