EXTRACTA MATHEMATICAE Vol. **24**, Núm. 3, 243 – 249 (2009)

On Extreme Points of the Dual Ball of a Polyhedral Space

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Presented by Pier L. Papini   Received February 3, 2009

Abstract: We prove that every separable polyhedral Banach space X is isomorphic to a polyhedral Banach space Y such that, the set $\operatorname{ext} B_{Y^*}$ cannot be covered by a sequence of balls $B(y_i, \epsilon_i)$ with $0 < \epsilon_i < 1$ and $\epsilon_i \to 0$. In particular $\operatorname{ext} B_{Y^*}$ cannot be covered by a sequence of norm compact sets. This generalizes a result from [7] where an equivalent polyhedral norm $|||\cdot|||$ on c_0 was constructed such that $\operatorname{ext} B_{(c_0,|||\cdot|||)^*}$ is uncountable but can be covered by a sequence of norm compact sets.

Key words: Polyhedral Banach space, boundary, extreme points.

AMS *Subject Class.* (2000): 46B20.

In [8] V. Klee introduced the following definition of a polyhedral Banach space.

Definition 1. A Banach space X is called *polyhedral* if the unit ball of every finite dimensional subspace of X is a polytope.

Recall that a subset $B \subseteq S_{X^*}$ of the unit sphere of the dual Banach space X^* is called a *boundary* of X if for any $x \in X$ there is $f \in B$ with $f(x) = \|x\|$. In [3] (see also [5] and [10]), it was proved that any separable polyhedral space has a countable boundary. The converse is true under a suitable renorming (see [2]).

By the Krein-Milman Theorem, the set $\operatorname{ext} B_{X^*}$ is a boundary for any Banach space X. In [7], a separable polyhedral Banach space X was constructed (actually X is isomorphic to c_0) such that $\operatorname{ext} B_{X^*}$ is uncountable. Of course, being separable polyhedral, X admits a countable boundary. However, it is easily seen from the construction in [7] that the set $\operatorname{ext} B_{X^*}$ can be covered by a sequence of norm compact sets, i.e. although $\operatorname{ext} B_{X^*}$ is uncountable it is in a sense "close" to a countable set.

Definition 2. Let L be a Banach space and $C \subset E$. We say that C has property (A) if for each sequence $\epsilon_i \to 0$, $0 < \epsilon_i < 1$ and any sequence of balls $B(z_i, \epsilon_i) = \{x \in L : \ ||x - z_i|| \le \epsilon_i\}$, we have $C \nsubseteq \bigcup_{i=1}^{\infty} B(z_i, \epsilon_i)$.

Clearly, if C has (A) then C cannot be covered by a sequence of norm compact sets.

The main result of this paper is the following

THEOREM 1. *Let Y be a separable polyhedral Banach space. Then Y is isomorphic to a polyhedral Banach space Z such that the set* $\operatorname{ext} B_{Z^*}$ *has property* (A).

Remark. It follows from Theorem 3 [4], that if a Banach space Y is not isomorphic to a polyhedral space then $\operatorname{ext} B_{Y^*}$ has property (A) in any equivalent norm on Y.

We prove Theorem 1 in two steps. First we prove Theorem 1 for $Y = c_o$. Here we use some ideas from [7]. Then, by using that any polyhedral space contains an isomorphic copy of c_0 (see [3]) we finish the proof.

THEOREM 2. *There exists a separable polyhedral Banach space X, isomorphic to c_0, such that the set* $\operatorname{ext} B_{X^*}$ *has property* (A).

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be the natural basis of c_0 and $\{e_i^*\}_{i=1}^{\infty}$ be its biorthogonal sequence in $l_1 = c_0^*$. Fix $\varrho \in (0, \frac{1}{2})$ and denote

$$\lambda_i = \frac{1}{2^i}, \quad i = 1, 2, \ldots, \quad a = \frac{1}{\lambda_1}, \quad a_n = \frac{a \sum_{i=1}^{n} \lambda_i}{1 - \varrho \sum_{i=n+1}^{\infty} \lambda_i}, \quad n = 1, 2, \ldots.$$

Let $\mathcal{G}_m$ be the family of all injective, non-decreasing mappings from $\{1, \ldots, m\}$ to $\mathbb{N}$ and $\mathcal{G}_\infty$ be the family of all injective, non-decreasing mappings from $\mathbb{N}$ to $\mathbb{N}$. Next define:

$$A_m = \left\{ a_m \left(\sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{k=1}^{m} \epsilon_k \lambda_k e_{g(k)}^* : \ \epsilon_k = \pm 1, \ g \in \mathcal{G}_m \right\}.$$

Clearly, each A_m is countable. Denote

$$B = \bigcup_{m=1}^{\infty} A_m, \quad U^* = \overline{\operatorname{conv}}^{w^*} B,$$

and define a new norm on c_0 as follows

$$|||x||| = \sup\{ f(x) : \ f \in U^* \}, \quad x \in c_0.$$

It is easily seen that the norm $|||\cdot|||$ on c_0 is equivalent to the original one (note that $A_1 = \{\pm a_1 e_k^* : k = 1, 2 \ldots\}$). Put $X = (c_0, |||\cdot|||)$. Also a standard argument shows that $B_{X^*} = U^*$.

For every subset A of X^*, denote A' the set of all w^*-limit points of the set A.

Claim 1. Every $f \in B'$ with $|||f||| = 1$ (if any) does not attain its norm $|||f|||$ at an element of the unit ball of X.

Proof. Take $f \in B'$, $f \neq 0$. We first assume that $f \in A'_m$ for some $m \geq 2$. Since $e_n^* \to^{w^*} 0$ we get

$$f = a_m \left(\sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^*,$$

for some $n < m$ and $g \in \mathcal{G}_n$.

$$\begin{aligned} |||f||| &= \left|\left|\left| a_m \left(\sum_{i=1}^{m} \lambda_i \right)^{-1} \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^* \right|\right|\right| \\ &= \left|\left|\left| \frac{a_m \left(\sum_{i=1}^{m} \lambda_i \right)^{-1}}{a_n \left(\sum_{i=1}^{n} \lambda_i \right)^{-1}} a_n \left(\sum_{i=1}^{n} \lambda_i \right)^{-1} \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^* \right|\right|\right| < 1. \end{aligned}$$

Next assume that $f \in B'$ and $f \notin A'_m$, $m = 1, 2, \ldots$. It is easy to see that either f is of the form

$$f = a \sum_{k=1}^{\infty} \epsilon_k \lambda_k e_{g(k)}^*, \quad \epsilon_k = \pm 1, \; g \in \mathcal{G}_\infty, \tag{1}$$

or

$$f = a \sum_{k=1}^{n} \epsilon_k \lambda_k e_{g(k)}^*, \quad \epsilon_k = \pm 1, \; g \in \mathcal{G}_n \tag{2}$$

If f satisfies (2) then $|||f||| < 1$. So we assume that f satisfies (1). Assume to the contrary, that there is $x \in c_0$, $|||x||| = 1$, such that $f(x) = 1$. Choose s so large that $a \cdot \max\{|x_{g(k)}|\}_{k=s+1}^{\infty} < \frac{\varrho}{2}$. Then the definition of $|||\cdot|||$ implies

$$\begin{aligned}1 = f(x) &= a\sum_{k=1}^{s}\epsilon_k\lambda_k x_{g(k)} + a\sum_{k=s+1}^{\infty}\epsilon_k\lambda_k x_{g(k)}\\ &\le \frac{a}{a_s}\left[a_s\left(\sum_{i=1}^{s}\lambda_i\right)^{-1}\sum_{k=1}^{s}\lambda_k|x_{g(k)}|\right]\sum_{i=1}^{s}\lambda_i + \Big(a\cdot\max_{k>s}|x_{g(k)}|\Big)\sum_{k=s+1}^{\infty}\lambda_k\\ &< \frac{a}{a_s}\cdot\sum_{i=1}^{s}\lambda_i + \frac{\varrho}{2}\sum_{i=s+1}^{\infty}\lambda_i < 1.\end{aligned}$$

The last inequality follows from the following equality:

$$\frac{a}{a_s}\sum_{i=1}^{s}\lambda_i + \varrho\sum_{i=s+1}^{\infty}\lambda_i = 1.$$

∎

Claim 2. B is a countable boundary for X and X is polyhedral.

Proof. Since each A_m is countable and $B = \bigcup_{m=1}^{\infty} A_m$, it follows that B is countable. The rest of the claim is a direct result of Claim 1 and Proposition 6.11 from [6]. We give a proof for the sake of completeness. Since $U^* = \overline{conv}^{w^*}B$, $\overline{B}^{w^*} = B \cup B'$ is a boundary for X. As a result of Claim 1, none of the elements in B' attain their norm at B_X hence B is a boundary for X. Now let F be a finite dimensional subspace of X and assume F^* has infinitely many extreme points, By Milman's theorem, these would be restrictions to F of elements of $\overline{B}^{w^*}$. Since F is finite-dimensional, any w^*-cluster point of the set of the extreme points of B_{F^*} attains its norm at an element of B_F. But this contradicts Claim 1. Hence F^* has only finitely many extreme points, and F is polyhedral. ∎

Claim 3. For any $g \in \mathcal{G}_\infty$ and $\{\epsilon_i\}_{i=1}^{\infty}$ a sequence of signs, we have $f = a\sum_{k=1}^{\infty}\epsilon_k\lambda_k e^*_{g(k)} \in \operatorname{ext} U^*$.

Proof. First note that from the definition of the norm $|||\cdot|||$ (the supremum over the set B) follows that

$$\left|\left|\left|\sum_{i=1}^{n}\epsilon_i e_{g(i)}\right|\right|\right| \le 2$$

Next the series $\sum_{i=1}^{\infty} \epsilon_i e_{g(i)}$ converges in the w^*-topology of $X^{**} \cong \ell_\infty$ and it follows that $|||\sum_{i=1}^{\infty} \epsilon_i e_{g(i)}||| \leq 2$. Moreover, setting $z^{**} = \sum_{i=1}^{\infty} \epsilon_i e_{g(i)}$ and $b^* = a\sum_{i=1}^{\infty} \epsilon_i \lambda_i e^*_{g(i)}$ we see that $b^* \in B_{X^*}$ and $z^{**}(b^*) = 2$. Therefore z^{**} attains its norm at the element $b^* \in B_{X^*}$ and $|||z^{**}||| = 2$. By a classical result [1], since X^* is separable, z^{**} attains its norm at an extreme point of B_{X^*} too. The latter set of points, in view of Milman's theorem, is contained in $\overline{B}^{w^*}$. It is easy to check that z^{**} does not attain its norm at a finitely supported (with respect to (e_i^*)) element of $\overline{B}^{w^*}$. Among the infinitely supported members of $\overline{B}^{w^*}$, it is clear that only b^* satisfies $z^{**}(b^*) = 2$, hence b^* is an extreme point of B_{X^*}. ∎

Claim 4. The set $\operatorname{ext} U^*$ has property (A).

Proof. Denote $E = \left\{a\sum_{i=1}^{\infty} \lambda_i e^*_{g(i)} : g \in \mathcal{G}_\infty\right\}$. By Claim 3, $E \subseteq \operatorname{ext} U^*$. So it is enough to prove that E has property (A). Our proof relies on the following easily verified fact.

Fact 1. For each two elements $u, v \in E$, if $u = a\sum_{i=1}^{\infty} \lambda_i e^*_{g_u(i)}$, $v = a\sum_{i=1}^{\infty} \lambda_i e^*_{g_v(i)}$ and $g_u(j) \neq g_v(j)$ then $\|u - v\| > \frac{1}{2^j}$.

Assume to the contrary that

$$E \subseteq \bigcup_{i=1}^{\infty} B_{X^*}(x_i, \epsilon_i), \quad \epsilon_i \to 0.$$

Since $B_{X^*} \subseteq 2B_{\ell_1}$ it follows that

$$E \subseteq \bigcup_{i=1}^{\infty} B_{\ell_1}(x_i, 2\epsilon_i).$$

Obviously, we can suppose that each $B_{\ell_1}(x_i, 2\epsilon_i)$ intersects E. For each i choose a representative $y_i \in B_{\ell_1}(x_i, 2\epsilon_i) \cap E$.

Choose m_0 sufficiently large so that for $m > m_0$ it holds that $2\epsilon_m < \frac{1}{4}$. Choose n_0 sufficiently large so that if $y \in E$ and $g_y(1) > n_0$ then

$$\max\{4\epsilon_1, \ldots, 4\epsilon_{m_0}\} < \|y - y_j\|,$$

for each $j \leq m_0$ (this is possible since $4\epsilon_i < 4$ and $E \subseteq 2S_{\ell_1}$). Denote by G_0 the set $\{1, 2, \ldots, n_0\}$. Choose $m_1 > m_0$ sufficiently large such that if $m > m_1$

then $2\epsilon_m < \frac{1}{8}$. Denote by G_1 the set $\{g_{y_{m_0+1}}(1), \dots, g_{y_{m_1}}(1)\}$. By Fact 1 if $x \in E$ and $g_x(1) \notin G_1$ then $\|x - y_j\| > \frac{1}{2}$ for $m_0 < j \le m_1$. Hence, $x \notin \cup_{i=m_0+1}^{m_1} B_{\ell_1}(x_i, 2\epsilon_i)$. Next we define inductively m_n and G_n such that

1) For every $m > m_n$, $2\epsilon_m < \frac{1}{2^{n+2}}$.
2) G_n is finite.
3) If $g_x(n) \notin G_n$ then $x \notin \cup_{i=m_{n-1}+1}^{m_n} B_{\ell_1}(x_i, 2\epsilon_i)$.

Choose m_{n+1} so that for $m > m_{n+1}$ it holds that $2\epsilon_m < \frac{1}{2^{n+3}}$. Denote by G_{n+1} the set $\{g_{y_{m_n+1}}(n+1), \dots, g_{y_{m_{n+1}}}(n+1)\}$. For every $x \in E$ and $m_n < j \le m_{n+1}$ if $g_x(n+1) \notin G_{n+1}$ then by Fact 1 $\|x - y_j\| > \frac{1}{2^{n+1}} > 4\epsilon_j$ and $x \notin \cup_{m_n+1}^{m_{n+1}} B_{\ell_1}(x_i, 2\epsilon_i)$. Define $b_1 = \max(G_0 \cup G_1) + 1$ and b_n to be $\max\big(\cup_{i=0}^{n} G_n \cup \{b_1, \dots, b_{n-1}\}\big) + 1$. Next define $g \in \mathcal{G}_\infty$ to be $g(n) = b_n$, $n = 1, 2, \dots$, and $x = \sum_{i=1}^{\infty} \lambda_i e^*_{g(i)}$. From our construction follows that $x \notin \bigcup_{i=1}^{\infty} B_{\ell_1}(x_i, 2\epsilon_i)$, a contradiction. ∎

The proof of Theorem 2 is complete. ∎

Proof of Theorem 1. By [3] Y contains c_0 (actually Y is c_0-saturated). Since Y is separable it follows [9] that c_0 is complemented in Y. Hence Y is isomorphic to the direct sum of Y_1 and c_0, where Y_1 is isometric to some subspace of Y and hence polyhedral. By Theorem 2, c_0 is isomorphic to a polyhedral Banach space X with the set $\operatorname{ext} B_{X^*}$ having property (A). Put $Z = (Y_1 \oplus_\infty X)$. Clearly, Z is polyhedral and $Y \cong Z$. Since $\operatorname{ext} B_{Z^*} = \operatorname{ext} B_{Y_1^*} \cup \operatorname{ext} B_{X^*}$ it follows that the set $\operatorname{ext} B_{Z^*}$ has property (A). The proof is complete. ∎

Acknowledgements

This article is a part of the author's MS thesis under the supervision of Professor V. P. Fonf. The author wishes to thank Professor Fonf for his encouragement and useful comments. He would also like to thank the referees for their valuable suggestions which improved the paper.

References

[1] C. Bessaga, A. Pełczyński, On extreme points in separable conjugate spaces, *Israel J. Math.* **4** (1966), 262 – 264 .

[2] V. P. Fonf,, Some properties of polyhedral Banach spaces, *Funktsional Anal. i Prilozhen,* **14** (4) (1980), 89 – 90, (English trans. in *Funct. Anal. Appl.* **14** (1980)).

[3] V. P. Fonf, Polyhedral Banach spaces, *Mat. Zametki.* **30** (4) (1981), 627 – 634, (English translation in *Math. Notes Acad. Sci. USSR* **30** (1981), 809 – 813).

[4] V. P. Fonf, Three characterizations of polyhedral Banach spaces, *Ukrainian Math. J.* **42** (9) (1990), 1145 – 1148.

[5] V. P. Fonf, On the boundary of a polyhedral Banach Space, *Extracta Math.* **15** (1) (2000), 145 – 154.

[6] V. P. Fonf, J. Lindenstrauss, R. R. Phelps, Infinite dimensional convexity, in "Handbook of the Geometry of Banach Spaces, Vol. I", North-Holland, Amsterdam, 2001, 599 – 670.

[7] V. P. Fonf, L. Vesely, Infinite dimensional polyhedrality, *Canad. J. Math.* **56** (3) (2004), 472 – 494.

[8] V. Klee, Polyhedral sections of convex bodies, *Acta Math.* **103** (1960), 243 – 267.

[9] A. Sobczyk, Projections of the space m on its subspace c_0, *Bull. Amer. Math. Soc.* **47** (1941), 937 – 947.

[10] L. Veselý, Boundary of polyhedral spaces: an alternative proof, *Extracta Math.* **15** (1) (2000), 213 – 217.