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Linear Mapping Preserving the Kernel or the Range of Operators *

Mourad Oudghiri

13 *Département de Mathématiques et Informatiques, Faculté des Sciences d'Oujda, Oujda, Maroc, Mourad.Oudghiri@math.univ-lille1.fr*

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Abstract: Let X and Y be two infinite dimensional real or complex Banach spaces. In this note we determine the forms of surjective additive maps $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ preserving the kernel's dimension or the range's codimension. As consequence, we establish that $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserves the kernel (respectively, the range) if and only if there exists an invertible operator $A \in \mathcal{L}(X)$ such that $\phi(T) = AT$ (respectively, $\phi(T) = TA$) for all $T \in \mathcal{L}(X)$.

Key words: Additives preservers, kernel operator, range operator.

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Introduction and statement of main results

Let X be a Banach space, and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X. For $T \in \mathcal{L}(X)$, write T^* for its adjoint, $\mathrm{N}(T)$ for its kernel and $\mathrm{R}(T)$ for its range. Recall that an operator $T \in \mathcal{L}(X)$ is called *semi-Fredholm* if $\mathrm{R}(T)$ is closed and either $\dim \mathrm{N}(T)$ or $\operatorname{codim} \mathrm{R}(T)$ is finite. The *index* of such operator is defined by

$$\operatorname{ind}(T) = \dim \mathrm{N}(T) - \operatorname{codim} \mathrm{R}(T)\,,$$

and if $\operatorname{ind}(T)$ is finite then T is said to be *Fredholm*.

In [9] it is shown that a surjective linear map $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$, where X is an infinite-dimensional complex Banach space, is unital, i.e., $\phi(I) = I$, and preserves injective operators in both direction if and only if there is an invertible operator $A \in \mathcal{L}(X)$ such that $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{L}(X)$. Moreover, if X is assumed to be a Hilbert space, then it is proved that the surjective unital linear maps ϕ preserving surjective operators take the above

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mentioned form. These results are extended to the case of unital surjective additive maps $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ where X and Y are a complex Banach spaces, see [1].

Let X and Y be an infinite-dimensional Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The purpose of this note is to determine the forms of all surjective additive maps, non-necessary unital, $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ preserving the kernel's dimension or the range's codimension. We establish also that $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ preserves the kernel (respectively, the range) if and only if there exists an invertible operator $A \in \mathcal{L}(X)$ such that $\phi(T) = AT$ (respectively, $\phi(T) = TA$) for all $T \in \mathcal{L}(X)$.

Theorem 1. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be an additive surjective mapping. The following assertions are equivalent.*

(i) $\dim \mathrm{N}(\phi(T)) = \dim \mathrm{N}(T)$ *for all* $T \in \mathcal{L}(X)$*;*

(ii) *there is two bijective bounded linear, or conjugate linear, mappings $U : X \to Y$ and $V : Y \to X$ such that $\phi(T) = UTV$ for all $T \in \mathcal{L}(X)$.*

Theorem 2. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be an additive surjective mapping such that* $\operatorname{codim} \mathrm{R}(\phi(T)) = \operatorname{codim} \mathrm{R}(T)$ *for all $T \in \mathcal{L}(X)$. Then one of the following assertions holds:*

(i) *There exist a bijective linear or conjugate linear mappings $U : X \to Y$ and $V : Y \to X$ such that $\phi(T) = UTV$ for all $T \in \mathcal{L}(X)$.*

(ii) *There exist a bijective linear or conjugate linear mappings $U' : X^* \to Y$ and $V' : Y \to X^*$ such that $\phi(T) = U'T^*V'$ for all $T \in \mathcal{L}(X)$. In this case, X and Y are reflexive.*

Notice that the case (ii) in the above theorem can occur in some special Banach spaces. More precisely, it is shown in [2, 3, 4] that there exists an infinite-dimensional complex reflexive Banach space X such that every bounded operator $T \in \mathcal{L}(X)$ is of the form $T = \lambda I + S$ where $\lambda \in \mathbb{C}$ and S is strictly singular; the essential spectrum of such operator is $\sigma_{\mathrm{e}}(T) = \{\lambda\}$. Consider the linear map $\phi(T) = T^*$ for all $T \in \mathcal{L}(X)$. Then ϕ preserves the range's codimension. In fact, for Fredholm operators T, we have $\operatorname{ind}(T) = 0$ and so $\operatorname{codim} \mathrm{R}(T) = \operatorname{codim} \mathrm{R}(\phi(T))$. If T is not Fredholm, then it is strictly singular and $\sigma_{\mathrm{e}}(T) = \{0\}$. Hence, the continuity of the index implies that T and T^* are not semi-Fredholm, and consequently $\operatorname{codim} \mathrm{R}(T) = \operatorname{codim} \mathrm{R}(\phi(T) = \infty$.

Let $x \in X$ and let f be in the dual space X^* of X, we denote, as usual, by $x \otimes f$ the rank one operator given by $(x \otimes f)z = f(z)x$ for $z \in X$. The spectrum of such operator is $\sigma(x \otimes f) = \{0, f(x)\}$.

As consequence of Theorem 1 and Theorem 2, we derive the following two results.

THEOREM 3. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective additive map. Then the following assertions are equivalent:*

(i) $\mathrm{N}(\phi(T)) = \mathrm{N}(T)$ *for all* $T \in \mathcal{L}(X)$*;*

(ii) *there is an invertible operator* $A \in \mathcal{L}(X)$ *such that* $\phi(T) = AT$ *for all* $T \in \mathcal{L}(X)$.

Proof. Assume that ϕ preserves the kernel, then, obviously, it preserves the kernel's dimension, and by Theorem 1, it takes the form $\phi(T) = UTV$ for all $T \in \mathcal{L}(X)$. Let us show that $V = \lambda I$. Suppose, on the contrary, that there exists $x \in X$ such that x and Vx are linearly independent, and let $f \in X^*$ satisfy $f(x) = 1$ and $f(Vx) = 0$. It follows that

$$x \in \mathrm{N}(I - x \otimes f) = \mathrm{N}(U(I - x \otimes f)V) = \mathrm{N}(V - x \otimes fV)\,,$$

and hence $Vx = 0$, a contradiction. Thus $\phi(T) = AT$ for all $T \in \mathcal{L}(X)$, where $A = \lambda U = \phi(I) \in \mathcal{L}(Y)$. This completes the proof. ∎

THEOREM 4. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(X)$ be a surjective additive map. Then the following assertions are equivalent:*

(i) $\mathrm{R}(\phi(T)) = \mathrm{R}(T)$ *for all* $T \in \mathcal{L}(X)$*;*

(ii) *there exists an invertible operator* $B \in \mathcal{L}(X)$ *such that* $\phi(T) = TB$ *for all* $T \in \mathcal{L}(X)$.

Proof. Assume that ϕ preserves the range. Then ϕ preserves the range's dimension. Observe that ϕ can not take the second form in Theorem 1, because otherwise, for $T = x \otimes f$ such that $U'(f)$ and x are linearly independent, we will get

$$\mathrm{Vect}\{x\} = \mathrm{R}(T) = \mathrm{R}(\phi(T)) = \mathrm{Vect}\{U'(f)\}\,,$$

a contradiction. Hence, ϕ takes the form $\phi(T) = UTV$ for all $T \in \mathcal{L}(X)$. Now, for an arbitrary $a \in X$ and $g \in X^*$ such that $g(a) \neq 0$, we have

$$\mathrm{R}(a \otimes g) = \mathrm{R}(U(a \otimes g)V) = \mathrm{R}(Ua \otimes g)\,,$$

and so $\{a, Ua\}$ is linearly dependent. This shows that $\phi(T) = TB$ for all $T \in \mathcal{L}(X)$, where $B = \lambda V = \phi(I) \in \mathcal{L}(X)$, as desired. ∎

Before giving the proof of Theorem 1 and Theorem 2, some lemmas are to be established first.

It is well known that the set of semi-Fredholm operators remains invariant under perturbation by finite rank operators.

Lemma 5. *Let T be a non-zero operator in $\mathcal{L}(X)$. Then the following assertions are equivalent:*

(i) $\mathrm{rg}(T) = 1$;

(ii) *If $S \in \mathcal{L}(X)$, then the map $\lambda \to \dim \mathrm{N}(S + \lambda T)$ is constant on $\mathbb{Q}$ minus at most one point;*

(iii) *If $S \in \mathcal{L}(X)$, then the map $\lambda \to \operatorname{codim} \mathrm{R}(S + \lambda T)$ is constant on $\mathbb{Q}$ minus at most one point.*

Proof. (i) ⇒ (ii) : Let $x \in X$ and $f \in X^*$ be such that $T = x \otimes f$, and $S \in \mathcal{L}(X)$. Suppose that there exists a scalar μ such that $\mathrm{N}(S+\mu T)\backslash \mathrm{N}(S) \neq \emptyset$. Then we get easily that $x = Sa$ for some $a \in X$, and so $S+\lambda T = S(I+\lambda a\otimes f)$ for all λ. Therefore, if $\dim \mathrm{N}(S+\lambda T) \neq \dim \mathrm{N}(S)$, $I+\lambda a\otimes f$ is not invertible, and consequently $\lambda f(a) = -1$. This shows that the map $\lambda \to \dim \mathrm{N}(S + \lambda T)$ is constant on $\mathbb{Q} \setminus \{- f(a)^{-1}\}$.

Now, if $\mathrm{N}(S+\lambda T) \subseteq \mathrm{N}(S)$ for all λ, then $\mathrm{N}(S+\lambda T) \subseteq \mathrm{N}(S)\cap \mathrm{N}(f)$ for $\lambda \neq 0$. But, since $\mathrm{N}(S)\cap \mathrm{N}(f) \subseteq \mathrm{N}(S+\lambda T)$, we get that $\mathrm{N}(S+\lambda T) = \mathrm{N}(S)\cap \mathrm{N}(f)$ for all $\lambda \neq 0$, as desired.

(i) ⇒ (iii) : Let $S \in \mathcal{L}(X)$. Without loss of generality we can suppose the existence of some $\mu \in \mathbb{Q}$ for which $\operatorname{codim} \mathrm{R}(S + \mu T)$ is finite, and it follows in this case that $S + \mu T$ is semi-Fredholm. Hence $S + \lambda T$ is semi-Fredholm for all λ. Consequently, $S^* + \lambda T^*$ is semi-Fredholm and so $\operatorname{codim} \mathrm{R}(S + \lambda T) = \dim \mathrm{N}(S^* + \lambda T^*)$ for all λ. Finally, since T^* is rank one, the first implication implies that the map $\lambda \to \dim \mathrm{N}(S^* + \lambda T^*) = \operatorname{codim} \mathrm{R}(S + \lambda T)$ is constant on $\mathbb{Q}$ minus at most one point.

[(ii) or (iii)] ⇒ (i) : Let δ denote the kernel's dimension or the range's codimension. Assume, on the contrary, that $\mathrm{R}(T)$ contains two linearly independent vectors $u = Tx$ and $v = Ty$, and let N be a closed subspace such that $X = \mathrm{Vect}\{u, v\} \oplus N$. Then it follows easily that $X = \mathrm{Vect}\{x, y\} \oplus M$ where $M = T^{-1}N$. Now, let S be a bounded operator satisfying $Sx = Tx$,

$Sy = -Ty$ and $S : M \mapsto N$ is invertible. Then S is invertible, and

$$\delta(S - \lambda T) = \delta\big(I - \lambda S^{-1}T\big) = 0 \qquad \text{for } \lambda^{-1} \notin \sigma\big(S^{-1}T\big)\,.$$

Hence, $\delta(S-\lambda T) = 0$ for all λ in $\mathbb{Q}$ minus at most one point. This contradicts the fact that $S - T$ and $S + T$ are neither injective nor surjective because $(S - T)x = (S + T)y = 0$, $u \notin \mathrm{R}(S - T)$ and $v \notin \mathrm{R}(S + T)$. Thus, T is rank one operator. ∎

LEMMA 6. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map preserving the kernel's dimension or the range's codimension. Then ϕ is injective.*

Proof. Let $T \in \mathcal{L}(X)$ be such that $\phi(T) = 0$. Then, by Lemma 5, T is of rank less than one. Assume that $Tx = y \neq 0$ for some $x, y \in X$, and let S be an invertible operator such that $Sx = y$. It follows that $\phi(S - T) = \phi(S)$ is either injective or surjective. But, since $x \in \mathrm{N}(S - T)$ and $y \notin \mathrm{R}(S - T)$, $\dim \mathrm{N}(S - T)$ and $\operatorname{codim} \mathrm{R}(T - S)$ are non-zero, a contradiction. ∎

Let τ be a ring automorphism of $\mathbb{K}$. An additive map $A : X \to Y$ will be called τ-quasilinear if $A(\lambda x) = \tau(\lambda)Ax$ holds for all numbers $\lambda \in \mathbb{C}$ and $x \in X$. Notice that in the real case all the quasilinear maps are linear because the identity is the only ring automorphism of $\mathbb{R}$, while in the complex case the ring continuous automorphisms are the identity and the complex conjugation.

From Lemmas 5 and 6 it follows that ϕ preserves in both direction the set of operators of rank one, and consequently it takes one of the following forms:

$$\phi(x \otimes f) = Gx \otimes Hf \qquad \text{for all } x \in X \text{ and } f \in X^*, \tag{1}$$

or

$$\phi(x \otimes f) = Kf \otimes Lx \qquad \text{for all } x \in X \text{ and } f \in X^*, \tag{2}$$

where $G : X \to Y$, $H : X^* \to Y^*$, $K : X^* \to Y$ and $L : X \to Y^*$ are τ-quasilinear bijective maps, and $\tau : \mathbb{K} \to \mathbb{K}$ is a ring automorphism, see [8].

LEMMA 7. *Let $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective additive map preserving the kernel's dimension or the range's codimension. Then $\phi(I)$ is invertible.*

Proof. Let $S = \phi(I)$. Suppose that ϕ preserves the kernel's dimension, then in particular S is injective. To show that S is surjective, let y be a non-zero vector in Y. By (1) and (2) we obtain the existence of $x \in X$, $f \in X^*$ and $g \in Y^*$ such that $f(x) = 1$ and $\phi(x \otimes f) = y \otimes g$. Since

$$\dim \mathrm{N}(S - y \otimes g) = \dim \mathrm{N}(I - x \otimes f) = 1\,,$$

$S - y \otimes g$ is not injective, and hence $y \in \mathrm{R}(S)$ because S is injective.

Now assume that ϕ preserves the range's codimension, then S is surjective. Suppose that $\mathrm{N}(S)$ contains a non-zero vector y, and using (1) or (2) one can find $x \in X$, $f \in X^*$ and $y \in Y$ such that $f(x) = 1$, $g(y) \neq 0$ and $\phi(x \otimes f) = y \otimes g$. Hence, $\operatorname{codim}(S - y \otimes g) = \operatorname{codim}(I - x \otimes f) = 1$. But, since

$$S = (S - y \otimes g)\big(I - g(y)^{-1} y \otimes g\big)\,,$$

$S - y \otimes g$ is surjective, a contradiction. ∎

Lemma 8. *Let S, T be two bounded invertible operators on X, and denote by δ and δ^* the dimension of the kernel or the codimension of the range. If $\delta(T + F) = \delta^*(S + F)$ for all rank one operator F, then $S = T$.*

Proof. Let $x \in X$, and consider an arbitrary $f \in X^*$ such that $f\big(T^{-1}x\big) = 1$. It follows that $\delta\big(I - T^{-1}x \otimes f\big) = 1$, and

$$\begin{aligned} \delta^*\big(I - S^{-1}x \otimes f\big) &= \delta^*(S - x \otimes f) = \delta(T - x \otimes f) \\ &= \delta\big(I - T^{-1}x \otimes f\big) = 1\,. \end{aligned}$$

Therefore, $I - S^{-1}x \otimes f$ is not invertible, and so $f\big(S^{-1}x\big) = 1$. This shows that $T^{-1}x = S^{-1}x$ for all x. Consequently, $T = S$. ∎

Proof of Theorem 1 *and Theorem* 2. Let δ denote the kernel's dimension or the range's codimension, and suppose that ϕ preserves δ. Then, by Lemma 7, $\phi(I)$ is invertible, and the unital map $\tilde{\phi} = \phi(I)^{-1}\phi$ preserves δ.

We first treat the case when $\tilde{\phi}$ takes the form (1), i.e., $\tilde{\phi}(x \otimes f) = Gx \otimes Hf$ for all $x \in X$ and $f \in X^*$. Observe that for every non-zero scalar λ, $x \in X$ and $f \in X^*$, we have $\delta(I - \lambda x \otimes f) = \delta(I - \tau(\lambda)Gx \otimes Hf)$, and so $I - \lambda x \otimes f$ is invertible if and only if $I - \tau(\lambda)Gx \otimes Hf$ is invertible. This shows that $H(f)(Gx) = \tau(f(x)) = \big(\tau \circ f \circ G^{-1}\big)(Gx)$ for all x and f. Hence $H(f) = \tau \circ f \circ G^{-1}$, and consequently $\tilde{\phi}(x \otimes f) = G(x \otimes f)G^{-1}$ for all x and f. Arguing as in [1, 8] we get that τ and G are bounded, and so τ is either an identity or the complex conjugation.

Let $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{R} \setminus \big(\sigma(T) \cup \sigma\big(\tilde{\phi}(T)\big)\big)$. For an arbitrary rank one operator F we have

$$\begin{aligned} \delta(T - \lambda I + F) &= \delta\big(\tilde{\phi}(T) - \lambda I + GFG^{-1}\big) \\ &= \delta\big(G^{-1}\big(\tilde{\phi}(T) - \lambda I\big)G + F\big)\,. \end{aligned}$$

Hence, according to Lemma 8, $\tilde{\phi}(T) = GTG^{-1}$. Therefore $\phi(T) = UTV$ for all $T \in \mathcal{L}(X)$ where $U = \phi(I)G$ and $V = G^{-1}$.

Now assume that $\tilde{\phi}$ is of the second form $\tilde{\phi}(x \otimes f) = Kf \otimes Lx$. By an argument similar to the previous case one can establish that K and L are a bounded linear, or conjugate linear, operators and that $\tilde{\phi}(F) = KF^*K^{-1}$ for every rank one operator F. Moreover, in this case, the spaces X and Y are reflexive, see [1]. Let $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{R} \setminus \big(\sigma(T) \cup \sigma\big(\tilde{\phi}(T)\big)\big)$. Consider an arbitrary rank one operator F, then it follows that

$$\begin{aligned}\delta(T - \lambda I + F) &= \delta\big(\tilde{\phi}(T) - \lambda I + KF^*K^{-1}\big)\\ &= \delta\big(K^{-1}\big(\tilde{\phi}(T) - \lambda I\big)K + F^*\big)\,.\end{aligned}$$

But, since $K^{-1}\big(\tilde{\phi}(T) - \lambda I\big)K$ is invertible, $K^{-1}\big(\tilde{\phi}(T) - \lambda I\big)K + F^*$ is semi-Fredholm and hence it has a closed range. Then, using Lemma 8 we get that $T = K^{-1}\tilde{\phi}(T)K$. Therefore, $\phi(T) = U'T^*V'$ for all $T \in \mathcal{L}(X)$ where $U' = \phi(I)K$ and $V' = K^{-1}$.

To complete the proof it remains to show that ϕ cannot take the second form when ϕ preserves the kernel's dimension. Assume on the contrary that $\phi(T) = U'T^*V'$ for all $T \in \mathcal{L}(X)$. Since Y is reflexive, there exists a non-invertible injective operator $S \in \mathcal{L}(Y)$, see [9, 1]. As ϕ is surjective, $S = \phi(T)$ where $T \in \mathcal{L}(X)$ is injective. Consider a nonzero vector $y \in Y$, and let $f = U'^{-1}y$ and $x \in X$ be such that $f(x) = 1$. It follows that

$$\dim \mathrm{N}\left(S - U'(Tx \otimes f)^*V'\right) = \dim \mathrm{N}(T - Tx \otimes f) = 1\,.$$

Consequently, $S - U'(Tx \otimes f)^*V'$ is not injective, and since S is injective, we obtain that

$$\mathrm{Vect}\{U'f\} = \mathrm{R}\left(U'(Tx \otimes f)^*V'\right) \subseteq \mathrm{R}(S)\,.$$

Thus, $y \in \mathrm{R}(S)$ and so S is surjective, a contradiction. ∎

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References

[1] J. Cui, J. Hou, Additive maps on standard operator algebras preserving invertibilities or zero divisors, *Linear Algebra Appl.* **359** (2003), 219–233.

[2] M. Gonzalez, Banach spaces with small Calkin algebras, in "Perspectives in Operator Theory", Banach Center Publ., 75, Polish Acad. Sci., Warsaw, 2007, 159–170.

[3] W.T. Gowers, B. Maurey, Banach spaces with small spaces of operators, *Math. Ann.* **307** (1997), 543–568.

[4] W.T. Gowers, B. Maurey, The unconditional basic sequence problem, *J. Amer. Math. Soc.* **6** (1993), 851–874.

[5] A. Jafarian, A.R. Sourour, Spectrum-preserving linear maps, *J. Funct. Anal.* **66** (1986), 255–261.

[6] M. Mbekhta, L. Rodman, P. Šemrl, Linear maps preserving generalized invertibility, *Integral Equations Operator Theory* **55** (2006), 93–109.

[7] M. Mbekhta, Linear maps preserving a set of Fredholm operators, *Proc. Amer. Math. Soc.* **135** (2007), 3613–3619.

[8] M. Omladic, P. Šemrl, Additive mappings preserving operators of rank one, *Linear Algebra Appl.* **182** (1993), 239–256.

[9] P. Šemrl, Two characterizations of automorphisms on $B(X)$, *Studia Math.* **150** (1993), 143–149.