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Quillen-Suslin Rings

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Abstract: In this paper we introduce the *Quillen-Suslin rings* and investigate its relation with some other classes of rings as Hermite rings (each stably free module is free), *PSF* rings (each finitely generated projective module is stably free), *PF* rings (each finitely generated projective module is free), etc. Quillen-Suslin rings are induced by the famous Serre's problem formulated by J.P. Serre in 1955 ([30]) and solved independently by Quillen ([28]) and Suslin ([31]) in 1976. The solution is known as *the Quillen-Suslin theorem* and states that every finitely generated projective module over the polynomial ring $K[x_1, \ldots, x_n]$ is free, where K is a field. There are algorithmic proofs and some generalizations of this important theorem that we will also study in this paper. In particular, we will consider extended modules and rings, and the Bass-Quillen conjecture.

Key words: Quillen-Suslin theorem, Hermite rings, extended modules and rings, Bass-Quillen conjecture.

AMS *Subject Class.* (2000): 13C10, 13B25, 13E15.

1. Introduction

The present paper is divided in four sections. The second section is dedicated to define the Quillen-Suslin rings (*QS*) and present some other rings close related with them (see [18]); in particular, we will prove a theorem about some matrix characterizations of Hermitian rings that could help to study an old conjecture about polynomial rings over Hermitian rings. The third section is focused into computational aspects of *QS* rings. In particular, we will discuss the most recent algorithmic proofs of the Quillen-Suslin theorem (see [10], [11], [12], [21], [22], [24] and [27]). There are many generalizations of the Quillen-Suslin theorem that we will also study in this paper. In particular, we will consider in the last section extended modules and rings and the Bass-Quillen conjecture.

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2. QS rings and some related properties

From now on, S represents an arbitrary commutative ring, $S[x_1, \ldots, x_n]$ is the polynomial ring over S in $n \geq 1$ variables, and $\mathrm{GL}_n(S)$ is the general linear group of invertible matrices over S of size $n \times n$.

Definition 1. Let S be a commutative ring.

(i) S is a *PF* ring if every finitely generated projective S-module is free (a module M is projective if M is a direct summand of a S-free module).

(ii) S is a *PSF* ring if every finitely generated projective S-module is stably free (a module M is stably free if there exist integers $r, s \geq 0$ such that $S^r \cong S^s \oplus M$).

(iii) S is a *FFR* ring (finite free resolutions) if each f.g. S-module has a finite free resolution

$$0 \to S^{t_k} \xrightarrow{F_k} S^{t_{k-1}} \xrightarrow{F_{k-1}} \cdots \xrightarrow{F_1} S^{t_0} \xrightarrow{F_0} M \to 0\,,$$

for some $k \geq 0$.

(iv) S is an Hermite ring, denoted H, if any stably free S-module is free.

(v) Let $n \geq 1$, S is a QS_n ring if $S[x_1, \ldots, x_n]$ is *PF*.

(vi) S is a Quillen-Suslin ring, denoted QS, if S is QS_n for each $n \geq 1$.

From the above definition is obvious that

$$PSF \cap H = PF\,, \tag{2.1}$$

$$QS = \bigcap_{n \geq 1} QS_n\,. \tag{2.2}$$

Examples 2. (i) Any principal ideal domain (*PID*) is *PF* (see [29]).

(ii) Any Bézout domain is *PF* (a domain D is *Bézout* if any f.g. ideal of D is principal, see [4]).

(iii) Any local ring is *PF*.

(iv) Semilocal rings (ring with finite many maximal ideals) are not always *PF*. In fact, $\mathbb{Z}_6$ is a semilocal ring and $\mathbb{Z}_6 = \langle 3 \rangle \oplus \langle 4 \rangle$. Thus, $\langle 3 \rangle$ is a finitely generated projective $\mathbb{Z}_6$-module, but is not free. Since 6 is square free, $\mathbb{Z}_6$ is semisimple; thus, semisimple rings are not always *PF*. We observe that this example illustrate also that hereditary rings (each ideal is projective) are not always *PF*, and consequently, semihereditary rings (each finitely generated ideal is projective) are not always *PF* (see [29]).

In the next theorem we present some characterizations of stably free modules that we will use later for proving some interesting results about *PSF* rings (compare with [19] and [24]). We start recalling the definition of Fitting ideals of a matrix and also the concept of unimodular matrix.

DEFINITION 3. Let S be a commutative ring and F a matrix over S of size $n \times m$. For each integer r, the r-th Fitting ideal of F, denoted by $\mathrm{F}_r^S(F)$, is defined in the following way:

(i) $\mathrm{F}_r^S(F)$ is the ideal of S generated by all minors of F of size $(n-r)\times(n-r)$, if $1 \leq n-r \leq \min\{n,m\}$.

(ii) $\mathrm{F}_r^S(F) := S$, if $n-r \leq 0$.

(iii) $\mathrm{F}_r^S(F) := 0$, if $n-r > \min\{n,m\}$.

DEFINITION 4. Let S be a commutative ring and F a matrix over S of size $r \times s$. F is unimodular if $\mathrm{F}_t^S(F) = S$ with $t = r - \min\{r,s\}$.

Thus, F is unimodular if and only if the maximal minors of the matrix F generate the unit ideal in S (see [24]). Unimodular matrices can be characterized in the following way.

PROPOSITION 5. *Let S be a commutative ring and F a matrix over S of size $r \times s$. Then,*

(i) *Let $s \geq r$. F is unimodular if and only if F has a right inverse.*

(ii) *Let $r \geq s$. F is unimodular if and only if F has a left inverse.*

Proof. (i) We note that $\min\{r,s\} = r$. If F is unimodular, then $\mathrm{F}_0^S(F) = S$ and it is well known that the linear system $FX = \boldsymbol{e}_i$, for $1 \leq i \leq r$, has solution $C^{(i)} \in S^s$, where $\boldsymbol{e}_i$ is the canonical basis column vector of S^r (see [5, Corollary 5.35]). Hence the matrix $C = \begin{bmatrix} C^{(1)} & \cdots & C^{(r)} \end{bmatrix}$ satisfies $AC = I_r$. Conversely, if C is a matrix over S of size $s \times r$ such that $AC = I_r$, then by the Binet-Cauchy theorem (see [25]) we conclude that the ideal generated by all minors of size $r \times r$ of F is S, i.e., $\mathrm{F}_0^S(F) = S$, so F is unimodular.

(ii) In this case $\min\{r,s\} = s$, F is unimodular if and only if F^T is unimodular if and only if F^T has a right inverse (by (i)), if and only if F has a left inverse. ∎

THEOREM 6. *Let S be a commutative ring and M an S-module. Then, the following conditions are equivalent*

(i) *M is stably free.*

(ii) *M is projective and has a finite free resolution.*

(iii) *There exist matrices P of size $s \times r$ and Q of size $r \times s$ such that $r \geq s$, $PQ = I_s$ and $M \cong \ker(P)$, i.e., M is isomorphic to the kernel of an unimodular matrix. In other words, M is isomorphic to the kernel of an S-module epimorphism of free modules of finite dimension.*

(iv) *M is projective and has a finite presentation $S^s \xrightarrow{F_1} S^r \xrightarrow{F_0} M \to 0$ where $\ker(F_0)$ is stably free.*

(v) *M is projective and has a finite presentation $S^s \xrightarrow{F_1} S^r \xrightarrow{F_0} M \to 0$ where $r \geq s$ and F_1 has a left inverse.*

(vi) *M is projective and has a finite presentation $S^s \xrightarrow{F_1} S^r \xrightarrow{F_0} M \to 0$ where $r \geq s$ and F_1 is unimodular.*

Proof. (i) $\Rightarrow$ (ii): If $S^r \cong S^s \oplus M$ for some integers $r, s \geq 0$, then M is projective and we have the finite free resolution

$$0 \to S^s \xrightarrow{\iota} S^r \xrightarrow{\pi} M \to 0\,,$$

where ι is the canonical inclusion and π is the canonical projection on M.

(ii) $\Rightarrow$ (i): Let

$$0 \to S^{t_k} \xrightarrow{F_k} S^{t_{k-1}} \xrightarrow{F_{k-1}} \cdots \xrightarrow{F_2} S^{t_1} \xrightarrow{F_1} S^{t_0} \xrightarrow{F_0} M \to 0$$

be a finite free resolution of M. By induction on k we will prove that M is stably free.

If $k = 0$ then M is free of finite dimension, and hence, stably free. Let $k \geq 1$ and let $M_1 = \ker(F_0)$. We get the exact sequence

$$0 \to M_1 \xrightarrow{\iota} S^{t_0} \xrightarrow{F_0} M \to 0\,,$$

and hence $S^{t_0} \cong M \oplus M_1$ since M is a projective module. This implies that M_1 is also projective and then we have the finite free resolution of M_1

$$0 \to S^{t_k} \xrightarrow{F_k} S^{t_{k-1}} \xrightarrow{F_{k-1}} \cdots \xrightarrow{F_2} S^{t_1} \xrightarrow{F_1} M_1 \to 0\,.$$

By induction, there exist integers $p, q \geq 0$ such that $S^p \cong S^q \oplus M_1$, and hence, $S^{t_0} \oplus S^q \cong M \oplus M_1 \oplus S^q \cong M \oplus S^p$, i.e., $S^{t_0+q} \cong M \oplus S^p$.

(i) $\Rightarrow$ (iii): There exist integers $r, s \geq 0$ such that $S^r \cong S^s \oplus M$, and hence $M \cong \ker(\pi)$, where π is the canonical projection of S^r on S^s. We observe that $r \geq s$; let P be the matrix of π in the canonical bases; since S^s is projective there exists a matrix Q of size $r \times s$ such that $PQ = I_s$; moreover, $M \cong \ker(P)$.

(iii) $\Rightarrow$ (i): Let $S^r \xrightarrow{A} S^s$ be an epimorphism such that $M \cong \ker(A)$. Then we have the exact sequence

$$0 \to M \xrightarrow{\iota} S^r \xrightarrow{A} S^s \to 0\,,$$

but S^s is projective and hence $S^r \cong S^s \oplus M$.

(i) $\Rightarrow$ (iv): Let $S^r \cong S^s \oplus M$ for some integers $r, s \geq 0$, then M is projective and we have the exact sequence

$$0 \to S^s \xrightarrow{F_1} S^r \xrightarrow{F_0} M \to 0\,,$$

and also the finite presentation

$$S^s \xrightarrow{F_1} S^r \xrightarrow{F_0} M \to 0\,,$$

where F_0 is the canonical projection and F_1 is the canonical injection of S^s in S^r. But $\ker(F_0) = \operatorname{Im}(F_1) \cong S^s$, thus $\ker(F_0)$ is free, and hence, stably free.

(iv) $\Rightarrow$ (i): Let M be projective and

$$S^s \xrightarrow{F_1} S^r \xrightarrow{F_0} M \to 0$$

a finite presentation of M with $\ker(F_0)$ stably free. Then $S^r \cong M \oplus \ker(F_0)$. There exist some integers $p, q \geq 0$ such that $S^p \cong S^q \oplus \ker(F_0)$ and hence $S^{r+q} \cong M \oplus S^p$.

(i) $\Rightarrow$ (v): Let $S^r \cong S^s \oplus M$ for some integers $r, s \geq 0$, then $r \geq s$, M is projective and we have the exact sequence

$$0 \to S^s \xrightarrow{F_1} S^r \xrightarrow{F_0} M \to 0\,,$$

where F_0 is the canonical projection and F_1 is the canonical injection of S^s in S^r. Since M is projective there exists $H_0 : M \to S^r$ such that $F_0 H_0 = i_M$, and hence,

$$S^r = \ker(F_0) \oplus \operatorname{Im}(H_0) = \operatorname{Im}(F_1) \oplus \operatorname{Im}(H_0)\,.$$

For $x \in S^r$ we have $x = F_1(y) + H_0(z)$ with $y \in S^s$ and $z \in M$, we note that y and z are unique for x since F_1 and H_0 are injective, so we define $G_1 : S^r \to S^s$ by $G_1(x) = y$. It is clear that G_1 is an S-homomorphism and $G_1 F_1 = I_s$.

(v) $\Rightarrow$ (i): Let $G_1 : S^r \to S^s$ such that $G_1F_1 = I_s$, then F_1 is injective and M has the finite free resolution

$$0 \to S^s \xrightarrow{F_1} S^r \xrightarrow{F_0} M \to 0\,.$$

By (ii) and (i) M is stably free.

(v) $\Leftrightarrow$ (vi): This is a direct consequence of Proposition 5. ∎

Example 7. Theorem 6, part (iii), gives a method for constructing stably free modules. In fact, if $\boldsymbol{f}$ is a row unimodular matrix of size $1 \times r$, then the module $M := \ker(\boldsymbol{f}) = \mathrm{Syz}(\boldsymbol{f})$ is stably free. For example, consider the row matrix $\boldsymbol{f} = (xy + y, x + y, x^2y + xy + 1)$ over $\mathbb{Z}[x, y]$, then its right inverse is $\boldsymbol{g} = (y, -xy - y, 1)^T$, hence $\boldsymbol{f}$ is unimodular and

$$\begin{aligned}\mathrm{Syz}(\boldsymbol{f}) = \Big\langle &\big(x + y, -xy - y, 0\big), \big(- xy - y^2, x^2y + xy^2 + xy + y^2 + 1, -x - y\big),\\ &\big(- y^3 + y^2 - 1, xy^3 - xy^2 + y^3 - y^2 + y, -y^2 + y\big)\Big\rangle\end{aligned}$$

is stably free. In a similar way, the row matrix $\boldsymbol{v} = \big(xy-x+y, -y^2, x+y, x^2y+2xy+1\big)$ over $\mathbb{Q}[x, y]$ is unimodular with right inverse $\boldsymbol{u} = (y, 0, -xy - y, 1)^T$, hence $\boldsymbol{v}$ is unimodular and

$$\begin{aligned}\mathrm{Syz}(\boldsymbol{v}) = \Big\langle &\big(- y, y - 2, y^2 - y, 0\big), \big(- x, x + 2, xy - x + 2y, 0\big),\\ &\big(y, x + 2, y, 0\big), \big(- 2y^2 + 1, -1, 2xy^2 + 2y^2 - y + 1, -2y\big),\\ &\big(- 2xy - 1, 1, 2x^2y + 2xy + y + 1, -2x\big)\Big\rangle\end{aligned}$$

is stably free. $\mathrm{Syz}(\boldsymbol{v})$ and $\mathrm{Syz}(\boldsymbol{f})$ were computed with `CoCoA` (see [15]).

A direct consequence of previous theorem is the following characterization of *PSF* rings.

Corollary 8. *A ring S is PSF if and only if each f.g. projective S-module has a finite free resolution.*

Corollary 9. *If S is a Noetherian FFR ring, then $S[x_1, \dots, x_n]$ is a PSF ring.*

Proof. By [29, Theorem 9.44], $S[x_1, \dots, x_n]$ is a *FFR* ring, for each $n \geq 1$. Thus, the result is a direct consequence of previous corollary. ∎

Corollary 10. (Serre's theorem) *If K is a field, then for each $n \geq 1$, $K[x_1, \dots, x_n]$ is PSF.*

Proof. In [1, Theorem 3.10.4], there is a constructive proof (using Gröbner bases) of Hilbert's Syzygy theorem that says that $K[x_1, \dots, x_n]$ is *FFR*. Thus, Serre's theorem is a direct consequence of previous corollary. ∎

Matrix descriptions of H rings are presented in the following theorem (compare with [6], [18] and [24]).

Theorem 11. *Let S be a commutative ring. Then, the following conditions are equivalent.*

(i) *S is H.*

(ii) *Any unimodular column matrix $\mathbf{v}$ over S of size $r \times 1$ can be completed to an invertible matrix of $\mathrm{GL}_r(S)$ adding $r-1$ new columns.*

(ii)′ *Any unimodular row matrix $\mathbf{v}$ over S of size $1 \times r$ can be completed to an invertible matrix of $\mathrm{GL}_r(S)$ adding $r-1$ new rows.*

(iii) *Given a unimodular column matrix $\mathbf{v}$ over S of size $r \times 1$ there exists a matrix $U \in \mathrm{GL}_r(S)$ such that $U\mathbf{v} = \boldsymbol{e}_1$.*

(iii)′ *Given a unimodular row matrix $\mathbf{v}$ over S of size $1 \times r$ there exists a matrix $U \in \mathrm{GL}_r(S)$ such that $\mathbf{v}U = (1, 0, \dots, 0)$.*

(iv) *Given a unimodular matrix F of size $r \times s$, $r \geq s$, there exists $U \in \mathrm{GL}_r(S)$ such that*
$$UF = \begin{bmatrix} I_s \\ 0 \end{bmatrix}.$$

(iv)′ *Given a unimodular matrix F of size $s \times r$, $r \geq s$, there exists $U \in \mathrm{GL}_r(S)$ such that*
$$FU = [\, I_s \mid 0 \,].$$

Proof. We recall that the elements of S^r are columns vectors of size $r \times 1$. It is clear that (ii) $\Leftrightarrow$ (ii)′, (iii) $\Leftrightarrow$ (iii)′ and (iv) $\Leftrightarrow$ (iv)′.

(i) $\Rightarrow$ (ii): Let $\boldsymbol{v} = [\, v_1 \cdots v_r \,]^T$ be an unimodular matrix of size $r \times 1$, there exists $\boldsymbol{u} = [\, u_1 \cdots u_r \,]$ such that $\boldsymbol{u}\boldsymbol{v} = 1$, i.e., $u_1 v_1 + \cdots + u_r v_r = 1$; we define
$$\begin{array}{ccc} S^r & \xrightarrow{\alpha} & S \\ \boldsymbol{e}_i & \longmapsto & u_i \end{array}$$

where $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_r\}$ is the canonical basis of S^r. We observe that α is a surjective homomorphism since $\alpha(\boldsymbol{v})=1$. There exists $\beta: S\to S^r$ such that $\alpha\beta=i_S$ and $S^r=\mathrm{Im}(\beta)\oplus\ker(\alpha)$; in fact, we define $\beta(1):=\boldsymbol{v}$ and β is injective, so $\mathrm{Im}(\beta)\cong S$ is free with basis $\{\boldsymbol{v}\}$. This implies that $S^r\cong S\oplus\ker(\alpha)$, i.e., $\ker(\alpha)$ is stably free, so by hypothesis, $\ker(\alpha)$ is free of dimension $r-1$; let $\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_{r-1}\}$ be a basis of $\ker(\alpha)$, then $\{\boldsymbol{v},\boldsymbol{x}_1,\dots,\boldsymbol{x}_{r-1}\}$ is a basis of S^r. This means that $[\,\boldsymbol{v}\;\boldsymbol{x}_1\;\cdots\;\boldsymbol{x}_{r-1}\,]\in\mathrm{GL}_r(S)$.

(ii) $\Rightarrow$ (i): Let M be an stably free S-module, then there exist integers $r,s\geq 0$ such that $S^r\cong S^s\oplus M$. It is enough to prove that M is free for the case when $s=1$. In fact, $S^s\oplus M=S\oplus(S^{s-1}\oplus M)$ is free and hence $S^{s-1}\oplus M$ is free; repeating this reasoning we conclude that $S\oplus M$ is free, so M is free.

Let $r\geq 1$ such that $S^r\cong S\oplus M$, let $\pi:S^r\to S$ be the canonical projection with kernel isomorphic to M and let $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_r\}$ be the canonical basis of S^r; there exists $\mu:S\to S^r$ such that $\pi\mu=i_S$ and $S^r=\ker(\pi)\oplus\mathrm{Im}(\mu)$. Let $\mu(1)=\boldsymbol{v}=[\,v_1\;\cdots\;v_r\,]^T\in S^r$, then $\pi(\boldsymbol{v})=1=v_1\pi(\boldsymbol{e}_1)+\cdots+v_r\pi(\boldsymbol{e}_r)$, i.e., $\boldsymbol{v}$ is a unimodular matrix over S of size $r\times 1$, moreover $S^r=\ker(\pi)\oplus\langle\boldsymbol{v}\rangle$. By hypothesis, there exists $U\in\mathrm{GL}_r(S)$ such that $U\boldsymbol{e}_1=\boldsymbol{v}$.

Let $f:S^r\to S^r$ be the isomorphism defined by U in the canonical basis of S^r, then $f(\boldsymbol{e}_1)=\boldsymbol{v}$ and $f(\boldsymbol{e}_i)=\boldsymbol{v}_i$, $i\geq 2$, where $\boldsymbol{v}_2,\dots,\boldsymbol{v}_r$ are the others columns of U.

If we prove that $f(\boldsymbol{e}_i)\in\ker(\pi)$ for each $i\geq 2$, then $\ker(\pi)$ is free, and consequently, M is free. In fact, let f' be the restriction of f to $\langle\boldsymbol{e}_2,\dots,\boldsymbol{e}_r\rangle$, i.e., $f':\langle\boldsymbol{e}_2,\dots,\boldsymbol{e}_r\rangle\to\ker(\pi)$. Then f' is bijective: of course f' is injective; let $\boldsymbol{w}$ be any vector of S^r, then there exists $\boldsymbol{x}\in S^r$ such that $f(\boldsymbol{x})=\boldsymbol{w}$, we write $\boldsymbol{x}=(x_1,x_2,\dots,x_r)=x_1\boldsymbol{e}_1+\boldsymbol{z}$, with $\boldsymbol{z}=x_2\boldsymbol{e}_2+\cdots+x_r\boldsymbol{e}_r$. We have $f(\boldsymbol{x})=f(x_1\boldsymbol{e}_1+\boldsymbol{z})=x_1f(\boldsymbol{e}_1)+f(\boldsymbol{z})=x_1\boldsymbol{v}+f(\boldsymbol{z})=\boldsymbol{w}$. In particular, if $\boldsymbol{w}\in\ker(\pi)$, then $\boldsymbol{w}-f(\boldsymbol{z})\in\ker(\pi)\cap\langle\boldsymbol{v}\rangle=0$, so $\boldsymbol{w}=f(\boldsymbol{z})$ and hence $\boldsymbol{w}=f'(\boldsymbol{z})$, i.e., f' is surjective.

In order to conclude the proof we will show that $f(\boldsymbol{e}_i)\in\ker(\pi)$ for each $i\geq 2$. Since f was defined by U, the idea is to change U in a such way that its first column was $\boldsymbol{v}$ and for the others columns were $\boldsymbol{v}_i\in\ker(\pi)$, $2\leq i\leq r$. Let $\pi(\boldsymbol{v}_i)=r_i\in S$, $i\geq 2$ and $\boldsymbol{v}'_i=\boldsymbol{v}_i-r_i\boldsymbol{v}$; then adding to column i of U the first column multiplied by $-r_i$ we get a new matrix U such that its first column is again $\boldsymbol{v}$ and for the others we have $\pi(\boldsymbol{v}'_i)=\pi(\boldsymbol{v}_i)-r_i\pi(\boldsymbol{v})=r_i-r_i=0$, i.e., $\boldsymbol{v}'_i\in\ker(\pi)$.

(ii) $\Rightarrow$ (iii): $\boldsymbol{v}$ can be completed to an invertible matrix of $\mathrm{GL}_r(S)$ if and only if there exists $V\in\mathrm{GL}_r(S)$ such that $V\boldsymbol{e}_1=\boldsymbol{v}$ if and only if $\boldsymbol{e}_1=V^{-1}\boldsymbol{v}$;

thus $U := V^{-1}$.

(iii)$'$ $\Rightarrow$ (iv)$'$: The proof will be done by induction on s. For $s = 1$ the result is trivial. We assume that (iv)$'$ is true for unimodular matrices with $l \leq s - 1$ rows. Let F be a unimodular matrix of size $s \times r, r \geq s$, then there exists a matrix B such that $FB = I_s$. This implies that the first row $\boldsymbol{v}$ of F is unimodular; by (iii)$'$ there exists $U' \in \mathrm{GL}_r(S)$ such that $\boldsymbol{v}U' = (1, 0, \ldots, 0) = \boldsymbol{e}_1^T$, and hence $FU' = F''$,

$$F'' = \begin{bmatrix} \boldsymbol{e}_1^T \\ F' \end{bmatrix},$$

with F' a matrix of size $(s-1) \times r$. Since $FB = I_s$, then $I_s = F''(U'^{-1}B)$, i.e., F'' is a unimodular matrix; let F''' be the matrix eliminating the first column of F', then F''' is unimodular of size $(s-1) \times (r-1)$, with $r - 1 \geq s - 1$; by induction, there exists a matrix $C \in \mathrm{GL}_{r-1}(S)$ such that $F'''C = [\, I_{s-1} \mid 0\,]$. From this we get,

$$FU' = F'' = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a'_{11} & a'_{12} & \cdots & a'_{1r} \\ \vdots & \vdots & & \vdots \\ a'_{s-11} & a'_{s-12} & \cdots & a'_{s-1r} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ * & F''' \end{bmatrix},$$

and hence

$$FU' \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ * & F''' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ * & I_{s-1} & 0 \end{bmatrix}.$$

Multiplying the last matrix on the right by elementary matrices we get (iv)$'$.

(iv)$'$ $\Rightarrow$ (iii)$'$: Taking $s = 1$ and $F = \boldsymbol{v}$ in (iv)$'$ we get (iii)$'$. ∎

A useful result for checking freeness for stably free modules (see Theorem 6) is given by the following theorem.

THEOREM 12. *Let S be a commutative ring and M a stably free S-module given by the kernel of a unimodular matrix F of size $s \times r$, $r \geq s$, with right inverse B. Then the following conditions are equivalent:*

(i) *M is free of dimension $r - s$.*

(ii) *There exists a matrix $U \in \mathrm{GL}_r(S)$ such that $FU = [\, I_s \mid 0\,]$. In such case, the last $r - s$ columns of U conform a basis for M. Moreover, the first s columns of U conform B.*

(iii) *There exists a matrix $V \in \mathrm{GL}_r(S)$ such that F coincides with the first s rows of V, i.e., F can be completed to an invertible matrix V of $\mathrm{GL}_r(S)$.*

Proof. (i) $\Rightarrow$ (ii): Let B be a matrix of size $r \times s$ such that $FB = I_s$, moreover let $S^r \xrightarrow{F} S^s$, then $S^r = \mathrm{Im}(B) \oplus \ker(F)$, thus, we are assuming that $\ker(F)$ is free. If $s = r$ then F is invertible and $U = F^{-1} = B$ and the result is trivially true. Let $r > s$ and let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_p\}$ be a basis of $\ker(F)$ with $p := r - s$. If $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_s\}$ is the canonical basis of S^s, then $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_s\}$ is basis of $\mathrm{Im}(B)$ with $\boldsymbol{u}_i := B\boldsymbol{e}_i$, $1 \leq i \leq s$, thus $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_p, \boldsymbol{u}_1, \dots, \boldsymbol{u}_s\}$ is a basis of S^r. We define $S^r \xrightarrow{U} S^r$ by $U\ \boldsymbol{e}_i := \boldsymbol{u}_i$ for $1 \leq i \leq s$, and $U\boldsymbol{e}_{s+j} := \boldsymbol{v}_j$ for $1 \leq j \leq p$. Clearly U is bijective; moreover, $FU\boldsymbol{e}_i = F\boldsymbol{u}_i = FB\boldsymbol{e}_i = \boldsymbol{e}_i$ and $FU\boldsymbol{e}_{s+j} = A\boldsymbol{v}_j = \boldsymbol{0}$, i.e., $FU = [\,I_s \mid 0\,]$. Additionally, by the definition of U we observe that the first s columns of U form the matrix B.

(ii) $\Rightarrow$ (i): Let $U^{(k)}$ the k-th column of U, then $FU = F[\,U^{(1)} \cdots U^{(s)} \cdots U^{(r)}] = [\,I_s \mid 0\,]$, so $FU^{(i)} = \boldsymbol{e}_i$, $1 \leq i \leq s$, $FU^{(s+j)} = \boldsymbol{0}$, $1 \leq j \leq p$ with $p := r - s$. This means that $U^{(s+j)} \in \ker(F)$ and hence $\big\langle U^{(s+j)} : 1 \leq j \leq p\big\rangle \subseteq \ker(F)$. On the other hand, let $\boldsymbol{c} \in \ker(F) \subseteq S^r$, then $F\ \boldsymbol{c} = \boldsymbol{0}$ and $FUU^{-1}\boldsymbol{c} = \boldsymbol{0}$, thus $[\,I_s \mid 0\,]U^{-1}\boldsymbol{c} = \boldsymbol{0}$ and hence $U^{-1}\boldsymbol{c} \in \ker([\,I_s \mid 0\,])$; let $\boldsymbol{d} = [d_1, \dots, d_r]^T \in \ker(\,I_s \mid 0\,])$, then $[\,I_s \mid 0\,]\boldsymbol{d} = \boldsymbol{0}$ and from this we conclude that $d_1 = \cdots = d_s = 0$, i.e., $\ker([\,I_s \mid 0\,]) = \langle \boldsymbol{e}_{s+1}, \boldsymbol{e}_{s+2}, \dots, \boldsymbol{e}_{s+p}\rangle$, in other words, $\ker([\,I_s \mid 0\,])$ coincides with the column module of the matrix

$$\begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

From $U^{-1}\boldsymbol{c} \in \ker([\,I_s \mid 0\,])$ we get that $\boldsymbol{c}$ is in the column module of matrix

$$U \begin{bmatrix} 0 \\ I_p \end{bmatrix} = \big[\,U^{(s+1)} \cdots U^{(s+p)}\,\big].$$

This proves that $\ker(F) = \big\langle U^{(s+j)} : 1 \leq j \leq p\big\rangle$; but since U is invertible, then $ker(F) \cong M$ is free of dimension $p = r - s$. We also has proved that the last $r - s$ columns of U conform a basis for M.

(ii) $\Leftrightarrow$ (iii): $FU = [\,I_s \mid 0\,]$ if and only if $F = [\,I_s \mid 0\,]U^{-1}$, but the first s rows of $[\,I_s \mid 0\,]U^{-1}$ coincides with the first s rows of U^{-1}; taking $V := U^{-1}$ we get the result. ∎

Some examples of H rings are presented next.

EXAMPLE 13. Semilocal rings are H. This can be proved in the following way: finite product of H rings is a H ring; S is a H ring if and only if $S/\operatorname{Rad}(S)$ is a H ring ($\operatorname{Rad}(S)$ is the Jacobson radical of S); any field is a H ring, so we conclude the proof applying the chinese remainder theorem. On the other hand, from (iv) of Example 2 we get that

$$PF \subsetneqq H\,.$$

Moreover, from Examples 2 and (2.1) we get that semilocals rings are not always *PSF*.

EXAMPLES 14. (i) If R is a Dedekind domain (hereditary integral domain), then $R[x_1,\dots,x_n]$ is H, for any $n \geq 1$ (see [18, Theorem V.2.11]).

(ii) If S is a commutative ring of Krull dimension 0, then $S[x_1,\dots,x_n]$ is H, for any $n \geq 1$ (see [18, Proposition V.2.13]).

(iii) If S is a local ring, in [3] Bhatwadekar and Rao have proved that $S[x]$ is H if and only if $S\langle x\rangle$ is H, where $S\langle x\rangle$ is the localization of $S[x]$ at the multiplicative set of monic polynomials (see also [18, Theorem 5.9]).

EXAMPLE 15. Now we will exhibit a ring that is not H (see [6]); this example also shows that if S is H not always S/I is H, where I is a proper ideal of S: let $S := \mathbb{R}[x,y,z]/I$ and $I := \langle x^2+y^2+z^2-1\rangle$, then $\boldsymbol{f} := (\overline{x},\overline{y},\overline{z})$ is unimodular with right inverse $\boldsymbol{f}^T$, however $\boldsymbol{f}$ cannot be completed to a unimodular matrix.

Related with the H condition there are two well known conjectures (see [18]), probably not solved yet, that could be investigated with the results of Theorem 11:

CONJECTURES 16. (i) If S is H, then $S[x]$ is H.
(ii) If S is local, then $S[x]$ is H.

3. A constructive proof of the Quillen-Suslin's theorem

The most famous example of *QS* ring is given by the Quillen-Suslin theorem proved not only for coefficients in a field but also for coefficients in a *PID*:

> *Let D be a PID, then for $n \geq 1$ every finitely generated projective $D[x_1,\dots,x_n]$-module is free, i.e., $D[x_1,\dots,x_n]$ is PF.*

Thus, the Quillen-Suslin theorem stays that any *PID* is *QS*. A complete study of the Quillen-Suslin's theorem could be found in [18]. A non-algorithmic proof of this key theorem could be found in [28], [31], [17], [18], [19] and [29].

In this section we present a clear and constructive proof of the theorem in the classical case, i.e., when the coefficients are in a field (compare with [24]). More exactly, if $M \subseteq (K[x_1, \dots, x_n])^m$ is a f.g. projective module, K a field, the procedure that will exhibit in the following two theorems shows how to construct a free basis for M. For this purpose we will adapt the Logar-Sturmfels' algorithm of [24] and also the ideas in [27].

The first theorem (Theorem 20) proves that $K[x_1, \dots, x_n]$ is an Hermite ring; the second theorem (Theorem 21) constructs a finite free basis for M. In order to prove these two theorems we need some preliminary lemmas.

Lemma 17. (Noether normalization) *Let $p(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$ and $m := \deg(p(x_1, \dots, x_n)) + 1$, where $\deg(p(x_1, \dots, x_n))$ is the total degree of $p(x_1, \dots, x_n)$. Consider the following automorphism of $K[x_1, \dots, x_n]$*

$$y_n := x_n\,, \qquad\qquad y_i := x_i - x_n^{m^{n-i}}, \quad 1 \le i \le n-1\,.$$

Then, $p(y_1, \dots, y_n) = aq(y_n)$, where $a \in K - \{0\}$ and $q(y_n) \in R[y_n]$ is monic, with $R := K[y_1, \dots, y_{n-1}]$. In the case where K is an infinite field, the automorphism could be taken linear, i.e., $y_i := \sum_{j=1}^{n} m_{ij} x_j$, where $M = [m_{ij}]$ is an invertible matrix over K.

Proof. See [29, Lemma 4.58] and [13, Theorem 3.4.1]. ∎

Lemma 18. *Let S be a commutative ring and let $f_1, f_2, b, d \in S[x]$. Let $s := \mathrm{Res}_x(f_1, f_2) \in S$ be the resultant of f_1 and f_2 with respect to x. Then, there exists $\widetilde{U} \in \mathrm{GL}_2(S[x])$ such that*

$$\begin{bmatrix} f_1(b) & f_2(b) \end{bmatrix} \widetilde{U} = \begin{bmatrix} f_1(b + sd) & f_2(b + sd) \end{bmatrix}.$$

Proof. The proof in [27] of this lemma is constructive and we will include it. For the resultant of two polynomials consult [5] or [19]. Using Gröbner bases we can find $p_1, p_2 \in S[x]$ such that $s = f_1 p_1 + f_2 p_2$. Let $s_1, s_2, t_1, t_2 \in S[x, y, z]$

be polynomials defined by

$$
\begin{aligned}
f_1(x+yz) &= f_1(x) + ys_1(x,y,z)\,,\\
f_2(x+yz) &= f_2(x) + ys_2(x,y,z)\,,\\
p_1(x+yz) &= p_1(x) + yt_1(x,y,z)\,,\\
p_2(x+yz) &= p_2(x) + yt_2(x,y,z)\,.
\end{aligned}
$$

We note that

$$
\begin{aligned}
s_1(b,s,d) &:= \frac{f_1(b+sd)-f_1(b)}{s}\,,\\
s_2(b,s,d) &:= \frac{f_2(b+sd)-f_2(b)}{s}\,,\\
t_1(b,s,d) &:= \frac{p_1(b+sd)-p_1(b)}{s}\,,\\
t_2(b,s,d) &:= \frac{p_2(b+sd)-p_2(b)}{s}\,;
\end{aligned}
$$

and we define

$$
\begin{aligned}
U_{11} &:= 1 + s_1(b,s,d)p_1(b) + t_2(b,s,d)f_2(b)\,,\\
U_{21} &:= s_1(b,s,d)p_2(b) - t_2(b,s,d)f_1(b)\,,\\
U_{12} &:= s_2(b,s,d)p_1(b) - t_1(b,s,d)f_2(b)\,,\\
U_{22} &:= 1 + s_2(b,s,d)p_2(b) + t_1(b,s,d)f_1(b)\,;
\end{aligned}
$$

then the matrix

$$
\widetilde{U} := \begin{bmatrix} U_{11} & U_{12}\\ U_{21} & U_{22}\end{bmatrix}
$$

has determinant 1 and satisfies the identity of the lemma. ∎

Lemma 19. *Let* $\boldsymbol{f}(x) := (f_1,\dots,f_r) \in (R[x])^r$ *be a unimodular row matrix, with* $R := K[x_1,\dots,x_{n-1}]$, $x := x_n$ *and* f_1 *monic in* x. *Then, there exist a matrix* $U \in \mathrm{GL}_r(R[x])$ *such that* $\boldsymbol{f}U = \boldsymbol{f}(0)$.

Proof. We include the constructive proof given in [27]. Let $\boldsymbol{a}_1 := (0,\dots,0) \in K^{n-1}$, we define $M_1 := \{g \in R \,|\, g(\boldsymbol{a}_1) = 0\}$, then M_1 is a maximal ideal of R and $K_1 := R/M_1 \cong K$ (see [9]); by hypothesis $\boldsymbol{f} \in (R[x])^r$ is unimodular and its image $\overline{\boldsymbol{f}} \in (K_1[x])^r = ((R/M_1)[x])^r$ is also unimodular. Since $K_1[x]$ is a principal ideal domain, by the Smith canonical form we can construct matrices $\overline{V'} \in \mathrm{GL}_1(K_1[x])$ and $\overline{U_1'} \in \mathrm{GL}_{r-1}(K_1[x])$ such that $\overline{V'}(\overline{f_2}, \ldots, \overline{f_r})\overline{U_1'} = [\,\overline{g_1}\ 0\ \cdots\ 0\,]$, with $\overline{g_1} \in K_1[x]$, but then $\overline{V'}$ is a nonzero element of K_1 and we can assume that $(\overline{f_2}, \ldots, \overline{f_r})\overline{U_1'} = [\,\overline{g_1}\ 0\ \cdots\ 0\,]$. Additionally we observe that $\langle \overline{g_1} \rangle = \langle \overline{f_2}, \ldots, \overline{f_r} \rangle$ and since $\langle \overline{f_1}, \ldots, \overline{f_r} \rangle = K_1[x]$, then $\langle \overline{g_1}, \overline{f_1} \rangle = K_1[x]$. Since $K_1 \cong K$ is a subring of R, we may lift $\overline{U_1'} := U_1'$ as an element of $\mathrm{GL}_{r-1}(R[x])$ and $\overline{g_1} := g_1$ as an element of $R[x]$. Then,

$$\boldsymbol{f}\begin{bmatrix} 1 & 0 \\ 0 & U_1' \end{bmatrix} = [\, f_1 \quad g_1 + q_{12} \quad q_{13} \quad \cdots \quad q_{1r}\,],$$

where $q_{12}, \ldots, q_{1r} \in M_1[x]$. We define $r_1 := \mathrm{Res}_x(f_1, g_1 + q_{12}) \in R$, and we can find $p_1, h_1 \in R[x]$ such that $p_1 f_1 + h_1(g_1 + q_{12}) = r_1$. Since f_1 is monic, and $\overline{f_1}, \overline{g_1} \in K_1[x]$ generate the unit ideal, we have

$$\overline{r_1} = \overline{\mathrm{Res}_x(f_1, g_1 + q_{12})} = \mathrm{Res}_x(\overline{f_1}, \overline{g_1}) \neq \overline{0}\,,$$

i.e., $r_1 \notin M_1$. Let $\overline{K}$ be the algebraic closure of K; for $j = 2$, let $\boldsymbol{a}_2 \in (\overline{K})^{n-1}$ be a zero of r_1 and $M_2 := \{g \in R \mid g(\boldsymbol{a}_2) = 0\}$ the corresponding maximal ideal of R, note that $r_1 \in M_2$; as above we can construct $r_2 \in R - M_2$, $U_2' \in \mathrm{GL}_{r-1}(R[x])$, $g_2, p_2, h_2 \in R[x]$ and $q_{22}, \ldots, q_{2r} \in M_2[x]$; or in general, for $j \geq 2$, let $\boldsymbol{a}_j \in (\overline{K})^{n-1}$ be a common zero of $r_1, \ldots, r_{j-1}$, M_j the corresponding maximal ideal of R, $r_j \in R - M_j$, $U_j' \in \mathrm{GL}_{r-1}(R[x])$, $g_j, p_j, h_j \in R[x]$ and $q_{j2}, \ldots, q_{jr} \in M_j[x]$; we observe that $r_1, \ldots, r_{j-1} \in M_j$ but $r_j \notin r_1R + \cdots + r_{j-1}R$. Since R is Noetherian, there exists a finite l such that $r_1R + \cdots + r_lR = R$ and using Gröbner bases we can find $w_1, \ldots, w_l \in R$ such that $r_1w_1 + \cdots + r_lw_l = 1$. We define $b_0, b_1, \ldots, b_l \in R[x]$ as

$$\begin{aligned}
b_0 &:= 0\,,\\
b_1 &:= r_1w_1x\,,\\
b_2 &:= r_1w_1x + r_2w_2x\,,\\
&\ \vdots\\
b_l &:= r_1w_1x + r_2w_2x + \cdots + r_lw_lx = x\,.
\end{aligned}$$

Note that for each $1 \leq i \leq l$

$$b_i = b_{i-1} + r_iw_ix\,.$$

Claim. For each $1 \le i \le l$, there exists a matrix $U_i \in \mathrm{GL}_r(R[x])$ such that $\boldsymbol{f}(b_i) = \boldsymbol{f}(b_{i-1})U_i$.

From previous claim we inductively get $\boldsymbol{f}(x) = \boldsymbol{f}(b_l) = \boldsymbol{f}(b_{l-1})U_l = \cdots = \boldsymbol{f}(0)U_1U_2\cdots U_l$, so $\boldsymbol{f}U = \boldsymbol{f}(0)$, with $U := U_l^{-1}U_{l-1}^{-1}\cdots U_1^{-1}$.

In order to complete the proof we must prove the above claim. For $1 \le i \le l$, let

$$\widetilde{g}_i := g_i + q_{i2}\,,$$

then

$$\boldsymbol{f}(x)\begin{bmatrix} 1 & 0 \\ 0 & U_i'(x) \end{bmatrix} = \begin{bmatrix} f_1(x) & \widetilde{g}_i(x) & q_{i3}(x) & \cdots & q_{ir}(x) \end{bmatrix}.$$

For $3 \le j \le r$, we have $q_{ij}(b_i) - q_{ij}(b_{i-1}) \in (b_i - b_{i-1})R[x] = r_iw_ixR[x]$ since $b_i - b_{i-1} = r_iw_ix$ for each $1 \le i \le l$. Since r_i does not depend on x, we have $r_i = p_i(x)f_1(x) + h_i(x)\widetilde{g}_i(x) = p_i(b_{i-1})f_1(b_{i-1}) + h_i(b_{i-1})\widetilde{g}_i(b_{i-1}) =$ a linear combination of $f_1(b_{i-1})$ and $\widetilde{g}_i(b_{i-1})$ over $R[x]$; therefore, for $3 \le j \le r$, we have $q_{ij}(b_i) = q_{ij}(b_{i-1}) +$ a linear combination of $f_1(b_{i-1})$ and $\widetilde{g}_i(b_{i-1})$ over $R[x]$. From this we conclude that there exists a matrix $C_i \in \mathrm{GL}_r(R[x])$ such that

$$\begin{aligned}\boldsymbol{f}(b_{i-1})\begin{bmatrix} 1 & 0 \\ 0 & U_i'(b_{i-1}) \end{bmatrix} C_i &= \begin{bmatrix} f_1(b_{i-1}) & \widetilde{g}_i(b_{i-1}) & q_{i3}(b_{i-1}) & \cdots & q_{ir}(b_{i-1}) \end{bmatrix} C_i \\ &= \begin{bmatrix} f_1(b_{i-1}) & \widetilde{g}_i(b_{i-1}) & q_{i3}(b_i) & \cdots & q_{ir}(b_i) \end{bmatrix}.\end{aligned}$$

By Lemma 18, we can construct a matrix $\widetilde{U}_i \in \mathrm{GL}_2(R[x])$ such that

$$\begin{bmatrix} f_1(b_{i-1}) & \widetilde{g}_i(b_{i-1}) \end{bmatrix}\widetilde{U}_i = \begin{bmatrix} f_1(b_i) & \widetilde{g}_i(b_i) \end{bmatrix}.$$

Finally, we define $U_i \in \mathrm{GL}_r(R[x])$ as

$$U_i := \begin{bmatrix} 1 & 0 \\ 0 & U_i'(b_{i-1}) \end{bmatrix} C_i \begin{bmatrix} \widetilde{U}_i & 0 \\ 0 & I_{r-2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_i'(b_i)^{-1} \end{bmatrix},$$

then $\boldsymbol{f}(b_{i-1})U_i = \boldsymbol{f}(b_i)$. This conclude the proof of the claim and also the proof of the lemma. ∎

THEOREM 20. *Let K be a field. Then,*

(i) *Given a unimodular matrix F over $K[x_1, \ldots, x_n]$ of size $s \times r$, $r \ge s$, with right inverse B, there exists $U \in \mathrm{GL}_r(K[x_1, \ldots, x_n])$ such that*

$$FU = [\,I_s \mid 0\,].$$

In such case, the last $r-s$ columns of U conform a basis for $\ker(F)$*. Moreover, the first s columns of U conform B.*

(ii) *Given a unimodular matrix F over $K[x_1,\dots,x_n]$ of size $r\times s$, $r\geq s$, with left inverse B, there exists $U\in \mathrm{GL}_r(K[x_1,\dots,x_n])$ such that*

$$UF=[\,I_s\mid 0\,]\,.$$

In such case, the last $r-s$ rows of U conform a basis for $\ker(F)$*. Moreover, the first s rows of U conform B.*

Proof. Taking the transposes of matrices involved we observe that (i) and (ii) are equivalent, so we only need to prove (i). The second part of (i) was proven in Theorem 12. Moreover, by induction, we only need to consider the case $s=1$ (see also the proof of Theorem 11).

Thus, given a unimodular row matrix $\boldsymbol{f}=(f_1,\dots,f_r)$ over $K[x_1,\dots,x_n]$ of size $1\times r$ we will construct a matrix $U\in \mathrm{GL}_r(K[x_1,\dots,x_n])$ such that $\boldsymbol{f}U=(1,0\dots,0)$.

Case 1. For $r=1$ the property is trivially true. For $r=2$ the property is valid for any commutative ring R: in fact, let $\boldsymbol{g}=(g_1,g_2)\in R^2$ such that $\boldsymbol{f}\boldsymbol{g}^T=[1]$, i.e., $f_1g_1+f_2g_2=1$, then in this case the matrix U is

$$U:=\begin{bmatrix} g_1 & -f_2\\ g_2 & f_1\end{bmatrix}$$

since $\det(U)=1$ and $\boldsymbol{f}U=(1,0)$.

Case 2. We can assume that $r\geq 3$. For $n=1$ the matrix U is computable since $K[x_1]$ is a principal ideal domain: in fact, by the Smith canonical form we can construct matrices $V\in \mathrm{GL}_1(K[x_1])$ and $U\in \mathrm{GL}_r(K[x_1])$ such that $V\boldsymbol{f}U=[\,d\ 0\ \cdots\ 0\,]$, with $d\in K[x_1]$, but then V is a nonzero element of K and we can assume that $\boldsymbol{f}U=[\,d\ 0\ \cdots\ 0\,]$. Since U is invertible $\langle d\rangle=\langle f_1,\dots,f_r\rangle=K[x_1]$ and hence d is a nonzero constant of K, so we can assume that $d=1$ and $\boldsymbol{f}U=[\,1\ 0\ \cdots\ 0\,]$.

The rest of the proof is as in [27]. We assume that the result is true for $k\leq n-1$ variables and let $R:=K[x_1,\dots,x_{n-1}]$ and $x:=x_n$; let $\boldsymbol{f}(x):=\boldsymbol{f}=(f_1,\dots,f_r)\in K[x_1,\dots,x_n]=R[x]$ be a unimodular row matrix. Permuting some columns of $\boldsymbol{f}$ (if it is necessary) and by Lemma 17 we can assume that f_1 is monic. By Lemma 19 we can construct a matrix $U'\in \mathrm{GL}_r(R[x])$ such that

$$\boldsymbol{f}U'=\boldsymbol{f}(0)\in R\,.$$

Since $\boldsymbol{f}(0)$ is unimodular over R, by induction there exists a matrix $U'' \in \mathrm{GL}_r(R)$ such that $\boldsymbol{f}(0)U'' = (1,0,\dots,0)$, and then, $\boldsymbol{f}U = (1,0,\dots,0)$, with $U := U'U'' \in \mathrm{GL}_r(R[x])$. ∎

THEOREM 21. (QUILLEN-SUSLIN) *Let K be a field. If M is a f.g. projective $K[x_1,\dots,x_n]$-module, then M is free and a basis for M is effectively computable.*

Proof. By [1, Theorem 3.10.4], we can construct a finite free resolution for M,

$$0 \to A^{t_k} \xrightarrow{F_k} A^{t_{k-1}} \xrightarrow{F_{k-1}} \cdots \xrightarrow{F_1} A^{t_0} \xrightarrow{F_0} M \to 0, \tag{3.1}$$

where $A := K[x_1,\dots,x_n]$. Since M is projective the short exact sequence

$$0 \to \ker(F_0) \to A^{t_0} \xrightarrow{F_0} M \to 0$$

splits, so $\ker(F_0) = \mathrm{Im}(F_1)$ is projective. By induction we get that $\mathrm{Im}(F_i)$ is projective for $1 \le i \le k$; in particular, $\mathrm{Im}(F_{k-1})$ is projective and the exact sequence

$$0 \to A^{t_k} \xrightarrow{F_k} A^{t_{k-1}} \xrightarrow{F_{k-1}} \mathrm{Im}(F_{k-1}) \to 0$$

splits; then F_k has a left inverse L_k, i.e., $L_kF_k = I_{t_k}$, and also F_{k-1} has right inverse H_{k-1}, and hence we have the exact sequence

$$0 \to \mathrm{Im}(F_{k-1}) \xrightarrow{H_{k-1}} A^{t_{k-1}} \xrightarrow{L_k} A^{t_k} \to 0\,.$$

We note that L_k is effective computable from F_k: in fact, by Theorem 20, part (ii), we compute a matrix U' such that $U'F_k = [\,I_{t_k} \mid 0\,]^T$ and the first t_k rows of U' conform L_k. Since L_k has a right inverse we can apply Theorem 20, part (i), and compute an invertible matrix U_{k-1} of size $t_{k-1} \times t_{k-1}$ such that

$$L_kU_{k-1} = [\,I_{t_k} \mid 0\,]$$

and its first t_k columns conform the matrix F_k; the remaining $t_{k-1}-t_k$ columns of U_{k-1} form a free basis for $\ker(L_k)$; we denote this submatrix of U_{k-1} by V_{k-1}. Let

$$C_{k-1} := F_{k-1}V_{k-1}\,,$$

then the size of C_{k-1} is $t_{k-2} \times (t_{k-1} - t_k)$. We have proved that C_{k-1} is effective computable.

We claim that the following sequence is exact:

$$0 \to A^{t_{k-1}-t_k} \xrightarrow{C_{k-1}} A^{t_{k-2}} \xrightarrow{F_{k-2}} A^{t_{k-3}} \xrightarrow{F_{k-3}} \cdots \xrightarrow{F_1} A^{t_0} \xrightarrow{F_0} M \to 0\,. \tag{3.2}$$

In fact, $V_{k-1} : A^{t_{k-1}-t_k} \to A^{t_{k-1}}$ is injective since the columns of V_{k-1} are linearly independent, moreover $\mathrm{Im}(V_{k-1}) = \ker(L_k) = \mathrm{Im}(H_{k-1})$; hence, let $x \in \ker(C_{k-1})$, then $C_{k-1}(x) = 0 = F_{k-1}V_{k-1}(x)$, this means that $V_{k-1}(x) \in \ker(F_{k-1}) = \mathrm{Im}(F_k)$, so there exists $z \in A^{t_k}$ such that $V_{k-1}(x) = F_k(z)$, and hence $L_k(V_{k-1}(x)) = 0 = L_k(F_k(z)) = z$, i.e., $V_{k-1}(x) = 0$, so $x = 0$. This prove that C_{k-1} is injective. $F_{k-2}C_{k-1} = F_{k-2}F_{k-1}V_{k-1} = 0$, so $\mathrm{Im}(C_{k-1}) \subseteq \ker(F_{k-2})$; finally, if $w \in \ker(F_{k-2}) = \mathrm{Im}(F_{k-1})$, then $w = F_{k-1}(u)$ with $u \in A^{t_{k-1}}$, but $u - F_kL_k(u) \in \ker(L_k)$, so $u - F_kL_k(u) = V_{k-1}(y)$ and consequently

$$\begin{aligned} w &= F_{k-1}(u) = F_{k-1}(V_{k-1}(y) + F_kL_k(u)) \\ &= F_{k-1}(V_{k-1}(y)) + F_{k-1}(F_kL_k(u)) = F_{k-1}(V_{k-1}(y)) = C_{k-1}(y)\,, \end{aligned}$$

thus, $\ker(F_{k-2}) \subseteq \mathrm{Im}(C_{k-1})$. This prove that (3.2) is exact.

The finite free resolution (3.2) is shorter than the resolution (3.1), hence, repeating this procedure we effectively construct a matrix $C_0 := F_0V_0$ such that the sequence

$$0 \to A^t \xrightarrow{C_0} M \to 0\,.$$

is exact. Then, the columns of C_0 form a free basis for M. ∎

Example 22. Let

$$\begin{aligned} M = \Big\langle &\big(-y, y-2, y^2-y, 0\big)\,,\ \big(-x, x+2, xy-x+2y, 0\big)\,, \\ &\big(y, x+2, y, 0\big)\,,\ \big(-2y^2+1, -1, 2xy^2+2y^2-y+1, -2y\big)\,, \\ &\big(-2xy-1, 1, 2x^2y+2xy+y+1, -2x\big)\Big\rangle \subseteq \big(\mathbb{Q}[x,y]\big)^4\,. \end{aligned}$$

According to Example 7, M is projective, and with `CoCoA` or `Singular` (see [16]), we computed the finite free resolution

$$0 \to A^2 \xrightarrow{F_1} A^5 \xrightarrow{F_0} M \to 0\,,$$

where $A := \mathbb{Q}[x,y]$,

$$F_0 = \begin{bmatrix} -y & -x & y & -2y^2+1 & -2xy-1 \\ y-2 & x+2 & x+2 & -1 & 1 \\ y^2-y & xy-x+2y & y & 2xy^2+2y^2-y+1 & 2x^2y+2xy+y+1 \\ 0 & 0 & 0 & -2y & -2x \end{bmatrix}$$

and

$$F_1 = \begin{bmatrix} x+2 & -x^2-2x+\frac{1}{2} \\ -y & xy+\frac{1}{2} \\ 2 & -2x \\ 0 & \frac{1}{2}x \\ 0 & -\frac{1}{2}y \end{bmatrix}.$$

A left inverse of F_1 can be computed directly,

$$L_1 := \begin{bmatrix} 0 & 0 & \frac{1}{2} & 2 & 0 \\ 1 & 1 & -\frac{1}{2}x+\frac{1}{2}y-1 & -y & -x \end{bmatrix},$$

and according to the proof of Theorem 21 the matrix U_0 is given by the columns of F_1 and a free basis for the kernel of L_1; with `CoCoA` we computed the kernel of L_1 and we get

$$\ker(L_1) = \begin{bmatrix} 1 & -y+2 & 0 \\ -1 & x-\frac{1}{2}y & x \\ 0 & 2 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \ker(\ker(L_1)) = 0\,,$$

i.e., the columns of the previous matrix form a free basis for the kernel of L_1. Consequently,

$$U_0 := \begin{bmatrix} x+2 & -x^2-2x+\frac{1}{2} & 1 & -y+2 & 0 \\ -y & xy+\frac{1}{2} & -1 & x-\frac{1}{2}y & x \\ 2 & -2x & 0 & 2 & 0 \\ 0 & \frac{1}{2}x & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2}y & 0 & 0 & 1 \end{bmatrix},$$

$$V_0 := \begin{bmatrix} 1 & -y+2 & 0 \\ -1 & x-\frac{1}{2}y & x \\ 0 & 2 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, the columns of $C_0 = F_0V_0$ form a basis for M, with C_0 given by

$$C_0 = \begin{bmatrix} x-y & -x^2+\frac{1}{2}xy+2y^2-\frac{1}{2} & -x^2-2yx-1 \\ y-x-4 & x^2-\frac{1}{2}xy+4x-y^2+3y+\frac{1}{2} & x^2+2x+1 \\ \alpha & \beta & \gamma \\ 0 & y & -2x \end{bmatrix},$$

where

$$\alpha = x-3y-xy+y^2\,,$$
$$\beta = x^2y-x^2-\frac{3}{2}xy^2+\frac{5}{2}xy-y^3+y^2+\frac{1}{2}y-\frac{1}{2}\,,$$
$$\gamma = y+3x^2y+4xy-x^2+1\,.$$

With `CoCoA` we checked that $\ker(C_0)=0$ and M coincides with the column module of C_0:

$UseR ::= Q[x,y];$

$\mathrm{Syz}([Vector(x-y, y-x-4, x-3y-xy+y^2, 0),$

$Vector(-x^2+1/2xy+2y^2-1/2, x^2-1/2xy+4x-y^2+3y+1/2,$

$x^2y-x^2-3/2xy^2+5/2xy-y^3+y^2+1/2y-1/2, y),$

$Vector(-x^2-2yx-1, x^2+2x+1, y+3x^2y+4xy-x^2+1, -2x)]);$

$Module([0])$

$UseR ::= Q[x,y];$

$G := ReducedGBasis(Module(Vector(x-y, y-x-4, x-3y-xy+y^2, 0),$

$Vector(-x^2+1/2xy+2y^2-1/2, x^2-1/2xy+4x-y^2+3y+1/2,$

$x^2y-x^2-3/2xy^2+5/2xy-y^3+y^2+1/2y-1/2, y),$

$Vector(-x^2-2yx-1, x^2+2x+1, y+3x^2y+4xy-x^2+1, -2x)));$

$G;$

$[Vector(y, x+2, y, 0), Vector(-x-y, 0, xy-x+y, 0),$

$Vector(x^2-1/2, 1/2, x^2+1/2y+1/2, -x), Vector(-y, y-2, y^2-y, 0),$

$Vector(xy+x+y+1/2, -1/2, x-3/2y+1/2, -y)]$

$UseR ::= Q[x,y];$

$G := ReducedGBasis(Module(Vector(-y, y-2, y^2-y, 0),$

$Vector(-x, x+2, xy-x+2y, 0),$

$Vector(y, x+2, y, 0), Vector(-2y^2+1, -1, 2xy^2+2y^2-y+1, -2y),$

$Vector(-2xy-1, 1, 2x^2y+2xy+y+1, -2x)));$

$G;$

$[Vector(-y, y-2, y^2-y, 0),$

$Vector(xy+x+y+1/2, -1/2, x-3/2y+1/2, -y),$

$Vector(x^2-1/2, 1/2, x^2+1/2y+1/2, -x), Vector(y, x+2, y, 0),$

$Vector(-x-y, 0, xy-x+y, 0)]$

In the previous example the computation of matrices L_1 and U_0 (see the notation in the proof of Theorem 21) was trivial, i.e., Theorem 20 was not applied. We present next an example that illustrates the procedure for computing the matrix U of Theorem 20, part (i). As we observe in the proof of that theorem, it is enough to consider the case $s = 1$.

Example 23. We will consider the example given by A. van den Essen and presented in [6], i.e.,

$$\boldsymbol{f} = \left(2txz + ty^2, 2txy + t^2, tx^2\right) \in \left(\mathbb{Q}[t,x,y,z]\right)^3.$$

We illustrate the procedure given in Theorem 20, Lemma 19 and Lemma 18, dividing the computations in some steps. We will use the package `CoCoA` for some computations.

Step 0. With CoCoA we check that $\boldsymbol{f}$ is unimodular:

$Use\ R ::= Q[t,x,y,z];$

$G := ReducedGBasis(Ideal(2txz+ty^2+1, 2txy+t^2, tx^2));$

$G;$

$[1]$

Step 1. Noether normalization: the automorphism in this case is given by $t \to z$, $x \to t$, $y \to x$, $z \to y$; so $f_1 = (x^2+2ty)z+1$, $f_2 = z^2+2txz$ and

$f_3 = t^2z$. Permuting f_1 and f_2 we have a new unimodular row matrix, and we can assume that

$$\boldsymbol{f} = \big(z^2 + 2txz, (x^2 + 2ty)z + 1, t^2z\big) \in \big(\mathbb{Q}[t, x, y, z]\big)^3$$

where $f_1 = z^2 + 2txz$ is monic in z with coefficients in $\mathbb{Q}[t, x, y]$.

Step 2. Now we apply Lemma 19 in order to construct a matrix $U' \in \mathrm{GL}_3(R[z])$ such that $\boldsymbol{f}\, U' = \boldsymbol{f}(0)$, with $R := \mathbb{Q}[t, x, y]$; we will use the notation of Lemma 19.

Step 2.1. $\boldsymbol{a}_1 := (0, 0, 0)$, $M_1 = \langle t, x, y\rangle$ is a maximal ideal of R, $K_1 := R/M_1 \cong \mathbb{Q}$, $\overline{\boldsymbol{f}} = \big(z^2, \overline{1}, \overline{0}\big) \in (K_1[z])^3 \cong (\mathbb{Q}[z])^3$,

$$\overline{U_1'} = U_1' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$\overline{g_1} = \overline{1}$; thus, $g_1 = 1$, $q_{12} = (x^2 + 2ty)z \in M_1[z]$ and $q_{13} = t^2z \in M_1[z]$. With `CoCoA` we compute $r_1 := \mathrm{Res}_z(f_1, \widetilde{g}_1) = \mathrm{Res}_z(z^2 + 2txz, 1 + (x^2 + 2ty)z)$,

$$\begin{aligned} &Use\, R ::= Q[t, x, y, z];\\ &F := z^2 + 2txz; G := 1 + (x^2 + 2ty)z;\\ &Resultant(F, G, z);\\ &-2tx^3 - 4t^2xy + 1, \end{aligned}$$

then

$$r_1 = -2tx^3 - 4t^2xy + 1\,.$$

Step 2.2. $\boldsymbol{a}_2 := \big(\frac{1}{2}, 1, 0\big)$ is a zero of r_1, $M_2 = \big\langle t - \frac{1}{2}, x - 1, y\big\rangle$ is a maximal ideal of R, $K_2 := R/M_2 \cong \mathbb{Q}$, $\overline{\boldsymbol{f}} = \big(z^2 + z, z + \overline{1}, \overline{\frac{1}{4}}z\big) \in (K_2[z])^3 \cong (\mathbb{Q}[z])^3$,

$$\overline{U_2'} = U_2' = \begin{bmatrix} 1 & -\frac{1}{4}z \\ -4 & z + 1 \end{bmatrix}$$

$\overline{g_2} = \overline{1}$; thus, $g_2 = 1$, $q_{22} = \big(-4t^2 + 2ty + x^2\big)z \in M_2[z]$ and $q_{23} = \big(t^2 - \frac{1}{2}ty - \frac{1}{4}x^2\big)z^2 + \big(t^2 - \frac{1}{4}\big)z \in M_2[z]$, moreover, $\mathrm{Res}_z\, \big(z^2 + 2txz, 1 + (-4t^2 + 2ty + x^2)z\big) = 8t^3x - 2tx^3 - 4t^2xy + 1$, i.e.,

$$r_2 = 8t^3x - 2tx^3 - 4t^2xy + 1\,.$$

Step 2.3. With `CoCoA` we computed

$$Use\,R ::= Q[t,x,y];$$
$$I := Ideal(-2tx^3-4t^2xy+1, 8t^3x-2tx^3-4t^2xy+1);$$
$$GenRepr(1,I);$$
$$[4t^2x^6-x^8-2tx^6y+2tx^3+4t^2xy-2x^3y-2txy^2+1,$$
$$x^8+2tx^6y+2x^3y+2txy^2],$$

hence $\langle r_1,r_2\rangle = R$ and $1 = r_1w_1 + r_2w_2$ with

$$w_1 = 4t^2x^6-x^8-2tx^6y+2tx^3+4t^2xy-2x^3y-2txy^2+1\,,$$
$$w_2 = x^8+2tx^6y+2x^3y+2txy^2\,.$$

Step 2.4. We have

$$b_0 = 0\,, \qquad b_1 = r_1w_1z\,, \qquad b_2 = r_1w_1z + r_2w_2z = z\,.$$

Step 2.5. According to the proof of Lemma 19, $\boldsymbol{f}(b_2) = \boldsymbol{f}(z) = \boldsymbol{f} = \boldsymbol{f}(b_1)U_2$ and $\boldsymbol{f}(b_1) = \boldsymbol{f}(b_0)U_1 = \boldsymbol{f}(0)U_1$, where U_2 and U_1 must be computed with Lemma 18. Hence, $\boldsymbol{f}U' = \boldsymbol{f}(0)$, with $U' := U_2^{-1}U_1^{-1}$. With the notation of Lemma 19 we have

$$U_1 := \begin{bmatrix} 1 & 0 \\ 0 & U_1'(0) \end{bmatrix} C_1 \begin{bmatrix} \widetilde{U}_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_1'(b_1)^{-1} \end{bmatrix}$$

and

$$U_2 := \begin{bmatrix} 1 & 0 \\ 0 & U_2'(b_1) \end{bmatrix} C_2 \begin{bmatrix} \widetilde{U}_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U_2'(b_2)^{-1} \end{bmatrix}.$$

Thus,

$$U_1'(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U_1'(b_1)^{-1}\,;$$

moreover,

$$\boldsymbol{f}(0) \begin{bmatrix} 1 & 0 \\ 0 & U_1'(0) \end{bmatrix} C_1 = \begin{bmatrix} f_1(0) & \widetilde{g}_1(0) & q_{13}(b_1) \end{bmatrix},$$

but $\boldsymbol{f}(0) = (0,1,0)$, so $(0,1,0)C_1 = \begin{bmatrix} 0 \ 1 \ t^2b_1 \end{bmatrix}$ since $f_1(0) = 0$, $\widetilde{g}_1(0) = 1$ and $q_{13}(b_1) = t^2b_1$. We can take

$$C_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t^2b_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, for $\widetilde{U}_1$ we have

$$\begin{bmatrix} f_1(0) & \widetilde{g}_1(0) \end{bmatrix} = \begin{bmatrix} f_1(b_1) & \widetilde{g}_1(b_1) \end{bmatrix},$$

Lemma 18 gives a procedure for computing $\widetilde{U}_1$, let

$$\widetilde{U}_1 = \begin{bmatrix} \widetilde{u}_{11} & \widetilde{u}_{12} \\ \widetilde{u}_{21} & \widetilde{u}_{22} \end{bmatrix};$$

using the proof of Lemma 18 with $b = b_0 = 0$, $s = r_1$, $d = w_1 z$, we have

$$\begin{aligned}
\widetilde{u}_{11} &= 1 + s_1(0, r_1, w_1 z) p_1(0) + t_2(0, r_1, w_1 z) \widetilde{g}_1(0), \\
\widetilde{u}_{21} &= s_1(0, r_1, w_1 z) h_1(0) - t_2(0, r_1, w_1 z) f_1(0), \\
\widetilde{u}_{12} &= s_2(0, r_1, w_1 z) p_1(0) - t_1(0, r_1, w_1 z) \widetilde{g}_1(0), \\
\widetilde{u}_{22} &= 1 + s_2(0, r_1, w_1 z) h_1(0) + t_1(0, r_1, w_1 z) f_1(0),
\end{aligned}$$

where $p_1 f_1 + h_1(g_1 + q_{12}) = r_1$ and $p_1, h_1, s_1, s_2, t_1, t_2$ are some polynomials that we must compute. With `CoCoA` we computed

$$\begin{aligned}
p_1 &= x^4 + 4tx^2 y + 4t^2 y^2 \\
h_1 &= -2tx^3 - 4t^2 xy - x^2 z - 2tyz + 1.
\end{aligned}$$

Since $\widetilde{g}_1(0) = 1$, $f_1(0) = 0$, $p_1(0) = p_1$ and $h_1(0) = r_1$, by Lemma 18

$$\begin{aligned}
s_1 &= \frac{f_1(b_1) - f_1(b_0)}{r_1} = w_1 z(r_1 w_1 z + 2tx), \\
s_2 &= \frac{\widetilde{g}_1(b_1) - \widetilde{g}_1(b_0)}{r_1} = w_1 z(x^2 + 2ty), \\
t_1 &= \frac{p_1(b_1) - p_1(b_0)}{r_1} = 0, \\
t_2 &= \frac{h_1(b_1) - h_1(b_0)}{r_1} = -w_1 z(x^2 + 2ty).
\end{aligned}$$

From all of these computations we conclude that

$$\begin{aligned}
\widetilde{u}_{11} &= 1 + w_1 z(r_1 w_1 z + 2tx)(x^2 + 2ty)^2 - w_1 z(x^2 + 2ty), \\
\widetilde{u}_{21} &= r_1 w_1 z(r_1 w_1 z + 2tx), \\
\widetilde{u}_{12} &= w_1 z(x^2 + 2ty)^3, \\
\widetilde{u}_{22} &= 1 + r_1 w_1 z(x^2 + 2ty),
\end{aligned}$$

and

$$U_1 = \begin{bmatrix} \widetilde{u}_{11} & \widetilde{u}_{12} & 0 \\ \widetilde{u}_{21} & \widetilde{u}_{22} & t^2 b_1 \\ 0 & 0 & 1 \end{bmatrix};$$

we observe that $\det(U_1) = 1$ and

$$U_1^{-1} = \begin{bmatrix} \widetilde{u}_{22} & -\widetilde{u}_{12} & \widetilde{u}_{12} t^2 b_1 \\ -\widetilde{u}_{21} & \widetilde{u}_{11} & -\widetilde{u}_{11} t^2 b_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we will compute U_2^{-1}. We start with C_2, we have

$$\begin{bmatrix} f_1(b_1) & \widetilde{g}_2(b_1) & q_{23}(b_1) \end{bmatrix} C_2 = \begin{bmatrix} f_1(b_1) & \widetilde{g}_2(b_1) & q_{23}(b_2) \end{bmatrix};$$

since $q_{23}(b_2) = q_{23}(b_1)+$ a linear combination of $f_1(b_1)$ and $\widetilde{g}_2(b_1)$ over $R[z]$, we conclude that the form of C_2 is

$$C_2 = \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix},$$

where p, q are polynomials that we must compute; we get that

$$q_{23}(b_2) = p f_1(b_1) + q \widetilde{g}_2(b_1) + q_{23}(b_1).$$

Expressing the previous relation in terms of t, x, y, z and using CoCoA we found that
$p = 16t^6x^8z^2 - 12t^4x^{10}z^2 + 3t^2x^{12}z^2 - \frac{1}{4}x^{14}z^2 + 32t^7x^6yz^2 - 48t^5x^8yz^2 + 18t^3x^{10}yz^2 - 2tx^{12}yz^2 - 48t^6x^6y^2z^2 + 36t^4x^8y^2z^2 - 6t^2x^{10}y^2z^2 + 24t^5x^6y^3z^2 - 8t^3x^8y^3z^2 - 4t^4x^6y^4z^2 + 16t^6x^8z - 8t^4x^{10}z + t^2x^{12}z + 32t^7x^6yz - 32t^5x^8yz + 6t^3x^{10}yz - 32t^6x^6y^2z + 12t^4x^8y^2z + 8t^5x^6y^3z + 32t^6x^3yz^2 - 24t^4x^5yz^2 + 6t^2x^7yz^2 - \frac{1}{2}x^9yz^2 + 32t^7xy^2z^2 - 72t^5x^3y^2z^2 + 30t^3x^5y^2z^2 - \frac{7}{2}tx^7y^2z^2 - 48t^6xy^3z^2 + 48t^4x^3y^3z^2 - 9t^2x^5y^3z^2 + 24t^5xy^4z^2 - 10t^3x^3y^4z^2 - 4t^4xy^5z^2 + 32t^6x^3yz - 16t^4x^5yz + 2t^2x^7yz + 32t^7xy^2z - 48t^5x^3y^2z + 10t^3x^5y^2z - 32t^6xy^3z + 16t^4x^3y^3z + 8t^5xy^4z$,
or factoring, we have

$$p = \frac{1}{4}xz(x^7 + 2tx^5y + 2x^2y + 2ty^2)(4t^2z - x^2z - 2tyz + 4t^2)(4t^2 - x^2 - 2ty)^2$$

and

$$q = -\frac{1}{4}xz(x^7 + 2tx^5y + 2x^2y + 2ty^2)q'$$

with

$q' = 128t^7x^9z^2 - 96t^5x^{11}z^2 + 24t^3x^{13}z^2 - 2tx^{15}z^2 + 256t^8x^7yz^2 - 384t^6x^9yz^2 + 144t^4x^{11}yz^2 - 16t^2x^{13}yz^2 - 384t^7x^7y^2z^2 + 288t^5x^9y^2z^2 - 48t^3x^{11}y^2z^2 + 192t^6x^7y^3z^2 - 64t^4x^9y^3z^2 - 32t^5x^7y^4z^2 + 128t^7x^9z - 64t^5x^{11}z + 8t^3x^{13}z + 256t^8x^7yz - 256t^6x^9yz + 48t^4x^{11}yz - 256t^7x^7y^2z + 96t^5x^9y^2z + 64t^6x^7y^3z + 16t^4x^8z^2 - 8t^2x^{10}z^2 + x^{12}z^2 + 256t^7x^4yz^2 - 160t^5x^6yz^2 + 16t^3x^8yz^2 + 2tx^{10}yz^2 + 256t^8x^2y^2z^2 - 576t^6x^4y^2z^2 + 208t^4x^6y^2z^2 - 16t^2x^8y^2z^2 - 384t^7x^2y^3z^2 + 384t^5x^4y^3z^2 - 64t^3x^6y^3z^2 + 192t^6x^2y^4z^2 - 80t^4x^4y^4z^2 - 32t^5x^2y^5z^2 + 16t^4x^8z - 4t^2x^{10}z + 256t^7x^4yz - 96t^5x^6yz + 256t^8x^2y^2z - 384t^6x^4y^2z + 64t^4x^6y^2z - 256t^7x^2y^3z + 128t^5x^4y^3z + 64t^6x^2y^4z + 32t^4x^3yz^2 - 16t^2x^5yz^2 + 2x^7yz^2 + 32t^5xy^2z^2 - 48t^3x^3y^2z^2 + 10tx^5y^2z^2 - 32t^4xy^3z^2 + 16t^2x^3y^3z^2 + 8t^3xy^4z^2 + 32t^4x^3yz - 8t^2x^5yz + 32t^5xy^2z - 24t^3x^3y^2z - 16t^4xy^3z - 32t^5xz + 16t^3x^3z - 2tx^5z + 32t^4xyz - 8t^2x^3yz - 8t^3xy^2z - 32t^5x + 8t^3x^3 + 16t^4xy - 16t^4z^2 + 8t^2x^2z^2 - x^4z^2 + 16t^3yz^2 - 4tx^2yz^2 - 4t^2y^2z^2 - 16t^4z + 4t^2x^2z + 8t^3yz + 8t^3x - 2tx^3 - 4t^2xy - 4t^2z + x^2z + 2tyz - 4t^2 + 1.$

For $\widetilde{U}_2$, let

$$\widetilde{U}_2 = \begin{bmatrix} \widetilde{v}_{11} & \widetilde{v}_{12} \\ \widetilde{v}_{21} & \widetilde{v}_{22} \end{bmatrix},$$

by the proof of Lemma 18 with $b = b_1$, $s = r_2$ and $d = w_2z$, we have

$$\begin{aligned} \widetilde{v}_{11} &= 1 + s_1'(b_1, r_2, w_2z)p_1'(b_1) + t_2'(b_1, r_2, w_2z)\widetilde{g}_2(b_1)\,, \\ \widetilde{v}_{21} &= s_1'(b_1, r_2, w_2z)h_1'(b_1) - t_2'(b_1, r_2, w_2z)f_1(b_1)\,, \\ \widetilde{v}_{12} &= s_2'(b_1, r_2, w_2z)p_1'(b_1) - t_1'(b_1, r_2, w_2z)\widetilde{g}_2(b_1)\,, \\ \widetilde{v}_{22} &= 1 + s_2'(b_1, r_2, w_2z)h_1'(b_1) + t_1'(b_1, r_2, w_2z)f_1(b_1)\,, \end{aligned}$$

where $p_1'f_1 + h_1'(g_2 + q_{22}) = r_2$ and $p_1', h_1', s_1', s_2', t_1', t_2'$ are some polynomials that we must compute. With `CoCoA` we found that

$$\begin{aligned} p_1' &= 16t^4 - 8t^2x^2 + x^4 - 16t^3y + 4tx^2y + 4t^2y^2 = (4t^2 - x^2 - 2ty)^2\,, \\ h_1' &= 8t^3x - 2tx^3 - 4t^2xy + 4t^2z - x^2z - 2tyz + 1\,. \end{aligned}$$

Moreover, with `CoCoA` we computed

$$s_1' = \frac{f_1(b_2) - f_1(b_1)}{r_2} = -xz(x^7 + 2tx^5y + 2x^2y + 2ty^2)s_1''\,,$$

where

$$\begin{aligned} s_1'' := \ & 8t^3x^9z - 2tx^{11}z + 16t^4x^7yz - 8t^2x^9yz - 8t^3x^7y^2z + x^8z \\ & + 16t^3x^4yz - 2tx^6yz + 16t^4x^2y^2z - 12t^2x^4y^2z \\ & - 8t^3x^2y^3z + 2x^3yz + 2txy^2z - 2tx - 2z \,. \end{aligned}$$

In a similar way we get that

$$\begin{aligned} s_2' &= \frac{\widetilde{g}_2(b_2) - \widetilde{g}_2(b_1)}{r_2} = -xz(x^7 + 2tx^5y + 2x^2y + 2ty^2)(4t^2 - x^2 - 2ty) \,, \\ t_1' &= \frac{p_1'(b_2) - p_1'(b_1)}{r_2} = 0 \,, \\ t_2' &= \frac{h_1'(b_2) - h_1'(b_1)}{r_2} = xz(x^7 + 2tx^5y + 2x^2y + 2ty^2)(4t^2 - x^2 - 2ty) \,. \end{aligned}$$

We have proved that $\widetilde{U}_2$ is effectively computable, and consequently, we have computed U_2^{-1}: the columns of U_2^{-1} are

$$C_1'' = \begin{bmatrix} \widetilde{v}_{22} \\ -\widetilde{v}_{21} \\ 4\widetilde{v}_{21} \end{bmatrix}, \qquad C_2'' = \begin{bmatrix} -(b_1 - 4q + 1)\widetilde{v}_{12} - 4p\widetilde{v}_{22} \\ (b_1 - 4q + 1)\widetilde{v}_{11} + 4p\widetilde{v}_{21} - z \\ -4((b_1 - 4q + 1)\widetilde{v}_{11} + 4p\widetilde{v}_{21} - z - 1) \end{bmatrix},$$

and

$$C_3'' = \begin{bmatrix} -\frac{1}{4}((b_1 - 4q)\widetilde{v}_{12} + 4p\widetilde{v}_{22}) \\ \frac{1}{4}((b_1 - 4q)\widetilde{v}_{11} + 4p\widetilde{v}_{21} - z) \\ -((b_1 - 4q)\widetilde{v}_{11} + 4p\widetilde{v}_{21} - z - 1) \end{bmatrix}.$$

Then $U' = U_2^{-1}U_1^{-1}$ and its columns are

$$C_1' = \begin{bmatrix} \widetilde{u}_{22}\widetilde{v}_{22} + \widetilde{u}_{21}(4p\widetilde{v}_{22} + \widetilde{v}_{12}(b_1 - 4q + 1)) \\ -\widetilde{u}_{22}\widetilde{v}_{21} - \widetilde{u}_{21}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1)) \\ 4(\widetilde{u}_{22}\widetilde{v}_{21} + \widetilde{u}_{21}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1) - 1)) \end{bmatrix},$$

$$C_2' = \begin{bmatrix} -\widetilde{u}_{12}\widetilde{v}_{22} - \widetilde{u}_{11}(4p\widetilde{v}_{22} + \widetilde{v}_{12}(b_1 - 4q + 1)) \\ \widetilde{u}_{12}\widetilde{v}_{21} + \widetilde{u}_{11}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1)) \\ -4(\widetilde{u}_{12}\widetilde{v}_{21} + \widetilde{u}_{11}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1) - 1)) \end{bmatrix},$$

$$C_3' = \begin{bmatrix} t^2b_1\widetilde{u}_{12}\widetilde{v}_{22} - \frac{1}{4}\widetilde{v}_{12}(b_1 - 4q) - p\widetilde{v}_{22} + t^2b_1\widetilde{u}_{11}(4p\widetilde{v}_{22} + \widetilde{v}_{12}(b_1 - 4q + 1)) \\ p\widetilde{v}_{21} - \frac{1}{4}z + \frac{1}{4}\widetilde{v}_{11}(b_1 - 4q) - t^2b_1\widetilde{u}_{12}\widetilde{v}_{21} - t^2b_1\widetilde{u}_{11}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1)) \\ z - 4p\widetilde{v}_{21} - \widetilde{v}_{11}(b_1 - 4q) + 4t^2b_1\widetilde{u}_{12}\widetilde{v}_{21} + 4t^2b_1\widetilde{u}_{11}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1) - 1) + 1 \end{bmatrix}.$$

With `CoCoA` we checked that $\det(U') = 1$.

Step 3. Since $\boldsymbol{f}(0) = (0, 1, 0)$, then with notation in the proof of Theorem 20,

$$U'' := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and hence,

$$U = U'U'' = \begin{bmatrix} C'_2 & C'_1 & C'_3 \end{bmatrix}.$$

Thus, if

$$U := \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix},$$

then

$$\begin{aligned}
u_{11} &= -\widetilde{u}_{12}\widetilde{v}_{22} - \widetilde{u}_{11}(4p\widetilde{v}_{22} + \widetilde{v}_{12}(b_1 - 4q + 1))\,, \\
u_{21} &= \widetilde{u}_{12}\widetilde{v}_{21} + \widetilde{u}_{11}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1))\,, \\
u_{31} &= -4(\widetilde{u}_{12}\widetilde{v}_{21} + \widetilde{u}_{11}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1) - 1)) \\
&= -4u_{21} + 4\widetilde{u}_{11}\,, \\
u_{12} &= \widetilde{u}_{22}\widetilde{v}_{22} + \widetilde{u}_{21}(4p\widetilde{v}_{22} + \widetilde{v}_{12}(b_1 - 4q + 1))\,, \\
u_{22} &= -\widetilde{u}_{22}\widetilde{v}_{21} - \widetilde{u}_{21}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1))\,, \\
u_{32} &= 4(\widetilde{u}_{22}\widetilde{v}_{21} + \widetilde{u}_{21}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1) - 1)) \\
&= -4u_{22} - 4\widetilde{u}_{21}\,, \\
u_{13} &= t^2 b_1 \widetilde{u}_{12}\widetilde{v}_{22} - \frac{1}{4}\widetilde{v}_{12}(b_1 - 4q) - p\widetilde{v}_{22} \\
&\qquad + t^2 b_1 \widetilde{u}_{11}(4p\widetilde{v}_{22} + \widetilde{v}_{12}(b_1 - 4q + 1))\,, \\
u_{23} &= p\widetilde{v}_{21} - \frac{1}{4}z + \frac{1}{4}\widetilde{v}_{11}(b_1 - 4q) - t^2 b_1 \widetilde{u}_{12}\widetilde{v}_{21} \\
&\qquad - t^2 b_1 \widetilde{u}_{11}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1))\,, \\
u_{33} &= z - 4p\widetilde{v}_{21} - \widetilde{v}_{11}(b_1 - 4q) + 4t^2 b_1 \widetilde{u}_{12}\widetilde{v}_{21} \\
&\qquad + 4t^2 b_1 \widetilde{u}_{11}(4p\widetilde{v}_{21} - z + \widetilde{v}_{11}(b_1 - 4q + 1) - 1) + 1 \\
&= -4u_{23} - 4t^2 b_1 \widetilde{u}_{11} + 1\,.
\end{aligned}$$

EXAMPLE 24. In the previous example we can take $y = 0$ and calculate the concrete entries of U. Thus,

$$\boldsymbol{f} = \big(z^2 + 2txz, x^2z + 1, t^2z\big) \in \big(\mathbb{Q}[t, x, z]\big)^3.$$

In this case we have

$$r_1 = -2tx^3+1\,, \quad r_2 = 8t^3x-2tx^3+1\,, \quad w_1 = 4t^2x^6-x^8+2tx^3+1\,, \quad w_2 = x^8\,.$$

We observe that $r_1w_1 + r_2w_2 = 1$. Moreover,

$$b_1 = r_1w_1z = \big(-2tx^3 + 1\big)\big(4t^2x^6 - x^8 + 2tx^3 + 1\big)z\,.$$

From this we get

$$\begin{aligned}
\widetilde{u}_{11} &= -32t^5x^{19}z^2 + 16t^3x^{21}z^2 - 2tx^{23}z^2 - 16t^4x^{16}z^2 \\
&\quad + x^{20}z^2 - 8t^3x^{13}z^2 + 8t^3x^{11}z - 2tx^{13}z + 4t^2x^{10}z^2 \\
&\quad - 2x^{12}z^2 + x^{10}z + 2tx^7z^2 + x^4z^2 - x^2z + 1\,, \\
\widetilde{u}_{21} &= 64t^6x^{18}z^2 - 32t^4x^{20}z^2 + 4t^2x^{22}z^2 \\
&\quad + 16t^3x^{17}z^2 - 4tx^{19}z^2 + x^{16}z^2 - 16t^4x^{10}z + 4t^2x^{12}z \\
&\quad - 16t^3x^9z^2 + 4tx^{11}z^2 - 2tx^9z - 2x^8z^2 + 2txz + z^2\,, \\
\widetilde{u}_{12} &= 4t^2x^{12}z - x^{14}z + 2tx^9z + x^6z\,, \\
\widetilde{u}_{22} &= -8t^3x^{11}z + 2tx^{13}z - x^{10}z + x^2z + 1\,.
\end{aligned}$$

With `CoCoA` we checked that $\widetilde{u}_{11}\widetilde{u}_{22} - \widetilde{u}_{21}\widetilde{u}_{12} = 1$.

On the other hand,

$$\begin{aligned}
p &= \frac{1}{4}x^8z\big(4t^2z - x^2z + 4t^2\big)\big(4t^2 - x^2\big)^2\,, \\
q &= -\frac{1}{4}x^8zq'\,,
\end{aligned}$$

with
$q' = 128t^7x^9z^2 - 96t^5x^{11}z^2 + 24t^3x^{13}z^2 - 2tx^{15}z^2 + 128t^7x^9z - 64t^5x^{11}z + 8t^3x^{13}z + 16t^4x^8z^2 - 8t^2x^{10}z^2 + x^{12}z^2 + 16t^4x^8z - 4t^2x^{10}z - 32t^5xz + 16t^3x^3z - 2tx^5z - 32t^5x + 8t^3x^3 - 16t^4z^2 + 8t^2x^2z^2 - x^4z^2 - 16t^4z + 4t^2x^2z + 8t^3x - 2tx^3 - 4t^2z + x^2z - 4t^2 + 1$.

Moreover,

$$\begin{aligned}
\widetilde{v}_{11} &= 32t^5x^9z - 16t^3x^{11}z + 2tx^{13}z + 16t^4x^8z^2 - 8t^2x^{10}z^2 \\
&\quad + x^{12}z^2 + 4t^2x^8z - x^{10}z + 1\,, \\
\widetilde{v}_{21} &= -64t^6x^{18}z^2 + 32t^4x^{20}z^2 - 4t^2x^{22}z^2 - 32t^5x^{17}z^3 \\
&\quad + 16t^3x^{19}z^3 - 2tx^{21}z^3 - 16t^3x^{17}z^2 + 4tx^{19}z^2 - 4t^2x^{16}z^3 \\
&\quad + x^{18}z^3 - x^{16}z^2 + 16t^4x^{10}z - 4t^2x^{12}z + 16t^3x^9z^2 - 4tx^{11}z^2 \\
&\quad + 4t^2x^8z^3 - x^{10}z^3 + 2tx^9z + 2x^8z^2\,, \\
\widetilde{v}_{12} &= -64t^6x^8z + 48t^4x^{10}z - 12t^2x^{12}z + x^{14}z, \\
\widetilde{v}_{22} &= 128t^7x^{17}z^2 - 96t^5x^{19}z^2 + 24t^3x^{21}z^2 - 2tx^{23}z^2 \\
&\quad + 16t^4x^{16}z^2 - 8t^2x^{18}z^2 + x^{20}z^2 - 32t^5x^9z + 16t^3x^{11}z - 2tx^{13}z \\
&\quad - 16t^4x^8z^2 + 8t^2x^{10}z^2 - x^{12}z^2 - 4t^2x^8z + x^{10}z + 1\,.
\end{aligned}$$

With `CoCoA` we also checked that $\widetilde{v}_{11}\widetilde{v}_{22} - \widetilde{v}_{21}\widetilde{v}_{12} = 1$. Finally, with the notation of the previous example we have

$$\begin{aligned}
u_{11} &= 64t^6x^{18}z^2 - 32t^4x^{20}z^2 + 4t^2x^{22}z^2 + 32t^5x^{15}z^2 - 8t^3x^{17}z^2 \\
&\quad + 16t^4x^{12}z^2 - 4t^2x^{14}z^2 - 16t^4x^{10}z + 4t^2x^{12}z - 2tx^9z - x^6z\,, \\
u_{21} &= 16t^4x^{18}z^3 - 4t^2x^{20}z^3 + 8t^3x^{15}z^3 \\
&\quad + 4t^2x^{12}z^3 + 2tx^7z^2 + x^4z^2 - x^2z + 1\,, \\
u_{31} &= -128t^5x^{19}z^2 + 64t^3x^{21}z^2 - 8tx^{23}z^2 - 64t^4x^{18}z^3 \\
&\quad + 16t^2x^{20}z^3 - 64t^4x^{16}z^2 + 4x^{20}z^2 - 32t^3x^{15}z^3 - 32t^3x^{13}z^2 \\
&\quad - 16t^2x^{12}z^3 + 32t^3x^{11}z - 8tx^{13}z + 16t^2x^{10}z^2 - 8x^{12}z^2 + 4x^{10}z\,.
\end{aligned}$$

We have checked that $f_1u_{11} + f_2u_{21} + f_3u_{31} = 1$.

$$\begin{aligned}
u_{12} &= 128t^7x^{17}z^2 - 64t^5x^{19}z^2 + 8t^3x^{21}z^2 + 16t^4x^{16}z^2 \\
&\quad - 4t^2x^{18}z^2 - 32t^5x^9z + 8t^3x^{11}z - 16t^4x^8z^2 \\
&\quad + 4t^2x^{10}z^2 - 4t^2x^8z + x^2z + 1\,, \\
u_{22} &= 32t^5x^{17}z^3 - 8t^3x^{19}z^3 + 4t^2x^{16}z^3 - 4t^2x^8z^3 - 2txz - z^2\,,
\end{aligned}$$

$$\begin{aligned} u_{32} = &-256t^6x^{18}z^2 + 128t^4x^{20}z^2 - 16t^2x^{22}z^2 - 128t^5x^{17}z^3 \\ &+ 32t^3x^{19}z^3 - 64t^3x^{17}z^2 + 16tx^{19}z^2 - 16t^2x^{16}z^3 \\ &- 4x^{16}z^2 + 64t^4x^{10}z - 16t^2x^{12}z + 64t^3x^9z^2 \\ &- 16tx^{11}z^2 + 16t^2x^8z^3 + 8tx^9z + 8x^8z^2\,. \end{aligned}$$

We have checked that $f_1u_{12} + f_2u_{22} + f_3u_{32} = 0$.

$$\begin{aligned} u_{13} = {}&512t^{11}x^{27}z^3 - 384t^9x^{29}z^3 + 96t^7x^{31}z^3 - 8t^5x^{33}z^3 + 256t^{10}x^{24}z^3 \\ &- 64t^8x^{26}z^3 - 16t^6x^{28}z^3 + 4t^4x^{30}z^3 + 128t^9x^{21}z^3 - 32t^7x^{23}z^3 \\ &- 128t^9x^{19}z^2 + 64t^7x^{21}z^2 - 8t^5x^{23}z^2 - 64t^8x^{18}z^3 + 48t^6x^{20}z^3 \\ &- 8t^4x^{22}z^3 - 32t^6x^{18}z^2 + 8t^4x^{20}z^2 - 32t^7x^{15}z^3 + 8t^5x^{17}z^3 \\ &- 8t^5x^{15}z^2 - 16t^6x^{12}z^3 + 4t^4x^{14}z^3 + 16t^6x^{10}z^2 - 4t^4x^{12}z^2 \\ &- t^2x^{14}z^2 - 16t^6x^8z + 8t^4x^{10}z - t^2x^{12}z + 2t^3x^9z^2 + t^2x^6z^2\,, \\ u_{23} = {}&128t^9x^{27}z^4 - 64t^7x^{29}z^4 + 8t^5x^{31}z^4 + 64t^8x^{24}z^4 - 4t^4x^{28}z^4 \\ &+ 32t^7x^{21}z^4 - 16t^6x^{18}z^4 + 8t^4x^{20}z^4 + 16t^6x^{16}z^3 - 4t^4x^{18}z^3 \\ &- 8t^5x^{15}z^4 + 8t^5x^{13}z^3 - 4t^4x^{12}z^4 - 8t^5x^{11}z^2 + 2t^3x^{13}z^2 \\ &+ t^2x^{12}z^3 - 4t^4x^8z^2 - 2t^3x^7z^3 - t^2x^4z^3 + t^2x^2z^2 - t^2z\,, \\ u_{33} = {}&-1024t^{10}x^{28}z^3 + 768t^8x^{30}z^3 - 192t^6x^{32}z^3 + 16t^4x^{34}z^3 - 512t^9x^{27}z^4 \\ &+ 256t^7x^{29}z^4 - 32t^5x^{31}z^4 - 512t^9x^{25}z^3 + 96t^5x^{29}z^3 - 16t^3x^{31}z^3 \\ &- 256t^8x^{24}z^4 + 16t^4x^{28}z^4 - 256t^8x^{22}z^3 + 4t^2x^{28}z^3 - 128t^7x^{21}z^4 \\ &+ 256t^8x^{20}z^2 - 128t^6x^{22}z^2 + 16t^4x^{24}z^2 + 256t^7x^{19}z^3 - 192t^5x^{21}z^3 \\ &+ 24t^3x^{23}z^3 + 64t^6x^{18}z^4 - 32t^4x^{20}z^4 + 64t^5x^{19}z^2 - 16t^3x^{21}z^2 \\ &+ 64t^6x^{16}z^3 + 16t^4x^{18}z^3 - 12t^2x^{20}z^3 + 32t^5x^{15}z^4 + 4t^2x^{18}z^2 \\ &+ 32t^5x^{13}z^3 + 16t^4x^{12}z^4 - 32t^5x^{11}z^2 + 8t^3x^{13}z^2 - 16t^4x^{10}z^3 \\ &+ 8t^2x^{12}z^3 + 32t^5x^9z - 8t^3x^{11}z + 16t^4x^8z^2 - 8t^2x^{10}z^2 + 4t^2x^8z + 1\,. \end{aligned}$$

We have checked that $f_1u_{13} + f_2u_{23} + f_3u_{33} = 0$.

The above computations show that $\boldsymbol{f}U = (1, 0, 0)$.

Remark 25. In [7] has been recently implemented the package `QUILLENSUSLIN` developed in the computer algebra system `MAPLE`, that will appear soon. The main functions of the package `QUILLENSUSLIN` are: compute a unimodular matrix U which transforms a row vector admitting a right-inverse into a matrix of the form $[\,I \;\; 0\,]$; complete a matrix admitting a right-inverse to a unimodular matrix; compute a basis of a free module finitely presented by a given matrix.

We conclude this section commenting some recent generalizations of the Quillen-Suslin theorem: Gago-Vargas in [11] extended the algorithmic proofs of the Quillen-Suslin theorem to coefficients in a *PID* with some additional computational conditions. In [14] Gubeladze presented a non algorithmic proof that the monoid ring $D[M]$ is *PF*, where D is a *PID* and M is a certain type of commutative monoid; in [22] is presented an algorithmic proof of the Gubeladze's generalization for fields. In [20] is presented an algorithmic proof for quotients rings of $K[x_1, \ldots, x_n]$ by monomials ideals. When D is a *PID*, quotients rings of $D[M]$ by monomials ideals are also *PF*. A non algorithmic proof of this fact is given in [32]. According to these results arise the following problem-conjecture.

Conjecture 26. The constructive proofs in [11] and [20] can be extended to $D[M]/I$, where D is a *PID* and M is a commutative, seminormal, finitely generated monoid, which is torsion free, cancellative, and has no nontrivial units.

4. Extended rings and the Bass-Quillen conjecture

If we consider finitely generated projective modules over arbitrary commutative polynomial rings, is natural to ask if the Quillen-Suslin theorem also holds, i.e., if S is an arbitrary commutative ring, we ask if S is a *QS* ring. Related with this question are defined the extended modules and the correspondent extended rings (see [18]). In this section we will study these topics and some related conjectures.

Definition 27. Let S be a commutative ring and B a S-algebra. Let M be a B-module, M es extended from S if there exists a S-module M_0 such that $M \cong M_0 \otimes_S B$.

With the notation of the previous definition and setting

$$S[X] := S[x_1, \ldots, x_n] \qquad \text{and} \qquad \langle X \rangle := \langle x_1, \ldots, x_n \rangle \,,$$

we have the following properties.

Proposition 28. (i) *If M is free over B, then M is extended from S.*

(ii) *If $B = S[X]$ and M is extended from S, then*

$$M_0 \cong M/\langle X\rangle M\,.$$

Moreover, if M is finitely generated (projective) as B-module, then M_0 is finitely generated (projective) as S-module.

Proof. (i) If $M \cong B^{(Y)}$, then $M \cong S^{(Y)} \otimes_S B$.
(ii) If $M \cong M_0 \otimes_S S[X]$ then

$$\begin{aligned} M \otimes_{S[X]} S[X]/\langle X\rangle &\cong M_0 \otimes_S S[X] \otimes_{S[X]} S[X]/\langle X\rangle\,,\\ M/\langle X\rangle M &\cong M_0 \otimes_S S[X]/\langle X\rangle\,,\\ M/\langle X\rangle M &\cong M_0 \otimes_S S\,,\\ M/\langle X\rangle M &\cong M_0\,. \end{aligned}$$

Let $M = \langle z_1, \dots, z_t\rangle$ and $w \in M_0$, then $w = \overline{z}$ with $z \in M$; there exist polynomials $p_1(X), \dots, p_t(X) \in S[X]$ such that $w = \overline{z} = \overline{z_1 p_1(X) + \cdots + z_t p_t(X)} = \overline{z_1} p_{01} + \cdots + \overline{z_t} p_{0t}$, where p_{0i} is the independent term of $p_i(X)$, $1 \le i \le t$. Hence, $M_0 = \langle \overline{z_1}, \dots, \overline{z_t}\rangle$.

Finally, let $M \oplus M' = (S[X])^{(Y)}$, then

$$\begin{aligned} (M \oplus M') \otimes_{S[X]} S[X]/\langle X\rangle &\cong (S[X])^{(Y)} \otimes_{S[X]} S[X]/\langle X\rangle\,,\\ M_0 \oplus M'/\langle X\rangle M' &\cong S^{(Y)}\,. \end{aligned}$$

∎

Extended modules are close related with QS rings as is showed in the following results (see [18]).

Definition 29. Let S be a commutative ring.

(i) Let $n \ge 1$, S is a E_n ring if every f.g. projective $S[x_1, \dots, x_n]$-module is extended from S.

(ii) S is an extended ring E, if S is E_n for each $n \ge 1$.

From Proposition 28 we get the following consequences:

$$E = \bigcap_{n\geq 1} E_n\,, \tag{4.1}$$

$$QS_n \subseteq E_n \quad \text{for each } n \geq 1\,, \tag{4.2}$$

$$QS \subseteq E\,. \tag{4.3}$$

The following results are announced without proof in [18, p. 166].

Theorem 30. (i) *$QS = PF \cap E = PSF \cap H \cap E$.*

(ii) *Let S a PF ring. Then, for each $n \geq 1$, S is E_n if and only if $S[x_1, \ldots, x_n]$ is PF. In other words, for PF rings $E_n = QS_n$ for each $n \geq 1$, and consequently, $E = QS$.*

(iii) *For each $n \geq 1$, if S is E_{n+1}, then S and $S[x]$ are E_n.*

(iv) *$E \subseteq \cdots \subseteq E_{n+1} \subseteq E_n \subseteq \cdots \subseteq E_1$.*

Proof. (i) First we will prove that $QS \subseteq PF \cap E$. By (4.3), $QS \subseteq E$; let M be a S-f.g. projective module, then $M \oplus M' \cong S^m$ and hence

$$(M \oplus M') \otimes_S S[X] \cong S^m \otimes_S S[X]\,,$$
$$(M \otimes_S S[X] \oplus M' \otimes_S S[X]) \cong (S \otimes_S S[X])^m \cong S[X]^m\,;$$

this means that $M \otimes_S S[X]$ is a $S[X]$-f.g. projective module, so by the hypothesis $M \otimes_S S[X]$ is a $S[X]$-free module, i.e., $M \otimes_S S[X] \cong S[X]^k$, for some $k \geq 0$. From this we get

$$M \otimes_S S[X] \otimes_{S[X]} S[X]/\langle X\rangle \cong S[X]^k \otimes_{S[X]} S[X]/\langle X\rangle\,,$$
$$M \otimes_S (S[X] \otimes_{S[X]} S[X]/\langle X\rangle) \cong (S[X] \otimes_{S[X]} S[X]/\langle X\rangle)^k\,,$$
$$M \otimes_S S[X]/\langle X\rangle \cong (S[X]/\langle X\rangle)^k\,,$$
$$M \otimes_S S \cong M \cong S^k\,.$$

This means that S is PF. Thus, $QS \subseteq PF$, and hence, $QS \subseteq PF \cap E$.

Now we will prove that $PF \cap E \subseteq QS$: let M be a $S[X]$-f.g. projective module, then M is extended from S and there exists a S-f.g. projective module M_0 such that $M \cong M_0 \otimes_S S[X]$. By the hypothesis, M is S-free, i.e., $M_0 \cong S^m$, for some $m \geq 0$. Hence, $M \cong S^m \otimes_S S[X] \cong S[X]^m$, i.e., M is $S[X]$-free. This prove that S is QS.

The second equality follows from (2.1).

(ii) "$\Rightarrow$" Let M be a f.g. projective $S[X]$-module, then M is extended from S and theres exists a f.g. projective S-module M_0 such that $M \cong M_0 \otimes_S S[X]$. We are assuimng that S is PF, then M_0 is S-free, and hence, M is $S[X]$-free. This proves that $E_n \subseteq QS_n$ for each $n \geq 1$. The converse is thrue because of (4.2).

(iii) Let M be a f.g. projective $S[X]$-module, the there exists a $S[X]$-module M' such that $M \oplus M' \cong (S[X])^m$ for some $m \geq 1$. From this we get

$$(M \oplus M') \otimes_{S[X]} S[X, x_{n+1}] \cong S[X]^m \otimes_{S[X]} S[X, x_{n+1}]\,,$$
$$M \otimes_{S[X]} S[X, x_{n+1}] \oplus M' \otimes_{S[X]} S[X, x_{n+1}] \cong S[X, x_{n+1}]^m\,.$$

This means that $M \otimes_{S[X]} S[X, x_{n+1}]$ is a f.g. projective $S[X, x_{n+1}]$-module; since S is E_{n+1}, there exists a S-module M_0 such that

$$M \otimes_{S[X]} S[X, x_{n+1}] \cong M_0 \otimes_S S[X, x_{n+1}]\,,$$

and from this we get

$$\begin{aligned} &M \otimes_{S[X]} S[X, x_{n+1}] \otimes_{S[X,x_{n+1}]} S[X, x_{n+1}]/\langle x_{n+1}\rangle \\ &\qquad \cong M_0 \otimes_S S[X, x_{n+1}] \otimes_{S[X,x_{n+1}]} S[X, x_{n+1}]/\langle x_{n+1}\rangle\,, \\ &M \otimes_{S[X]} \Big(S[X, x_{n+1}] \otimes_{S[X,x_{n+1}]} S[X, x_{n+1}]/\langle x_{n+1}\rangle\Big) \\ &\qquad \cong M_0 \otimes_S \Big(S[X, x_{n+1}] \otimes_{S[X,x_{n+1}]} S[X, x_{n+1}]/\langle x_{n+1}\rangle\Big)\,, \end{aligned}$$

i.e.,

$$M \otimes_{S[X]} S[X] \cong M \cong M_0 \otimes_S S[X]\,.$$

This means that M is extended from S, and hence, S is E_n.

Now let $B := S[x]$ and M be a f.g. projective $B[X]$-module, the there exists a $B[X]$-module M' such that $M \oplus M' \cong (B[X])^m$ for some $m \geq 1$. From this we get

$$\big(M \oplus M'\big) \otimes_{B[X]} S[X, x] \cong B[X]^m \otimes_{B[X]} S[X, x]\,,$$
$$M \otimes_{B[X]} S[X, x] \oplus M' \otimes_{B[X]} S[X, x] \cong \big(B[X] \otimes_{B[X]} S[X, x]\big)^m \cong S[X, x]^m\,.$$

But $B[X] = S[X, x]$, so
$$M \oplus M' \cong S[X, x]^m ,$$
i.e., M is a f.g. projective $S[X, x]$-module. This implies that M is extended from S; thus, there exists M_0 a f.g. projective S-module such that $M \cong M_0 \otimes_S S[X, x]$. Hence,
$$M \cong M_0 \otimes_S \big(S[x] \otimes_{S[x]} S[X, x]\big) ,$$
$$M \cong \big(M_0 \otimes_S S[x]\big) \otimes_{S[x]} S[x][X] ,$$
$$M \cong M_0' \otimes_B B[X] ,$$
with $M_0' := M_0 \otimes_S S[x] = M_0 \otimes_S B$.

(iv) This is consequence of (iii) and (4.1). ∎

From this theorem arise the following conjectures.

Conjectures 31. (i) For each $n \geq 1$, if S is E_n, then $S[x]$ is E_n.
(ii) If S is E, then $S[x]$ is E.
(iii) $E_1 = E_2$.
(iv) $E_1 = E_m$ for some $m \geq 2$.
(v) $E_1 = E$.

Related with these questions are the following properties (see [18]).

Proposition 32. *For a fixed integer $n \geq 1$, the following four statements are equivalent:*

(i) *Any ring satisfying E_n also satisfies E_{n+1}.*

(ii) *Any ring satisfying E_n also satisfies E_{n+r} for all $r \geq 1$.*

(iii) *If a ring S satisfies E_n, then so does $S[x]$.*

(iv) *If a ring S satisfies E_n, then so does $S[x_1, \dots, x_r]$ for all $r \geq 1$.*

Proof. (i) $\Rightarrow$ (iii): Let S be an E_n ring, then S is an E_{n+1} ring. Thus, any f.g. projective $S[x][X]$-module is extended from S, so $S[x]$ is E_n.

(iii) $\Rightarrow$ (i): Let S be an E_n ring, and let M be an $S[x_1, \dots, x_{n+1}]$-f.g. proyective module; since $S[x_1]$ is E_n, then M is extended from $S[x_1]$, i.e.,
$$M \cong M_1 \otimes_{S[x_1]} S[x_1][x_2, \dots, x_{n+1}] ,$$

where M_1 is a f.g. projective $S[x_1]$-module. Since S is E_n, so by Theorem 30 (iv), S is E_1, hence M_1 is extended form S, and consequently, M is extended from S. This means that S is E_{n+1}.

(i) $\Rightarrow$ (ii): Let $S \in E_n$, then $S \in E_{n+1}$, thus the result is obvious for $r = 1$. Since $S \in E_{n+1}$, by Theorem 30, $S[x_1] \in E_n$ and again by (i), $S[x_1] \in E_{n+1}$. We have proved (iii) for $n+1$, but (iii) is equivalent to (i) for a fixed integer, in this case for the integer $n+1$, then by (i) $S \in E_{n+2}$. Thus, we have proved that $E_{n+1} = E_{n+2}$. Since $S \in E_{n+2}$, then by Theorem 30, $S[x] \in E_{n+1}$, so $S[x] \in E_{n+2}$. We have proved (iii) for $n+2$, but (iii) is equivalent to (i) for a fixed integer, in this case for the integer $n+2$, then by (i) $S \in E_{n+3}$. We can repeat this reasoning and we get that $S \in E_{n+r}$, for each $r \geq 1$.

(ii) $\Rightarrow$ (i): Obvious.

(iii) $\Rightarrow$ (iv): If $S \in E_n$ then by (iii) $S[x_1] \in E_n$, and again by (iii) $S[x_1, x_2] \in E_n$. By induction on r we complete the proof of this part.

(iv) $\Rightarrow$ (iii): Obvious. ∎

Corollary 33. *With the notation of Conjectures* 31, *it holds:*

(a) (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v).
(b) (v) $\Rightarrow$ (i) $\Rightarrow$ (ii).

Proof. (a) (iii) $\Rightarrow$ (iv): It is clear that $E_m \subseteq E_1$; let $S \in E_1$, then by (iii), $S \in E_2$. Using Proposition 32 with $n = 1$, we get that $S \in E_r$ for $r \geq 1$, i.e., $S \in E_m$.

(iv) $\Rightarrow$ (iii): It is clear that $E_2 \subseteq E_1$; let $S \in E_1$, then by (iv), $S \in E_m \subseteq E_2$.

(iii) $\Rightarrow$ (v): This proof is similar to previous proof.

(b) (v) $\Rightarrow$ (i): Let $S \in E_n$, then $S \in E_1 = E$, so $S \in E_{n+1}$; by Proposition 32, $S[x] \in E_n$.

(i) $\Rightarrow$ (ii): Obvious. ∎

Some non trivial examples of E rings are the followings.

Examples 34. (i) Any Prüfer domain (every f.g. ideal is projective) is a E ring (see [23] and [8]). We observe that the Quillen-Suslin theorem is a consequence of this result: in fact, if D is a *PID*, then D is a Prüfer domain; let M be a f.g. projective $D[X]$-module, by the result M is extended from D, i.e., $M \cong M_0 \otimes_D D[X]$, where M_0 is a f.g. projective D-module. But since D is a *PID*, M_0 is D-free, and consequently, M is $D[X]$-free.

(ii) Bézout domains are Prüfer domains. Thus, Bézout domains are E; however, if D is a Bézout domain, any f.g. projective $D[X]$-module is free, i.e., the Bézout domains are *QS* (see [26]).

(iii) Let S be a commutative ring of Krull dimension 0. Then, S is E (see [18])

(iv) A Noetherian generalization of the Quillen-Suslin theorem has been given also by Quillen and Suslin (see [18]). *Let S be a commutative regular ring of Krull dimension ≤ 2. Then, S is E.* We note that the Quillen-Suslin theorem is a consequence of this result. In fact, any *PID* satisfies the conditions of the result: we recall that a commutative ring S is *regular* if S is Noetherian and the global dimension of S is finite (see [18] and [29]). But any *PID* is Noetherian and any *PID* has global dimension ≤ 1. Thus, any *PID* is regular; moreover, the Krull dimension of any *PID* is ≤ 1. Thus, if D is a *PID*, then every finitely generated projective $D[X]$-module M is extended from D, and as we saw above, this implies that M is $D[X]$-free.

Related with the results of previous example, Bass ([2]) and Quillen ([28]) formulated the following conjecture.

Conjecture 35. (BQ_d: The Bass-Quillen conjecture) *Let S be a commutative regular ring of Krull dimension $\leq d$. Then, every finitely generated projective $S[X]$-module is extended from S, i.e., S is E.*

The H property and the BQ_d conjecture are related in the following way.

Theorem 36. *The following conjectures are equivalent:*

(C1) *If S is H then $S[x]$ is H.*

(C2) *If S is local, then $S[x]$ is H.*

(C3) *If S is a commutative ring and M is a stably free $S[x]$-module, then M is extended from S.*

(C4) *If S is local and M is a stably free $S[x]$-module, then M is extended from S.*

(C5) *If S is a commutative ring and $\boldsymbol{f} = (f_1(x), \ldots, f_n(x))^T$ is an unimodular column matrix over $S[x]$ such that $\boldsymbol{f}(0)$ can be completed to a matrix of $\mathrm{GL}_n(S)$, then $\boldsymbol{f}$ can be completed to a matrix of $\mathrm{GL}_n(S[x])$.*

(C6) *If S is local and $\boldsymbol{f} = (f_1(x), \ldots, f_n(x))^T$ is an unimodular column matrix over $S[x]$ such that $\boldsymbol{f}(0)$ can be completed to a matrix of $\mathrm{GL}_n(S)$, then $\boldsymbol{f}$ can be completed to a matrix of $\mathrm{GL}_n(S[x])$.*

Morever, the truth of any of these conjectures will imply the truth of BQ_d for all d.

Proof. (C1) $\Rightarrow$ (C2): Let M be a stably free S-module, then M is a f.g. projective S-module, but since S is local then M is S-free. Thus, S has the H property. By (C1), $S[x]$ is H.

(C2) $\Rightarrow$ (C4): Let M be a stably free $S[x]$-module, with S local; by (C2), $S[x]$ is H, then M is $S[x]$-free, and hence, M is extended from S (see Proposition 28).

(C4) $\Rightarrow$ (C2): Let M be a stably free $S[x]$-module, then M is a f.g. projective $S[x]$-module; by (C4), M is extended from S, $M \cong M_0 \otimes_S S[x]$, where M_0 is a f.g. projective S-module. But since S is local, then M_0 is S-free, so M is $S[x]$-free.

(C4) $\Rightarrow$ (C3): This is a consequence of the *Quillen Patching theorem: Let S be a commutative ring. Let M be a finitely presented $S[X]$-module. Then, M is extended from S if and only if for every maximal ideal $P \in Max(S)$, M_P is extended from S_P* (see [18]). In fact, let M be a stably free $S[x]$-module, then M is a finitely presented module; moreover, M_P is a stably free $S_P[x]$-module for each maximal ideal P of S: $S[x]^m \cong M \oplus S[x]^n$ for some $m, n \geq 0$, then $S_p[x]^m \cong M_p \oplus S_p[x]^n$. By (C4), M_P is extended from S_P, and by the Quillen Patching theorem, M is extended from S.

(C3) $\Rightarrow$ (C1): Let S be a H ring; let M be a stably free $S[x]$-module; by (C3), M is extended from S, $M \cong M_0 \otimes_S S[x]$, where M_0 is a f.g. projective S-module. We need to prove that M is $S[x]$-free. We have $S[x]^p \cong M \oplus S[x]^q$ for some $p, q \geq 0$; then

$$S[x]^p \cong (M_0 \otimes_S S[x]) \oplus S[x]^q \,,$$

and hence

$$S[x]^p \otimes_{S[x]} S[x]/\langle x\rangle \cong (M_0 \otimes_S S[x]) \otimes_{S[x]} S[x]/\langle x\rangle \oplus S[x]^q \otimes_{S[x]} S[x]/\langle x\rangle \,,$$

i.e.,

$$S^p \cong M_0 \oplus S^q \,,$$

This means that M_0 is a stably free S-module, but since S is H, then M_0 is S-free, and hence, M is $S[x]$-free.

(C3) $\Rightarrow$ (C5): There exists $\boldsymbol{g} = (g_1(x), \ldots, g_n(x))$ such that $g_1(x)f_1(x) + \cdots + g_n(x)f_n(x) = 1$, we define

$$\begin{array}{rcl} S[x]^n & \xrightarrow{\alpha} & S[x] \\ \boldsymbol{e}_i & \longmapsto & g_i(x) \end{array}$$

where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the canonical basis of $S[x]^n$. We observe that α in a surjective homomorphism since $\alpha(\boldsymbol{f}) = 1$. There exists $\beta : S[x] \to S[x]^n$ such that $\alpha\beta = i_{S[x]}$ and $S[x]^n = \operatorname{Im}(\beta) \oplus \ker(\alpha)$. In fact, β is defined by $\beta(1) := \boldsymbol{f}$ and β is injective, hence, $\operatorname{Im}(\beta) \cong S[x]$ is free with basis $\{\boldsymbol{f}\}$. This implies that $\ker(\alpha)$ is stably free, and by hypothesis, $\ker(\alpha)$ is extended from S. So, there exists a S-module K_0 such that $\ker(\alpha) \cong K_0 \otimes_S S[x]$, moreover $K_0 \cong \ker(\alpha)/\langle x\rangle \ker(\alpha)$. If we prove that $\ker(\alpha)/\langle x\rangle \ker(\alpha)$ is S-free, then $\ker(\alpha)$ is $S[x]$-free, and consequently, $\{\boldsymbol{f}, \boldsymbol{h}_1, \ldots, \boldsymbol{h}_{n-1}\}$ is a basis if $S[x]^n$, where $\{\boldsymbol{h}_1, \ldots, \boldsymbol{h}_{n-1}\}$ is a basis of $\ker(\alpha)$. This means that $B := [\boldsymbol{f}\ \boldsymbol{h}_1 \cdots \boldsymbol{h}_{n-1}] \in \mathrm{GL}_n(S[x])$ and (C5) holds.

Thus, we must prove that $\ker(\alpha)/\langle x\rangle \ker(\alpha)$ is S-free. We define

$$\begin{array}{ccc} S^n & \xrightarrow{\alpha_0} & S \\ \boldsymbol{e}_i & \longmapsto & g_i(0) \end{array}$$

where $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the canonical basis of S^n. There exists $\beta_0 : S \to S^n$ such that $\alpha_0\beta_0 = i_S$ and $S^n = \operatorname{Im}(\beta_0) \oplus \ker(\alpha_0)$. In fact, β_0 is defined by $\beta_0(1) := \boldsymbol{f}(0)$ and β_0 is injective, hence, $\operatorname{Im}(\beta_0) \cong S$ is free with basis $\{\boldsymbol{f}(0)\}$. Thus,

$$S^n = \ker(\alpha_0) \oplus \langle \boldsymbol{f}(0)\rangle\,;$$

but by hypothesis $\boldsymbol{f}(0)$ is completable to a square matrix of $\mathrm{GL}_n(S)$, then as we saw in the proof of Theorem 11, $\ker(\alpha_0) \cong S^{n-1}$. The idea is to prove that $\ker(\alpha)/\langle x\rangle \ker(\alpha) \cong \ker(\alpha_0)$. We have the following commutative diagram:

$$\begin{array}{ccc} S[x]^n \otimes \big(S[x]/\langle x\rangle\big) & \xrightarrow{\alpha\otimes i} & S[x] \otimes \big(S[x]/\langle x\rangle\big) \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ S^n & \xrightarrow{\alpha_0} & S \end{array}$$

where i is the identical map of $S[x]/\langle x\rangle$ and the vertical arrows ϕ and φ are the natural isomorphisms defined by

$$\phi\big((h_1(x), \ldots, h_n(x)) \otimes \overline{1}\,\big) = (h_1(0), \ldots, h_n(0))\,,$$
$$\varphi\big(h(x) \otimes \overline{1}\,\big) = h(0)\,.$$

Then we have

$$\ker(\alpha_0) \cong \ker(\alpha \otimes i) = \ker(\alpha) \otimes (S[x]/\langle x\rangle) = \ker(\alpha)/\langle x\rangle \ker(\alpha)\,.$$

(C5) $\Rightarrow$ (C6): Obvious.

(C6) $\Rightarrow$ (C2): We will apply again Theorem 11. If $\boldsymbol{f} = (f_1(x), \ldots, f_n(x))^T$ is an unimodular column matrix over $R[x]$, then there exists $g_i(x) \in S[x]$, $1 \leq i \leq n$, such that $g_1(x)f_1(x) + \cdots + g_n(x)f_n(x) = 1$, so $g_1(0)f_1(0) + \cdots + g_n(0)f_n(0) = 1$. Since S is local there exists i such that $f_i(0) \in S^*$, and hence, by elementary operations on the rows of $\boldsymbol{f}(0)$ we find an invertible matrix $B \in \mathrm{GL}_n(S)$ such that $B\mathbf{f}(0) = \boldsymbol{e}_1$, i.e., $\boldsymbol{f}(0)$ can be completed to an invertible matrix of $\mathrm{GL}_n(S)$. By (C6), $\boldsymbol{f}$ can be completed to an invertible matrix of $\mathrm{GL}_n(S[x])$.

The proof of the second part of theorem is very extensive and need many preliminaries; this proof can be read in [18]. ∎

According to previous theorem, if the conjecture (C6) is true then the Bass-Quillen conjecture is true, and also, the Conjecture 16. The conjecture (C6) can be formulated in the following way: in a local ring S if $f = (f_1(x), \ldots, f_n(x))^T$ is an unimodular column matrix over $R[x]$, then

$$\begin{bmatrix} f_1(0) \\ \vdots \\ f_n(0) \end{bmatrix} = B\boldsymbol{e}_1 \quad \Rightarrow \quad \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \widetilde{B}\boldsymbol{e}_1\,,$$

where $B \in \mathrm{GL}_n(S)$ and $\widetilde{B} \in \mathrm{GL}_n(S[x])$.

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