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Weyl Type Theorems for Polaroid Operators *

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Abstract: This paper concerns two variants of Weyl's theorem, for bounded linear operators defined on Banach spaces, the property (w) and the property (b). We study the relationship between these two property in the framework of polaroid and a-polaroid operators.

Key words: Weyl's theorems, property (w), property (b), SVEP.

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1. Introduction

H. Weyl [23] studied the spectra of all compact perturbations of a self-adjoint operator defined on a Hilbert space and found that their intersection consisted precisely of those points of the spectrum where are not isolated eigenvalues of finite multiplicity. Later, the property established by Weyl for self-adjoint operators has been observed for several other classes of operators, for instance hyponormal operators on Hilbert spaces, Toeplitz operators [12], convolution operators on group algebras [10], and many other classes of operators defined on Banach spaces. In the literature, a bounded operator defined on a Banach space which satisfies this property is said to satisfy *Weyl's theorem*. Weaker variants of Weyl's theorem have been discussed by Harte and Lee [15], while two approximate-point spectrum versions of Weyl's theorem have been introduced by Rakočević, *a-Weyl's theorem* [21], and the so-called *property* (w) [20]. Recently, the last property has been studied in several articles ([9], [4], [3], [8] and [6]). In this paper we investigate the relationship between property (w) and another variant of Weyl's theorem, the *property* (b) introduced very recently by Berkani and Zariouh [11]. In particular, most of our results are established in the framework of polaroid or a-polaroid opera-

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tors. We shall also consider operators T for which the single-valued extension property holds for T, or for its dual T^*.

Throughout this paper, X will denote an infinite-dimensional complex Banach space, $L(X)$ the Banach algebra of all bounded linear operators. A bounded operator $T \in L(X)$ is said to be an *upper semi-Fredholm* operators, $T \in \Phi_+(X)$, if $\alpha(T) := \dim \ker T < \infty$ and $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm*, $T \in \Phi_-(X)$, if $\beta(T) := \operatorname{codim} T(X) < \infty$. The *index* of a semi-Fredholm operator is defined as $\operatorname{ind} T := \alpha(T) - \beta(T)$. $T \in L(X)$ is said to be a *Fredholm operator* if $T \in \Phi_+(X) \cap \Phi_-(X)$. A bounded operator $T \in L(X)$ is said to be *upper semi-Weyl*, $T \in W_+(X)$, if $T \in \Phi_+(X)$ and $\operatorname{ind} T \leq 0$. $T \in L(X)$ is said to be *lower semi-Weyl*, $T \in W_-(X)$, if $T \in \Phi_-(X)$ and $\operatorname{ind} T \geq 0$. The class of Weyl operators is defined by $W(X) := W_+(X) \cap W_-(X) = \{T \in \Phi(X) : \operatorname{ind} T = 0\}$. These classes of operators generate the following spectra: the *Weyl spectrum* defined by

$$\sigma_{\mathrm{w}}(T) := \big\{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\big\},$$

the *upper semi-Weyl spectrum* (in literature called also *Weyl essential approximate point spectrum*) defined by

$$\sigma_{\mathrm{uw}}(T) := \big\{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\big\},$$

and the *lower semi-Weyl spectrum* (in literature called also *Weyl essential surjectivity spectrum*) defined by

$$\sigma_{\mathrm{lw}}(T) := \big\{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\big\}.$$

The *approximate point spectrum* is canonically defined by

$$\sigma_{\mathrm{a}}(T) := \big\{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\big\},$$

where an operator is said to be *bounded below* if it is injective and has closed range. Note that $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{a}}(T^*)$, where $\sigma_{\mathrm{s}}(T)$ denotes the *surjectivity spectrum*, and dually $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{s}}(T^*)$. Two other important quantities in Fredholm theory are the *ascent* of $T \in L(X)$, defined as $p := p(T) = \inf\{n \in \mathbb{N} : \ker T^n = \ker T^{n+1}\}$, and the *descent* of T, defined as let $q := q(T) = \inf\{n \in \mathbb{N} : T^n(X) = T^{n+1}(X)\}$, the infimum over the empty set is taken ∞. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$ (see [16, Proposition 38.3]). Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T, see [16, Proposition 50.2] of Heuser. The class of all *upper semi-Browder operators* is defined $B_+(X) := \{T \in$

$\Phi_+(X) : p(T) < \infty\}$, while the class of all *lower semi-Browder operators* is defined $B_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}$. The class of all *Browder operators* is defined

$$B(X) := B_+(X) \cap B_-(X) = \{T \in \Phi(X) : p(T) = q(T) < \infty\}.$$

It is well-known that

$$B(X) \subseteq W(X), \qquad B_+(X) \subseteq W_+(X), \qquad B_-(X) \subseteq W_-(X).$$

The *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\},$$

the *upper semi-Browder spectrum* and the *lower semi-Browder spectrum* are defined, respectively, by

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\}$$

and

$$\sigma_{\mathrm{lb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_-(X)\}.$$

The operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}$ centered at λ_0, the only analytic function $f : \mathbb{D} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, every operator T, as well as its dual T^*, has SVEP at every point in the boundary of the spectrum $\sigma(T)$, in particular at every isolated point of $\sigma(T)$.

The following implications hold:

$$p(\lambda I - T) < \infty \qquad \Longrightarrow \qquad T \text{ has SVEP at } \lambda \tag{1}$$

and, dually,

$$q(\lambda I - T) < \infty \qquad \Longrightarrow \qquad T^* \text{ has SVEP at } \lambda \tag{2}$$

(see [1, Theorem 3.8]). Furthermore, from definition of SVEP we have

$$\lambda \operatorname{acc} \sigma_{\mathrm{a}}(T) \qquad \Longrightarrow \qquad T \text{ has SVEP at } \lambda\,, \tag{3}$$

and dually

$$\lambda \notin \operatorname{acc} \sigma_{\mathrm{s}}(T) \qquad \Longrightarrow \qquad T^* \text{ has SVEP at } \lambda\,. \tag{4}$$

In particular, if the point spectrum $\sigma_{\rm p}(T)$ (= the set of all eigenvalues of T) is empty then T satisfies SVEP. An important subspace in local spectral theory is the *quasi-nilpotent part* of T defined by

$$H_0(T) := \left\{x \in X \,:\, \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\right\}.$$

Obviously, $\ker T^n \subseteq H_0(T)$ for all $n \in \mathbb{N}$. We also have [1, Theorem 2.31].

$$H_0(\lambda I - T) \text{ closed} \qquad \Longrightarrow \qquad T \text{ has SVEP at } \lambda\,. \tag{5}$$

Remark 1.1. All the implications (1), (2), (3), (4) and (5) are actually equivalences if we assume that $\lambda I - T$ is semi-Fredholm, see [1, Chapter 3].

Another important subspace in local spectral theory is the *analytic core* $K(T)$ defined as the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \le c^n \|x\|$ for all $n \in \mathbb{N}$. $K(T)$ is T invariant, precisely $T(K(T)) = K(T)$; see [1] for further information on $K(T)$.

2. Weyl type theorems

If $T \in L(X)$, define $p_{00}(T) := \sigma(T) \setminus \sigma_{\rm b}(T)$ and $p_{00}^a(T) := \sigma_{\rm a}(T) \setminus \sigma_{\rm ub}(T)$. It is easy to check that $p_{00}(T) \subseteq p_{00}^a(T)$ for all $T \in L(X)$ and, obviously, every point of $p_0 0(T)$ is an isolated point of $\sigma(T)$ and $\sigma_{\rm a}(T)$. According Harte and W.Y. Lee [15] we say that a bounded operator $T \in L(X)$ satisfies *Browder's theorem* if $\sigma_{\rm w}(T) = \sigma_{\rm b}(T)$, or equivalently $\sigma(T) \setminus \sigma_{\rm w}(T) = p_{00}(T)$, while we say that T satisfies *a-Browder's theorem* if $\sigma_{\rm uw}(T) = \sigma_{\rm ub}(T)$, or equivalently $\sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T) = p_{00}^a(T)$. We have:

$$\text{a-Browder's theorem for } T \qquad \Longrightarrow \qquad \text{Browder's theorem for } T\,.$$

Define

$$\pi_{00}(T) := \big\{\lambda \in \operatorname{iso}\sigma(T) \,:\, 0 < \alpha(\lambda I - T) < \infty\big\}.$$

and

$$\pi_{00}^a(T) := \big\{\lambda \in \operatorname{iso}\sigma_{\rm a}(T) \,:\, 0 < \alpha(\lambda I - T) < \infty\big\}.$$

It is easily seen that

$$p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T) \qquad \text{and} \qquad p_{00}^a(T) \subseteq \pi_{00}^a(T)\,. \tag{6}$$

DEFINITION 2.1. A bounded operator $T \in L(X)$ is said to be *left Drazin invertible* if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed. We say that λ is a *left pole* if $\lambda \in \sigma_{\mathrm{a}}(T)$ and $\lambda I - T$ is left Drazin invertible. A left pole λ is said to be of *finite rank* if $\alpha(\lambda I - T) < \infty$.

THEOREM 2.2. *Let $T \in L(X)$. Then $\lambda \in p_{00}^{a}(T)$ if and only if λ is a left pole of finite rank.*

Proof. Let $\lambda \in p_{00}^{a}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T)$. We may assume $\lambda = 0$. Then $p := p(T) < \infty$, $T \in \Phi_{+}(X)$, so $\alpha(T) < \infty$. From classical Fredholm theory we know that $T^n \in \Phi_{+}(X)$ for all $n \in \mathbb{N}$, so $T^{p+1}(X)$ is closed. Therefore, 0 is a left pole of finite rank. Conversely, suppose that 0 is a left pole of finite rank. Then, $0 \in \sigma_{\mathrm{a}}(T)$, T is left Drazin invertible, so $p(T) < \infty$. The condition of left Drazin invertibility is equivalent to saying that T is upper semi B-Browder, i.e., there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and the restriction $T_n := T|T^n(X)$ is upper semi-Browder [5], in particular upper semi-Fredholm. Since $\alpha(T) < \infty$ then $\alpha(T^n) < \infty$, hence $T^n \in \Phi_{+}(X)$, from which obtain that $T \in \Phi_{+}(X)$. Since $p(T) < \infty$ we then conclude that $T \in B_{+}(X)$, so $0 \notin \sigma_{\mathrm{ub}}(T)$, and consequently $0 \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = p_{00}^{a}(T)$. ∎

Following Coburn [12], we say that *Weyl's theorem holds* for $T \in L(X)$ if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T) .$$

There is a precise relationship between Browder's theorem and Weyl's theorem:

THEOREM 2.3. ([2]) *If $T \in L(X)$ then Weyl's theorem for T holds precisely when T satisfies Browder's theorem and $\pi_{00}(T) = p_{00}(T)$.*

The following first two variants of Weyl's theorem have been introduced by Rakočević ([20], [21]), while the third variant has been introduced in [11].

DEFINITION 2.4. A bounded operator $T \in L(X)$ is said to satisfy a-Weyl's theorem if

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}^{a}(T) ,$$

$T \in L(X)$ is said to satisfy property (w) if

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T) ,$$

$T \in L(X)$ is said to satisfy property (b) if

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = p_{00}(T) .$$

The relationships between property (w), a-Weyl's theorem, property (b) and a-Browder's theorem is established by the following theorem.

THEOREM 2.5. *Suppose that $T \in L(X)$. Then we have:*

(i) *T satisfies a-Weyl's theorem if and only if a-Browder's theorem holds for T and $p_{00}^a(T) = \pi_{00}^a(T)$ ([2]);*

(ii) *T satisfies property (w) if and only if a-Browder's theorem holds for T and $p_{00}^a(T) = \pi_{00}(T)$ ([9]);*

(iii) *T satisfies property (b) if and only if a-Browder's theorem holds for T and $p_{00}^a(T) = p_{00}(T)$ ([11]).*

The following diagram resume the relationships between Weyl's theorems, a-Browder's theorem and property (w):

$$\begin{array}{ccc} \text{Property } (w) & \Rightarrow & a\text{-Browder's theorem} \\ \Downarrow & & \Uparrow \\ \text{Weyl's theorem} & \Leftarrow & a\text{-Weyl's theorem} \end{array}$$

(see [20] and [9]). Examples of operators satisfying Weyl's theorem but not property (w) may be found in [9]. Property (w) is not intermediate between Weyl's theorem and a-Weyl's theorem, see [9] for examples. Note that property (w) is satisfied by a certain number of Hilbert space operators, see [9]. By contrast, propery (b) does not entails Weyl's theorem and conversely, Weyl's theorem does not imply property (b), see the next Example 2.6 and Example 2.14. However, see [11],

$$\text{property } (w) \quad \Longrightarrow \quad \text{property } (b) \quad \Longrightarrow \quad a\text{-Browder's theorem}\,.$$

The following two examples show that a-Weyl's theorem and property (b) for $T \in L(X)$ are independent. The first example shows that a-Weyl's theorem does not imply property (b).

EXAMPLE 2.6. Let R be the canonical unilateral right shift on $\ell^2(\mathbb{N})$ and let P denote the projection defined by

$$P(x_1, x_2, \dots) := (0, x_2, \dots) \qquad \text{for all} \quad x := (x_1, x_2, \dots) \in \ell^2(\mathbb{N})\,.$$

Consider $T := R \oplus P$ on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Then $\sigma(T) = \mathbb{D}_1$, where $\mathbb{D}_1$ is the closed unit disc of $\mathbb{C}$, so that $\sigma(T)$ has no isolated points and hence

$p_{00}(T) = \emptyset$. Furthermore, $\sigma_{\rm a}(T) = \Gamma \cup \{0\}$, where Γ is the closed unit circle, and $\sigma_{\rm uw}(T) = \Gamma$. Therefore, T does not satisfies property (b), since

$$\sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T) = \{0\} \neq p_{00}(T) .$$

On the other hand, T satisfies a-Weyl's theorem, since $\pi^a_{00}(T) = \{0\}$. Note that T satisfies Weyl's theorem.

The second example shows that property (b) does not imply a-Weyl's theorem.

Example 2.7. Let $R \in L(\ell^2(\mathbb{N}))$ be right shift and let L be the weighted unilateral left shift defined by

$$L(x_1, x_2, \dots) = \Big(\frac{x_2}{2}, \frac{x_3}{3}, \dots\Big) \qquad \text{for all} \quad x = (x_1, x_2, \dots) \in \ell^2(\mathbb{N}) ,$$

If $T := R \oplus L$, then $\sigma(T) = \mathbb{D}_1$, hence there is no isolated point in $\sigma(T)$ and consequently $p_{00}(T) = \emptyset$. Moreover, $\sigma_{\rm a}(T) = \sigma_{\rm uw}(T) = \Gamma \cup \{0\}$, thus

$$\sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T) = \emptyset = p_{00}(T) ,$$

hence T satisfies property (b). On the other hand, $\pi^a_{00}(T) = \{0\}$ so T does not satisfy a-Weyl's theorem.

Definition 2.8. An operator $T \in L(X)$ is said to be *polaroid* if $\operatorname{iso} \sigma(T)$ is empty or every isolated point of $\sigma(T)$ is a pole of the resolvent.

The next result shows that the polaroid condition, as well as the equality $p_{00}(T) = \pi_{00}(T)$, may be described in terms of quasi-nilpotent part.

Theorem 2.9. *Let $T \in L(X)$. Then we have:*

(i) *T is polaroid if and only if there exists $p := p(\lambda) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker (\lambda I - T)^p \qquad \text{for all} \quad \lambda \in \operatorname{iso} \sigma(T) ; \tag{7}$$

(ii) *$p_{00}(T) = \pi_{00}(T)$ if and only if there exists $p := p(\lambda) \in \mathbb{N}$ such that*

$$H_0(\lambda I - T) = \ker (\lambda I - T)^p \qquad \text{for all} \quad \lambda \in \pi_{00}(T) . \tag{8}$$

Proof. (i) Suppose T satisfies (7) and that λ is an isolated point of $\sigma(T)$. Then there exists $p \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Since λ is isolated in $\sigma(T)$ then, by [1, Theorem 3.74],

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^p \oplus K(\lambda I - T)\,,$$

from which we obtain

$$(\lambda I - T)^p(X) = (\lambda I - T)^p(K(\lambda I - T)) = K(\lambda I - T)\,,$$

so

$$X = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X)\,,$$

which implies, by [1, Theorem 3.6], that $p(\lambda I - T) = q(\lambda I - T) \leq p$, hence λ is a pole of the resolvent, so that T is polaroid.

Conversely, suppose that T is polaroid and λ is an isolated point of $\sigma(T)$. Then λ is a pole, and if p is its order then $H_0(\lambda I - T) = \ker(\lambda I - T)^p$, see [1, Theorem 3.74].

(ii) Suppose T satisfies (8) and that $\lambda \in \pi_{00}(T)$. Since $\pi_{00}(T) \subseteq \operatorname{iso}\sigma(T)$ then from the proof of part (i) we know that $p(\lambda I - T) = q(\lambda I - T) \leq p$. By definition of $\pi_{00}(T)$ we also know that $\alpha(\lambda I - T) < \infty$ and this implies by [1, Theorem 3.4] that $\beta(\lambda I - T)$ is also finite. Therefore $\lambda \in p_{00}(T)$ and hence $\pi_{00}(T) \subseteq p_{00}(T)$. Since the opposite inclusion holds for every operator we then conclude that $p_{00}(T) = \pi_{00}(T)$.

Conversely, if $p_{00}(T) = \pi_{00}(T)$ and $\lambda \in \pi_{00}(T)$ then $p := p(\lambda I - T) = q(\lambda I - T) < \infty$. By [1, Theorem 3.16] it then follows that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. ∎

THEOREM 2.10. *Suppose that $T \in L(X)$ is polaroid. Then T satisfies property (b) if and only if T satisfies property (w). Analogously, T^* satisfies property (b) if and only if T^* satisfies property (w).*

Proof. The implication "$(w) \Rightarrow (b)$" holds for every $T \in L(X)$, so we have only to show that "$(b) \Rightarrow (w)$". Let T satisfy property (b). Then $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = p_{00}(T)$. Now, let $\lambda \in \pi_{00}(T)$. Then λ is an isolated point of $\sigma(T)$, so that λ is a pole and consequently, $p(\lambda I - T) = q(\lambda I - T) < \infty$. Since $\alpha(\lambda I - T) < \infty$ by [1, Theorem 3.4] it then follows that $\beta(\lambda I - T) < \infty$, hence $\lambda I - T$ is Browder and consequently $\lambda \in p_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$. Therefore, $\pi_{00}(T) \subseteq p_{00}(T)$. The opposite inclusion holds for every operator, so $\pi_{00}(T) = p_{00}(T)$ and hence $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$, i.e., T satisfies property (w). The second statement is clear: if T is polaroid then T^* is polaroid ([8]), so the first part applies. ∎

DEFINITION 2.11. $T \in L(X)$ is said to be *a-polaroid* if iso $\sigma_a(T)$ is empty or every isolated point of $\sigma_a(T)$ is a pole of the resolvent.

Clearly,

$$T \text{ } a\text{-polaroid} \quad \Longrightarrow \quad T \text{ polaroid}.$$

Observe that if T^* has SVEP then $\sigma(T) = \sigma_a(T)$, see [1, Corollary 2.45], so that

$$T^* \text{ has SVEP and } T \text{ polaroid} \quad \Longrightarrow \quad T \text{ } a\text{-polaroid}. \tag{9}$$

If T is polaroid then T^* is polaroid [8]. Moreover, if T has SVEP then $\sigma(T) = \sigma_s(T) = \sigma_a(T^*)$, see [1, Corollary 2.45], hence

$$T \text{ has SVEP and } T \text{ polaroid} \quad \Longrightarrow \quad T^* \text{ } a\text{-polaroid}. \tag{10}$$

The equivalences of Theorem 2.10 cannot be extended to a-Weyl's theorem, i.e., for polaroid operators property (b) and property (w) are not equivalent For instance, the operator T defined in Example 2.6 is polaroid, since iso $\sigma(T) = \emptyset$, T satisfies a-Weyl's theorem but not property (b). Also the operator T defined in Example 2.7 is polaroid, T satisfies (b) but not a-Weyl's theorem. However, the next result shows that under the stronger assumption that T is a-polaroid, properties (b) and (w) are equivalent to a-Weyl's theorem.

THEOREM 2.12. *Let $T \in L(X)$. If T is a-polaroid then the following statements are equivalent:*

(i) *T satisfies property (w);*

(ii) *T satisfies a-Weyl's theorem;*

(iii) *T satisfies property (b).*

Proof. We show first that if T is a-polaroid then $p_{00}(T) = \pi_{00}(T) = \pi_{00}^a(T)$. The inclusions $p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T)$ have been already observed for all operators. We show that $\pi_{00}^a(T) \subseteq p_{00}(T)$. Let $\lambda \in \pi_{00}^a(T)$. Then λ is isolated in $\sigma_a(T)$ and hence the a-polaroid condition entails that $p(\lambda I - T) = q(\lambda I - T) < \infty$. Moreover, $\alpha(\lambda I - T) < \infty$, so by [1, Theorem 3.4] we have $\beta(\lambda I - T) < \infty$, hence $\lambda \in p_{00}(T)$.

(i) $\Rightarrow$ (ii) If T satisfies property (w) then $\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T) = \pi_{00}^a(T)$, thus T satisfies a-Weyl's theorem.

(ii) $\Rightarrow$ (iii) Suppose that T satisfies a-Weyl's theorem. Then $\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T) = p_{00}(T)$, so T satisfies property (b).

(iii) $\Rightarrow$ (i) This follows from Theorem 2.10, since every a-polaroid operator is polaroid. ∎

The following result has been proved in [9].

Theorem 2.13. *Let $T \in L(X)$.*

(i) *If T^* has SVEP, then property (w), Weyl's theorem and a-Weyl's theorem for T are equivalent.*

(ii) *If T has SVEP, then property (w), Weyl's theorem and a-Weyl's theorem for T^* are equivalent.*

It has some sense to ask if property (w) for T, (or, equivalently, a-Weyl's theorem) is equivalent to property (b) for T whenever T^* has SVEP. The following example shows that in general this is not true.

Example 2.14. Let $L \in L(\ell^2(\mathbb{N}))$ be the weighted unilateral left shift defined in Example 2.7. L is quasi-nilpotent, hence the dual L^* has SVEP. On the other hand, the property (b) holds for L, since $\sigma_{\rm a}(L) = \sigma_{\rm uw}(L) = \{0\}$ and $p_{00}(L) = \emptyset$, while property (w) does not hold for L since $\pi_{00}(L) = \{0\}$.

It should be noted that the operator L provides an example of operator satisfying property (b), but not Weyl's theorem. In fact, L does not satisfies property (w) and hence, by Theorem 2.13, L does not satisfy Weyl's theorem since L^* has SVEP.

Lemma 2.15. *If $T \in L(X)$ and T^* has SVEP then $\sigma_{\rm w}(T) = \sigma_{\rm uw}(T)$ and $\sigma(T) = \sigma_{\rm a}(T)$.*

Proof. Evidently, the inclusion $\sigma_{\rm uw}(T) \subseteq \sigma_{\rm w}(T)$ holds for all $T \in L(X)$. To see the opposite inclusion, suppose that $\lambda \notin \sigma_{\rm uw}(T)$. Then $\lambda I - T$ is upper semi-Fredholm with $\operatorname{ind}(\lambda I - T) \leq 0$, and by Remark 1.1 the SVEP of T^* implies that $q(\lambda I - T) < \infty$. By [1, Theorem 3.4] then $\operatorname{ind}(\lambda I - T) \geq 0$, hence $\operatorname{ind}(\lambda I - T) = 0$ and, consequently, $\lambda \notin \sigma_{\rm w}(T)$. A proof of the equality $\sigma(T) = \sigma_{\rm a}(T)$ may be found in [1, Corollary 2.45]. ∎

Theorem 2.16. *Let $T \in L(X)$ such that*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \qquad \text{for all} \quad \lambda \in \pi_{00}^a(T)\,. \tag{11}$$

If T^* satisfies SVEP then property (b) holds for T.

Proof. Since $\pi_{00}(T) \subseteq \pi_{00}^a(T)$, by part (i) of Theorem 2.9 it then follows that $p_{00}(T) = \pi_{00}(T)$. The SVEP for T^* implies that Browder's theorem holds for T and from the equality $p_{00}(T) = \pi_{00}(T)$ it then follows that Weyl's theorem holds for T, see [2]. By Lemma 2.15 then we have

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T) = p_{00}(T)\,,$$

thus T satisfies property (b). ∎

THEOREM 2.17. *Suppose that $T \in L(X)$ is polaroid. Then we have:*

(i) *If T^* has SVEP then property (b), (or equivalently property (w), Weyl's theorem, a-Weyl's theorem) holds for T.*

(ii) *If T has SVEP then property (b), (or equivalently property (w), Weyl's theorem, a-Weyl's theorem) holds for T^*.*

Proof. (i) If T^* has SVEP and T is polaroid, by [9, Theorem 2.24] then T satisfies property (w), or equivalently, by Theorem 2.10, T satisfies property (b). Moreover, by Theorem 2.13, the SVEP for T^* entails that a-Weyl's and Weyl's theorem for T are equivalent to property (w).

(ii) Suppose that T has SVEP and T is polaroid. By [9, Theorem 2.24] then T^* satisfies property (w) and this is equivalent, by Theorem 2.13, to saying that Weyl's theorem T, or a-Weyl's theorem holds for T^*. Since T^* is polaroid, see [8], by Theorem 2.10 then property (b) and property (w) for T^* are equivalent. ∎

An operator $U \in L(X, Y)$ between the Banach spaces X and Y is said to be a *quasi-affinity* if U is injective and has dense range. The operator $S \in L(Y)$ is said to be a *quasi-affine transform* of $T \in L(X)$, notation $S \prec T$, if there is a quasi-affinity $U \in L(Y, X)$ such that $TU = US$. If both $S \prec T$ and $T \prec S$ hold, then S, T are said *quasi-similar*.

THEOREM 2.18. *Suppose that $T \in L(X)$ and $S \prec T$. If T is polaroid and T has SVEP, then property (b), or equivalently property (w), holds for S.*

Proof. The SVEP for S is inherited by the SVEP for T. Indeed, $f : \mathcal{U} \to Y$ be an analytic function defined on an open disc $\mathcal{U}$ of λ_0 such that $(\mu I - S)f(\mu) = 0$ for all $\mu \in \mathcal{U}$. Then

$$U(\lambda I - S)f(\mu) = (\mu I - T)Uf(\mu) = 0\,,$$

and the SVEP of T at λ_0 entails that

$$Uf(\mu) = 0 \qquad \text{for all} \quad \mu \in \mathcal{U}\,.$$

Since U is injective then $f(\mu) = 0$ for all $\mu \in \mathcal{U}$, hence S has the SVEP at λ_0.

We show now that S is polaroid. Let $\lambda \in \operatorname{iso} \sigma(S)$. Then there exists $p \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Let $x \in H_0(\lambda I - S)$. Then

$$\begin{aligned}\lim_{n\to\infty} \|(\lambda I - T)^n U x\|^{1/n} &= \lim_{n\to\infty} \|U(\lambda I - T)^n U x\|^{1/n} \\ &\le \lim_{n\to\infty} \|(\lambda I - S)x\|^{1/n} = 0\end{aligned}$$

so $Ux \in H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Therefore,

$$U(\lambda I - S)^p x = (\lambda I - S)^p U x = 0\,,$$

and since U is injective it then follows that $x \in \ker(\lambda I - S)^p$. This shows that $H_0(\lambda I - S) \subseteq \ker(\lambda I - S)^p$ and since the opposite inclusion is true we then conclude that $H_0(\lambda I - S) = \ker(\lambda I - S)^p$. By Theorem 2.9 then S is polaroid. Property (b), or equivalently property (w) then follows from Theorem 2.17. ∎

Remark 2.19. Recall that $T \in L(X)$ is said to be a *Riesz operator* if $\lambda I - T \in \Phi(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Evidently, every quasi-nilpotent operator is a Riesz operator. It is well-known that if Q is a quasi-nilpotent operator commuting with T then

$$\sigma(T+Q) = \sigma(T) \qquad \text{and} \qquad \sigma_{\mathrm{a}}(T+Q) = \sigma_{\mathrm{a}}(T)\,.$$

Moreover, if R is a Riesz operator commuting with T then $T \in W_+(X)$ if and only if $T + R \in W_+(X)$, see [18, Lemma 2.2], and this obviously implies the equality $\sigma_{\mathrm{uw}}(T+Q) = \sigma_{\mathrm{uw}}(T)$ for every operator quasi-nilpotent Q commuting with T. A similar result holds for $\sigma_{\mathrm{b}}(T)$, i.e., $\sigma_{\mathrm{b}}(T+Q) = \sigma_{\mathrm{b}}(T)$ for every quasi-nilpotent operator Q commuting with T (see [18, Proposition 4]).

Generally, property (w) is not transmitted from T to a quasi-nilpotent perturbation $T + Q$. In fact, if $Q \in L(\ell^2(\mathbb{N}))$ is defined by

$$Q(x_1, x_2, \dots) = \Big(\frac{x_2}{2}, \frac{x_3}{3}, \dots\Big) \qquad \text{for all} \quad (x_n) \in \ell^2(\mathbb{N})\,,$$

Then Q is quasi-nilpotent and

$$\{0\} = \pi_{00}(Q) \ne \sigma_{\mathrm{a}}(Q) \setminus \sigma_{\mathrm{uw}}(Q) = \emptyset\,.$$

Take $T = 0$. Clearly, T satisfies property (w) but $T+Q = Q$ fails this property. The next result shows that property (b) is invariant under commuting quasi-nilpotent perturbations.

THEOREM 2.20. *Let $T \in L(X)$ and let $Q \in L(X)$ be a quasi-nilpotent operator such that $TQ = QT$. If T satisfies property (b) then $T + Q$ satisfies property (b).*

Proof. From Remark 2.19 we have

$$p_{00}(T+Q) = \sigma(T+Q) \setminus \sigma_{\mathrm{b}}(T+Q) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = p_{00}(T)\,.$$

Hence, if T satisfies property (b) then

$$\sigma_{\mathrm{a}}(T+Q) \setminus \sigma_{\mathrm{uw}}(T+Q) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = p_{00}(T) = p_{00}(T+Q)\,,$$

thus $T + Q$ satisfies property (b). ∎

Generally, property (w), as well as Weyl's theorem and a-Weyl's theorem, is not preserved under a commuting finite-rank perturbation K. Precisely, Weyl's theorem is preserved if T isoloid (i.e., every isolated point in $\sigma(T)$ is an eigenvalue of T), a-Weyl's theorem is preserved if we assume that T is a-isoloid (i.e., every isolated point in $\sigma_{\mathrm{a}}(T)$ is an eigenvalue of T). Property (w) is preserved if we assume that T is a-isoloid and that $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T+K)$ [4]. The preservation of property (b) does not require that T is a-isoloid.

THEOREM 2.21. *Let $T \in L(X)$ and let $K \in L(X)$ be a finite-rank operator such that $TK = KT$. If T satisfies property (b) and $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T+K)$ then $T + K$ satisfies property (b).*

Proof. Since K is a Riesz operator, from Remark 2.19 we have

$$p_{00}(T+K) = \sigma(T+K) \setminus \sigma_{\mathrm{b}}(T+K) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = p_{00}(T)\,.$$

Hence, if T satisfies property (b) then

$$\sigma_{\mathrm{a}}(T+K) \setminus \sigma_{\mathrm{uw}}(T+K) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = p_{00}(T) = p_{00}(T+K)\,,$$

thus $T + K$ satisfies property (b). ∎

The condition $\sigma_{\mathrm{a}}(T+K) = \sigma_{\mathrm{a}}(T)$, K of finite rank commuting with T, is satisfied in some special cases, for instance if $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \emptyset$, see [3]. The condition $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \emptyset$ is satisfied by every not quasi-nilpotent unilateral right shift T on $\ell^p(\mathbb{N})$, with $1 \leq p < \infty$, see [17, Proposition 1.6.15].

Theorem 2.22. *Suppose that $T \in L(X)$ is polaroid, and let $K \in L(X)$ be a finite-rank operator such that $TK = KT$. If T^* has SVEP and T satisfies property (b) then $T + K$ satisfies property (b).*

Proof. Observe first that the SVEP for T^* entails that T is a-polaroid, and hence a-isoloid. By Theorem 2.12 the assumption that T is a-polaroid entails that a-Weyl's theorem holds for T. a-Weyl's theorem for a-isoloid operators is transmitted by commuting finite rank perturbations, see [13], so a-Weyl's theorem holds for $T + K$. Since $T^* + K^* = (T + K)^*$ has SVEP, see [3], a-Weyl's theorem for $T + K$ is equivalent to property (w), and consequently $T + K$ satisfies property (b). ∎

The class of polaroid operators is rather large. In the sequel we list some classes of polaroid operators and a-polaroid operators.

(a) A bounded operator is said to belong to the class $H(p)$ if there exists a natural $p := p(\lambda)$ such that:

$$H_0(\lambda I - T) = \ker (\lambda I - T)^p \qquad \text{for all} \quad \lambda \in \mathbb{C}\,. \tag{12}$$

The class $H(p)$ has been introduced in [19] and in [10] this class of operators has been studied for $p := p(\lambda) = 1$ for all $\lambda \in \mathbb{C}$. Property $H(p)$ is satisfied by every generalized scalar operator, and in particular for p-hyponormal, log-hyponormal, M-hyponormal operators on Hilbert spaces, see [19]. From the implication (5) we see that every operator T which belongs to the class $H(p)$ has SVEP. By Theorem 2.9 we have

$$T \in H(p) \quad \Longrightarrow \quad T \text{ polaroid} \quad \Longrightarrow \quad p_{00}(T) = \pi_{00}(T)\,.$$

Consequently, by Theorem 2.10, $T \in H(p)$ satisfies property (w) if and only if T satisfies property (b). It should be noted that every generalized scalar operator T is decomposable [17], so both T and T^* have SVEP and hence every generalized scalar operator T, as well as its dual T^*, satisfies property (b). Of course, a generalized scalar operator is a-polaroid.

(b) A bounded operator $T \in L(X)$ on a Banach space X is said to be *paranormal* if

$$\|Tx\|^2 \leq \|T^2x\|\|x\| \qquad \text{holds for all} \quad x \in X\,.$$

An operator $T \in L(X)$ for which there exists a complex nonconstant polynomial h such that $h(T)$ is paranormal is said to be *algebraically paranormal.*

Every algebraic paranormal operator defined on a Hilbert space is polaroid and has SVEP, see [7].

(c) Every multiplier T of a semi-simple commutative Banach algebra is polaroid, in particular every convolution T_μ operator of $L^1(G)$, $L^1(G)$ the group algebra of a locally compact Abelian group G, see [1, Theorem 4.36]. Every multiplier has property $H(1)$, hence has SVEP and is polaroid [10]. If $L^1(G)$ is regular and Tauberian then $\sigma(T) = \sigma_{\mathrm{a}}(T)$, see [1, Corollary 5.88], so T is a-polaroid. In particular, a convolution operator T_μ on $L^1(G)$ is a-polaroid whenever G is compact. Another example of a-polaroid operator is given by the multipliers of a Banach algebra with an orthogonal basis. In fact, by [1, Theorem 4.46] for these operators we have $\sigma(T) = \sigma_{\mathrm{a}}(T)$.

(d) A bounded operator is said to be *meromorphic* if every $\lambda \in \mathbb{C} \setminus \{0\}$ is a pole. The spectrum of a meromorphic operator is finite or a countable set which clusters at 0 [16]. Every meromorphic operator T is polaroid and both T and T^* have SVEP, so T is a-polaroid.

(e) Let H be a Hilbert space and denote by $d_{A,B} \in L(L(H))$ either the *generalized derivation* defined by

$$\delta_{A,B}(S) := AS - SB \qquad \text{for all} \quad S \in L(H)\,,$$

or the *elementary operator* defined by

$$\Delta_{A,B}(S) := ASB - S \qquad \text{for all} \quad S \in L(H)\,.$$

If A and B^* are hyponormal then $p(\lambda I - d_{A,B}) < \infty$ for all $\lambda \in \mathbb{C}$, so $d_{A,B}$ is polaroid, see [14].

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