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Limits and the Ext Functor

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Abstract: We show the identity $\mathrm{Ext}(\lim_{\leftarrow} X_\alpha, \mathbb{R}) = \lim_{\rightarrow} \mathrm{Ext}(X_\alpha, \mathbb{R})$ for projective limits of quasi-Banach spaces X_α. The proof is derived from a pull-back lemma asserting that a topologically exact sequence $0 \to \mathbb{R} \to E \to Z \to 0$ of locally pseudoconvex spaces is the pull-back of an exact sequence of quasi-Banach spaces. Among the consequences we show that exact sequences $0 \to \mathbb{R} \to E \to Z \to 0$ of locally pseudoconvex spaces come induced by quasi-linear maps, which extends a result of Kalton for Fréchet spaces; and that projective limits of K-spaces are K-spaces.

Key words: Twisted sums, Ext functor, K-space.

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1. Introduction

The purpose of this paper is to obtain the identity

$$\mathrm{Ext}(\lim_{\leftarrow} X_\alpha, \mathbb{R}) = \lim_{\rightarrow} \mathrm{Ext}(X_\alpha, \mathbb{R})$$

for projective limits of quasi-Banach spaces X_α. This is interesting since it shows that the Ext functor behaves, in the category of quasi-Banach spaces, in a similar way as the $\mathfrak{L}$ functor in the following sense: for each fixed A the contravariant functor $\mathfrak{L}(\cdot, A)$ is right-adjoint of itself and therefore transforms inverse limits into direct limits. The functor $\mathrm{Ext}(\cdot, \mathbb{R})$ does the same with respect to projective limits.

We will base our proof in a pull-back lemma of independent interest:

Theorem 1.1. *A topologically exact sequence*

$$0 \to \mathbb{R} \to E \to Z \to 0$$

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of topological vector spaces in which Z is locally pseudoconvex (in the terminology of [6]) comes induced by a single quasi-linear map $F : Z \to \mathbb{R}$.

This extends a result of Kalton [7, Thm. 10.1] who proved the analogous result when Z is a Fréchet space. We think that what could deserve some interest is the method of proof, homological approach based on a pull-back lemma plus a suitable extension of the 3-lemma to the category of topological vector spaces. More precisely:

Pull-back lemma. *A topologically exact sequence $0 \to \mathbb{R} \to E \to Z \to 0$ of locally pseudoconvex spaces is the pull-back sequence of an exact sequence of quasi-Banach spaces.*

The homological approach to the result we present consists of two steps: 1) To obtain a version of the 3-lemma suitable to work with topological vector spaces (where the open mapping can fail); from this we can derive an answer to a problem posed in [4, pag. 186]; namely, that a topological vector space Q such that every topologically exact sequence $0 \to \mathbb{R} \to E \to Q \to 0$ splits is such that every twisted sum with a locally convex space is locally convex. In turn, this result is the natural extension of a theorem of Dierolf [2] for quasi-Banach spaces. 2) To prove a "pull-back lemma" asserting that if one has a topologically exact sequence $0 \to \mathbb{R} \to E \to Q \to 0$ in which Q verifies some minimal assumption (to be pseudo-convex, in the language of Jarchow [6]) then no essential information is lost when one simply considers the sequence localized around a neighborhood of zero.

2. Preliminaries

General background on homology can be found in [5]. A background on exact sequences of quasi-Banach spaces sufficient for our purposes can be seen in [1]. An exact sequence in a suitable category (vector spaces and linear maps, topological vector spaces and linear continuous maps, etc) is a diagram $0 \to Y \to X \to Z \to 0$ in the category with the property that the kernel of each arrow coincides with the image of the preceding. When an open mapping theorem exists (e.g., in quasi-Banach or Fréchet spaces) then it guarantees that Y is a subspace of X and the corresponding quotient X/Y is isomorphic to Z. Since we shall work in categories where no open mapping theorem exists we shall say that an exact sequence $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ is topologically exact when j is an into embedding and q is a continuous open map. For a more general background about twisted sums of quasi-Banach spaces and the theory of quasi-linear maps the reader is addressed to the monograph [1]. Here we are interested in the following facts:

Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be equivalent if there exists an arrow $T : X \to X_1$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \to & Y & \to & X_1 & \to & Z & \to & 0. \end{array}$$

This definition makes sense in the categories of quasi-Banach or Fréchet spaces where the open map theorem works,making T an isomorphism. An exact sequence is said to split if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$.

Push-out construction. Let $A : K \to Y$ and $B : K \to X$ be two arrows in a given category. The *push-out* of $\{A, B\}$ is an object Λ and two arrows $u : Y \to \Lambda$ and $v : Y \to \Lambda$ in the category such that $uA = vB$; and with the property that given another object Γ and two arrows $\alpha : Y \to \Gamma$ and $\beta : X \to \Gamma$ in the category verifying $\alpha A = \beta B$ then there exists a unique arrow $\gamma : \Lambda \to \Gamma$ such that $\beta = \gamma v$ and $\alpha = \gamma u$.

In the category of Hausdorff topological vector spaces push-outs exist. The push-out of two arrows $A : Y \to M$ and $B : Y \to X$ is the quotient space $\Lambda = M \oplus X/\overline{\Delta}$, where $\overline{\Delta}$ is the closure of $\{(Ay, -y) : y \in Y\}$, together with the restriction of the canonical quotient map $M \oplus X \to \Lambda$ to, respectively, M and X.

We are interested in the following property of the push-out construction

Lemma 2.1. *Given a topologically exact sequence $0 \to Y \xrightarrow{j} X \xrightarrow{p} Z \to 0$ of topological vector spaces and a linear continuous map $T : Y \to M$ and if PO denotes the push-out of the couple $\{j, T\}$ then there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & Y & \xrightarrow{j} & X & \xrightarrow{p} & Z & \to & 0 \\ & & T \downarrow & & \downarrow u & & \| & & \\ 0 & \to & M & \xrightarrow{J} & PO & \xrightarrow{Q} & Z & \to & 0 \end{array}$$

with topologically exact (lower) row.

Proof. Observe that since $0 \to Y \xrightarrow{j} X \xrightarrow{p} Z \to 0$ is topologically exact, Δ is closed, and the sequence $0 \to Y \to M \oplus X \to PO \to 0$, with injection

$i(y) = (Ty, y)$ and quotient map $q(m, x) = (m, x) + \Delta$ is topologically exact and PO is Hausdorff. The only undefined arrow is $Q : PO \to Z$, which is given by $Q((m, x) + \Delta) = px$. The commutativity of the right square and the continuity of the induced arrow $u : X \to PO$ yield that $Q : PO \to Z$ is open: if V is a neighborhood of 0 in PO then $q(u^{-1}(V)) = Q(uu^{-1}(V)) \subset Q(V)$. So, we prove that $J : M \to PO$ is an into isomorphism. The continuity of J is by definition, so we prove that it is open. Let thus U_M be a neighborhood of zero in M. We choose V_M a neighborhood of zero in M such that $V_M - V_M \subset U_M$ and then U_X, a neighborhood of 0 in X, such that $U_X \cap Y \subset T^{-1}(V_M)$. Let us show that $q(V_M \times U_X) \cap M \subset U_M$. Since $q(V_M \times U_X) \cap M$ coincides with

$$\{(m, x) + \Delta : m \in V_M, x \in U_X, \exists n \in M : (m, x) - (n, 0) \in \Delta\}$$

then $m = n + Ty$ and $x = y$ for some $y \in U_X \cap Y$; hence $Ty \in V_M$, and since $m \in V_M$ then $m - Ty \in V_M - V_M \subset U_M$ and the proof is complete. ∎

Pull-back construction. The dual notion to that of push-out is the pull-back. Let $A : X \to Z$ and $B : M \to Z$ be two arrows in a given category. The *pull-back* of $\{A, B\}$ is an object Λ and two arrows $u : \Lambda \to X$ and $v : \Lambda \to M$ in the category such that $Au = Bv$; and with the property that given another object Γ and two arrows $\alpha : \Gamma \to X$ and $\beta : \Gamma \to M$ in the category verifying $A\alpha = B\beta$ then there exists a unique arrow $\gamma : \Gamma \to \Lambda$ such that $\beta = v\gamma$ and $\alpha = u\gamma$.

In the category of Hausdorff topological vector spaces pull-backs exist. The pull-back of two arrows $A : X \to Z$ and $B : M \to Z$ is the subspace $PB = \{(x, m) \in X \times M : Ax = Bm\}$ endowed with the product topology induced by $X \oplus M$ and the corresponding restrictions of the canonical projections. The inclusion $Y \to PB$ is given by $y \to (y, 0)$. We are interested in the following property of the pull-back contruction; the proof is left to the reader.

Lemma 2.2. *Given a topologically exact sequence* $0 \to Y \xrightarrow{j} X \xrightarrow{p} Z \to 0$ *of topological vector spaces and a linear continuous map* $T : M \to Z$ *and if* PB *denotes the pull-back of the couple* $\{q, T\}$ *then there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \to & Y & \to & X & \to & Z & \to & 0 \\
 & & \| & & \uparrow & & \uparrow T & & \\
0 & \to & Y & \to & PB & \to & M & \to & 0
\end{array}$$

in which the lower row is topologically exact.

2.1. Twisted sums of quasi-Banach spaces. Exact sequences $0 \to Y \to X \to Z \to 0$ of quasi-Banach spaces correspond to homogeneous maps $F : Z \curvearrowright Y$ (we use this notation to stress the fact they are not usually linear) with the property that there exists a constant K such that for each two points $x, y \in Z$
$$\|F(x+y) - F(x) - F(y)\| \leq K(\|x\| + \|y\|).$$
Such maps are called *quasi-linear*.

Indeed, if one has an exact sequence $0 \to Y \to X \to Z \to 0$ then a quasi-linear map $F : Z \to Y$ can be obtained by considering a homogenous bounded selection $B : Z \to Y$ for the quotient map, then a linear (non-continuous) selection $L : Z \to Y$ for the quotient map, and making their difference $F = B - L$. Conversely, if one has a quasi-linear map $F : Z \to Y$ then endowing the product space $Y \times Z$ with the quasi-norm

$$\|(y, z)\| = \|y - Fz\| + \|z\|$$

one obtains a quasi-Banach space denoted $Y \oplus_F Z$ for which there exists an exact sequence $0 \to Y \to Y \oplus_F Z \to Z \to 0$. The quasi-Banach space $Y \oplus_F Z$ is called a *twisted sum of* Y *and* Z. Of course, the two processes are, in a very specific sense, inverse one of the other. This is the theory created by Kalton [7] and Kalton and Peck [9]. The reason to consider non-locally convex spaces is that twisted sums of locally convex spaces are not necessarily locally convex: Ribe [11] and Kalton [8] showed the existence of an exact sequence $0 \to \mathbb{R} \to E \to l_1 \to 0$ that does not split. Since $\mathbb{R}$ is uncomplemented, the space E cannot be locally convex.

Working with topological vector spaces, efforts have been made to mimicry this part of the theory in a non-locally bounded ambient (see [4]). Nevertheless, when $Y = \mathbb{R}$ Kalton [7] defines *quasi-linear* map $F : Q \curvearrowright \mathbb{R}$ as a homogeneous map so that for some continuous seminorm $n(\cdot)$ on Q one has:

$$|F(x+y) - F(x) - f(y)| \leq C(n(x) + n(y)).$$

With a quasi-linear map $F : Q \curvearrowright \mathbb{R}$ one can construct an exact sequence $0 \to \mathbb{R} \to \mathbb{R} \oplus_F Q \to Q \to 0$ endowing the product space $\mathbb{R} \times Q$ with the family of quasi-seminorms

$$q_\alpha(r, x) = |r - Fx| + p_\alpha(x)$$

where $\{p_\alpha\}$ runs through the gauge functionals of a fundamental system of neighborhoods of Q. The inclusion map $r \to (r, 0)$ is clearly continuous while the surjective map $q(r,x) = x$ is continuous and open. Hence $\mathbb{R} \oplus_F Q$ is a topological vector space which is complete when so is Q.

Nonetheless, it is by no means clear that a topologically exact sequence $0 \to \mathbb{R} \to E \to Q \to 0$ can be defined by a single quasi-linear map $F : Q \curvearrowright \mathbb{R}$. Nevertheless, Kalton [7] succeeds in showing that a sequence $0 \to \mathbb{R} \to E \to Q \to 0$ can be defined by a single quasi-linear map when Q and E are Fréchet spaces. Given a topological vector space E then $\mathcal{U}(E)$ denotes a fundamental system of closed balanced neighborhoods of zero. If the topology of E comes defined by a family of semi-quasi-norms, then given such a semi-quasi-norm p with unit ball U we denote by E_U the quotient vector space $E/\ker p_U$ endowed with the quasi-norm $\|\phi_U(x)\|_U = p_U(x)$; obviously, $\phi_U : E \to E_U$ is the quotient map.

3. Two algebraic lemmata

3.1. The 3-lemma for topological vector spaces. In the quasi-Banach or Fréchet space setting a simple consequence of the open mapping theorem and the 3-lemma is that twisted sums giving equivalent exact sequences are isomorphic. In topological vector spaces no open mapping exists, in general; nevertheless, the 3-lemma still works.

Proposition 3.1. (The 3-lemma for topological vector spaces) *Assume that one has a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \to & Y & \to & X_1 & \to & Z & \to & 0 \end{array}$$

in the category TVS of topological vector spaces and linear continuous maps, with topologically exact rows. Then T is a topological isomorphism.

Proof. Consider on X the initial vector space topology τ_X induced by T (namely, the vector space topology in which a typical basic neighborhood of 0 has he form $T^{-1}(U)$ for some neighborhood of zero U in X_1). Since T is continuous, $\tau \leq \tau_X$. Since $T|Y = id_Y$ it turns out that $\tau_X|Y \leq \tau|Y$. And since the right square is commutative and the arrows $X \to Z$ and $X_1 \to Z$ are quotient maps, $\tau_X/Y \leq \tau/Y$. Thus, using the following result of Dierolf and Schwanengel [3] *Let G be a group and $H \subset G$ be a subgroup. Let τ, τ_1 be group topologies on G such that $\tau_1 \subset \tau$. If $\tau|H = \tau_1|H$ and $\tau/H = \tau_1/H$ then $\tau = \tau_1$;* we get $\tau = \tau_X$, which makes T open. ∎

As we have already said, examples of Ribe and Kalton [11, 8] show that a twisted sum of locally convex spaces can be non-locally convex. A theorem of Dierolf [2] ensures that if all twisted sums of $\mathbb{R}$ and X are locally convex then all twisted sum of any Banach space Y and X are locally convex as well. As a consequence we give the following result that extends Dierolf's theorem (the question of whether such extension was possible was posed in [4] under the form: do locally convex K-spaces coincide with TSC-spaces?

Theorem 3.1. *Let Q be a locally convex space such that every topologically exact sequence $0 \to \mathbb{R} \to E \to Q \to 0$ splits if and only if every twisted sum of a locally convex space Y and Q is locally convex.*

Proof. Assume that Q is a locally convex topological vector space such that every topologically exact sequence $0 \to \mathbb{R} \to E \to Q \to 0$ splits. Let Y be a locally convex space and let $0 \to Y \to G \to Q \to 0$ be a topologically exact sequence. If $f : Y \to \mathbb{R}$ is a linear continuosu functional then the push-out diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & G & \to & Q & \to & 0 \\ & & \downarrow f & & \downarrow & & \| & & \\ 0 & \to & \mathbb{R} & \to & PO & \to & Q & \to & 0 \end{array}$$

and the fact that the lower sequence splits show that f can be extended to a linear continuous functional on G. Thus Y is a topological vector subspace of G endowed with its Mackey (G, G^*)-topology. It is a simple matter to verify that the induced quotient topology on Q is the original topology of Q. So the Mackey and the starting topology must coincide on G since they induce the same topologies in both Y and G/Y. ∎

3.2. A pull-back lemma.

Lemma 3.1. *Let $0 \to \mathbb{R} \to E \to Q \to 0$ be a topologically exact sequence of topological vector spaces in which the topology of Q comes defined by a family of semi-quasi-norms. Then there exists a neighborhood $U \in \mathcal{U}(Q)$, a quasi-Banach space X and an operator $\tau : X \to \hat{Q}_U$ such that*

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & E & \xrightarrow{q} & Q & \to & 0 \\ & & \| & & \downarrow & & \downarrow \phi_U & & \\ 0 & \to & \mathbb{R} & \to & X & \xrightarrow{\tau} & \hat{Q}_U & \to & 0 \end{array}$$

is a pull-back diagram.

Proof. Assume that $\mathbb{R} =< u >$. Let $U \in \mathcal{U}(E)$ such that $p_U(u) = 1$. Since q is open, $q(U)$ is a neighborhood of zero in Q. Let (U_n) be a chain of neighborhoods of zero in E starting with U; i.e., a sequence of neigborhoods of zero such that $U_{n+1} + U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. Let $\cap = \cap_{n\in\mathbb{N}} U_n$. It is easy to see that $\cap$ is a vector space. Moreover, the application $\overline{q}_U : E/\cap \to Q_{qU}$ given by $\overline{q}_U(x + \cap) = \phi_{qU} q(x)$ is well defined and gives the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & E & \to & Q & \to & 0 \\ & & j \downarrow & & \downarrow \phi_\cap & & \downarrow \phi_{qU} & & \\ 0 & \to & \ker \overline{q}_U & \to & E/\cap & \stackrel{\overline{q}}{\to} & Q_{qU} & \to & 0. \end{array}$$

If we endow $E/\cap$ with the quotient topology induced by the chain $\{U_n\}_{nin\mathbb{N}}$ then the diagram can be considered in the category of Hausdorff topological vector spaces. The lower row is topologically exact and the map $j : \mathbb{R} \to \ker \overline{q_U}$ is an into isomorphism since $p_U(u) \neq 0$. To simplify notation let us call $V = U_2$ and $W = U_3$. Observe that the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & E & \to & Q & \to & 0 \\ & & j \downarrow & & \downarrow & & \downarrow \phi_{qV} & & \\ 0 & \to & \ker \overline{q}_V & \to & E/\cap & \stackrel{\overline{q}_V}{\to} & Q_{qV} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \phi_{qV,qU} & & \\ 0 & \to & \ker \overline{q}_U & \to & E/\cap & \stackrel{\overline{q}_U}{\to} & Q_{qU} & \to & 0. \end{array}$$

Observe now that the points x such that $x + \cap \in \ker \overline{q_V}$ are those satisfying that for all $\varepsilon > 0$ there exists some $\lambda > 0$ and some $v \in V$ such that $x \varepsilon v + \lambda u$. If a point x can be written in two different forms

$$x = \varepsilon v_1 + \lambda_1 u = \varepsilon v_2 + \lambda_2 u$$

then $\varepsilon(v_1 - v_2) = (\lambda_2 - \lambda_1)u$ and thus

$$|\lambda_2 - \lambda_1| = p_U\big((\lambda_2 - \lambda_1)u\big) = p_U\big(\varepsilon(v_1 - v_2)\big) \leq \varepsilon p_U(v_1 - v_2) \leq \varepsilon.$$

This implies that if $\lambda(\varepsilon, x)$ is a family of scalars such that $x = \varepsilon v_\varepsilon + \lambda(\varepsilon, x)u$ then $\lim_{\varepsilon\to 0} \lambda(\varepsilon, x)$ exists for $x + \cap \in \ker \overline{q_V}$. Moreover, such limit is independent of the choice of the neighborhood $W \subset V$ (as long as $x + \cap \in \ker \overline{q}_W$). We can define a linear projection $L : \ker \overline{q_V} \to < p >$ by the formula

$$L(x) = \lim_{\varepsilon \to 0} \lambda(\varepsilon, x)u.$$

The map L is continuous restricted to $\ker \overline{q_W} \to < p >$ since for small ε

$$\begin{aligned}(Lx)u &= (Lx)u - \lambda(\varepsilon,x)u + \lambda(\varepsilon,x)u - x + x\\ &= \big(Lx - \lambda(\varepsilon,x)\big)u + \big(\lambda(\varepsilon,x)u - x\big) + x\\ &\in \varepsilon_1 U_{n+2} + \varepsilon U_{n+2} + p_{U_{n+1}}(x)U_{n+1}\\ &\subset \varepsilon U_{n+1} + p_{U_{n+1}}(x)U_{n+1}\\ &\subset 2p_{U_{n+1}}(x)U_n;\end{aligned}$$

and thus $p_{U_n}(L(x)u) \leq 2p_{U_{n+1}}(x)$ and L is continuous.

Consider now the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & E & \to & Q & \to & 0\\ & & j\downarrow & & \downarrow & & \downarrow \phi_{qW} & & \\ 0 & \to & \ker \overline{q_W} & \to & E/\cap & \xrightarrow{\overline{q_W}} & Q_{qW} & \to & 0\\ & & L\downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathbb{R} & \to & PO & \to & Q_{qW} & \to & 0\\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{R} & \to & \widehat{PO} & \to & \widehat{Q}_{qW} & \to & 0.\end{array}$$

The exactness of the last sequence (completion of the previous line) is a rather standard consequence of the open mapping theorem (see [9]; or else [1]). Thus, if $\phi_q W$ is understood as a map $Q \to \widehat{Q_{qW}}$, one can construct the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & \widehat{PO} & \to & \widehat{Q}_{qW} & \to & 0\\ & & \| & & \uparrow & & \uparrow \phi_{qW} & & \\ 0 & \to & \mathbb{R} & \to & PB & \to & Q & \to & 0.\end{array}$$

It only remains to prove that this last sequence is topologically equivalent to the starting one. But the universal property of the pull-back gives a connecting map $E \to PB$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & PB & \to & Q & \to & 0\\ & & \| & & \uparrow & & \| & & \\ 0 & \to & \mathbb{R} & \to & E & \to & Q & \to & 0.\end{array}$$

Now, the 3-lemma we obtained at 3.1 shows that the two sequences are topologically equivalent. ∎

4. Using the pull-back lemma

From here we obtain the result we wanted.

Theorem 4.1. *A topologically exact sequence* $0 \to \mathbb{R} \to E \to Q \to 0$ *of topological vector spaces in which the topology of* Q *comes induced by a family of semi-quasi-norms is defined by a quasi-linear map* $F : Q \to \mathbb{R}$

Proof. By the standard theory of exact sequences of quasi-Banach spaces, the sequence

$$0 \to \mathbb{R} \to \widehat{PO} \to \widehat{Q_{qW}} \to 0$$

is equivalent to some sequence

$$0 \to \mathbb{R} \to \mathbb{R} \oplus_F \widehat{Q_{qW}} \to \widehat{Q_{qW}} \to 0$$

defined by some quasi-linear map $F : \widehat{Q_{qW}} \to \mathbb{R}$. It only remains to observe that in a pull-back square

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & E & \to & Q & \to & 0 \\ & & \| & & \downarrow & & \downarrow \phi_{qW} & & \\ 0 & \to & \mathbb{R} & \to & \mathbb{R} \oplus_F Q_{qW} & \to & Q_{qW} & \to & 0 \end{array}$$

the pull-back sequence is defined by the quasi-linear map $F\phi_{qW}$. We state this in a separate lemma:

Lemma 4.1. *Let* $0 \to \mathbb{R} \to E \to Q \to 0$ *be a topologically exact sequence of topological vector spaces defined by a quasi-linear map* $G : Q \curvearrowright \mathbb{R}$ *and let* $T : V \to Q$ *be a linear continuous map. Then the pull-back sequence is equivalent to the sequence defined by the quasi-linear map* GT.

Proof. One only has to appeal to the 3-lemma for topological vector spaces once observed that there exists a linear continuous map u making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R} & \to & \mathbb{R} \oplus_G Q & \to & Q & \to & 0 \\ & & \| & & \uparrow u & & \uparrow T & & \\ 0 & \to & \mathbb{R} & \to & \mathbb{R} \oplus_{GT} V & \to & V & \to & 0. \end{array}$$

The definition of u is $u(r, v) = (r, Tv)$. It clearly makes the diagram commutative. As for the continuity, if A is a neighborhood in F and B is a neighborhood in V so that $p_A(Tv) \leq c(A, B)p_B(v)$ then

$$|r - GTv| + p_A(Tv) \leq |r - GTv| + c_{AB}p_B(v) \leq c(A, B)\big(|r - GTv| + p_B(v)\big).$$

This completes the proof of the lemma and the theorem. ∎

A topological vector space X is said to be a K-space when every exact sequence $0 \to \mathbb{R} \to E \to X \to 0$ splits (i.e., every quasi Banach space E such that $E/\mathbb{R} = X$ is locally convex). We have:

Proposition 4.1. *A projective limit of quasi-Banach K-spaces is a K-space.*

Proof. If Q is a projective limit of quasi-Banach K-spaces then every topologically exact sequence $0 \to \mathbb{R} \to E \to Q \to 0$ of topological vector spaces is the pull-back sequence of some sequence $0 \to \mathbb{R} \to X \to \widehat{Q_U} \to 0$ of quasi-Banach spaces. One can also choose $U \subset V$ with $\widehat{Q_V}$ a K-space. Hence $0 \to \mathbb{R} \to X \to \widehat{Q_U} \to 0$ splits, and so does $0 \to \mathbb{R} \to E \to Q \to 0$. ∎

From this and the pull back it immediately follows

Theorem 4.2. *Let* $\lim_{\leftarrow} X_\alpha$ *be a projective limit of quasi-Banach spaces. Then*

$$\operatorname{Ext}(\lim_{\leftarrow} X_\alpha, \mathbb{R}) = \lim_{\rightarrow} \operatorname{Ext}(X_\alpha, \mathbb{R}) .$$

Proof. The pull-back lemma yields for every element $F \in \operatorname{Ext}(\lim_{\leftarrow} X_\alpha, \mathbb{R})$ an inductive family (F_α) with $F_\alpha \in \operatorname{Ext}(X_\alpha, \mathbb{R})$. The converse is clear. ∎

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