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The $SL(2,\mathbb{C})$ Character Variety of a Class of Torus Knots

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Abstract: Let $K_{\frac{m}{2}}$ be the torus knot of type $(m,2)$. It is well-known that the fundamental group of $S^3 \setminus K_{\frac{m}{2}}$ is $G = \langle A, B \mid A^m = B^2 \rangle$. In this paper we obtain a defining polynomial of the character variety $X(G)$ which allows us to give an easy geometrical description of it.

Key words: Torus knot, character variety

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1. The character variety of a finitely presented group

Let G be a group, a *representation* $\rho : G \longrightarrow SL(2,\mathbb{C})$ is just a group homomorphism. We say that two representations ρ and ρ' are equivalent if there exists $P \in SL(2,\mathbb{C})$ such that $\rho'(g) = P^{-1}\rho(g)P$ for every $g \in G$. A representation ρ is *reducible* if the elements of $\rho(G)$ all share a common eigenvector, otherwise we say ρ is *irreducible*.

Now, let us consider a finitely presented group $G = \langle x_1, \dots, x_k \mid r_1, \dots, r_s \rangle$ and let $\rho : G \longrightarrow SL(2,\mathbb{C})$ be a representation. It is clear that ρ is completely determined by the k-tuple $(\rho(x_1), \dots, \rho(x_k))$ and thus we can identify

$$R(G) = \{(\rho(x_1), \dots, \rho(x_k)) \mid \rho \text{ is a representation of } G\} \subseteq \mathbb{C}^{4k}$$

with the set of all representations of G into $SL(2,\mathbb{C})$, which is therefore (see [1]) a well-defined affine algebraic set, up to canonical isomorphism.

Recall that given a representation $\rho : G \longrightarrow SL(2,\mathbb{C})$ its character $\chi_\rho : G \longrightarrow \mathbb{C}$ is defined by $\chi_\rho(g) = \operatorname{tr} \rho(g)$. Note that two equivalent representations ρ and ρ' have the same character, and the converse is also true if ρ or ρ' is

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irreducible [1, Prop. 1.5.2]. Now choose any $g \in G$ and define $t_g : R(G) \longrightarrow \mathbb{C}$ by $t_g(\rho) = \chi_\rho(g)$. Let T denote the ring generated by $\{t_g \mid g \in G\}$, then ([1, Prop. 1.4.1]) T is finitely generated ring and, moreover, it can be shown using the well-known identities

$$\begin{aligned} \operatorname{tr} A &= \operatorname{tr} A^{-1} \\ \operatorname{tr} AB &= \operatorname{tr} BA \\ \operatorname{tr} AB &= \operatorname{tr} A \operatorname{tr} B - \operatorname{tr} AB^{-1} \end{aligned}$$

which hold in $SL(2,\mathbb{C})$ (see [2, Cor. 4.1.2]) that T is generated by the set:

$$\{t_{x_i}, t_{x_i x_j}, t_{x_i x_j x_h} \mid 1 \le i < j < h \le k\}.$$

Now choose $\gamma_1, \dots, \gamma_\nu \in G$ such that $T = \langle t_{\gamma_i} \mid 1 \le i \le \nu \rangle$ and define the map $t : R(G) \longrightarrow \mathbb{C}^\nu$ by $t(\rho) = (t_{\gamma_1}(\rho), \dots, t_{\gamma_\nu}(\rho))$. Observe that $\nu \le \frac{k(k^2+5)}{6}$. Put $X(G) = t(R(G))$, then $X(G)$ is an algebraic variety which is well defined up to canonical isomorphism [1, Cor. 1.4.5] and is called the *character variety* of the group G in $SL(2,\mathbb{C})$. Note that $X(G)$ can be identified with the set of all characters χ_ρ of representations $\rho \in R(G)$.

For every $1 \le j \le k$ and for every $1 \le i \le s$ we have that $p_{ij} = t_{r_i x_j} - t_{x_j}$ is a polynomial with rational coefficients in the variables $\{t_{x_{i_1} \dots x_{i_m}} \mid m \le 3\}$, (see [2, Cor. 4.1.2]). Then, we have the following explicit description of $X(G)$.

Theorem 1.1. ([2, Theor. 3.2]) $X(G) = \{\overline{x} \in X(F_k) \mid p_{ij}(\overline{x}) = 0, \ \forall i, j\}$, where F_k is the free group in k generators.

2. Torus knots

Recall that $\mathbb{R}^2$ is the universal covering of the torus T^2. We define the action $\Phi : (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $\Phi((m,n),(x,y)) = (x+m, y+n)$, this action induces an isomorphism $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z}) \cong T^2$ that we shall denote by ϕ. If we now consider the family $\{r_p : y = px \mid p \in \mathbb{R}\}$ of straight lines passing through the origin, it is easily seen that if p is irrational then $\phi(r_p)$ is dense in T^2 and if $p = m/n$ with g.c.d.$(m,n) = 1$, then $\phi(r_p) \subseteq T^2 \subseteq \mathbb{R}^3$ is a knot. We denote this knot by $K_{\frac{m}{n}}$ and call it the torus knot of type (m,n) (see [5, Chapter 3] for further considerations).

If we denote, as usual, by $G(K)$ the fundamental group of the exterior of any knot K it is well-known that

$$G(K_{\frac{m}{n}}) \cong \langle A, B \mid A^m = B^n \rangle.$$

Now, if $m \geq 1$ is an odd integer, let us define the following group:

$$H_m = \langle x, y \mid \underbrace{xyxy\dots yx}_{\text{length } m} = \overbrace{yxyx\dots xy}^{\text{length } m} \rangle.$$

Where length m means that there are m letters counting y's and x's together. Note that since m is odd, the word starts and ends with the same letter. Now, the following isomorphism will be useful in the sequel.

LEMMA 2.1. *Let $m \geq 1$ be an odd integer. Then $G(K_{\frac{m}{2}}) \cong H_m$.*

Proof. We define $\varphi : H_m \longrightarrow G(K_{\frac{m}{2}})$ given by $\varphi(x) = B^{-1}A^{\frac{m+1}{2}}$, $\varphi(y) = A^{-\frac{m-1}{2}}B$ and $\psi : G(K_{\frac{m}{2}}) \longrightarrow H_m$ given by $\psi(A) = yx$, $\psi(B) = \overbrace{yxyx\dots y}^{\text{length } m}$. The result follows from some easy computations. ∎

3. SOME FAMILIES OF POLYNOMIALS

We will start this section by defining recursively the following family of polynomials:

$$q_1(T) = T - 2,$$
$$q_2(T) = T + 2,$$
$$\prod_{1 \neq d \mid n} q_d\left(X + \frac{1}{X}\right) = \frac{X^{n-1} + X^{n-2} + \cdots + X + 1}{X^{\frac{n-1}{2}}} \quad \text{if } n \text{ is odd,}$$
$$\prod_{1,2 \neq d \mid n} q_d\left(X + \frac{1}{X}\right) = \frac{X^{n-2} + X^{n-4} + \cdots + X^2 + 1}{X^{\frac{n-2}{2}}} \quad \text{if } n \text{ is even.}$$

Remark 1. If we recall the recursive definition of the cyclotomic polynomials (see [3, Chapter 5]) by

$$\prod_{d \mid n} g_d(T) = T^n - 1,$$

then it is easily seen that for $n > 1$

$$g_n(T) = T^{\frac{\varphi(n)}{2}} q_n\left(T + \frac{1}{T}\right)$$

where φ is the Euler function.

Now we introduce another family of polynomials:

$$p_1(X) = X,$$
$$p_2(X) = X^2 - 2,$$
$$p_n(X) = Xp_{n-1}(X) - p_{n-2}(X),\ \forall n \geq 3.$$

Remark 2. Let G be a group and $\rho : G \longrightarrow SL(2,\mathbb{C})$ a representation. Then $p_n(\mathrm{tr}\rho(x)) = \mathrm{tr}\rho(x^n)$ for every $n \geq 1$. For the sake of completeness we will set, where necessary, $p_0(X) = 1$.

We have the following relationship between the families we have just defined:

Proposition 3.1. *$p_n(X) - 2 = q_1(X)\prod_{1\neq d|n} q_d^2(X)$ if n is odd, and $p_n(X) - 2 = q_1(X)q_2(X)\prod_{1,2\neq d|n} q_d^2(X)$ if n is even.*

Proof. We will just show the odd case, the even case being completely analogous.

Consider the cyclic group $G = \langle x\rangle$ and a representation $\rho : G \longrightarrow SL(2,\mathbb{C})$. We can suppose, conjugating if necessary, that

$$\rho(x) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

In such case it must be

$$\rho(x^n) = \rho(x)^n = \begin{pmatrix} a^n & c \\ 0 & a^{-n} \end{pmatrix}.$$

Set $X = \mathrm{tr}(\rho(x)) = a + a^{-1}$, then

$$\begin{aligned} p_n(X) - 2 &= \mathrm{tr}(\rho(x^n)) - 2 = a^n + a^{-n} - 2 = \frac{(a^n-1)^2}{a^n} \\ &= \frac{1}{a^n}\left(\prod_{d|n} g_d(a)\right)^2 = \frac{(a-1)^2}{a^n}\left(\prod_{1\neq d|n} g_d(a)\right)^2 \\ &= \frac{(a+a^{-1}-2)a}{a^n}\left(\prod_{1\neq d|n} a^{\frac{\varphi(d)}{2}} q_d(a+a^{-1})\right)^2 \\ &= q_1(X)\prod_{1\neq d|n} q_d^2(X). \end{aligned}$$

where the identity $\sum_{d|n} \varphi(d) = n$ was used. ∎

Remark 3. The roots of $p_n(X) - 2$ are precisely the possible values of $\mathrm{tr}(\rho(x))$ if $\rho : G \longrightarrow SL(2,\mathbb{C})$ is a representation and $x^n = 1$.

Let R be any ring and consider a polynomial $g(T) = \sum_{i=0}^{n} a_i T^i \in R[T]$. We define

$$* : R[T] \longrightarrow R[T] \quad \text{by} \quad g^*(T) = \sum_{i=0}^{n} (-1)^{n-i} a_i T^i.$$

In the next lemma we show some useful properties of this application.

LEMMA 3.2. *Given $g, h \in R[T]$ we have:*

a) $g^{**} = g$.

b) $(gh)^* = g^* h^*$.

c) *If $g(T) = \sum_{i=0}^{n} a_i T^i$, then $g^* = g$ if and only if $a_i = 0$ for every i such that $(n - i) \equiv 1$ (mod 2).*

Proof. a) and b) follow from the identity $g^*(T) = (-1)^{\deg(g)} g(-T)$. c) is straightforward. ∎

We can use the involution just defined to show another relation between our two families of polynomials.

PROPOSITION 3.3. *If $s \geq 1$ is an integer, then*

$$\sum_{i=0}^{s} (-1)^i p_{s-i}(Z) = \prod_{1 \neq d | 2s+1} q_d^*(Z).$$

Proof. We observe that the degree of every term in $p_s(Z)$ has the same parity as $s = \deg p_s(Z)$. This fact together with the definition of $*$ shows that

$$\left(\sum_{i=0}^{s} (-1)^i p_{s-i}(Z) \right)^* = \sum_{i=0}^{s} p_i(Z).$$

Now, we claim that

$$\sum_{i=0}^{s} p_i(Z) = \prod_{1 \neq d | 2s+1} q_d(Z).$$

We will prove this by induction on s, the case $s = 1$ being trivial since $p_0(Z) + p_1(Z) = 1 + Z = q_3(Z)$. Now let $s > 1$ be an odd integer (the even case is similar), by hypothesis we have

$$\sum_{i=0}^{s} p_i(Z) = \sum_{i=0}^{s-1} p_i(Z) + p_s(Z) = \prod_{1 \neq d | 2s-1} q_d(Z) + p_s(Z)$$

and thus, setting $Z = X + \frac{1}{X}$ one obtains:

$$\begin{aligned}
\sum_{i=0}^{s} p_i\left(X + \frac{1}{X}\right) &= \prod_{1 \neq d | 2s-1} q_d\left(X + \frac{1}{X}\right) + p_s\left(X + \frac{1}{X}\right) \\
&= \frac{\sum_{i=0}^{2s-2} X^i}{X^{s-1}} + q_1\left(X + \frac{1}{X}\right) \prod_{1 \neq d | s} q_s^2\left(X + \frac{1}{X}\right) + 2 \\
&= \frac{\sum_{i=0}^{2s-2} X^i}{X^{s-1}} + \frac{(X-1)^2}{X} \frac{\left(\sum_{i=0}^{s-1} X^i\right)^2}{X^{s-1}} + 2 \\
&= \frac{\sum_{i=0}^{2s-2} X^i}{X^{s-1}} + \frac{X^{2s}+1}{X^s} = \frac{\sum_{i=0}^{2s} X^i}{X^s} = \prod_{1 \neq d | 2s+1} q_d\left(X + \frac{1}{X}\right).
\end{aligned}$$

The proof is now completed by applying 3.2 a), b). ∎

4. The $SL(2, \mathbb{C})$ character variety of the knots $K_{\frac{m}{2}}$

The aim of this section is to give a generating family of polynomials for $X(G)$ with $G = G(K_{\frac{m}{2}})$ ($m > 1$, odd) as well as a geometric description of this variety. Since we know that $G(K_{\frac{m}{2}}) \cong H_m$ we will work with $X(H_m)$ instead.

Before going into our main result we have to introduce another polynomial. We set $h(X, Z) = X^2 - Z$ and $k(X) = X^2 - 2$. Now we define

$$\alpha_l(X, Z) = \begin{cases} h(X, Z) & \text{if } l \text{ is even.} \\ k(X) & \text{if } l \text{ is odd.} \end{cases}$$

and finally we write for $s \geq 1$

$$f_s(X,Z) = p_s(Z)(h(X,Z)-1) + \sum_{i=1}^{s}(-1)^i p_{s-i}(Z)\alpha_i(X,Z).$$

With these definitions we can prove the following result.

PROPOSITION 4.1. *If $m > 1$ is an odd integer, then*

$$X(H_m) = \{(X,Z) \in \mathbb{C}^2 \mid f_{\frac{m-1}{2}}(X,Z) = 0\}.$$

Proof. We set $w = \underbrace{xyxy\dots yx}_{length\ m}\underbrace{y^{-1}x^{-1}y^{-1}x^{-1}\dots y^{-1}}_{length\ m}$. Then, using Theorem 3.2 in [2], we have

$$X(H_m) = \{(X,Y,Z) \in \mathbb{C}^3 \mid p_0(X,Y,Z) = p_1(X,Y,Z) = p_2(X,Y,Z) = 0\}$$

where

$$\begin{aligned} X &= \tau_x, & p_0(X,Y,Z) &= \tau_w - \tau_1 \\ Y &= \tau_y, & p_1(X,Y,Z) &= \tau_{wx} - \tau_x \\ Z &= \tau_{xy}, & p_2(X,Y,Z) &= \tau_{wy} - \tau_y \end{aligned}$$

Now, $wy = \underbrace{xyx\dots y}_{length\ m-1}\, x(\underbrace{xyx\dots y}_{length\ m-1})^{-1}$ so we have $\tau_{wy} = \tau_x$ obtaining that $p_2(X,Y,Z) = X - Y$.

On the other hand $\tau_{wx} = \tau_w\tau_x - \tau_{wx^{-1}}$ and $wx^{-1} = \underbrace{xy\dots x}_{length\ m}\, y^{-1}(\underbrace{xy\dots x}_{length\ m})^{-1}$ so we get $\tau_{wx^{-1}} = \tau_{y^{-1}} = \tau_y$ and thus

$$\begin{aligned} p_1(X,Y,Z) &= \tau_{wx} - \tau_x = \tau_w\tau_x - \tau_y - \tau_x = \tau_x(\tau_w - 1) - \tau_y \\ &= Xp_0(X,Y,Z) + X - Y. \end{aligned}$$

Set now $w_1 = (xy)^{\frac{m-1}{2}}$ and $w_2 = (yx)^{\frac{m-1}{2}}yx^{-1}$. Since $w = 1$ if and only if $w_1 = w_2$, then it is easy to see that $p_0(X,Y,Z) = \tau_w - \tau_1$ vanishes if and only if $f(X,Y,Z) = \tau_{w_2} - \tau_{w_1}$ does. As a a consequence

$$\begin{aligned} X(H_m) &= \{(X,Y,Z) \in \mathbb{C}^3 \mid f(X,Y,Z) = 0 = X - Y\} \\ &\cong \{(X,Z) \in \mathbb{C}^2 \mid f(X,X,Z) = 0\} \end{aligned}$$

Let us compute now the polynomial $f(X,Y,Z)$.

Firstly it is obvious by definition that $\tau_{w_1} = p_{\frac{m-1}{2}}(Z)$. In addition we have

$$\tau_{w_2} = \tau_{(yx)^{\frac{m-1}{2}}}\tau_{yx^{-1}} - \tau_{(xy)^{\frac{m-3}{2}}xx} = p_{\frac{m-1}{2}}(Z)(XY-Z) - \tau_{(xy)^{\frac{m-3}{2}}xx}.$$

Thus, we have that

$$f(X,Y,Z) = \tau_{w_2} - \tau_{w_1} = p_{\frac{m-1}{2}}(Z)(XY-Z-1) - \tau_{(xy)^{\frac{m-3}{2}}xx}.$$

Now, we claim that if $X = Y$ then

$$\tau_{(xy)^{\frac{m-3}{2}}xx} = -\sum_{i=1}^{\frac{m-1}{2}}(-1)^i p_{\frac{m-1}{2}-i}(Z)\alpha_i(X,Z).$$

We will proceed by induction on m, the cases $m = 3, 5$ being an easy verification. If $n \geq 7$, then some straightforward computations and the use of the recursive definition of the family $\{p_n\}$ gives

$$\begin{aligned}
\tau_{(xy)^{\frac{m-3}{2}}xx} &= \tau_{xy}\tau_{(xy)^{\frac{m-5}{2}}xx} - \tau_{(xy)^{\frac{m-7}{2}}xx} \\
&= -Z\sum_{i=1}^{\frac{m-3}{2}}(-1)^i p_{\frac{m-3}{2}-i}(Z)\alpha_i(X,Z) + \sum_{i=1}^{\frac{m-5}{2}}(-1)^i p_{\frac{m-5}{2}-i}(Z)\alpha_i(X,Z) \\
&= -\sum_{i=1}^{\frac{m-7}{2}}(-1)^i[Zp_{\frac{m-3}{2}-i}(Z) - p_{\frac{m-3}{2}-i-1}(Z)]\alpha_i(X,Z) \\
&\quad -(-1)^{\frac{m-5}{2}}Zp_1(Z)\alpha_{\frac{m-5}{2}}(X,Z) - (-1)^{\frac{m-3}{2}}Zp_0(Z)\alpha_{\frac{m-3}{2}}(X,Z) \\
&\quad +(-1)^{\frac{m-5}{2}}p_0(Z)\alpha_{\frac{m-5}{2}}(X,Z) \\
&= -\sum_{i=1}^{\frac{m-1}{2}}(-1)^i p_{\frac{m-1}{2}-i}(Z)\alpha_i(X,Z).
\end{aligned}$$

Consequently, and recalling the definition of $f_s(X,Z)$ we get that $f(X,X,Z) = f_{\frac{m-1}{2}}(X,Z)$ and the proof is complete. ∎

In order to obtain a geometrical description of $X(H_m)$ we are interested in factorizing the polynomial $f_s(X,Z)$. We will start by rewriting it in a different way:

$$f_s(X,Z) = (X^2-Z-2)\left(\sum_{i=0}^{s}(-1)^i p_{s-i}(Z)\right) + p_s(Z) + \sum_{i=1}^{s}(-1)^i\beta_i(Z)p_{s-i}(Z),$$

where $\beta_k(Z) = \begin{cases} Z & \text{if } k \text{ is odd.} \\ 2 & \text{if } k \text{ is even.} \end{cases}$

LEMMA 4.2. $p_s(Z) + \sum_{i=1}^{s}(-1)^i \beta_i(Z) p_{s-i}(Z) = 0$

Proof. It is enough to use the fact that $p_s(Z) - Zp_{s-1}(Z) = -p_{s-2}(Z)$. ∎

COROLLARY 4.3. *If $m > 1$ is an odd integer, then*

$$X(H_m) \cong \{(X, Z) \in \mathbb{C}^2 \mid (X^2 - Z - 2) \prod_{1 \neq d | m} q_d^*(Z) = 0\}.$$

Proof. Just apply Proposition 3.3 and Lemma 4.2 to Proposition 4.1. ∎

Now, we will find the roots of $q_d(Z)$. This is done in the following lemma.

LEMMA 4.4. *Let $\{a_1, \overline{a}_1, \ldots, a_{\frac{\varphi(r)}{2}}, \overline{a}_{\frac{\varphi(r)}{2}}\}$ be set of the $\varphi(r)$ primitive rth roots of unity. Then*

$$q_r(Z) = \prod_{i=1}^{\frac{\varphi(r)}{2}} (Z - 2Re(a_i))$$

Proof. Recall that, for $r > 2$ we have $g_r(X) = X^{\frac{\varphi(r)}{2}} q_r(X + 1/X)$ with $g_r(X)$ being the rth cyclotomic polynomial. As for all $1 \leq j \leq \frac{\varphi(r)}{2}$ it holds that $\frac{1}{a_j} = \overline{a_j}$ we obtain that $q_r(Z)$ has exactly $\frac{\varphi(r)}{2}$ different roots, namely $\{2\mathrm{Re}(a_1), \ldots, 2\mathrm{Re}(a_{\frac{\varphi(r)}{2}})\}$. This together with the fact that the degree of $q_r(Z)$ is $\frac{\varphi(r)}{2}$ completes the proof. ∎

This lemma allows us to go one step further in our description of the curve $X(H_m)$.

COROLLARY 4.5. *Let $m > 1$ be an odd integer. In the complex plane (X, Z) the curve $X(H_m)$ consists of the parabola $Z = X^2 - 2$ and the union of $\frac{m-1}{2}$ horizontal lines of the form $Z = -2Re(w)$, being $1 \neq w$ an mth root of unity.*

Proof. It is enough to apply the previous lemma together with the fact that given a polynomial g, then a number a is a root of g if and only if $-a$ is a root of g^*. ∎

Remark 4. Recall that the genus of the torus knot $K_{\frac{m}{n}}$ is $\frac{(m-1)(n-1)}{2}$. In our case, where $n = 2$ the genus of $K_{\frac{m}{2}}$ is $\frac{m-1}{2}$, which precisely coincides with the number of straight lines in $X(H_m)$.

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