



On formal power series over topological algebras

M. WEIGT¹, I. ZARAKAS²

¹ *Department of Mathematics, Nelson Mandela University
Summerstrand Campus (South), Port Elizabeth, 6031, South Africa*

² *Department of Mathematics, Hellenic Military Academy, Athens, 19400, Greece*

martin.weigt@mandela.ac.za, weigt.martin@gmail.com, gzarak@hotmail.gr

Received April 21, 2021
Accepted February 9, 2022

Presented by D. Dikranjan

Abstract: We present a general survey on formal power series over topological algebras, along with some perspectives which are not easily found in the literature.

Key words: Topological algebra, formal power series, algebra homomorphism.

MSC (2020): 46H05, 46H10, 46H30.

1. INTRODUCTION

In our research course in [17], we came across the following question: Let $A[\tau]$ be a GB^* -algebra, δ a closed derivation of A with domain $D(\delta)$, $x \in D(\delta)$ and $0 \neq \lambda \in \mathbb{C}$ such that $(\lambda 1 - x)^{-1}$ exists in A . Does it follow that $(\lambda 1 - x)^{-1}$ lies in the domain of the derivation? The latter question is answered in the affirmative in the case of C^* -algebras by Bratelli and Robinson in [7], and more generally for pro- C^* -algebras in [17]. The basic tool used in the proof of Bratelli and Robinson's result is the Neumann series which is complemented with the use of an analytic continuation argument.

Given that a GB^* algebra is an algebra of possibly unbounded operators, the use of Neumann series in the GB^* -algebra setting fails and consequently the Bratelli-Robinson proof cannot be carried over.

We were thus led to ask ourselves whether it might be possible to answer our initial question, which for brevity we shall call the domain problem, with the use of formal power series. In particular, the following two questions naturally arise: (I) does there exist a unital injective algebra homomorphism $\tilde{\phi} : D(\delta) \rightarrow D(\delta)[[X]]$, with $\tilde{\phi}(x) = X$? (II) Does there exist a unital homomorphism $\psi : \mathbb{C}[[X]] \rightarrow D(\delta)$ such that $\psi(X) = x$? An affirmative answer to any of the above questions could have the potential to place $(\lambda 1 - x)^{-1}$ inside the domain of the derivation (see Section 3 for details).



However, observe that the existence of such a unital homomorphism in Question II would imply that x is an element of $D(\delta)$ such that $\text{Sp}_{D(\delta)}(x) \subseteq \{0\}$ (see Section 3). Although this is a severe restriction, this motivates the following more general questions:

- (I') Let $A[\tau]$ be a topological algebra and let $x \in A$. Does there exist a unital injective algebra homomorphism $\tilde{\phi} : A \rightarrow A[[X]]$, with $\tilde{\phi}(x) = X$?
- (II') Let $A[\tau]$ be a topological algebra and let $x \in A$. Does there exist a unital homomorphism $\psi : \mathbb{C}[[X]] \rightarrow A$ such that $\psi(X) = x$?

These two more general questions are interesting for their own sake, and are the main focus of this paper.

In the current expository paper, we visit some well-known results of relevance to our purposes in answering the two more general questions in the previous paragraph, which refer to formal power series on Banach and Fréchet algebras. The literature is rich in papers which deal with formal power series on topological algebras and what's more with their intriguing interplay with derivations. Indicatively, we refer the reader to [16, 4, 8]. Moreover, we present certain byproducts obtained along the way, for the case of generalized B^* -algebras, which give stimulus for further research on the topic, and which appear to be interesting for their own sake.

2. PRELIMINARIES

Throughout, all algebras are assumed to be over the complex numbers, i.e., all algebras are assumed to be \mathbb{C} -algebras.

A *topological algebra* $A[\tau]$ is an algebra which is also a topological vector space and has the property that multiplication is separately continuous. In some topological algebras, such as Fréchet algebras, multiplication is not only separately continuous, but jointly continuous [10]. By a Fréchet algebra, we mean a complete metrizable topological algebra.

A topological algebra is said to be a *locally convex algebra* if it is also a locally convex space. An *m -convex algebra* is a locally convex algebra $A[\tau]$ for which the topology τ on A is defined by a family of seminorms $\{p_\gamma : p_\gamma \in \Gamma\}$ such that $p_\gamma(xy) \leq p_\gamma(x)p_\gamma(y)$ for all $x, y \in A$ and for all $\gamma \in \Gamma$. For every $\gamma \in \Gamma$, let $N_\gamma = \{x \in A : p_\gamma(x) = 0\}$. Then A/N_γ is a normed algebra with respect to the norm $p_\gamma(x + N_\gamma) = p_\gamma(x)$ for all $x \in A$ and $A = \varprojlim_\gamma A/N_\gamma$. The completion A_γ of A/N_γ is therefore a Banach algebra for all $\gamma \in \Gamma$. If A is, in addition, complete, then $A = \varprojlim_\gamma A_\gamma$ up to a topological and algebraic

*-isomorphism, which is called the *Arens-Michael decomposition* of the complete m -convex algebra A . For details, we refer to [10].

A C^* -algebra is a Banach *-algebra A such that $\|x^*x\| = \|x\|^2$ for all $x \in A$, where $\|\cdot\|$ denotes the submultiplicative norm on A defining the topology on A .

DEFINITION 2.1. ([2]) Let $A[\tau]$ be a unital topological *-algebra and let \mathcal{B}_A^* (denoted by \mathcal{B}^* if no confusion arises as to which algebra is being considered) be a collection of subsets B of A satisfying the following:

- (i) B is absolutely convex, closed and bounded;
- (ii) $1 \in B$, $B^2 \subseteq B$ and $B^* = B$.

For every $B \in \mathcal{B}^*$, denote by $A[B]$ the linear span of B , which is a normed algebra under the gauge function $\|\cdot\|_B$ of B . If $A[B]$ is complete for every $B \in \mathcal{B}^*$, then $A[\tau]$ is called *pseudo-complete*.

An element $x \in A$ is called *bounded*, if there exists $0 \neq \lambda \in \mathbb{C}$ such that the set $\{(\lambda x)^n : n = 1, 2, 3, \dots\}$ is a bounded subset of A . We denote by A_0 the set of all bounded elements in A .

A unital topological *-algebra $A[\tau]$ is called *symmetric* if, for every $x \in A$, the element $(1 + x^*x)^{-1}$ exists and belongs to A_0 .

DEFINITION 2.2. ([2]) A symmetric pseudo-complete locally convex *-algebra $A[\tau]$, such that the collection \mathcal{B}^* has a greatest member, denoted by B_0 , is called a *GB*-algebra* over B_0 .

Every C^* -algebra is a GB^* -algebra. An example of a GB^* -algebra, which generally need not be a C^* -algebra, is a pro- C^* -algebra. By a *pro- C^* -algebra*, we mean a complete topological *-algebra $A[\tau]$ for which the topology τ is defined by a directed family of C^* -seminorms. Hence every pro- C^* -algebra is m -convex, and is therefore an inverse limit of C^* -algebras.

An example of a GB^* -algebra which is not a pro- C^* -algebra is the locally convex *-algebra $L^\omega([0, 1]) = \bigcap_{p \geq 1} L^p([0, 1])$ defined by the family of seminorms $\{\|\cdot\|_p : p \geq 1\}$, where $\|\cdot\|_p$ is the L^p -norm on $L^p([0, 1])$ for all $p \geq 1$.

A formal power series in the indeterminate X over a unital algebra A is an expression of the form $\sum_{n=0}^{\infty} a_n X^n$, where $a_n \in A$ for all $n \in \mathbb{N}$. The set $A[[X]]$ of all formal power series over A is an algebra with respect to addition and scalar multiplication defined in the obvious way, and multiplication defined as the usual Cauchy product as with complex power series. This is an immediate consequence of [12, Proposition 5.8].

We assume throughout this paper that all our algebras are unital, i.e., have an identity element 1, unless stated otherwise.

If A denotes an algebra, and $x \in A$, then the symbol $\text{Sp}_A(x)$ denotes the set

$$\{\lambda \in \mathbb{C} : \lambda 1 - x \text{ is not invertible in } A\}$$

throughout.

3. MAIN RESULTS

A *derivation* of an algebra A is a linear mapping $\delta : D(\delta) \rightarrow A$ such that $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in A$, where $D(\delta)$ denotes the domain of δ and is a subalgebra of A . If, in addition, A is equipped with an involution $*$, then we say that δ is a **-derivation* if $D(\delta)$ is a **-subalgebra* of A (i.e., $D(\delta)$ is closed under the involution $*$), and $\delta(x^*) = \delta(x)^*$ for all $x \in A$.

For a topological algebra $A[\tau]$, a **-derivation* $\delta : D(\delta) \rightarrow A$ is said to be a *closed *-derivation* if $D(\delta)$ is dense in A , and if (x_α) is a net in A with $x_\alpha \rightarrow x \in A$ and $\delta(x_\alpha) \rightarrow y \in A$, then $x \in D(\delta)$ and $y = \delta(x)$.

Let $A[\tau]$ be a GB^* -algebra, δ a closed **-derivation* of A with $1 \in D(\delta)$, $x \in D(\delta)$, $\lambda \in \mathbb{C} - \{0\}$ such that $(\lambda 1 - x)^{-1} \in A$. As already outlined in the introduction, in regards to the domain problem for a GB^* -algebra, the following two questions emerge:

- (I) Does there exist a unital injective algebra homomorphism $\tilde{\phi} : D(\delta) \rightarrow D(\delta)[[X]]$, with $\tilde{\phi}(x) = X$?
- (II) Does there exist a unital homomorphism $\psi : \mathbb{C}[[X]] \rightarrow D(\delta)$ such that $\psi(X) = x$?

Let us first assume that question (I) has an affirmative answer, and assume also that $\mathbb{C}[[X]] \subseteq \tilde{\phi}(D(\delta))$. Let $x \in D(\delta)$. Then, since $0 \in \text{Sp}_{\tilde{\phi}(D(\delta))}(X) \subseteq \text{Sp}_{\mathbb{C}[[X]]}(X) = \{0\}$ and $\tilde{\phi}$ is an injective algebra homomorphism, we get that $\text{Sp}_{D(\delta)}(x) = \{0\}$. So, for our initially assumed $\lambda \neq 0$ such that $(\lambda 1 - x)^{-1}$ exists in A , we conclude that $(\lambda 1 - x)^{-1} \in D(\delta)$. The condition $\text{Sp}_{D(\delta)}(x) = \{0\}$ is a severe restriction. Observe that the condition $\mathbb{C}[[X]] \subseteq \tilde{\phi}(D(\delta))$ is equivalent to the condition that $\tilde{\phi}(D(\delta)) \cap \mathbb{C}[[X]] = \mathbb{C}[[X]]$, and this is explored in Propositions 3.1 and 3.2 below.

Now let us assume that question (II) has a positive answer. Then, for the initially assumed non-zero $\lambda \in \mathbb{C}$, we have that

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{1}{\lambda} X \right)^k = \left(\lambda \left(1 - \frac{1}{\lambda} X \right) \right)^{-1} = (\lambda 1 - X)^{-1}.$$

Therefore, $\lambda 1 - X$ is invertible in $\mathbb{C}[[X]]$ for all $\lambda \in \mathbb{C} - \{0\}$. Given the assumption of the existence of the unital homomorphism ψ described in question (II), we consequently have that $(\lambda 1 - x)^{-1} \in D(\delta)$.

It is worth noting that once we assume the validity of questions (I) or (II), with the additional assumption that $\mathbb{C}[[X]] \subseteq \tilde{\phi}(D(\delta))$ in Question I, the domain problem can be immediately answered by straightforward algebraic considerations which hold for any algebra. It is therefore expected that the intrinsic nature of the topology of a GB*-algebra is to play its role in answering questions (I) and (II).

In Question II, however, the existence of such a unital homomorphism implies that x is an element of $D(\delta)$ having the property that

$$\text{Sp}_{D(\delta)}(x) = \text{Sp}_{D(\delta)}(\Psi(X)) \subseteq \text{Sp}_{\mathbb{C}[[X]]}(X) = \{0\}.$$

Although the restrictions above are severe, Questions I and II motivate the following more general questions, which appear to be interesting for their own sake.

- (I') Let $A[\tau]$ be a topological algebra and let $x \in A$. Does there exist a unital injective algebra homomorphism $\tilde{\phi} : A \rightarrow A[[X]]$, with $\tilde{\phi}(x) = X$?
- (II') Let $A[\tau]$ be a topological algebra and let $x \in A$. Does there exist a unital homomorphism $\psi : \mathbb{C}[[X]] \rightarrow A$ such that $\psi(X) = x$?

We start with some considerations with respect to question (I). In [8, Theorem 11.2], it is shown that if θ is an algebra homomorphism of a Fréchet algebra into the formal power series algebra $\mathbb{C}[[X]]$, then θ is either continuous or a surjection. In this regard, we have the following result.

PROPOSITION 3.1. *Let $A[\tau]$ be a topological algebra with identity element 1 and let $x \in A$ such that $(1 - x)^{-1}$ exists in A . Assume that $\tilde{\phi} : A \rightarrow A[[X]]$ is a unital algebra homomorphism such that $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is a Fréchet locally convex algebra in some topology τ_1 . Let ψ be the identity map of $\mathbb{C}[[X]]$ restricted to $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$. Let τ' denote the topology of convergence on $\mathbb{C}[[X]]$ with respect to all coefficients in $\mathbb{C}[[X]]$. Consider the following statements.*

- (i) *The map ψ is $\tau_1 - \tau'$ discontinuous.*
- (ii) *$\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is not m -convex with respect to τ_1 and is τ' -closed in $\mathbb{C}[[X]]$.*
- (iii) *$\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is not a Q -algebra with respect to τ_1 and is τ' -closed in $\mathbb{C}[[X]]$.*

Then (i) holds if and only if (ii) holds, and (i) implies (iii).

Proof. (i) \Rightarrow (ii): If ψ is $\tau_1 - \tau'$ discontinuous, then ψ is surjective [8], i.e., $\tilde{\phi}(A) \cap \mathbb{C}[[X]] = \mathbb{C}[[X]]$. Therefore $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is τ' -closed in $\mathbb{C}[[X]]$, and is commutative, m -convex, Fréchet with respect to the topology τ' , and Noetherian (this is due to $\mathbb{C}[[X]]$ being Fréchet, commutative, m -convex and Noetherian with respect to the topology τ' [9]). Since ψ is discontinuous, it is now immediate from [9, Theorem 2.5] that $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is not m -convex with respect to τ_1 .

(ii) \Rightarrow (i): Assume that (ii) holds, and suppose that ψ is $\tau_1 - \tau'$ continuous. Then, since ψ is a surjective homomorphism onto its range $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$, which is Fréchet with respect to τ' and τ_1 , it follows from the open mapping theorem that ψ is a $\tau_1 - \tau'$ topological isomorphism. This cannot be since $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is m -convex with respect to τ' , and $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is not m -convex with respect to τ_1 . Therefore ψ is $\tau_1 - \tau'$ discontinuous.

(i) \Rightarrow (iii): If ψ is $\tau_1 - \tau'$ discontinuous, then, as in the proof of (i) \Rightarrow (ii), it follows that $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is not m -convex with respect to τ_1 . By [18, Theorem 13.17], $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is not a Q -algebra with respect to τ_1 . ■

Consider a map $\tilde{\phi}$ as in question (I'). The following proposition depicts a certain positioning of the image of $\tilde{\phi}$, and the proof shows that the map ψ in Proposition 3.1 is always $\tau_1 - \tau'$ continuous, and therefore ψ is not surjective. This demonstrates that $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ does not satisfy condition (ii) of Proposition 3.1.

PROPOSITION 3.2. *Let $A[\tau]$ be a unital topological algebra with identity 1. Let $x \in A$ and $\tilde{\phi} : A \rightarrow A[[X]]$ a unital algebra homomorphism with $\tilde{\phi}(x) = X$, such that $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is a Fréchet algebra in some topology τ_1 . Then $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is properly contained in $\mathbb{C}[[X]]$.*

Proof. Let $\psi := id_{|\tilde{\phi}(A) \cap \mathbb{C}[[X]]} : \tilde{\phi}(A) \cap \mathbb{C}[[X]] \rightarrow \mathbb{C}[[X]]$. We have that

$$\tilde{\phi}(\lambda_0 1 + \lambda_1 x + \cdots + \lambda_n x^n) = \lambda_0 1 + \lambda_1 X + \cdots + \lambda_n X^n.$$

Therefore $\mathbb{C}[X] \subset \tilde{\phi}(A) \cap \mathbb{C}[[X]]$. Hence, $(\tilde{\phi}(A) \cap \mathbb{C}[[X]], \tau_1)$ is a Fréchet algebra which is a subalgebra of $\mathbb{C}[[X]]$ containing $\mathbb{C}[X]$. Then, by [8, Theorem 11.2], the map ψ is $\tau_1 - \tau'$ continuous, where τ' is the topology of convergence with respect to all coefficients in $\mathbb{C}[[X]]$. Hence, by [8, p. 3], we take that ψ is not onto and hence $\tilde{\phi}(A) \cap \mathbb{C}[[X]]$ is properly contained in $\mathbb{C}[[X]]$. ■

COROLLARY 3.3. *Let $A[\tau]$ be a unital topological algebra with identity 1 and $\delta : D(\delta) \rightarrow A$ a closed derivation of A such that $1 \in D(\delta)$. Let $x \in D(\delta)$ and $\tilde{\phi} : D(\delta) \rightarrow D(\delta)[[X]]$ a unital algebra homomorphism with $\tilde{\phi}(x) = X$, such that $\tilde{\phi}(D(\delta)) \cap \mathbb{C}[[X]]$ is a Fréchet algebra in some topology τ_1 . Then $\tilde{\phi}(D(\delta)) \cap \mathbb{C}[[X]]$ is properly contained in $\mathbb{C}[[X]]$.*

The following result is described in [16, first paragraph of p. 2145] and we give its proof for sake of completeness.

PROPOSITION 3.4. *Let A be a unital algebra and $x \in A$. If $\theta : A \rightarrow A[[X]]$ is a unital injective algebra homomorphism of A into the algebra $A[[X]]$ of all formal power series over A with indeterminate X such that $\theta(x) = X$, then x is not invertible in A and $\bigcap_{n=1}^{\infty} x^n A = \{0\}$.*

Proof. We first show that X is not invertible in $A[[X]]$. Observe that $X = \sum_{k=0}^{\infty} a_k X^k$, where $a_k = 0$ for all $k \neq 1$ and $a_1 = 1$. If we suppose that X is invertible in $A[[X]]$, then there is an element $\sum_{k=0}^{\infty} b_k X^k$ in $A[[X]]$ such that

$$X \left(\sum_{k=0}^{\infty} b_k X^k \right) = \left(\sum_{k=0}^{\infty} a_k X^k \right) \left(\sum_{k=0}^{\infty} b_k X^k \right) = \sum_{k=0}^{\infty} \left(\sum_{r=0}^k a_r b_{k-r} \right) X^k = 1.$$

Hence, $\sum_{r=0}^k a_r b_{k-r} = 0$ for all $k \geq 1$ and $a_0 b_0 = 1$. This last equation is in contradiction with the fact that $a_0 = 0$, thus X is not invertible in $A[[X]]$. So, since $\theta(x) = X$ and θ is unital, we conclude that x is not invertible in A .

Let now $a \in \bigcap_{n=1}^{\infty} x^n A$. Then, for all $n \in \mathbb{N}$, there exists $b_n \in A$ such that $a = x^n b_n$. Hence, $\theta(a) = \theta(x^n b_n) = \theta(x)^n \theta(b_n) = X^n \theta(b_n)$, for all $n \in \mathbb{N}$. Then, $X \theta(b_1) = X^2 \theta(b_2)$, implying that $\theta(b_1) = \theta(b_2) = 0$. Thus, $\theta(a) = X \theta(b_1) = 0$. So, $\bigcap_{n=1}^{\infty} x^n A \subset \ker \theta = \{0\}$. Therefore $a = 0$, implying that $\bigcap_{n=1}^{\infty} x^n A = \{0\}$. ■

The following corollary is a direct consequence of Proposition 3.4. It gives us necessary conditions under which question (I) in the beginning of this section has a negative answer.

COROLLARY 3.5. *Let $\delta : D(\delta) \rightarrow A$ be a closed $*$ -derivation of a GB*-algebra A such that $1 \in D(\delta)$. Let $x \in D(\delta)$ such that $(1 - x)^{-1}$ exists in A . If x is invertible in A , or $\bigcap_{n=1}^{\infty} x^n A \neq \{0\}$, then there is no injective unital algebra homomorphism $\tilde{\phi} : D(\delta) \rightarrow D(\delta)[[X]]$ with $\tilde{\phi}(x) = X$.*

The proof of the following result is similar to that of [16, Theorem 3.10], and we only give a sketch of the proof.

PROPOSITION 3.6. *Let $A[\tau]$ be a unital commutative topological algebra, and let $x \in A$ have the following properties:*

- (i) *there exists a derivation $D : A \rightarrow A$ such that $D(x) = 1$ and*

$$\sum_{k=0}^{\infty} \frac{(-1)^k D^k(a) x^k}{k!}$$

τ -converges for all $a \in A$,

- (ii) $\bigcap_{n=0}^{\infty} x^n A = \{0\}$.

Then there exists a unital injective algebra homomorphism $\tilde{\phi} : A \rightarrow A[[X]]$ such that $\tilde{\phi}(x) = X$.

Proof. By hypothesis we can form the map $\theta : A \rightarrow A$ by

$$\theta(a) = \sum_{k=0}^{\infty} \frac{(-1)^k D^k(a) x^k}{k!}$$

for all $a \in A$. It is easily seen that θ is a unital algebra homomorphism. Also, the kernel of θ is xA . Define the map $\tilde{\phi} : A \rightarrow A[[X]]$ by

$$\tilde{\phi}(a) = \sum_{n=0}^{\infty} \frac{\theta(D^n(a))}{n!} X^n$$

for all $a \in A$. Since the kernel of θ is xA and $D(x^n) = nx^{n-1}$, for all $n \in \mathbb{N}$, we get that $\tilde{\phi}$ is injective. Lastly, it is easily seen that $\tilde{\phi}(x) = X$. ■

Proposition 3.6 is an optimal result as far as sufficient conditions are concerned: Condition (i) of Proposition 3.6 cannot be dropped. The reason is that if condition (ii) is sufficient on its own, then it would imply that our map exists if x is nilpotent (for x nilpotent gives us condition (ii)). This cannot be, since $\tilde{\phi}(x) = X$ and X is not nilpotent in $A[[X]]$. So condition (i) is required

in addition to condition (ii). Also, the condition $D(x) = 1$ in (i) implies that x is not nilpotent: For if x is nilpotent and $x \neq 0$ (which we assume, since the case $x = 0$ is trivial), there is a smallest natural number $m > 1$ such that $x^m = 0$. Hence $0 = D(x^m) = mx^{m-1} \neq 0$, a contradiction. Furthermore, x not nilpotent and condition (ii) of Proposition 3.6 imply that x cannot have the property that $0 \neq Ax^m \subset \overline{Ax^{m+1}}$ (see [4, Corollary 2]). This is not in contradiction to [16, Proposition 3.10]. Also, recall from Proposition 3.4 that condition (ii) is a necessary condition for the existence of our map. Lastly, condition (i) is not redundant: No commutative C^* -algebra has a derivation D with $D(x) = 1$.

COROLLARY 3.7. *Let $\delta : D(\delta) \rightarrow A$ be a closed unbounded $*$ -derivation with $1 \in D(\delta)$, where $A[\tau]$ is a commutative GB^* -algebra. Let $x \in D(\delta)$ such that $(1 - x)^{-1} \in A$. Suppose that there exists a derivation $D : D(\delta) \rightarrow D(\delta)$ such that $D(x) = 1$ and*

$$\sum_{k=0}^{\infty} \frac{(-1)^k D^k(a)x^k}{k!}$$

τ -converges to an element in $D(\delta)$ for all $a \in D(\delta)$. If also $\bigcap_{n=0}^{\infty} x^n A = \{0\}$, then there exists a map $\tilde{\phi} : D(\delta) \rightarrow D(\delta)[[X]]$ which is a unital injective algebra homomorphism such that $\tilde{\phi}(x) = X$.

Remarks 3.8. (i) [16, Theorem 3.10] is stated for a commutative radical Fréchet m -convex algebra. It is interesting to note that the conditions of Proposition 3.6 above ensure that one does not require $A[\tau]$ to be a commutative radical Fréchet m -convex algebra. In the proof of [16, Theorem 3.10], one requires $A[\tau]$ to be a commutative radical Fréchet m -convex algebra in order to apply [16, Proposition 3.7 and Lemma 3.9], which is to ensure that condition (i) of Proposition 3.6 holds in some form.

(ii) The results obtained in [16], and therefore Propositions 3.4, 3.6 (above) and 3.9 (below), stem from the paper [15], in which M.P. Thomas proved that the image of a derivation of a commutative Banach algebra is contained in the Jacobson radical of the Banach algebra (commutative Singer-Wermer conjecture).

Based on Proposition 3.6 and [16, Lemma 3.9], the following result can be recorded.

PROPOSITION 3.9. *Let $A[\tau]$ be a Fréchet m -convex algebra and $\{\|\cdot\|_i\}$ a sequence of seminorms defining the topology τ . Let $x \in A$ and $D : A \rightarrow A$*

be a derivation such that $D(x) = 1$. Suppose that for every $i \in \mathbb{N}$ there is $m_i \in \mathbb{N}$ such that $x^{m_i} \in \overline{\bigcap_{n=0}^{\infty} x^n A}^{\|\cdot\|_i}$, where $\overline{\bigcap_{n=0}^{\infty} x^n A}^{\|\cdot\|_i}$ denotes the closure of $\bigcap_{n=0}^{\infty} x^n A$ with respect to the seminorm $\|\cdot\|_i$. Then there exists an algebra homomorphism $\tilde{\phi} : A \rightarrow A_0[[X]]$, where $A_0 = A / \bigcap_{n=0}^{+\infty} x^n A$, such that $\tilde{\phi}(x) = X$.

Proof. Based on the proof of [16, Theorem 3.10] there exists an injective algebra homomorphism $\phi : A / \bigcap_{k=0}^{\infty} x^k A \rightarrow A_0[[X]]$ such that

$$\phi \left(a + \bigcap_{k=0}^{\infty} x^k A \right) = \sum_{n=0}^{\infty} \frac{\theta(D^n(a) + \bigcap_{k=0}^{\infty} x^k A)}{n!} X^n,$$

where $\theta : A / \bigcap_{k=0}^{\infty} x^k A \rightarrow A / \bigcap_{k=0}^{\infty} x^k A$ is the following algebra homomorphism:

$$\theta \left(a + \bigcap_{k=0}^{\infty} x^k A \right) = \sum_{n=0}^{\infty} \frac{(-1)^n D^n(a) x^n}{n!} + \bigcap_{k=0}^{\infty} x^k A.$$

Then, by composing ϕ with the natural quotient map $\pi : A \rightarrow A / \bigcap_{k=0}^{\infty} x^k A$, we get the desired algebra homomorphism $\tilde{\phi}$.

The fact that $\tilde{\phi}(x) = X$ can be easily seen through the following considerations:

$$\tilde{\phi}(x) = (\phi \circ \pi)(x) = \phi \left(x + \bigcap_{k=0}^{+\infty} x^k A \right) = \sum_{n=0}^{+\infty} \frac{\theta(D^n(x) + \bigcap_{k=0}^{+\infty} x^k A)}{n!} X^n.$$

Since $D(x) = 1$, we get that $D^n(x) = 0$ for every $n \geq 2$. Therefore,

$$(\phi \circ \pi)(x) = \theta \left(x + \bigcap_{k=0}^{+\infty} x^k A \right) + \theta \left(1 + \bigcap_{k=0}^{+\infty} x^k A \right) \cdot X.$$

Moreover,

$$\begin{aligned} \theta \left(x + \bigcap_{k=0}^{+\infty} x^k A \right) &= \sum_{n=0}^{+\infty} \frac{(-1)^n D^n(x) x^n}{n!} + \bigcap_{k=0}^{+\infty} x^k A \\ &= (x - x) + \bigcap_{k=0}^{+\infty} x^k A \\ &= \bigcap_{k=0}^{+\infty} x^k A, \end{aligned}$$

and

$$\begin{aligned} \theta \left(1 + \bigcap_{k=0}^{+\infty} x^k A \right) &= \sum_{n=0}^{+\infty} \frac{(1)^n D^n(1) x^n}{n!} + \bigcap_{k=0}^{+\infty} x^k A \\ &= 1 + \bigcap_{k=0}^{+\infty} x^k A. \end{aligned}$$

So,

$$\tilde{\phi}(x) = (\phi \circ \pi)(x) = \bigcap_{k=0}^{+\infty} x^k A + \left(1 + \bigcap_{k=0}^{+\infty} x^k A \right) \cdot X = X,$$

given that on $A_0 (= A / \bigcap_{k=0}^{+\infty} x^k A)$, the elements $\bigcap_{k=0}^{+\infty} x^k A$ and $1 + \bigcap_{k=0}^{+\infty} x^k A$ are the zero and the unit element respectively. ■

With respect to a possible strategy for answering question (II) in the beginning of this section, we note the following two routes, denoted by (1) and (2) in what follows.

(1) In [3, Theorem 2], G.R. Allan proved the following result for a commutative Banach algebra.

THEOREM 3.10. *Let A be a commutative Banach algebra with identity and let x be in the Jacobson radical of A such that $0 \neq Ax^m \subset Ax^{m+1}$ for some $m \geq 1$. Then there is a unital injective algebra homomorphism $\psi : \mathbb{C}[[X]] \rightarrow A$ such that $\psi(X) = x$.*

The proof of Theorem 3.10 in [3] is entirely algebraic, except in the application of the Arens-Calderon theorem, which relies on A being a Banach algebra. The Arens-Calderon theorem is given by the following theorem.

THEOREM 3.11. ([11, Lemma 3.2.8]) *Let A be a commutative Banach algebra with identity and Jacobson radical R . If $n \in \mathbb{N}$ and $a_0, a_1, \dots, a_n \in A$ are such that $a_0 \in R$ and a_1 is invertible in A , then there exists $y \in R$ such that $\sum_{k=0}^n a_k y^k = 0$.*

The proof of Theorem 3.11, as given in [11], relies strongly on analytic functions of several complex variables and the implicit function theorem. If we could extend the Arens-Calderon theorem (Theorem 3.11) to commutative Fréchet algebras, then we will have extended Theorem 3.10 to the case where A is a Fréchet algebra, with exactly the same proof as that of [3, Theorem

2]. Therefore Question (II') would attain a positive answer in the setting of commutative Fréchet algebras.

In Theorem 3.10, the condition that x is in the radical of A is strong if A is semi-simple, especially if $x \neq 0$. Recall that a sequentially complete locally convex algebra is pseudo-complete, and hence has an abundance of Banach subalgebras (see [1, Proposition 4.2]). This motivates the following observation.

COROLLARY 3.12. *Let $A[\tau]$ be a commutative unital Fréchet locally convex algebra and $x \in A$. Assume that there is a unital Banach subalgebra B of A (in some norm) such that $x \in B$ and $0 \neq Bx^m \subset \overline{Bx^{m+1}}$ for some $m \geq 1$, and such that x is in the radical of B . Then there is an embedding $\psi : \mathbb{C}[[X]] \rightarrow B \subseteq A$ such that $\psi(X) = x$.*

If, in the above corollary, B is a commutative unital radical Banach subalgebra, then the condition x in the radical of B becomes more realistic. For example, a famous theorem of B.E. Johnson [13, Theorem 4] asserts that there are at least some commutative Banach algebras having at least one closed (hence Banach) radical subalgebra. Incidentally, this theorem of Johnson was used by M.P. Thomas in his proof of the commutative Singer-Wermer conjecture: He only had to settle the conjecture for commutative radical Banach algebras.

One word of caution: Not all Fréchet Jacobson semi-simple algebras have closed (hence Fréchet) radical subalgebras. For instance, a C^* -algebra is semi-simple, and every norm closed $*$ -subalgebra is again a C^* -algebra, and hence semi-simple.

An extension of Theorem 3.10 to Fréchet m -convex algebras is given in [4, Theorem 7].

(2) The following theorem is a special case of [9, Lemma 2], and is the holomorphic functional calculus for complete barrelled m -convex algebras.

THEOREM 3.13. *Let $A[\tau]$ be a commutative complete barrelled m -convex topological algebra with weakly compact character space. Let $a \in A$. Then there exists a continuous homomorphism $g : \mathcal{O}(sp_A(a)) \rightarrow A$, such that $g(f) = a$, where $\mathcal{O}(sp_A(a))$ denotes the algebra of all analytic functions on a neighbourhood of $sp_A(a)$ and $f(z) = z$ for all $z \in \mathbb{C}$.*

The proof of Theorem 3.13 relies on [5, Theorem 5.1], which is the Silov-Arens-Calderon theorem for Banach algebras. In what follows, we give a proof

of Theorem 3.13 for the case where all Banach algebras in the Arens-Michael decomposition of A are semi-simple.

Proof. By the Arens-Michael decomposition of A we have that $A[\tau]$ is topologically isomorphic to the inverse limit $\varprojlim_{\gamma} A_{\gamma}$, where all A_{γ} are semisimple by assumption.

The space $\mathcal{O}(\mathrm{sp}_A(a))$ can be represented as the following inductive limit

$$\mathcal{O}(\mathrm{sp}_A(a)) = \lim_{\rightarrow} H(\Omega)$$

where $H(\Omega)$ stands for the holomorphic functions on the open subset Ω of \mathbb{C} and Ω runs over all open subsets of \mathbb{C} which contain the weakly compact subset $\mathrm{sp}_A(a)$. The topology considered on each $H(\Omega)$ is that of the uniform convergence on compact subsets of Ω . The topology on $\mathcal{O}(\mathrm{sp}_A(a))$ is the inductive limit topology on $\mathcal{O}(\mathrm{sp}_A(a))$ induced by the $H(\Omega)$'s.

Let us fix an index $\gamma \in \Gamma$. We consider an open subset Ω of \mathbb{C} which contains $\mathrm{sp}_A(a)$. Since $\mathrm{sp}_A(a) = \bigcup_{\gamma \in \Gamma} \mathrm{sp}_{A_{\gamma}}(a_{\gamma})$ [10], we have that Ω contains $\mathrm{sp}_{A_{\gamma}}(a_{\gamma})$ for all $\gamma \in \Gamma$. By the Arens-Calderón theorem [6, Corollary II.20.6], there exists a homomorphism

$$g_{\Omega}^{\gamma} : H(\Omega) \longrightarrow A_{\gamma}$$

which is continuous with respect to the topology of uniform convergence on compact subsets of Ω , it extends the natural homomorphism of the complex polynomials $P(\mathbb{C})$ into A_{γ} and $g_{\Omega}^{\gamma}(f) = a_{\gamma}$, for $f(z) = z$, $z \in \mathbb{C}$.

By the proof of [6, Corollary II.20.6], we recall that if $f \in H(\Omega)$, and, say, $g_{\Omega}^{\gamma}(f) = l_{\gamma} \in A_{\gamma}$, then

$$\phi_{\gamma}(l_{\gamma}) = f(\phi_{\gamma}(a_{\gamma})), \quad \text{for all } \phi_{\gamma} \in \mathcal{M}(A_{\gamma}),$$

where $\mathcal{M}(A_{\gamma})$ is the character space of A_{γ} .

We then define the map

$$g_{\gamma} : \mathcal{O}(\mathrm{sp}_A(a)) \longrightarrow A_{\gamma}$$

as follows: let f be a function in $\mathcal{O}(\mathrm{sp}_A(a))$. Hence there is an open neighborhood Ω of $\mathrm{sp}_A(a)$ such that $f \in H(\Omega)$. We then define $g_{\gamma}(f) := g_{\Omega}^{\gamma}(f)$.

We show that the map g_{γ} is well-defined: for this, consider an open subset Ω' containing $\mathrm{sp}_A(a)$ such that $f \in H(\Omega')$. Then for every $\phi_{\gamma} \in \mathcal{M}(A_{\gamma})$, we have that

$$\phi_{\gamma}(g_{\Omega}^{\gamma}(f)) = f(\phi_{\gamma}(a_{\gamma})) = \phi_{\gamma}(g_{\Omega'}^{\gamma}(f)).$$

Hence, since A_γ is semisimple, we get that $g_\Omega^\gamma(f) = g_{\Omega'}^\gamma(f)$.

Hence, the map g_γ is well-defined. Moreover, if i_Ω is the canonical embedding of $H(\Omega)$ into $\mathcal{O}(\text{sp}_A(a))$, we clearly have that

$$g_\gamma \circ i_\Omega = g_\Omega^\gamma.$$

Therefore, for every open subset Ω containing $\text{sp}_A(a)$ we have that $g_\gamma \circ i_\Omega$ is continuous. So, by [14, (6.1), page 54] the map $g_\gamma : \mathcal{O}(\text{sp}_A(a)) \rightarrow A_\gamma$ is continuous for every $\gamma \in \Gamma$.

Next, we consider the map

$$g : \mathcal{O}(\text{sp}_A(a)) \longrightarrow A, \quad g(f) := (g_\gamma(f))_{\gamma \in \Gamma}.$$

In order to show that the map g is well-defined, we need to show that $\pi_{\gamma\delta}(g_\delta(f)) = g_\gamma(f)$, $\gamma \leq \delta$, $f \in \mathcal{O}(\text{sp}_A(a))$, where $\pi_{\gamma\delta} : A_\delta \rightarrow A_\gamma$ are the $\|\cdot\|$ -continuous connecting maps of the inverse system.

We fix one open neighborhood, say Ω , of the set $\text{sp}_A(a)$. By the preceding paragraphs, we have that $g_\gamma(f) = g_\Omega^\gamma(f) = l_\gamma$, $g_\delta(f) = g_\Omega^\delta(f) = l_\delta$. So, we shall show that $\pi_{\gamma\delta}(l_\delta) = l_\gamma$.

We recall that $a = (a_\lambda)_{\lambda \in \Gamma}$. Since Ω is an open neighborhood of $\text{sp}_{A_\delta}(a_\delta)$, by the proof of [6, Theorem II.20.5], we have that there are $a_{\delta_2}, a_{\delta_3}, \dots, a_{\delta_N} \in A_\delta$ and F_δ a holomorphic function on an open neighborhood of the polydisc

$$\Gamma_\delta = \{z \in \mathbb{C}^N : |z_k| \leq 1 + p_\delta(a_{\delta k}), k = 1, \dots, N\}$$

(in the above writing of Γ_δ , a_{δ_1} is identified with a_δ), such that

$$f(\phi_\delta(a_\delta)) = F_\delta(\phi_\delta(a_\delta), \phi_\delta(a_{\delta_2}), \dots, \phi_\delta(a_{\delta_N})), \quad \phi_\delta \in \mathcal{M}(A_\delta). \quad (3.1)$$

Moreover, by the proof of [6, Theorem II.20.5], we have that the series

$$\sum_k \alpha_k a_\delta^{k_1} a_{\delta_2}^{k_2} \cdots a_{\delta_N}^{k_N}$$

is $\|\cdot\|_\delta$ -convergent to the element l_δ and for every $\phi_\delta \in \mathcal{M}(A_\delta)$,

$$\phi_\delta(l_\delta) = F_\delta(\phi_\delta(a_\delta), \phi_\delta(a_{\delta_2}), \dots, \phi_\delta(a_{\delta_N})). \quad (3.2)$$

Now, given that $\pi_{\gamma\delta}$ is a $\|\cdot\|$ -continuous linear map, we have that

$$\pi_{\gamma\delta} \left(\sum_k \alpha_k a_\delta^{k_1} a_{\delta_2}^{k_2} \cdots a_{\delta_N}^{k_N} \right) = \sum_k \alpha_k a_\gamma^{k_1} a_{\gamma_2}^{k_2} \cdots a_{\gamma_N}^{k_N}$$

$\|\cdot\|_\gamma$ -converges to an element in A_γ , say ω_γ . For any $\phi_\gamma \in \mathcal{M}(A_\gamma)$ we have that

$$\begin{aligned} \phi_\gamma(\omega_\gamma) &= \phi_\gamma \left(\pi_{\gamma\delta} \left(\sum_k \alpha_k a_\delta^{k_1} a_{\delta_2}^{k_2} \cdots a_{\delta_N}^{k_N} \right) \right) \\ &= \phi_\gamma(\pi_{\gamma\delta}(l_\delta)) \\ &= F_\delta((\phi_\gamma \circ \pi_{\gamma\delta})(a_\delta), \dots, (\phi_\gamma \circ \pi_{\gamma\delta})(a_{\delta_N})) \\ &= f((\phi_\gamma \circ \pi_{\gamma\delta})(a_\delta)) \\ &= f(\phi_\gamma(a_\gamma)), \end{aligned}$$

where the second equality in the above string of relations derives from relation (3.2) and from the fact that $\phi_\gamma \circ \pi_{\gamma\delta} \in \mathcal{M}(A_\delta)$, and the third equality results from relation (3.1) above.

By the proof of [6, Corollary II.20.6], we have that for every $\phi_\gamma \in \mathcal{M}(A_\gamma)$:

$$\phi_\gamma(l_\gamma) = f(\phi_\gamma(a_\gamma)).$$

Therefore, we have that $\phi_\gamma(\omega_\gamma) = \phi_\gamma(l_\gamma)$ for every $\phi_\gamma \in \mathcal{M}(A_\gamma)$. Since A_γ is semisimple, we get that $\omega_\gamma = l_\gamma$, i.e., $\pi_{\gamma\delta}(l_\delta) = l_\gamma$.

The map g is clearly a homomorphism, such that

$$g(f) = (g_\gamma(f))_\gamma = (a_\gamma)_\gamma = a, \quad \text{for } f(z) = z, \quad z \in \mathbb{C}.$$

Moreover, if by π_γ we denote the canonical projections $\pi_\gamma : A \rightarrow A_\gamma$, then we have that $\pi_\gamma \circ g = g_\gamma$ and we have already seen that g_γ is continuous. Hence, by [14, (5.2), p. 51], we get that g is continuous. ■

Let now A be a commutative complete barrelled locally m -convex algebra which is also a \mathbb{Q} -algebra. Then, by [9, Lemma 0.1], $A[\tau]$ has weakly compact character space. Let $x \in A$, and $f(z) = z$ for all $z \in \mathbb{C}$. Then f is an entire function and thus analytic on $sp_A(x)$, that is $f \in \mathcal{O}(sp_A(x))$. Define a map $\phi_1 : \mathbb{C}[X] \rightarrow \mathcal{O}(sp_A(x))$ by

$$\phi_1 \left(\sum_{i=1}^n \lambda_i X^i \right) = g, \quad \text{where } g(z) = \sum_{i=1}^n \lambda_i z^i.$$

Then ϕ_1 is a homomorphism. By Theorem 3.13 there is a continuous homomorphism $\phi_2 : \mathcal{O}(sp_A(x)) \rightarrow A$ such that $\phi_2(f) = x$. Let $\psi_0 = \phi_2 \circ \phi_1$.

Then, $\psi_0(X) = x$. If ϕ_1 is continuous when $\mathbb{C}[X]$ is equipped with the coordinatewise topology coming from $\mathbb{C}[[X]]$, then ψ_0 is continuous and hence extends via continuity to an algebra homomorphism $\psi : \mathbb{C}[[X]] \rightarrow A$ such that $\psi(X) = x$.

For the case of a general commutative algebra A , the I -adic topology on A with respect to an ideal I of A plays an important role in embedding $\mathbb{C}[[X]]$ into A as the following result shows. Recall that a subset U of A is open with respect to the I -adic topology if and only if for all $x \in U$, there exists $n \in \mathbb{N}$ such that $x + I^n \subseteq U$.

THEOREM 3.14. *Let A be a unital commutative algebra, I an ideal in A and $x \in I$. If A is I -adic complete and Hausdorff with respect to the I -adic topology, then there is a unital homomorphism $\psi : \mathbb{C}[[X]] \rightarrow A$ such that $\psi(X) = x$.*

Proof. Let $A_n = A/I^n$, for all $n \in \mathbb{N}$. By [12, Theorem 5.5], there is a unital homomorphism $\psi_{0,m} : \mathbb{C}[X] \rightarrow A_m$ such that $\psi_{0,m}(X) = x + I^m$. Then

$$\psi_{0,m} \left(\sum_{i=0}^n \lambda_i X^i \right) = \sum_{i=0}^n \lambda_i (x + I^m)^i$$

for all $m \in \mathbb{N}$. This induces the map

$$\widetilde{\psi}_{0,m} : \mathbb{C}[X]/\langle X \rangle \longrightarrow A_m$$

for all $m \in \mathbb{N}$, where $\langle X \rangle$ is the two-sided ideal of $\mathbb{C}[X]$ generated by X . Since A is, by hypothesis, I -adic complete and Hausdorff, by taking the inverse limit of the aforementioned maps $\widetilde{\psi}_{0,m}$, we get the desired mapping ψ (see [12, Theorem 5.5] for details). ■

Remark 3.15. Note that the ideal I in Theorem 3.14 cannot be the whole algebra A . For then, there would exist a unital algebra homomorphism $\psi : \mathbb{C}[[X]] \rightarrow A$ such that $\psi(X) = 1$, i.e., $\psi(1 - X) = 0$. The latter equality contradicts the fact that $1 - X$ is invertible in $\mathbb{C}[[X]]$.

LEMMA 3.16. *Let A be a unital commutative algebra and I an ideal of A such that A is Hausdorff, has jointly continuous multiplication and is advertibly complete with respect to the I -adic topology. If $x \in A$, then $(1 - x)^{-1}$ exists and is in A .*

Proof. Let \tilde{A} be the completion of A with respect to the I -adic topology. Given that A has jointly continuous multiplication with respect to the I -adic topology, \tilde{A} is a Hausdorff topological algebra. By Theorem 3.14, there exists a unital homomorphism $\psi : \mathbb{C}[[X]] \rightarrow \tilde{A}$ such that $\psi(1 - X) = 1 - x$. Since $(1 - X)^{-1}$ exists in $\mathbb{C}[[X]]$, we get that $(1 - x)^{-1}$ exists in \tilde{A} . Therefore, since A is adverbly complete in the I -adic topology, by [10, Proposition 6.2] we conclude that $(1 - x)^{-1} \in A$. ■

ACKNOWLEDGEMENTS

This work is based on part of the research for which the first named author received financial support from the National Research Foundation of South Africa (NRF).

The authors thank the referees for their careful reading of the manuscript, along with helpful comments which improved the quality of the manuscript.

REFERENCES

- [1] G.R. ALLAN, A spectral theory for locally convex algebras, *Proc. London Math. Soc. (3)* **15** (1965), 399–421.
- [2] G.R. ALLAN, On a class of locally convex algebras, *Proc. London Math. Soc. (3)* **17** (1967), 91–114.
- [3] G.R. ALLAN, Embedding the algebra of formal power series in a Banach algebra, *Proc. London Math. Soc. (3)* **25** (1972), 329–340.
- [4] G.R. ALLAN, Fréchet algebras and formal power series, *Studia Math.* **119** (1996), 271–288.
- [5] R. ARENS, The analytic functional calculus in commutative topological algebras, *Pacific J. Math.* **11** (1961), 405–429.
- [6] F.F. BONSALL, J. DUNCAN, “Complete Normed Algebras”, Springer-Verlag, New York-Heidelberg, 1973.
- [7] O. BRATTELI, D.W. ROBINSON, Unbounded derivations of C^* -algebras, *Comm. Math. Phys.* **42** (1975), 253–268.
- [8] H.G. DALES, S.R. PATEL, C.J. READ, Fréchet algebras of formal power series, in “Banach Algebras 2009”, Banach Center Publications 91, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 2010, 123–158.
- [9] A.V. FERREIRA, G. TOMASSINI, Finiteness properties of topological algebras, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **5** (1978), 471–488.
- [10] M. FRAGOULOPOULOU, “Topological Algebras with Involution”, North-Holland Mathematics Studies 200, Elsevier Science B.V., Amsterdam, 2005.
- [11] L. HÖRMANDER, “An Introduction to Complex Analysis in Several Variables”, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966.

- [12] T. HUNGERFORD, “Algebra”, Graduate Texts in Mathematics 73, Springer-Verlag, New York-Berlin, 1980.
- [13] B.E. JOHNSON, Continuity of derivations on commutative algebras, *Amer. J. Math.* **91** (1969), 1–10.
- [14] H.H. SCHAEFER, “Topological Vector Spaces”, Springer Verlag, New York-Berlin, 1971.
- [15] M.P. THOMAS, The image of a derivation is contained in the radical, *Ann. of Math. (2)* **128** (1988), 435–460.
- [16] M.P. THOMAS, Local power series quotients of commutative Banach and Fréchet algebras, *Trans. Amer. Math. Soc.* **355** (2003), 2139–2160.
- [17] M. WEIGT, I. ZARAKAS, On domains of unbounded derivations of Generalized B^* -algebras, *Banach J. Math. Anal.* **12** (2018), 873–908.
- [18] W. ZELAZKO, Metric generalizations of Banach algebras, *Rozprawy Mat.* **47** (1965), 70 pp.