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Radial Projections of Bisectors in Minkowski Spaces

Horst Martini, Senlin Wu*

Faculty of Mathematics, Chemnitz University of Technology, 09107 *Chemnitz, Germany*
horst.martini@mathematik.tu-chemnitz.de, senlin_wu@hotmail.com

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Abstract: We study geometric properties of radial projections of bisectors in finite-dimensional real Banach spaces (i.e., in Minkowski spaces), especially the relation between the geometric structure of radial projections and Birkhoff orthogonality. As an application of our results it is shown that for any Minkowski space there exists a number, which plays somehow the role that $\sqrt{2}$ plays in Euclidean space. This number is referred to as the critical number of any Minkowski space. Lower and upper bounds on the critical number are given, and the cases when these bounds are attained are characterized. Some new characterizations of inner product spaces are also derived.

Key words: Birkhoff orthogonality, bisectors, characterizations of inner product spaces, critical number, isosceles orthogonality, Minkowski planes, Minkowski spaces, normed linear spaces, radial projection, Voronoi diagram.

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1. Introduction

It is well known that bisectors in finite-dimensional real Banach spaces (i.e., Minkowski spaces) have, in general, a complicated topological and geometric structure. It is interesting to observe that, due to this, even their radial projections (onto the unit sphere) still have a large variety of properties yielding interesting results, such as new characterizations of inner product spaces and basic relations to different orthogonality concepts.

Let X be a *Minkowski space* with *norm* $\|\cdot\|$, *origin* o, *unit sphere* S_X, and *unit ball* B_X (basic references to the geometry of Minkowski spaces are [18], [16], [17], and the monograph [19]). The *bisector* $B(p,q)$ of the linear segment with endpoints $p \neq q$ in X is defined by

$$B(p,q) := \{x \in X : \ \|x-p\| = \|x-q\|\}.$$

The notion of bisector is closely related to the construction of Voronoi diagrams and has been intensively studied in computational geometry (where

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most of the results are obtained without the assumption that the unit ball is centrally symmetric, cf. [6] and [7]) as well as in the geometry of Minkowski spaces (Minkowski geometry). We refer to [16] and [18] for a survey on many results about bisectors in the context of Minkowski geometry.

In the present paper, we study the structure of the *radial projection* $P(x)$ of $B(-x,x)$ for any point $x \in X\backslash\{o\}$, which is defined by

$$P(x) := \left\{ \frac{z}{\|z\|} : \ z \in B(-x,x)\backslash\{o\} \right\}.$$

It is evident that if X is the Euclidean plane, then $P(x)$ contains precisely two points for any $x \in X\backslash\{o\}$, and when X is an n-dimensional Euclidean space ($n \geq 3$), then $P(x)$ is the unit sphere of an $(n-1)$-dimensional subspace. As we shall see, the geometric properties of $P(x)$ in general Minkowski spaces are much more complicated and worth studying.

Among the various types of orthogonalities defined for normed linear spaces (cf. [3] and [4]), isosceles orthogonality and Birkhoff orthogonality are closely related to geometric properties of bisectors. We say that $x \in X$ is *isosceles orthogonal* to $y \in X$, denoted by $x \perp_I y$, if $\|x+y\| = \|x-y\|$ (cf. [12]); x is said to be *Birkhoff orthogonal* to y if $\|x+ty\| \geq \|x\|$ holds for any real number t, and in this case we write $x \perp_B y$ (cf. [13]). We refer to [12], [13], [3], and [4] for basic properties of Birkhoff orthogonality and isosceles orthogonality. As shown in [5, p. 26], a point z belongs to $B(p,q)$ if and only if $z - \frac{p+q}{2}$ is isosceles orthogonal to $\frac{p-q}{2}$, which means that the geometric structure of bisectors in Minkowski spaces is fully determined by geometric properties of isosceles orthogonality. (We also have that $B(-x,x) = \{z : \ x \perp_I z\}$ and $P(x) = \left\{\frac{z}{\|z\|} : \ x \perp_I z, \ z \neq o\right\}$.) Furthermore, it has been shown that in the planar case the bisector $B(-x,x)$ is fully contained in a bent strip determined by x and those points on S_X which are Birkhoff orthogonal to x (see Lemma 2.5 below). We show that much more can be said about the relation between Birkhoff orthogonality and the geometric structure of bisectors in Minkowski spaces.

In Section 2, we study geometric properties of bisectors in *Minkowski planes* and provide some detailed relation between Birkhoff orthogonality and the geometric structure of bisectors. Moreover, the intersection of radial projections of two bisectors is discussed. We lay special emphasis on planar results, since many of the results in higher dimensions can be directly obtained from their analogues in the planar case. One of the exceptions, namely the connectivity of $P(x)$ in higher dimensions, is presented in Section 5.

In Section 3 we prove the existence of a critical number $c(X)$ for any Minkowski space X, which plays the role that $\sqrt{2}$ plays in Euclidean space. Also, we derive lower and upper bounds on $c(X)$ and characterize the situations when $c(X)$ attains these bounds.

In the fourth section we derive a new characterization of Euclidean space, which says that if the bisector of the segment between any two points u, $v \in S_X$ $(u \neq -v)$ intersects S_X in $\frac{u+v}{\|u+v\|}$, then X is Euclidean. This characterization requires less information about properties of bisectors than other related characterizations.

For $x, y \in X$, with $x \neq y$, we denote by $[x, y]$ the *segment* between x and y, by $\langle x, y\rangle$ the *line* passing through x and y, and by $[x, y\rangle$ the *ray* with starting point x passing through y. Also we write $\overrightarrow{xy}$ for the *orientation* from x to y, and $\widehat{x}$ for $\frac{x}{\|x\|}$ $(x \neq o)$. The *convex hull*, *closure*, and *interior* of a set S are denoted by conv$S$, $\overline{S}$, and int$S$, respectively. The *distance* from a point x to a set S is denoted by $d(x, S)$. Several times we also need the *Monotonicity Lemma* (we refer to Proposition 31 in [18], which is a suitable generalization of it): *for $p \in S_X$ fixed and $x \in S_X$ variable in dimension two, the length $\|p - x\|$ is non-decreasing as x moves on S_X from p to $-p$.*

2. Radial projections of bisectors in Minkowski planes

Throughout this section, X is a Minkowski plane with a fixed orientation ω. For any $x \in X\backslash\{o\}$, let H_x^+ and H_x^- be the two open half-planes bounded by $\langle -x, x\rangle$ such that $\overrightarrow{(-x)z} = \overrightarrow{zx} = \omega$ holds for any point $z \in H_x^+$, and that $\overrightarrow{xz} = \overrightarrow{z(-x)} = \omega$ holds for any point $z \in H_x^-$. Set

$$P^+(x) = P(x) \cap H_x^+ \text{ and } P^-(x) = P(x) \cap H_x^-.$$

It is evident that for any $x \in X\backslash\{o\}$ and any number $\alpha > 0$

$$P(\alpha x) = P(x),\ P^+(\alpha x) = P^+(x) = P^-(-\alpha x)$$

and

$$P^-(\alpha x) = P^-(x) = P^+(-\alpha x).$$

Thus it suffices to study the geometric structure of $P(x) = P(-x)$ for each $x \in S_X$.

Theorem 2.1. *For any* $x \in S_X$, $P^+(x)$ *and* $P^-(x)$ *are two connected subsets of* S_X, *and* $P(x) = P^+(x) \cup P^-(x)$.

Proof. It is clear that $P(x) = P^+(x) \cup P^-(x)$, since $B(-x,x) \cap \langle -x, x\rangle = \{o\}$.

By $P^-(x) = -P^+(x)$ it suffices to show that $P^+(x)$ is connected. Let $y \in S_X \cap H_x^+$ be a point such that $y \perp_B x$. Then x and y are linearly independent. Set

$$\begin{array}{rccc} T: & X & \longrightarrow & \mathbb{R} \\ & z = \alpha x + \beta y & \longrightarrow & \beta. \end{array}$$

It is clear that T is continuous and $T(H_x^+) = \{t : t > 0\}$.

Now we show that $B(-x,x) \cap H_x^+$ is connected. Suppose the contrary, i.e., that $B(-x,x) \cap H_x^+$ can be partitioned into two disjoint nonempty subsets A_1 and A_2 which are open in the relative topology induced on $B(-x,x) \cap H_x^+$. Assume that there exists a number $t_0 \in T(A_1) \cap T(A_2)$. Then there exist two points $z_1 = \alpha_1 x + t_0 y \in A_1$ and $z_2 = \alpha_2 x + t_0 y \in A_2$. From the convexity of $B(-x,x)$ in the direction of x (i.e., if a line parallel to x intersects $B(-x,x)$ in two distinct points then the whole segment with these points as endpoints is contained in $B(-x,x)$, cf. [10, Lemma 1]) it follows that $[z_1, z_2] \subset B(-x,x) \cap H_x^+$. Thus $[z_1, z_2]$ can be partitioned into two disjoint nonempty sets $[z_1, z_2] \cap A_1$ and $[z_1, z_2] \cap A_2$ which are open in the subspace topology of $[z_1, z_2]$. This is impossible. Thus $T(A_1) \cap T(A_2) = \varnothing$. It is clear that $T(A_1)$ and $T(A_2)$ are open sets, and that

$$T(A_1) \cup T(A_2) = T(B(-x,x) \cap H_x^+) = \{t : t > 0\},$$

a contradiction to the fact that the set $\{t : t > 0\}$ is connected.

Then, as image of $B(-x,x) \cap H_x^+$ under the function $R(X) = \widehat{x}$ which is continuous on $X \backslash \{o\}$, $P^+(x)$ is connected. ∎

Theorem 2.2. *A Minkowski plane* X *is Euclidean if and only if for any* $x \in S_X$ *the set* $P^+(x)$ *is a singleton.*

Proof. The necessity is obvious. Conversely, for any $x \in S_X$ it follows from the assumption of the theorem that $B(-x,x)$ is contained in a line, which is a characteristic property of Euclidean planes (cf. [5, (3.3)]). ∎

Remark 2.3. R. C. James [12] provided an example to show that there exists a normed plane X_0 such that if $x \perp_I ty$ holds for any $t \in \mathbb{R}$, then either

$x = o$ or $y = o$. In other words, there exists a Minkowski plane X_0 such that $P^+(x)$ contains more than one point for any $x \in S_{X_0}$.

LEMMA 2.4. (cf. [18, Corollary 16]) *For any $x \in X\backslash\{o\}$, any line parallel to $\langle -x, x\rangle$ intersects $B(-x, x)$ in exactly one point if and only if S_X does not contain a non-trivial segment parallel to $\langle -x, x\rangle$.*

For any $x \in S_X$, we denote by $l(x)$ and $r(x)$ the two points such that $[r(x), l(x)]$ is a maximal segment parallel to $\langle -x, x\rangle$ on $S_X \cap H_x^+$ and that $r(x) - l(x)$ is a positive multiple of x. When there is no non-trivial segment on S_X parallel to $\langle -x, x\rangle$, the points $l(x)$ and $r(x)$ are chosen in such a way that $r(x) = l(x) \in S_X \cap H_x^+$ and $l(x) \perp_B x$ (cf. Figure 1 and Figure 2 below).

The following lemma, basic for the discussion after it, refers to the shape of bisectors in Minkowski planes.

LEMMA 2.5. (cf. [16, Proposition 22]) *For any $x \in S_X$, $B(-x, x)$ is fully contained in the bent strip bounded by the rays $[x, x + r(x)\rangle$, $[x, x - l(x)\rangle$, $[-x, -x + l(x)\rangle$, and $[-x, -x - r(x)\rangle$.*

THEOREM 2.6. *For any x, $y \in S_X$ we have that $y \in \overline{P(x)}$ whenever $y \perp_B x$.*

Proof. Case I: Suppose that there exists a non-trivial maximal segment $[a, b] \subset S_X$ parallel to $\langle -x, x\rangle$. It is trivial that if $y \in S_X$ is a point such that $y \perp_B x$, then either $y \in [a, b]$ or $-y \in [a, b]$. Thus it suffices to show that $[a, b] \subset \overline{P(x)}$.

For any $\lambda \in (0, 1)$, let α be an arbitrary number in the open interval $(0, \min\{\lambda, 1 - \lambda\})$. Then

$$\|\lambda a + (1 - \lambda)b + \alpha(b - a)\| = \|(\lambda - \alpha)a + (1 - \lambda + \alpha)b\| = 1$$

and

$$\|\lambda a + (1 - \lambda)b - \alpha(b - a)\| = \|(\lambda + \alpha)a + (1 - \lambda - \alpha)b\| = 1.$$

Thus $\lambda a + (1 - \lambda)b \in P(\alpha \|b - a\| x) = P(x)$, and therefore $[a, b] \subset \overline{P(x)}$.

Case II: If there exists a unique point $y \in S_X \cap H_x^+$ such that $y \perp_B x$, then, by Lemma 2.5, $B(-x, x)$ is bounded between the lines $\langle x, x + y\rangle$ and $\langle -x, -x + y\rangle$. On the other hand, by Lemma 2.4, for any integer $n > 0$ there exists a unique number $\lambda_n \in [0,1]$ such that $\lambda_n(x+ny)+(1-\lambda_n)(-x+ny) \perp_I x$, i.e.,

$$z_n := (2\lambda_n - 1)x + ny \in B(-x,x).$$

Moreover,

$$\begin{aligned}\|\widehat{z_n} - y\| &= \left\| \frac{(2\lambda_n - 1)x + ny}{\|(2\lambda_n - 1)x + ny\|} - y \right\| \\ &\le \left\| \frac{(2\lambda_n - 1)x}{\|(2\lambda_n - 1)x + ny\|} \right\| + \left\| \frac{ny}{\|(2\lambda_n - 1)x + ny\|} - y \right\|.\end{aligned}$$

Since

$$\lim_{n\to\infty} \left\| \frac{(2\lambda_n - 1)x}{\|(2\lambda_n - 1)x + ny\|} \right\| = \lim_{n\to\infty} \frac{1}{n} \left\| \frac{(2\lambda_n - 1)x}{\|\frac{2\lambda_n - 1}{n}x + y\|} \right\| = 0$$

and

$$\lim_{n\to\infty} \left\| \frac{ny}{\|(2\lambda_n - 1)x + ny\|} - y \right\| = \lim_{n\to\infty} \left| \frac{1 - \|\frac{2\lambda_n - 1}{n}x + y\|}{\|\frac{2\lambda_n - 1}{n}x + y\|} \right| = 0,$$

we have

$$\lim_{n\to\infty} \|\widehat{z_n} - y\| = 0.$$

It follows that $y \in \overline{P(x)}$, which completes the proof. ∎

One may expect that those points in S_X, to which x is Birkhoff orthogonal, are all in $\overline{P(x)}$. However, the following example shows that this is not true (see also Remark 2.10).

Example 1. Let X be the Minkowski plane on $\mathbb{R}^2$ with the maximum norm $\|(\alpha,\beta)\| = \max\{|\alpha|,|\beta|\}$ and $x = (1,1)$. Then $B(-x,x) = \langle(-1,1),(1,-1)\rangle$, and therefore $P(x) = \{(1,-1),(-1,1)\}$. It is clear that $(0,1) \notin \overline{P(x)}$ and $(1,0) \notin \overline{P(x)}$, while $x \perp_B (0,1)$ and $x \perp_B (1,0)$.

Lemma 2.7. (Uniqueness property of isosceles orthogonality, cf. [1, Corollary 4]) *For any $x \in S_X$ and $0 \le \alpha \le 1$, there exists a point $y \in \alpha S_X$ which is unique up to the sign and satisfies $x \perp_I y$.*

Let $x \in S_X$. By the uniqueness property of isosceles orthogonality, for any $t \in [0,1]$ there exists a unique point $F_x(t)$ such that

$$F_x(t) \in B(-x,x) \cap tS_X \cap \overline{H_x^+}.$$

For any $t \in (0,1]$, let
$$T_x(t) = \widehat{F_x(t)}.$$

LEMMA 2.8. *Let $\{t_n\} \subset (0,1]$ be a sequence such that $\lim_{n\to\infty} t_n = 0$ and that $\{T_x(t_n)\}$ is a Cauchy sequence. Then $x \perp_B \lim_{n\to\infty} T_x(t_n)$.*

Proof. From the compactness of S_X and the fact that $\{T_x(t_n)\}$ is a Cauchy sequence it follows that there exists a point $z \in S_X$ such that

$$z = \lim_{n\to\infty} T_x(t_n).$$

We show that $x \perp_B z$, and it suffices to prove that $\inf_{\lambda\in\mathbb{R}} \|x+\lambda z\| = 1$. In fact,

$$\begin{aligned}\inf_{\lambda\in\mathbb{R}} \|x+\lambda z\| &= \inf_{\lambda\in\mathbb{R}} \left\|x+\lambda \lim_{n\to\infty} T_x(t_n)\right\| = \lim_{n\to\infty} \inf_{\lambda\in\mathbb{R}} \left\|x+\frac{\lambda}{t_n} t_n T_x(t_n)\right\| \\ &= \lim_{n\to\infty} \inf_{\lambda\in\mathbb{R}} \|x+\lambda F_x(t_n)\| = \lim_{n\to\infty} \inf_{\lambda\in[-1,1]} \|x+\lambda F_x(t_n)\|,\end{aligned}$$

where the last equality follows from the fact that $\|x+F_x(t_n)\| = \|x-F_x(t_n)\|$. By the triangle inequality, we have for any $\lambda \in [-1,1]$

$$1-|\lambda|t_n = \|x\| - \|\lambda F_x(t_n)\| \le \|x+\lambda F_x(t_n)\| \le \|x\| + \|\lambda F_x(t_n)\| \le 1+|\lambda|t_n,$$

and therefore

$$\inf_{\lambda\in\mathbb{R}} \|x+\lambda z\| = \lim_{n\to\infty} \inf_{\lambda\in[-1,1]} \|x+\lambda F_x(t_n)\| = 1.$$

This completes the proof. ∎

THEOREM 2.9. *Let $x \in S_X$. If there exists a unique point $z \in S_X$ (except for the sign) such that $x \perp_B z$, then $z \in \overline{P(x)}$. And if there exists a point $z \in \overline{P(x)} \backslash P(x)$, then either $z \perp_B x$ or $x \perp_B z$.*

Proof. To prove the first statement, let $\{s_n\} \subset (0,1]$ be an arbitrary sequence such that $\lim_{n\to\infty} s_n = 0$. It is clear that $\{T_x(s_n)\}$ is a bounded subset of S_X, and therefore we can choose a convergent subsequence $\{T_x(s_{n_k})\}$. Let $t_k = s_{n_k}$. From Lemma 2.8 it follows that $x \perp_B \lim_{k\to\infty} T_x(t_k)$. Thus either $\lim_{k\to\infty} T_x(t_k) = z$ or $\lim_{k\to\infty} T_x(t_k) = -z$. Since $T_x(t_k) \in P(x)$ for each k, $z \in \overline{P(x)}$.

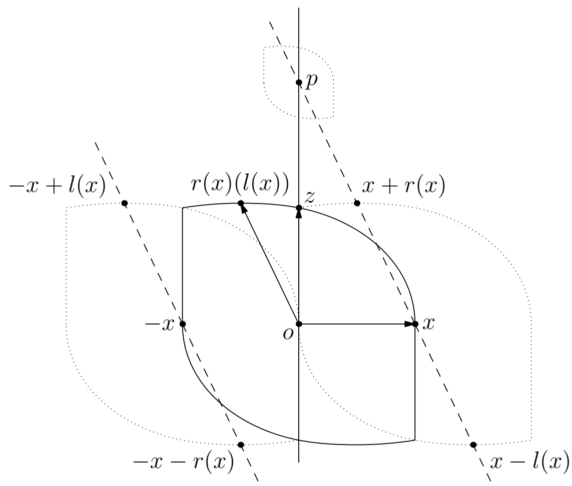

Figure 1: The proof of Theorem 2.9, Case I.

For proving the second statement, let $z \in \overline{P(x)} \backslash P(x)$. We consider the following two cases.

Case I: The line $\langle -z, z\rangle$ intersects one of the four rays

$$[x, x+r(x)\rangle,\ [x, x-l(x)\rangle,\ [-x, -x+l(x)\rangle,\ \text{and } [-x, -x-r(x)\rangle.$$

Without loss of generality, we can suppose that $[0, z\rangle$ intersects $[x, x+r(x)\rangle$ in some point p; see Figure 1. Since $z \in \overline{P(x)}$, there exists a sequence $\{z_n\} \subset P^+(x)$ such that $z_i \neq z_j$ $(i \neq j)$, $\lim_{n\to\infty} z_n = z$, and

$$(\langle -z_n, z_n\rangle \cap [x, x+r(x)\rangle) \in (p + \frac{1}{n} B_X).$$

By Lemma 2.5, for any number $t > \|p\| + 1$ we have $tz_n \notin B(-x, x)$. Thus, for each z_n there exists a number t_n being the largest positive number such that $t_n z_n \perp_I x$. It is clear that $\{t_n\}_{n=1}^{\infty}$ is bounded. Thus we can choose a subsequence $\{t_{n_k}\}$ such that

$$\lim_{k\to\infty} t_{n_k} = t_0.$$

Hence

$$\|t_0 z + x\| = \lim_{k\to\infty} \|t_{n_k} z_{n_k} + x\| = \lim_{k\to\infty} \|t_{n_k} z_{n_k} - x\| = \|t_0 z - x\| ,$$

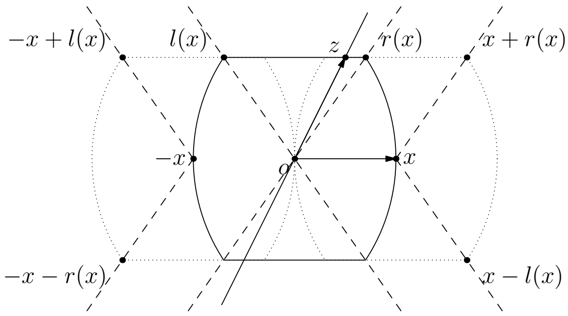

Figure 2: The proof of Theorem 2.9, Case II.

which means that $t_0 z \perp_I x$. Since $z \in \overline{P(x)} \backslash P(x)$, we see that $t_0 = 0$. Thus we can suppose, without loss of generality, that $\{t_{n_k}\}_{k=1}^{\infty} \subset (0,1]$. Hence

$$z = \lim_{k\to\infty} z_{n_k} = \lim_{k\to\infty} T_x(t_{n_k}).$$

By Lemma 2.8, $x \perp_B z$.

Case II: The line $\langle -z, z\rangle$ intersects none of the four rays

$$[x, x+r(x)\rangle,\ [x, x-l(x)\rangle,\ [-x, -x+l(x)\rangle, \text{ and } [-x, -x-r(x)\rangle;$$

see Figure 2. Then it is trivial that the line $\langle -z, z\rangle$ is fully contained in the double cone

$$\{\lambda l(x) + \mu r(x) :\ \lambda\mu \geq 0\}.$$

Thus $\langle -z, z\rangle$ intersects the segment $[l(x), r(x)]$, and therefore $z \perp_B x$. ∎

Remark 2.10. 1. The condition $y \in S_X$ together with $y \perp_B x$ does not imply that in general $y \in P(x)$. For example, take again the Minkowski plane on $\mathbb{R}^2$ with maximum norm, and let $x = (1,0)$. Then $y = (1,1)$ is a point such that $y \in S_X$ and $y \perp_B x$. But for any $t > 0$ we have

$$\|x+ty\| - \|x-ty\| = 1 + t - \max\{|1-t|, t\} > 0,$$

which means that $y \notin P(x)$.

2. In general, the condition that $z \in S_X$ is the unique point (except for the sign) satisfying $x \perp_B z$ does not imply $z \in P(x)$. Let X be the Minkowski plane on $\mathbb{R}^2$ with the norm $\|\cdot\|$, where for any point (α, β)

$$\|(\alpha,\beta)\| := \begin{cases} \sqrt{\alpha^2+\beta^2} & :\ \alpha\beta \geq 0; \\ \max\{|\alpha|,\ |\beta|\} & :\ \alpha\beta < 0. \end{cases}$$

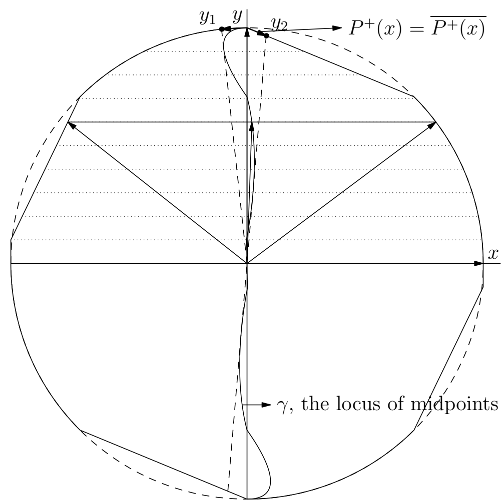

Figure 3: $\overline{P(x)}$ is not determined by points which are Birkhoff orthogonal to x or to which x is Birkhoff orthogonal.

Take $x = (1, 0)$ and $z = (0, 1)$. Then x, $z \in S_X$, and z is the unique point (except for the sign) in S_X such that $x \perp_B z$. But for any $t > 0$ we have

$$\|x + tz\| - \|x - tz\| = \sqrt{1 + t^2} - \max\{1, t\} > 0,$$

which implies that $z \notin P(x)$.

3. $\overline{P^+(x)}$ is an arc of S_X (possibly degenerate to a point) since $P^+(x)$ is connected. Theorem 2.9 says that if z is one of the endpoints of $\overline{P^+(x)}$ and $z \notin P^+(x)$, then either $x \perp_B z$ or $z \perp_B x$. We remark that, in general, the endpoints of $\overline{P^+(x)}$ have nothing to do with the points that are Birkhoff orthogonal to x or with the points to which x is Birkhoff orthogonal. For example, let X be a Minkowski plane on $\mathbb{R}^2$ (cf. Figure 3 with $x, y \in S_X$. Then it can be seen that y is the unique point (except for the sign) in S_X which is Birkhoff orthogonal to x, and it is also the unique point (except for the sign) in S_X to which x is Birkhoff orthogonal. However, y is contained in the arc between y_1 and y_2, which is a subset of $\overline{P^+(x)}$.

Now we study the distance $d(x, P(x))$ from a point x to $P(x)$, and we have to use the following lemma.

LEMMA 2.11. (cf. [12]) *If x and y are two points such that $x \perp_I y$, then*

(1) $\|x+ky\| \leq |k|\,\|x \pm y\|$ *and* $\|x \pm y\| \leq \|x+ky\|$, *if* $|k| \geq 1$.

(2) $\|x+ky\| \leq \|x \pm y\|$ *and* $|k|\,\|x \pm y\| \leq \|x+ky\|$, *if* $|k| \leq 1$.

A Minkowski plane X is said to be *rectilinear* if S_X is a parallelogram. One can easily verify that a Minkowski plane is rectilinear if and only if there exist two points x, $y \in S_X$ such that $\|x+y\| = \|x-y\| = 2$.

THEOREM 2.12. *For any $x \in S_X$ we have*

$$1 \leq d(x, P(x)) \leq 2,$$

with equality on the right only if X is rectilinear, and with equality on the left only if either there exists a segment parallel to $\langle -x, x\rangle$ on S_X whose length is not less than 1, or there exists a point $z \in S_X$ such that $\|z-x\| = 1$ and $[x, z] \subset S_X$.

Proof. It is trivial that $d(x, P(x)) \leq 2$. If $d(x, P(x)) = 2$, then for any $z \in P(x)$ we have $\|z-x\| = 2$. Let $z_0 \in P(x)$ be a point such that $z_0 \perp_I x$. Then

$$\|z_0 + x\| = \|z_0 - x\| = 2,$$

which implies that X is rectilinear.

For any $x \in S_X$ and $z \in P(x)$ there exists a number $t > 0$ such that $tz \perp_I x$. If $t \geq 1$, then $0 < \frac{1}{t} \leq 1$. By Lemma 2.11, we have

$$\|z-x\| = \left\|\frac{1}{t}tz - x\right\| \geq \frac{1}{t}\|tz+x\| = \frac{1}{2t}(\|tz+x\| + \|tz-x\|) \geq 1.$$

If $0 < t < 1$, then $\frac{1}{t} \geq 1$. Again, by Lemma 2.11 we have

$$\|z-x\| = \left\|\frac{1}{t}tz - x\right\| \geq \|tz+x\| = \frac{1}{2}(\|tz+x\| + \|tz-x\|) \geq 1.$$

Hence $d(x, P(x)) = \inf\{\|x-z\| : \; z \in P(x)\} \geq 1$.

Suppose now that $d(x, P(x)) = 1$, and without loss of generality we can assume that $d(x, P^+(x)) = 1$.

Case I: If there exists a point $z \in P^+(x)$ such that $\|z-x\| = 1$, then there exists a number $t > 0$ such that $tz \perp_I x$, which yields

$$\max\{t, 1\} \leq \frac{1}{2}(\|tz+x\| + \|tz-x\|) = \|tz-x\|. \tag{2.1}$$

If $0 < t < 1$, then it follows from the convexity of the function $f(s) = \|x - sz\|$ and $f(0) = f(1) = 1 \leq f(-t) = f(t)$ that $\|x - \lambda z\| = 1$ for any $\lambda \in [-t, 1]$, which implies that $[x - z, x] \subset S_X$.

If $t \geq 1$, then we have

$$\|tz\| = \|tz - z\| + \|z\| = \|tz - z\| + \|z - x\| \geq \|tz - x\| .$$

From (2.1) it follows that the convex function $g(s) = \|z - sx\|$ satisfies $g(0) = g(1) = g(-\frac{1}{t}) = g(\frac{1}{t}) = 1$, and then $\|z - \lambda x\| = 1$ for $-\frac{1}{t} \leq \lambda \leq 1$, which implies that $[z - x, z] \subset S_X$.

Case II: If $\|z' - x\| > 1$ for any $z' \in P^+(x)$, then there exists a point $z \in \overline{P^+(x)} \backslash P^+(x)$ such that $\|z - x\| = 1$. By Theorem 2.9, either $z \perp_B x$ or $x \perp_B z$. It can be proved in a similar way as in Case I that either $[x - z, x] \subset S_X$ or $[z - x, z] \subset S_X$. The proof is complete. ∎

Corollary 2.13. *For any $x \in S_X$ there exist two points $u, v \in S_X \backslash \big((x + d(x, P(x))\mathrm{int}B_X) \cup (-x + d(x, P(x))\mathrm{int}B_X)\big)$ such that*

$$B(-x, x) \subset \{\alpha u + \beta v : \ \alpha\beta \geq 0\}.$$

Theorem 2.14. *Let $x \in S_X$. If there exists a segment $[a, b] \subset S_X$ parallel to $\langle -x, x\rangle$ and of length not less than 1, then $d(x, P(x)) = 1$.*

Proof. From Theorem 2.6 it follows that $[a, b] \subset \overline{P(x)}$. Since $\|b - a\| \geq 1$, we can assume, without loss of generality, that there exists a point $z \in [a, b]$ such that $z - a = x$. Then $\|z - x\| = \|a\| = 1$, which implies that $d(x, P(x)) = 1$. ∎

Remark 2.15. The fact that there exists a point z with $\|z - x\| = 1$ and $[x, z] \subset S_X$ does in general not imply that $d(x, P(x)) = 1$. Namely, take again the Minkowski plane on $\mathbb{R}^2$ with maximum norm, and let $x = (1, 1)$. Then x is contained in the segment $[(-1, 1), x]$ whose length is 2. But it is clear that $d(x, P(x)) = 2$.

Next, we examine properties of intersections of radial projections of bisectors of two distinct segments, and we start with a characteristic property of the Euclidean plane.

THEOREM 2.16. *A Minkowski plane X is Euclidean if and only if for any x, $y \in S_X$ with $x \neq \pm y$, $P(x) \cap P(y) = \varnothing$.*

Proof. We only need to show sufficiency. Suppose that X is not Euclidean. Then, by Theorem 2.2, there exists a point $x \in S_X$ such that $P^+(x)$ contains more than one point. Let $x' \in S_X \cap H_x^+$ be such that $x \perp_I x'$. Then $x' \in P^+(x)$. Assume that there exists a point $y' \in P^+(x)$, $y' \neq x'$, and let $y \in S_X \backslash \{\pm x\}$ be such that $y \perp_I y'$. Then $y' \in P(y)$ and $P(x) \cap P(y) \neq \varnothing$, a contradiction. ∎

It is possible that $P(x) = P(y)$ holds for two points x, $y \in S_X$ with $x \neq \pm y$; see the following example.

EXAMPLE 2. Let X be the Minkowski plane on $\mathbb{R}^2$ with maximum norm, and let $x = (1, \frac{1}{2})$, and $x' = (1, \frac{1}{3})$. We show that

$$P(x) = P(x') = [(-1,1),(0,1)] \cup [(0,-1),(1,-1)] \backslash \{(-1,1),(1,-1)\}. \tag{2.2}$$

On the one hand, we have

$$\left\| \frac{1}{2}(0,1) + x \right\| = \left\| \frac{1}{2}(0,1) - x \right\| \text{ and } \left\| \frac{2}{3}(0,1) + x' \right\| = \left\| \frac{2}{3}(0,1) - x' \right\|,$$

and therefore $\{(0,1),(0,-1)\} \subseteq P(x) \cap P(x')$ and

$$d(x, P(x)) = \|(0,1) - x\| = \|(0,1) - x'\| = d(x', P(x')) = 1.$$

On the other hand, it is evident that $\|z - x\| < 1$ for any point $z \in S_X$ strictly between $(0,1)$ and x, and that $\|z - x'\| < 1$ for any point $z \in S_X$ strictly between $(0,1)$ and x'.

Now we show that

$$\{(-1,1),(1,-1)\} \subseteq (\overline{P(x)} \backslash P(x)) \cap (\overline{P(x')} \backslash P(x')). \tag{2.3}$$

For any $t > 0$ we have

$$\|t(-1,1) + x\| - \|t(-1,1) - x\| = \|(-t+1, t+\tfrac{1}{2})\| - \|(-t-1, t-\tfrac{1}{2})\| \neq 0$$

and

$$\|t(-1,1) + x'\| - \|t(-1,1) - x'\| = \|(-t+1, t+\tfrac{1}{3})\| - \|(-t-1, t-\tfrac{1}{3})\| \neq 0.$$

On the other hand, for any integer $n > 0$ we have

$$\begin{aligned}\left\|(1-n, n-\tfrac{1}{2})+x\right\| &= \|(2-n,n)\| = n = \|(-n,n-1)\| \\ &= \left\|(1-n,n-\tfrac{1}{2})-x\right\|\end{aligned}$$

and

$$\begin{aligned}\left\|(1-n, n-\tfrac{1}{3})+x'\right\| &= \|(2-n,n)\| = n = \left\|(-n,n-\tfrac{2}{3})\right\| \\ &= \left\|(1-n,n-\tfrac{1}{3})-x'\right\|.\end{aligned}$$

It is evident that

$$\lim_{n\to\infty}\frac{(1-n,n-\frac{1}{2})}{\left\|(1-n,n-\frac{1}{2})\right\|} = \lim_{n\to\infty}\frac{(1-n,n-\frac{1}{3})}{\left\|(1-n,n-\frac{1}{3})\right\|} = (-1,1).$$

Thus (2.3) holds, and therefore (2.2) holds.

Remark 2.17. This example shows also that $d(x, P^+(x))$ is not necessarily equal to $d(-x, P^+(x))$.

Next, we derive a sufficient condition for the property that two radial projections satisfy $P(x) \cap P(y) = \varnothing$.

LEMMA 2.18. *Let $x, y \in S_X$. If $x \perp_I y$, then for any number $t > 1$ the inequality*

$$\|x+ty\| > \|x+y\|$$

holds.

Proof. Suppose the contrary, i.e., that there exists a number $t_0 > 1$ such that $\|x+t_0y\| \leq \|x+y\|$. Then, since the function $f(t) = \|x+ty\|$ is convex,

$$\|x+y\| = \|x-y\| = \|x+t_0y\| = f(0) = 1.$$

This implies that $[x+t_0y, x-y]$ is a segment on S_X having length larger than 2, which is impossible. ∎

THEOREM 2.19. *For any x, $y \in S_X$ with $x \perp_I y$, $P(x) \cap P(y) = \varnothing$.*

Proof. First we show that $\widehat{x+y} \notin P(x)$. Suppose that there exists a number $t > 0$ such that $\|t(x+y)+x\| = \|t(x+y)-x\|$. Then

$$\left\|(1+\tfrac{1}{t})x+y\right\| = \left\|(1-\tfrac{1}{t})x+y\right\|.$$

If $t \geq \frac{1}{2}$, then $|1-\frac{1}{t}| \leq 1$. Thus, from Lemma 2.18 and the convexity of the function $\lambda \to \|\lambda x + y\|$ we get

$$\left\|(1+\tfrac{1}{t})x+y\right\| > \|x+y\| \geq \left\|(1-\tfrac{1}{t})x+y\right\|,$$

a contradiction. Hence $0 < t < \frac{1}{2}$. Then we have

$$\begin{aligned}\left\|(1-\tfrac{1}{t})x+y\right\| &= \left\|(\tfrac{1}{t}-1)x-y\right\| \\ &= \frac{1}{2}\big(\left\|(1+\tfrac{1}{t})x+y\right\| + \left\|(\tfrac{1}{t}-1)x-y\right\|\big) \geq \frac{1}{t}.\end{aligned}$$

On the other hand, we have

$$\left\|(\tfrac{1}{t}-1)x-y\right\| \leq \frac{1}{t}-1+1 = \frac{1}{t}$$

and, therefore,

$$\left\|(1+\tfrac{1}{t})x+y\right\| = \left\|(\tfrac{1}{t}-1)x-y\right\| = \frac{1}{t}.$$

Then the convex function $f(\lambda) = \|x+\lambda(x+y)\|$ satisfies $f(-1) = f(-t) = f(0) = f(t) = 1$ with $-1 < -t < 0 < t$. Therefore $f(\lambda) = 1$ for $-1 \leq \lambda \leq t$. In particular, we have $f(-\frac{1}{2}) = 1$ and then $\|x+y\| = \|x-y\| = 2$.

This implies that S_X is a parallelogram with $\pm x$ and $\pm y$ as vertices. Then $[x,y] \subseteq S_X$, and therefore $\left\|(1+\frac{1}{t})x+y\right\| = 2+\frac{1}{t}$, again a contradiction.

Since x and y are arbitrary, we also have $\widehat{x-y} \notin P(x)$.

Without loss of generality, we can suppose that $y \in H_x^+$. Then, since $P^+(x)$, $P^-(x)$, $P^+(y)$, and $P^-(y)$ are all connected sets, $P^+(x)$ lies strictly between $\widehat{x+y}$ and $\widehat{y-x}$, $P^-(x)$ lies strictly between $\widehat{-x-y}$ and $\widehat{x-y}$, $P^+(y)$ lies strictly between $\widehat{-x-y}$ and $\widehat{y-x}$, and $P^-(y)$ lies strictly between $\widehat{x+y}$ and $\widehat{x-y}$. Thus $P(x) \cap P(y) = \varnothing$, and this completes the proof. ∎

Remark 2.20. It is possible that there exist two points x, $y \in S_X$ with $x \perp_I y$ such that $\overline{P(x)} \cap \overline{P(y)} \neq \varnothing$. For example, let X be the Minkowski plane on $\mathbb{R}^2$ with maximum norm, and let $x = (1,0)$ and $y = (0,1)$. Then $(1,1) \in \overline{P(x)} \cap \overline{P(y)}$ (cf. [12, Example 4.1]).

3. A critical number for Minkowski spaces

The discussion in this section arises from the following natural problem: Determine the sign of the difference

$$\|x+y\| - \|x-y\| \tag{3.1}$$

when only the directions of the vectors x and y are known. We exclude the trivial case where one of the two vectors is o. In Euclidean case, this problem can be solved in different ways. For example, we know that the difference (3.1) is positive if and only if the angle between x and y is less than $\pi/2$. Equivalently, (3.1) is positive if and only if

$$\|\widehat{x} - \widehat{y}\| < \sqrt{2}. \tag{3.2}$$

From the discussion in the foregoing sections it can be seen that in general Minkowski spaces we cannot determine the sign of (3.1). The only thing we can probably do in this direction is to provide a sufficient condition for guaranteing that (3.1) is positive. As there is no natural definition of angular measure in Minkowski spaces, we would like to find a number which plays a role as the number $\sqrt{2}$ does in (3.2).

For the discussion in the sequel, we need to introduce the so called *non-square constants*

$$J(X) := \sup\{\min\{\|x+y\|, \|x-y\|\} : \ x, y \in S_X\}$$

and

$$S(X) := \inf\{\max\{\|x+y\|, \|x-y\|\} : \ x, y \in S_X\}.$$

Also we shall use the following equivalent representations of these two constants, which were provided in [14]:

$$J(X) = \sup\{\|x-y\| : \ x, y \in S_X, \ x \perp_I y\}$$

and

$$S(X) = \inf\{\|x-y\| : \ x, y \in S_X, \ x \perp_I y\}.$$

It has been shown (cf. [8], [9], and [11, Theorem 10]) that

$$1 \le S(X) \le \sqrt{2} \le J(X) \le 2$$

and

$$J(X) \cdot S(X) = 2.$$

Now we are going to define, for any Minkowski space, the so-called *critical number*

$$c(X) := \inf_{x \in S_X} d(x, P(x)).$$

Our first result on $c(X)$ is given by

THEOREM 3.1. *For any Minkowski space X we have*

$$1 \leq c(X) \leq \sqrt{2},$$

with equality on the left if and only if there exists a segment contained in S_X whose length is not less than 1, *and with equality on the right if and only if X is Euclidean.*

Proof. By Theorem 2.12, for any $x \in S_X$ the inequality $d(x, P(x)) \geq 1$ holds. Thus it is trivial that $c(X) \geq 1$. When $c(X) = 1$, by the compactness of the unit sphere there exists a point $x_0 \in S_X$ such that $d(x_0, P(x_0)) = 1$. Then, by Theorem 2.12, there exists a segment in S_X having length not less than 1.

Conversely, suppose that there exists a segment $[a, b] \subset S_X$ with $\|a - b\| \geq 1$. Then it follows from Theorem 2.14 that $d(\widehat{a-b}, P(\widehat{a-b})) = 1$.

On the other hand, for any x, $y \in S_X$ with $x \perp_I y$ we have

$$\|x + y\| = \|x - y\| \geq d(x, P(x)) \geq c(X).$$

Thus

$$\sqrt{2} \geq S(X) = \inf\{\|x - y\| : \ x, \ y \in S_X, \ x \perp_I y\} \geq c(X).$$

If $c(X) = \sqrt{2}$, then

$$\sup\{\|x - y\| : \ x, \ y \in S_X, \ x \perp_I y\} = c(X) = \sqrt{2}.$$

To prove that X is Euclidean, it suffices to show that each two-dimensional subspace of X is Euclidean, and therefore we can assume, without loss of generality, that $\dim X = 2$. Then, by Theorem 2.2, we only have to show that $P(x) = \{y, -y\}$ for any x, $y \in S_X$ with $x \perp_I y$. Suppose the contrary, i.e., that there exist some points x, y, $z \in S_X$ with $x \perp_I y$ such that $z \in P(x) \setminus \{y, -y\}$ and, without loss of generality, that z and y lie in the same half-plane bounded by $\langle -x, x \rangle$. It is clear that

$$\|z - x\| \geq d(x, P(x)) \geq c(X) = \sqrt{2} = \|y - x\|$$

and

$$\|z+x\| \geq d(-x, P(-x)) \geq c(X) = \sqrt{2} = \|y+x\| \,.$$

If one of $\|z+x\|$ and $\|z-x\|$ is $\sqrt{2}$ then, since $J(X) = S(X) = \sqrt{2}$, it follows from [2, Proposition 1] that $\|z+x\| = \|z-x\|$, which contradicts the uniqueness property of isosceles orthogonality (see Lemma 2.7). Thus we have $\min\{\|z+x\|\,, \|z-x\|\} > \sqrt{2}$, which contradicts the fact that $J(X) = S(X) = \sqrt{2}$. This completes the proof. ∎

Theorem 3.2. *For any Minkowski space X we have that*

$$c(X) = \sup\big\{c > 0 : \ x, y \in X\backslash\{o\}, \ \|\widehat{x} - \widehat{y}\| < c \textit{ implies } \|x-y\| < \|x+y\| \,\big\}.$$

Proof. Let x and y be arbitrary points from $X\backslash\{o\}$ and $\|\widehat{x} - \widehat{y}\| < c(X)$. We show that $\|x-y\| < \|x+y\|$. Suppose the contrary, i.e., that $\|x-y\| \geq \|x+y\|$. Let

$$f(t) = \|(tx+y)+x\| - \|(tx+y)-x\| \,.$$

Then $f(0) \leq 0$ and, by [12, Lemma 4.4],

$$\begin{aligned} \lim_{t\to+\infty} f(t) &= \lim_{t\to+\infty} \big(\, \|(tx+y)+x\| - \|(tx+y)-x\| \,\big) \\ &= \lim_{t\to+\infty} \big(\, \|((t-1)+2)x+y\| - \|(t-1)x+y\| \,\big) = 2\,\|x\| \,. \end{aligned}$$

Thus, by the continuity of $\|\cdot\|$, there exists a number $t_0 \geq 0$ such that $f(t_0) = 0$, and therefore $\widehat{t_0x+y} \in P(x)$. It is clear that $\widehat{t_0x+y}$ lies between $\widehat{x}$ and $\widehat{y}$. From the Monotonicity Lemma it follows that

$$c(X) \leq d(\widehat{x}, P(\widehat{x})) \leq \left\|\widehat{t_0x+y} - \widehat{x}\right\| \leq \|\widehat{y} - \widehat{x}\| < c(X),$$

which is impossible.

It is then sufficient to show that $c(X)$ is the largest number having the required properties. Suppose the contrary, i.e., that there exists a number $\alpha_0 > c(X)$ having the required properties. By the compactness of S_X, there exists a point $x_0 \in S_X$ such that $d(x_0, P(x_0)) = c(X)$. Since $P(x_0)$ is not empty, there exists a number $\varepsilon \geq 0$ such that $\varepsilon + c(X) < \alpha_0$ and that there exists a point $y \in P(x_0)$ with $\|y - x_0\| = \varepsilon + c(X)$. Then there exists a number $t > 0$ such that $\|ty + x_0\| = \|ty - x_0\|$, which is in contradiction to the assumption that $\alpha_0 > c(X)$ is a number having the required properties. ∎

4. A characterization of the Euclidean plane

The following result is well known.

Lemma 4.1. (cf. [15] and [5, 10.9]) *A Minkowski plane X is Euclidean if and only if the implication*

$$x \perp_I y \quad \Rightarrow \quad x \perp_B y$$

holds for any x, $y \in S_X$.

The next theorem strengthens some characterizations of inner product spaces collected in [5].

Theorem 4.2. *A Minkowski plane X is Euclidean if for any $u, v \in S_X$ with $u \neq -v$*

$$\left\|\widehat{u+v} - u\right\| = \left\|\widehat{u+v} - v\right\|.$$

Proof. For any $x \in S_X$ and $z \in B(-x, x)\backslash\{o\}$, let

$$G_x(z) = \frac{\|x+z\| - \|z\|}{\|z\|} z.$$

We show first that $x \perp_I G_x(z)$. Let

$$u = \widehat{x+z} \quad \text{and} \quad v = \widehat{z-x}.$$

Then

$$\left\|\widehat{u+v} - u\right\| = \left\|\frac{z}{\|z\|} - \frac{x+z}{\|x+z\|}\right\| = \left\|\left(\frac{1}{\|z\|} - \frac{1}{\|x+z\|}\right)z - \frac{1}{\|x+z\|}x\right\|, \tag{4.1}$$

$$\left\|\widehat{u+v} - v\right\| = \left\|\frac{z}{\|z\|} - \frac{z-x}{\|x+z\|}\right\| = \left\|\left(\frac{1}{\|z\|} - \frac{1}{\|x+z\|}\right)z + \frac{1}{\|x+z\|}x\right\|. \tag{4.2}$$

By the assumption of the theorem, $\left\|\widehat{u+v} - u\right\| = \left\|\widehat{u+v} - v\right\|$. Hence

$$\left\|\frac{\|x+z\| - \|z\|}{\|z\|} z - x\right\| = \left\|\frac{\|x+z\| - \|z\|}{\|z\|} z + x\right\|,$$

which means that $x \perp_I G_x(z)$. It is clear that

$$\|G_x(z)\| = \|x+z\| - \|z\| \leq \|x\| = 1.$$

Let $y \in S_X$ be a point such that $x \perp_I y$. Next we show that $x \perp_B y$.

Let $\{s_n\} \subset (0,1]$ be an arbitrary sequence such that $\lim_{n\to\infty} s_n = 0$. It is clear that $\{T_x(s_n)\}$ is a bounded subset of S_X, and therefore we can choose a convergent subsequence $\{T_x(s_{n_k})\}$. Let $t_k = s_{n_k}$. Then $\lim_{k\to\infty} t_k = 0$, and $\{T_x(t_k)\}$ is a Cauchy sequence. From Lemma 2.8 it follows that $x \perp_B \lim_{k\to\infty} T_x(t_k)$.

On the other hand, we have

$$\begin{aligned}\lim_{k\to\infty} G_x(t_kT_x(t_k)) &= \lim_{k\to\infty} \frac{\|x+t_kT_x(t_k)\| - \|t_kT_x(t_k)\|}{\|t_kT_x(t_k)\|} t_kT_x(t_k)\\ &= \lim_{k\to\infty} \big(\, \|x+t_kT_x(t_k)\| - \|t_kT_x(t_k)\|\,\big) T_x(t_k) = \lim_{k\to\infty} T_x(t_k).\end{aligned}$$

Then $\lim_{k\to\infty} T_x(t_k) \perp_I x$, since $G_x(t_kT_x(t_k)) \perp_I x$ for each t_k. From the uniqueness property of isosceles orthogonality it follows that either $\lim_{k\to\infty} T_x(t_k) = y$ or $\lim_{k\to\infty} T_x(t_k) = -y$. Thus $x \perp_B y$ and, by Lemma 4.1, X is Euclidean. ∎

In particular, Theorem 4.2 strengthens the following statement, which is used to derive many characterizations of inner product spaces in Chapter 3, Chapter 4, and Chapter 5 of the book [5].

Corollary. (cf. [12, Theorem 4.7]) *A Minkowski plane X is Euclidean if and only if the implication*

$$x \perp_I y \quad \Rightarrow \quad x \perp_I \alpha y \quad (\forall \alpha \in \mathbb{R})$$

holds for any $x, y \in X$.

Proof. For any u, $v \in S_X$ with $u \neq -v$ it is clear that $(u+v) \perp_I (u-v)$. Then, by the assumption of the theorem, we have

$$\frac{1}{2}(u-v) \perp_I \left(\frac{1}{\|u+v\|} - \frac{1}{2}\right)(u+v)$$

which implies that

$$\left\| \frac{1}{2}(u-v) + \left(\frac{1}{\|u+v\|} - \frac{1}{2}\right)(u+v) \right\| = \left\| \frac{1}{2}(u-v) - \left(\frac{1}{\|u+v\|} - \frac{1}{2}\right)(u+v) \right\|$$

or, equivalently,

$$\left\| \widehat{u+v} - u \right\| = \left\| \widehat{u+v} - v \right\|.$$

This completes the proof. ∎

5. Higher dimensions

In this short section one important property of radial projections of bisectors in dimensions $d \geq 3$ is proved.

THEOREM 5.1. *Let X be a Minkowski space with* $\dim X \geq 3$. *Then for any $x \in S_X$, $P(x)$ is a connected subset of S_X.*

Proof. For any $x \in S_X$, let H_x be a hyperplane through o such that $x \perp_B H_x$. We show first that $B(-x,x)\backslash\{o\}$ is connected. Let

$$\begin{array}{rccl} T: & X & \longrightarrow & H_x \\ & z = \alpha x + \beta y & \longrightarrow & \beta y. \end{array}$$

It is clear that T is continuous, $T(z) = o$ if and only if $z \in \langle -x, x\rangle$, and $T(B(-x,x)\backslash\{o\}) \subset H_x\backslash\{o\}$. On the other hand, from [12, Theorem 4.4] it follows that for any $y \in H_x\backslash\{o\}$ there exists a number α such that $\alpha x + y \in B(-x,x)\backslash\{o\}$. Thus $T(B(-x,x)\backslash\{o\}) = H_x\backslash\{o\}$.

Suppose that $B(-x,x)\backslash\{o\}$ can be partitioned into two disjoint nonempty subsets A_1 and A_2, which are open in the relative topology induced on $B(-x,x)\backslash\{o\}$. We show that $T(A_1) \cap T(A_2) = \varnothing$. Suppose the contrary, i.e., that there exists a point $y \in T(A_1) \cap T(A_2)$. Then it is evident that $y \neq o$. Let $\alpha_1 \neq \alpha_2$ be two numbers such that $\alpha_1 x + y \in A_1$ and $\alpha_2 x + y \in A_2$. Then, from the convexity of $B(-x,x)$ in the direction of x (cf. [10, Lemma 1]) it follows that $[\alpha_1 x + y, \alpha_2 x + y] \subset B(-x,x)\backslash\{o\}$, and therefore $[\alpha_1 x + y, \alpha_2 x + y]$ can be partitioned into two disjoint nonempty sets $[\alpha_1 x + y, \alpha_2 x + y] \cap A_1$ and $[\alpha_1 x + y, \alpha_2 x + y] \cap A_2$ which are open in the subspace topology of $[\alpha_1 x + y, \alpha_2 x + y]$. This is impossible. Thus $T(A_1) \cap T(A_2) = \varnothing$ and $T(A_1) \cup T(A_2) = H_x\backslash\{o\}$, which contradicts the fact that $H_x\backslash\{o\}$ is connected. Thus $B(-x,x)\backslash\{o\}$ is connected.

Then, as image of $B(-x,x)\backslash\{o\}$ under the continuous function $R(z) = \widehat{z}$ on $X\backslash\{o\}$, $P(x)$ is connected. ∎

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