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Solvability of a Recursive Functional Equation in the Sequence Banach Space l^2

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Abstract: The main aim of this paper is to study the existence of solutions of the following recursive functional equation

$$x(n) = f(n, x(n), x(n-1))$$

in the space l^2, under general assumptions. The main tools of our existence theorem are the characterization of the relatively compact sets in the space l^2 and Schauder Fixed point theorem. Moreover, our functional equation has as particular cases some integral equations of Urysohn type. Finally, we present some examples where our theorem can be applied.

Key words: Recursive functional equations, Banach sequence space, Fixed-point theorem.

AMS *Subject Class.* (2000): 45M99,47H09.

1. Introduction

In this paper we study the existence of solutions of the following recursive functional equation

$$x(n) = f(n, x(n), x(n-1)), \qquad x(0) = a. \tag{1}$$

We will prove that under some conditions the problem (1) has solution in the sequence Banach space l^2.

Using a characterization of the relatively compact sets of the space l^2 in conjunction with Schauder's fixed point theorem, we will be able to prove our existence result under rather general assumptions being convenient in applications.

We finally illustrate our result with some examples.

2. Notation and auxiliary facts

Assume E is a real Banach space with norm $\|\cdot\|$ and zero element Θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r and by B_r the ball $B(\Theta,r)$. If X is a nonempty subset of E we denote by $\overline{X}$, $Conv\,X$ the closure and the closed convex closure of X, respectively. The symbols λX and $X+Y$ denote the usual algebraic operations on sets. Finally, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We will accept the following definition of the concept of measure of noncompactness [2].

Definition 1. A mapping $\mu : \mathfrak{M}_E \longrightarrow [0,\infty)$ is said to be a *measure of noncompactness* in E if the following conditions are satisfied:

1. The family $\ker\mu = \{X \in \mathfrak{M}_E \ : \ \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.
2. $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$.
4. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
5. If $\{X_n\}_n$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1,2,\ldots$ and if $\lim_{n\to\infty}\mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ described above is called *the kernel of the measure of noncompactness* μ.

Further facts concerning measures of noncompactness and their properties may be found in [2].

In the sequel we will work in the Banach sequence space l^2 consisting of real sequences (x_n) which satisfy $\sum_{n=1}^{\infty}|x_n|^2 < \infty$. The space l^2 is furnished with the standard norm

$$\|(x_n)\| = \left(\sum_{n=1}^{\infty}|x_n|^2\right)^{\frac{1}{2}}.$$

Now we recollect the properties of the measure of noncompactness which will be used further on. We consider the operators R_m defined in l^2 by

$$R_m : l^2 \longrightarrow l^2$$

$$x = (x_n) \longmapsto R_m(x) = (y_k), \quad \text{where} \quad \begin{cases} y_k = 0, & \text{for } k \leq m-1 \\ y_k = x_k, & \text{for } k \geq m. \end{cases}$$

Moreover, for $X \in \mathfrak{M}_{l^2}$ let us put

$$\|R_m X\| = \sup\{\|R_m x\| : x \in X\}.$$

Then, we consider the function μ defined on the family $\mathfrak{M}_{l^2}$ by

$$\mu(X) = \lim_{m \to \infty} \|R_m X\|.$$

It is proved in [2, p. 22], that the function μ is a measure of noncompactness in the space l^2. Particularly, we have the following characterization of the relatively compact subset of l^2: Let $X \in \mathfrak{M}_{l^2}$, then $X \in \mathfrak{N}_{l^2}$ if and only if $\mu(X) = 0$.

3. Main Result

In this section, we will study the following infinite system of nonlinear numerical equations

$$x(n) = f(n, x(n), x(n-1)), \tag{2}$$

for $n \in \mathbb{N}$ and with the initial condition $x(0) = a$.

System (2) will be investigated under the following hypotheses:

- (i) The function $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that there exist two sequences $(c(n))$ and $(d(n)) \in l^2$ and there exists a function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that
$$|f(n, x, y)| \leq c(n) + d(n)\psi(|x|, |y|),$$
where ψ satisfies the following property : $(i-a)$ Let $r, r', s, s' \in \mathbb{R}_+$ such that $r \leq r'$ and $s \leq s'$, then $\psi(r, s) \leq \psi(r', s')$.
- (ii) For n fixed, $f(n, -, -) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
- (iii) The inequality
$$\|(c(n))\|_2 + \|(d(n))\|_2 \psi(r, r) \leq r,$$
has a positive solution r_0.

Now we present our main result.

Theorem 1. *Under assumptions $(i) - (iii)$, the system (2) has at least one solution $x = (x(n))$ in the space l^2.*

Proof. Let S be the set of all sequences on $\mathbb{R}$ and consider the operator A, defined by

$$A : l^2 \longrightarrow S, \quad ((Ax)(n)) = (f(n, x(n), x(n-1))).$$

Firstly, we will prove that if $(x(n)) \in l^2$, then $A(x(n)) \in l^2$. By using assumption (i) and the Cauchy-Schwarz's inequality, we can obtain that

$$\begin{aligned}
\|A(x(n))\|_2^2 &= \sum_{n=1}^{\infty} |f(n, x(n), x(n-1))|^2 \\
&\leq \sum_{n=1}^{\infty} (c(n) + d(n)\psi(|x(n)|, |x(n-1)|))^2 \\
&\leq \sum_{n=1}^{\infty} (c(n) + d(n)\psi(\|(x(n))\|_2, \|(x(n))\|_2))^2 \\
&= \sum_{n=1}^{\infty} (c(n)^2 + 2c(n)d(n)\psi(\|(x(n))\|_2, \|(x(n))\|_2) \\
&\quad + d(n)^2\psi(\|(x(n))\|_2, \|(x(n))\|_2))^2 \\
&\leq \sum_{n=1}^{\infty} c(n)^2 + 2\left(\sum_{n=1}^{\infty}(c(n)^2)\right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{n=1}^{\infty}(d(n)^2)\right)^{\frac{1}{2}} \psi(\|(x(n))\|_2, \|(x(n))\|_2) \\
&\qquad\qquad + \psi(\|(x(n))\|_2, \|(x(n))\|_2)^2 \sum_{n=1}^{\infty} d(n)^2.
\end{aligned}$$

As $(c(n)), (d(n)) \in l^2$ and $\psi(\|x(n)\|_2, \|x(n)\|_2)$ is a constant, this gives us that the operator A transforms the space l^2 into itself.

On the other hand, if $(x(n)) \in l^2$, the above estimate allows us to infer that

$$\begin{aligned}
\|(Ax(n))\|_2^2 \leq{}& \|(c(n))\|_2^2 + 2\psi(\|(x(n))\|_2, \|(x(n))\|_2 \cdot \|(c(n))\|_2\|(d(n))\|_2 \\
&+ \psi(\|(x(n))\|_2, \|(x(n))\|_2)^2 \cdot \|(d(n))\|_2^2 \\
={}& \left(\|(c(n))\|_2 + \psi(\|(x(n))\|_2, \|(x(n))\|_2) \cdot \|(d(n))\|_2\right)^2.
\end{aligned}$$

This inequality in conjunction with the assumption (iii) ensures that there exists a positive number r_0 for which, if $\|(x(n))\|_2 \leq r_0$ then $\|(Ax(n))\|_2 \leq r_0$, i.e. the operator A transforms the ball B_{r_0} into itself.

In what follows let us take a nonempty subset X of the ball B_{r_0}. We will prove that $A(X)$ is a relatively compact set of B_{r_0}. To do that we use the characterization of the relatively compact sets in l^2 that we have studied in the previous section. Let us fix $p_0 \in \mathbb{N}$, then using assumption (i) and the Cauchy-Schwartz's inequality, we get

$$\begin{aligned}\|R_{p_0}(Ax(n))\|_2^2 &= \|(0,\cdots,0,f(p_0,x(p_0),x(p_0-1),\cdots)\|_2^2\\ &= \sum_{n\geq p_0}|f(n,x(n),x(n-1))|^2\\ &\leq \sum_{n\geq p_0}c(n)^2+2\psi(r_0,r_0)\left(\sum_{n\geq p_0}c(n)^2\right)^{\frac{1}{2}}\cdot\left(\sum_{n\geq p_0}d(n)^2\right)^{\frac{1}{2}}\\ &\quad+\psi(r_0,r_0)^2\sum_{n\geq p_0}d(n)^2.\end{aligned}$$

Hence, taking limit when $p_0 \to \infty$,we get

$$\lim_{p_0\to\infty}\sup_{x\in X}\|R_{p_0}(Ax)\| = 0.$$

Consequently, AX is a relatively compact set of B_{r_0}.

Now, we consider the sequence of sets $(B_{r_0}^n)$, where $B_{r_0}^1 = Conv\,(A(B_{r_0}))$, $B_{r_0}^2 = Conv\,(A(B_{r_0}^1))$ and so on. Observe that all sets of this sequence are nonempty, bounded, closed and convex. Moreover $B_{r_0}^{n+1} \subset B_{r_0}^n$ for $n = 1, 2, \ldots$ As, in our case, $\mu(B_{r_0}^n) = 0$ for every n, we have that $\lim_{n\to\infty}\mu(B_{r_0}^n) = 0$ and thus from the axiom (5) of Definition 1, we infer that the set $C = \bigcap_{n=1}^{\infty} B_{r_0}^n$ is nonempty, bounded, closed and convex and moreover $\mu(C) = 0$. It is easily proved that the operator A maps the set C into itself.

Now we show that A is continuous on the set C. To do this we will consider the sequence $X_n = (x_n^j)$ in C and the element $X = (x^j)$ in C such that $\lim_{n\to\infty} X_n = X$. As C is relatively compact subset in l^2 and taking into account the characterization of such sets given in the previous section, fix $\varepsilon > 0$ there exists $p_0 \in \mathbb{N}$ such that for $p \geq p_0$, we have

$$\|R_p(AX_n)\|_2 < \sqrt{\frac{\varepsilon}{8}}, \qquad \|R_p(AX)\|_2 < \sqrt{\frac{\varepsilon}{8}}$$

(Note that AX_n and AX are elements of C).

On the other hand, in virtue of the uniform continuity of the function $f(p,-,-) : [-r_0, r_0] \times [-r_0, r_0] \longrightarrow \mathbb{R}$ for every $p \leq p_0$ we can obtain for the same fixed previously, $\varepsilon > 0$, the existence of $\delta > 0$ such that for $r, r', s, s' \in [-r_0, r_0]$ if $\|(r,s) - (r',s')\| < \delta$ then $|f(p,r,s) - f(p,r',s')| < \sqrt{\frac{\varepsilon}{2p_0}}$ for every $p \leq p_0$. Moreover, as $\lim_{n\to\infty} X_n = X$, for every $\delta > 0$ above mentioned there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have $\|X_n - X\|_2^2 < \delta$. Finally, for $n \geq n_0$ we get

$$\|AX_n - AX\|_2^2 = \sum_{j=1}^{\infty} |f(j, x_n^j, x_n^{j-1}) - f(j, x^j, x^{j-1})|^2$$

$$\leq \sum_{j=1}^{p_0-1} |f(j, x_n^j, x_n^{j-1}) - f(j, x^j, x^{j-1})|^2 + \sum_{j=p_0}^{\infty} |f(j, x_n^j, x_n^{j-1}) - f(j, x^j, x^{j-1})|^2$$

$$\leq \sum_{j=1}^{p_0-1} \frac{\varepsilon}{2p_0} + \|R_{p_0}AX_n\|_2^2 + \|R_{p_0}AX\|_2^2 + 2\|R_{p_0}AX_n\|_2 \cdot \|R_{p_0}AX\|_2$$

$$\leq p_0 \cdot \frac{\varepsilon}{2p_0} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + 2 \cdot \frac{\varepsilon}{8} < \varepsilon.$$

So we deduce that the operator A is continuous on the set C.

Finally, applying the Schauder fixed point theorem we infer that A has at least one fixed point in the set C. Obviously it is a solution of our equation (2). Thus the proof is complete. ∎

Remark 1. The condition $x(0) = a$ is used in order to give sense to our equation with $n = 1$ and it is not relevant in the proof of the theorem.

4. Examples

Firstly, in the first and second example, we will give some concrete examples of integral equations which illustrate the result contained in Theorem 1. The example 3 will show us the relevance of some assumptions of Theorem 1. And the example 4 will show us that our hypotheses are not necessary.

Example 1. Let us consider

$$f(n,x,y) = c(n) + \int_0^1 u(n,x,y,s)\,d\mu(s),$$

under the following assumptions

(*i*) $(c(n)) \in l^2$

(*ii*) $|u(n,x,y,s)| \leq d(n) + \psi(|x|,|y|)\varphi(s,n)$, with $(d(n)) \in l^2$, $\varphi(-,n) \in L^1[0,1]$, $\sum_{n=1}^{\infty} \|\varphi(-,n)\|_1^2 < \infty$ and the function ψ satisfying the assumptions of our main theorem.

Then

$$|f(n,x,y)| \leq |c(n)| + d(n) + \psi(|x|,|y|)\|\varphi(-,n)\|_1.$$

Under this conditions our equation is the following one

$$x(n) = c(n) + \int_0^1 u(n,x(n),x(n-1),s)d\mu(s),$$

which is an Uryshon type equation.

We conclude that our functional equation has as particular cases some integral equations of Urysohn type.

EXAMPLE 2. For every $s \in [0,1]$, let us consider the function

$$f(n,x,y) = \frac{1}{n} + \int_0^1 \frac{s^2}{n^2} + \frac{1}{2}\frac{\cos^2(n\cdot s)}{n}(x\cos s + y\sin s)d\mu(s).$$

In this case, our problem is

$$\begin{aligned} &x(n) = \frac{1}{n} + \int_0^1 \frac{s^2}{n^2} + \frac{1}{2}\frac{\cos^2(n\cdot s)}{n}(x(n)\cos s + x(n-1)\sin s)d\mu(s) \\ &x(0) = a. \end{aligned} \tag{3}$$

We get that

$$|f(n,x,y)| \leq \frac{1}{n} + \frac{1}{n^2}\mu([0,1]) + (|x|+|y|)\cdot\frac{1}{2}\int_0^1 \frac{|\cos^2(n\cdot s)|}{n}d\mu(s).$$

The elements of our main theorem are

$$(c(n)) = \left(\frac{1}{n} + \frac{1}{n^2}\right), \quad (d(n)) = \left(\frac{\sin(2n)}{8n^2} + \frac{1}{4n}\right), \quad \psi(|x|,|y|) = (|x|+|y|).$$

These elements satisfy the hypothesis of our main result, so applying theorem 1 we get that the problem (3) has a solution in l^2.

Example 3. We consider the function $f(n,x,y) = n + \frac{1}{n^2}(x+y)$.

In this case, our equation is the following one

$$x(n) = n + \frac{1}{n^2}(x(n) + x(n-1)).$$

Notice that

$$|f(n,x,y)| \le n + \frac{1}{n^2}(|x| + |y|),$$

so the elements of our theorem are

$$(c(n)) = (n), \quad (d(n)) = \left(\frac{1}{n^2}\right), \quad \psi(x,y) = x+y.$$

Assumptions $(i-a)$ and (ii) are satisfied, but hypotheses (i) and (iii) are not satisfy because $(c(n)) \notin l^2$.

Our equation can be written in the following way

$$\left(1 - \frac{1}{n^2}\right) x(n) = n + \frac{1}{n^2} x(n-1)). \tag{4}$$

If our equation will have a solution $(x(n)) \in l^2$, then $\lim_{n\to\infty} x(n) = 0$ and in this case, taking limit in (4) when $n \to \infty$ we will get $0 = \infty$ which is not possible.

Example 4. We consider the function $f(n,x,y) = \frac{1}{n}(y-x)$.

In this case, our equation is the following one

$$x(n) = \frac{1}{n}(x(n-1) - x(n)), \tag{5}$$

Notice that

$$|f(n,x,y)| \le \frac{1}{n}(|x-y|) \le \frac{1}{n}(|x| + |y|),$$

so the elements of our theorem are

$$(c(n)) = (0), \quad (d(n)) = \left(\frac{1}{n}\right), \quad \psi(x,y) = x+y.$$

Assumptions (i), $(i-a)$ and (ii) are satisfied, but the inequality of the hypothesis (iii) is

$$\frac{\pi^2}{6} \cdot 2 \cdot r \le r,$$

and it has not a positive and real solution. But we can write our equation (5) in the following way

$$x(n) = \frac{x(n-1)}{n+1},$$

and as $x(0) = a$, we get that

$$(x(n)) = \left(\frac{a}{(n+1)!} \right),$$

which belongs to l^2.

So, despite of not satisfying assumption (*iii*) our equation has a solution in l^2.

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