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Linear Maps Preserving the Generalized Spectrum *

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Abstract: Let H be an infinite-dimensional separable complex Hilbert space and $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. For an operator T in $\mathcal{B}(H)$, let $\sigma_g(T)$ denote the generalized spectrum of T. In this paper, we prove that if $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a surjective linear map, then ϕ preserves the generalized spectrum (i.e. $\sigma_g(\phi(T)) = \sigma_g(T)$ for every $T \in \mathcal{B}(H)$) if and only if there is $A \in \mathcal{B}(H)$ invertible such that either $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{B}(H)$, or $\phi(T) = AT^{tr}A^{-1}$ for every $T \in \mathcal{B}(H)$. Also, we prove that $\gamma(\phi(T)) = \gamma(T)$ for every $T \in \mathcal{B}(H)$ if and only if there is $U \in \mathcal{B}(H)$ unitary such that either $\phi(T) = UTU^*$ for every $T \in \mathcal{B}(H)$ or $\phi(T) = UT^{tr}U^*$ for every $T \in \mathcal{B}(H)$. Here $\gamma(T)$ is the reduced minimum modulus of T.

Key words: reduced minimum modulus, generalized spectrum, Jordan isomorphism,linear preserver problems.

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1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras over the complex field. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called a *Jordan homomorphism* if $\phi(a^2) = (\phi(a))^2$ for every $a \in \mathcal{A}$, or equivalently $\phi(ab+ba) = \phi(a)\phi(b)+\phi(b)\phi(a)$ for all a and $b \in \mathcal{A}$. It is obvious that every homomorphism and every anti-homomorphism is a Jordan homomorphism (a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called an anti-homomorphism if $\phi(ab) = \phi(b)\phi(a)$ for all $a, b \in \mathcal{A}$).

Over the last decade there has been a considerable interest in the so-called linear preserver problems (see the survey articles [3], [4], [11], [23], [18], [19], [20]). The goal is to study linear maps ϕ between two Banach algebras such that ϕ preserves a given class of elements of algebras (i.e., the invertible elements, regular elements, the nilpotents, the idempotents, the algebraic elements, finite rank operators, the spectral radius ...).

One of the most famous problems in this direction is Kaplansky's problem [12]: Let ϕ be a surjective linear map between two semi-simple Banach alge-

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bras $\mathcal{A}$ and $\mathcal{B}$. Suppose that $\sigma(\phi(x)) = \sigma(x)$ for all $x \in \mathcal{A}$. Is it true that ϕ is a Jordan isomorphism?

This problem has been first solved in the finite-dimensional case. J. Dieudonné [6], Marcus and Purves [14] have proved that every unital invertibility preserving linear map on a complex matrix algebra is either an inner automorphism, or an inner anti-automorphism. This result has been later extended to the algebra of all bounded linear operators on a Banach space by A.R. Sourour [27], and to von Neumann algebras by B. Aupetit [2].

Recently, M. Mbekhta, L. Rodman and P. Šemrl [18], studied bijective linear maps on $\mathcal{B}(H)$ that preserve generalized invertibility in both directions. Later, M. Mbekhta in [19] and [20] gave a characterization of linear maps from $\mathcal{B}(H)$ onto $\mathcal{B}(H)$ that preserve the set of Fredholm operators, and the minimum or surjection modulus of operators respectivety.

The aim of the present paper is to continue this study for the linear maps preserving the generalized spectrum and the reduced minimum modulus on a Hilbert space.

The paper is organized as follows. In the next section we recall some notations and known results concerning the reduced minimum modulus and the generalized spectrum of operators. In Section 3, we provide a characterization of all the linear maps from $\mathcal{B}(H)$ onto $\mathcal{B}(H)$ that preserve the generalized spectrum. Furthermore, in Section 4 we characterize the linear maps ϕ that preserve the reduced minimum modulus.

2. Preliminaries

Throughout, H denotes a separable complex Hilbert space and $\mathcal{B}(H)$ denotes the algebra of all bounded linear operators on H. For an operator $T \in \mathcal{B}(H)$ we write T^{tr} for the transpose of T with respect to an arbitrary but fixed orthonormal basis of H, $N(T)$ for its kernel and $R(T)$ for its range. The spectrum of T is denoted by $\sigma(T)$, and $|T| = (T^*T)^{1/2}$ denotes the positive square root of T.

Generalized inverses. A bounded linear operator $S \in \mathcal{B}(H)$ is said to be a *generalized inverse* of $T \in \mathcal{B}(H)$ if $TST = T$ and $STS = S$. In this case we will say that T has a generalized inverse. Notice that the first equality $TST = T$ is a necessary and sufficient condition for T to have a generalized inverse. Indeed, if $TST = T$, then $TS'T = T$ and $S'TS' = S'$ with $S' = STS$. For the properties of generalized inverse we refer to [8, 9, 17, 22].

Reduced minimum modulus. We define the *reduced minimum modulus* (also called *conorm*) of $T \in \mathcal{B}(H)$ by

$$\gamma(T) = \inf\{\|Tx\| \,:\, x \in N(T)^{\perp}\,,\ \|x\| = 1\}\,,$$
$$\gamma(T) = \infty \qquad \text{if}\ \ T = 0\,.$$

We refer the reader to [1, 8, 9, 13, 17, 21] for the properties of the reduced minimum modulus. In particular, the following useful facts are true. For $T \in \mathcal{B}(H)$, we have

$$\gamma(T)^2 \;=\; \gamma(TT^*) \;=\; \gamma(T^*T) \;=\; \gamma(T^*)^2\,, \tag{2.1}$$

$$\gamma(T) \;=\; \gamma(T^{tr}) \;=\; \gamma(T^*)\,, \tag{2.2}$$

and

$$\gamma(T) \;=\; \gamma(|T|) \;=\; \inf\{\sigma(|T|) \setminus \{0\}\}\,.$$

It is well-known that for $T \in \mathcal{B}(H)$,

$$\gamma(T) > 0 \iff R(T) \ \text{ is closed}$$
$$\iff T \ \text{ has a generalized inverse}\,.$$

In this case, if $S \in \mathcal{B}(H)$ is a generalized inverse of T, then we have

$$\gamma(T) \geq \|S\|^{-1},$$

and if T^+ is the *Moore-Penrose inverse*, i.e., the unique operator T^+ that satisfies $TT^+T = T$, $T^+TT^+ = T^+$, $(TT^+)^* = TT^+$ and $(T^+T)^* = T^+T$, then

$$\gamma(T) = \frac{1}{\|T^+\|}\,.$$

In particular, if T is invertible, then $T^+ = T^{-1}$ and we have

$$\gamma(T) = \frac{1}{\|T^{-1}\|}\,.$$

Clearly, if $A, B \in \mathcal{B}(H)$ are invertible and $T \in \mathcal{B}(H)$, then

$$\gamma(T) > 0 \qquad \iff \qquad \gamma(ATB) > 0\,.$$

On the other hand, it is not difficult to show that if U and V are unitary operators and $T \in \mathcal{B}(H)$, then

$$\gamma(UTV) = \gamma(T)\,. \tag{2.3}$$

Indeed, first, it is easy to see from the definition of reduced minimum modulus that, if U is an unitary operator then $\gamma(UT) = \gamma(T)$. Using (2.2), we have

$$\gamma(UTV) \;=\; \gamma(TV) \;=\; \gamma((TV)^*) \;=\; \gamma(V^*T^*) \;=\; \gamma(T^*) \;=\; \gamma(T)\,.$$

The generalized spectrum. In [16], a generalization of the spectrum $\sigma(T)$ of $T \in \mathcal{B}(H)$ was given by replacing the notion of invertibility which appears in the classical definition of the spectrum by the existence of analytic generalized inverses. More precisely, we denote by $\operatorname{reg}(T)$ the *regular set* of T, defined by:

$$\operatorname{reg}(T) := \left\{ \lambda \in \mathbb{C} \ : \begin{array}{c} \text{there is a neighborhood } U_\lambda \text{ of } \lambda \text{ and} \\ \text{an analytic function } R : U_\lambda \to \mathcal{B}(H) \\ \text{such that } R(\mu) \text{ is a generalized} \\ \text{inverse of } T-\mu \text{ for any } \mu \in U_\lambda \end{array} \right\}.$$

The complement $\sigma_{\mathrm{g}}(T) = \mathbb{C} \backslash \operatorname{reg}(T)$ of $\operatorname{reg}(T)$ in $\mathbb{C}$ is called the *generalized spectrum* of T.

We say that T is *regular* if T has a generalized inverse and $N^\infty(T) \subseteq R^\infty(T)$, where $N^\infty(T) = \bigcup_{n\geq 0} N(T^n)$ is the generalized kernel and $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$ is the generalized range.

According to [15, Théorème 2.6], T is *regular* if and only if $0 \notin \sigma_{\mathrm{g}}(T)$, that is, if 0 belongs to the regular set of T.

We refer to [1, 15, 16] for more information about the generalized spectrum. In particular, the following useful facts are true:

$$\sigma_{\mathrm{g}}(T) = \left\{ \lambda \in \mathbb{C} \ : \ \lim_{z \to \lambda} \gamma(T - z) = 0 \right\}, \tag{2.4}$$

$$\partial\sigma(T) \subseteq \sigma_{\mathrm{g}}(T) \subseteq \sigma(T), \tag{2.5}$$

$$\sigma_{\mathrm{g}}(T^{tr}) = \sigma_{\mathrm{g}}(T) \qquad \text{and} \qquad \sigma_{\mathrm{g}}(T^*) = \overline{\sigma_{\mathrm{g}}(T)}, \tag{2.6}$$

$$\sigma_{\mathrm{g}}(ATA^{-1}) = \sigma_{\mathrm{g}}(T) \qquad \text{for any invertible } A \in \mathcal{B}(H). \tag{2.7}$$

3. Linear maps preserving the generalized spectrum

Before formulation our results we need the following definition.

We will say that a linear map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ *preserves the generalized spectrum* if $\sigma_g(\phi(T)) = \sigma_g(T)$ for every $T \in \mathcal{B}(H)$.

Theorem 3.1. *Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective linear map. Then the following conditions are equivalent:*

(1) *$\sigma_g(\phi(T)) = \sigma_g(T)$ for all $T \in \mathcal{B}(H)$;*
(2) *ϕ is either an automorphism or an anti-automorphism;*

(3) *there is $A \in \mathcal{B}(H)$ invertible such that ϕ takes one of the following forms:*

$$\phi(T) = ATA^{-1} \quad \text{or} \quad \phi(T) = AT^{tr}A^{-1} \qquad \text{for all} \quad T \in \mathcal{B}(H)\,;$$

in particular, ϕ is bijective, unital and continuous.

Proof. We prove that (1) $\Rightarrow$ (2). Since the generalized spectrum satisfies (2.5), from [20, Proposition 3.4], ϕ is bijective, continuous, unital and preserves the set of idempotents in $\mathcal{B}(H)$.

We show that ϕ preserves the set of orthogonal idempotents in $\mathcal{B}(H)$. We argue as in [2]. If $E = E^2$ and $F = F^2$ such that $EF = FE = 0$, then $E + F$ is an idempotent in $\mathcal{B}(H)$. So, $\phi(E)$, $\phi(F)$ and $\phi(E+F)$ are idempotents in $\mathcal{B}(H)$. Consequently,

$$\begin{aligned}\phi(E) + \phi(F) &= (\phi(E) + \phi(F))^2 = \phi(E)^2 + \phi(F)^2 + \phi(E)\phi(F) + \phi(F)\phi(E)\\ &= \phi(E) + \phi(F) + \phi(E)\phi(F) + \phi(F)\phi(E)\,.\end{aligned}$$

Thus

$$\phi(E)\phi(F) + \phi(F)\phi(E) = 0\,.$$

Now, left multiplication by $\phi(F)$ gives

$$\phi(F)\phi(E)\phi(F) + \phi(F)\phi(E) = 0\,, \tag{3.1}$$

and right multiplication by $\phi(F)$ gives

$$2\phi(F)\phi(E)\phi(F) = 0\,,$$

therefore, formula (3.1) yields $\phi(F)\phi(E) = 0$. With similar reasoning, we also get $\phi(E)\phi(F) = 0$. Thus, ϕ preserves the set of orthogonal idempotents in $\mathcal{B}(H)$.

Consider now $P_n = \sum_{i=1}^{n} \lambda_i E_i$ a finite linear real combination of mutually orthogonal projections (i.e., $\lambda_i \in \mathbb{R}$ and $E_iE_j = \delta_{i,j}E_i$). Then

$$\Big(\phi(P_n)\Big)^2 = \left(\sum_{i=1}^{n} \lambda_i \phi(E_i)\right)^2 = \sum_{i=1}^{n} \lambda_i^2 \phi(E_i) = \phi\left(\left(\sum_{i=1}^{n} \lambda_i E_i\right)^2\right) = \phi\big(P_n^2\big)\,.$$

Hence

$$\Big(\phi(P_n)\Big)^2 = \phi\big(P_n^2\big)\,.$$

By the spectral theorem, the set of all finite real linear combinations of orthogonal projections is dense in the set of all self-adjoint elements of $\mathcal{B}(H)$. Since ϕ is continuous, we have

$$\phi(T)^2 = \phi(T^2)$$

for all self-adjoint operators $T \in \mathcal{B}(H)$. Since every operator is a linear combination of two self-adjoint operators, we obviously obtain that ϕ is a Jordan automorphism. Indeed, let A and B are self-adjoint, then $A + B$ is self-adjoint. Thus $\phi((A+B)^2) = (\phi(A+B))^2$ so $\phi(AB+BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$. Now, for any T, $T = A + \mathrm{i}B$, with $A = 1/2(T + T^*)$ and $B = 1/2\mathrm{i}(T - T^*)$ are self-adjoint. Then

$$\begin{aligned}
\phi(T^2) &= \phi\big((A+\mathrm{i}B)^2\big) = \phi\big(A^2 + (\mathrm{i}B)^2 + \mathrm{i}(AB+BA)\big) \\
&= \big(\phi(A)\big)^2 + \big(\mathrm{i}\phi(B)\big)^2 + \mathrm{i}\big(\phi(AB+BA)\big) \\
&= \big(\phi(A)\big)^2 + \big(\mathrm{i}\phi(B)\big)^2 + \mathrm{i}\big(\phi(A)\phi(B) + \phi(B)\phi(A)\big) \\
&= \big(\phi(A) + \mathrm{i}\phi(B)\big)^2 = \big(\phi(T)\big)^2 .
\end{aligned}$$

Therefore, for all T we have $\phi(T^2) = (\phi(T))^2$ (i.e., ϕ is a Jordan homomorphism). Recall further that an algebra $\mathcal{A}$ is a *prime algebra* if for every pair $a, b \in \mathcal{A}$ the relation $a\mathcal{A}b = \{0\}$ implies that $a = 0$ or $b = 0$. Standard arguments yield that $\mathcal{B}(H)$ is a prime algebra. It is well-known that every Jordan automorphism of a prime algebra is an automorphism or an anti-automorphism [10]. Thus, ϕ is either an automorphism, or an anti-automorphism.

The implication (2) $\Rightarrow$ (3) follows from the fundamental isomorphism theorem [24, Theorem 2.5.19] (see also [5]).

The implication (3) $\Rightarrow$ (1) follows from (2.8) and (2.9). ∎

Corollary 3.2. *Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective unital linear map. If there exists $M > 0$ such that*

$$\frac{1}{M}\gamma(T) \le \gamma(\phi(T)) \le M\gamma(T) \qquad (T \in \mathcal{B}(H)), \tag{3.2}$$

then there is $A \in \mathcal{B}(H)$ invertible such that ϕ takes one of the following forms:

$$\phi(T) = ATA^{-1} \qquad \text{or} \qquad \phi(T) = AT^{tr}A^{-1}.$$

Proof. It follows from the hypotheses (3.2), for $z \in \mathbb{C}$ and $T \in \mathcal{B}(H)$, that

$$\lim_{z\to\lambda} \gamma(\phi(T) - z) = 0 \qquad \Longleftrightarrow \qquad \lim_{z\to\lambda} \gamma(T - z) = 0\,.$$

Thus,

$$\sigma_g(\phi(T)) = \sigma_g(T) \qquad \text{for all } \ T \in \mathcal{B}(H)\,.$$

Now, to complete the proof apply the above theorem. ∎

We denote $\mathcal{S}(T) = \{\, STS^{-1} : S \in \mathcal{B}(H) \text{ invertible}\,\}$ and $\mathcal{S}_M(T) = \{\, STS^{-1} : S \in \mathcal{B}(H) \text{ invertible and } M(S) \leq M\,\}$, where $M > 0$ and $M(S) = \|S\|\|S^{-1}\|$.

We will say that a linear map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ *preserves strongly similarity* if $\phi(T) \in \mathcal{S}_M(T)$ for some $M > 0$ and for all $T \in \mathcal{B}(H)$.

In the following corollary, we characterize the linear maps preserving strongly similarity.

COROLLARY 3.3. *Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective linear map, then the following conditions are equivalent:*

(1) *$\phi(T) \in \mathcal{S}_M(T)$ for some $M > 0$ and for all $T \in \mathcal{B}(H)$;*
(2) *$\phi(T) \in \overline{\mathcal{S}_M(T)}$ for some $M > 0$ and for all $T \in \mathcal{B}(H)$;*
(3) *there is $A \in \mathcal{B}(H)$ invertible such that*

$$\phi(T) = ATA^{-1} \qquad (T \in \mathcal{B}(H))\,.$$

Proof. We prove only the difficult implication: (2) ⇒ (3). From (2), it is straightforward that ϕ is unital. Now from [7, Theorem 2.3] ϕ preserves the generalized spectrum that is $\sigma_g(\phi(T)) = \sigma_g(T)$ for all $T \in \mathcal{B}(H)$. By Theorem 3.1, ϕ is either an automorphism or an anti-automorphism.

We prove now that ϕ cannot be an anti-automorphism. Assume, on the contrary, that ϕ is an anti-automorphism and consider T left invertible but not invertible. Since $\phi(T) \in \overline{\mathcal{S}_M(T)}$, $\phi(T)$ is also left invertible but not invertible. Now, there is $S \in \mathcal{B}(H)$ such that $ST = I$ and $TS \neq I$. Hence $I = \phi(ST) = \phi(T)\phi(S)$. It follows that $\phi(T)$ is invertible, which is a contradiction. Therefore ϕ is an automorphism as desired. Consequently, there is $A \in \mathcal{B}(H)$ invertible such that $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{B}(H)$. ∎

Remark 3.4. A linear map ϕ on $\mathcal{B}(H)$ is said to *preserve similarity* if $\phi(S) \in \mathcal{S}(\phi(T))$ whenever $S \in \mathcal{S}(T)$. P. Šemrl [25] gave a characterization of bijective linear maps on $\mathcal{B}(H)$ which preserves similarity.

4. Linear maps preserving the reduced minimum modulus

The next theorem follows by the same arguments as in [20, Theorem 3.8], we omit its proof here. It gives a characterization of the unitary operators in terms of the reduced minimum modulus.

Theorem 4.1. *Let $A \in \mathcal{B}(H)$ be invertible, then the following conditions are equivalent:*

(i) *A is the product of a non-zero real scalar by a unitary operator;*

(ii) *$\gamma(ATA^{-1}) = \gamma(T)$ for every $T \in \mathcal{B}(H)$.*

Theorem 4.2. *Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective unital linear map. Then the following conditions are equivalent:*

(1) *$\gamma(\phi(T)) = \gamma(T)$ for every $T \in \mathcal{B}(H)$;*

(2) *there is $U \in \mathcal{B}(H)$ unitary such that ϕ takes one of the following forms:*

$$\phi(T) = UTU^* \quad \text{or} \quad \phi(T) = UT^{tr}U^* \qquad \text{for every } T \in \mathcal{B}(H)\,.$$

Proof. Suppose that (1) holds, then ϕ preserves the generalized spectrum. Indeed, for $z \in \mathbb{C}$ and $T \in \mathcal{B}(H)$, we have,

$$\gamma(\phi(T) - z) = \gamma(\phi(T - z)) = \gamma(T - z)\,,$$

thus

$$\lim_{z \to \lambda} \gamma(\phi(T) - z) = \lim_{z \to \lambda} \gamma(T - z)\,.$$

It follows from (2.6) that

$$\sigma_g(\phi(T)) = \sigma_g(T) \qquad \text{for all } \ T \in \mathcal{B}(H).$$

By Theorem 3.1 there is $A \in \mathcal{B}(H)$ invertible such that either $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{B}(H)$ or $\phi(T) = AT^{tr}A^{-1}$ for every $T \in \mathcal{B}(H)$. Now by the above theorem, A is unitary multiplied by a nonzero real number. Thus there is a unitary $U \in \mathcal{B}(H)$ such that $\phi(T) = UTU^*$ or $\phi(T) = UT^{tr}U^*$ for every $T \in \mathcal{B}(H)$ as desired.

The implication (2) $\Rightarrow$ (1) follows from (2.2) and (2.5). ∎

To conclude the paper, we leave open the interesting question of whether the condition that ϕ is unital (i.e., $\phi(I) = I$) can be omitted from the assumptions of the above theorem. Then (2.5) suggests the following formulation of this question.

Conjecture 4.3. Let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective linear map. Then the following conditions are equivalent:

(1) $\gamma(\phi(T)) = \gamma(T)$ for every $T \in \mathcal{B}(H)$;

(2) there are $U, V \in \mathcal{B}(H)$ unitary operators such that ϕ takes one of the following forms:

$$\phi(T) = UTV \qquad \text{or} \qquad \phi(T) = UT^{tr}V.$$

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