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Classification of 4-Dimensional Nilpotent Complex Leibniz Algebras

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1. Introduction

In [8] and [9] several classes of new algebras were introduced. Some of them have two generating operations and they are called dialgebras. The first motivation to introduce such algebraic structures (related with well known Lie and associative algebras) were problems in algebraic K-theory.

The categories of these algebras over their operads assemble into the commutative diagram which reflects the Koszul duality of those categories. The aim of the present paper is to study structural properties of one class of Loday's list, namely the so called Leibniz algebras.

Leibniz algebras present a "non-commutative" (to be more precise, a "non-antisymmetric") analogue of Lie algebras and they were introduced by J.-L.Loday [8], as algebras that satisfy the following identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

They appeared to be related in a natural way to several topics such as differential geometry, homological algebra, classic algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, etc. In fact, most papers concerning Leibniz algebras are devoted to the study of homological problems [5], [6], [11]. The structure theory of Leibniz algebras mostly remains unexplored and the extension of notions like simple, semisimple Leibniz algebras, radical, etc., have not been discussed. Some structural results concerning nilpotency, classification of low dimensional Leibniz algebras and related problems were, however, considered in [1]–[4]. The reader may find similar results for Lie algebras in [7].

The classification, up to isomorphism, of any class of algebras is a fundamental and very difficult problem. It is one of the first problems that one encounters when trying to understand the structure of a member of this class of algebras. The purpose of the present paper is to provide the classification of four dimensional complex nilpotent Leibniz algebras. Actually, it is a prelude of our paper [2], where the geometric classification problems of low dimensional complex nilpotent Leibniz algebras were discussed. From the geometrical point of view the classification of a class of algebras corresponds to a fibration of this class, the fiber being the isomorphic classes. We will give representatives of each isomorphism class for 4-dimensional nilpotent complex Leibniz algebras. By "classification" here we mean the algebraic classification, i.e., the determination of the types of isomorphic algebras, whereas geometric classification is the problem of finding generic structural constants in the sense of algebraic geometry. But the geometrical classification presupposes the algebraical classification.

We restrict our discussion to nilpotent Leibniz algebras of dimension four since all Leibniz algebras of dimension at most three already have been classified in [3], [7], [8].

Definition 1.1. An algebra L over a field F is said to be a Leibniz algebra if it satisfies the following Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

where $[\ ,\ ]$ denotes the multiplication in L.

Remark 1.2. If a Leibniz algebra has the additional property of antisymmetry $[x,y] = -[y,x]$ for all $x, y \in L$ then the Leibniz identity can be easily reduced to the Jacobi identity:

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0.$$

Therefore a Leibniz algebra is a generalization of well known Lie algebras.

Another generalization of Lie algebras was given by A. Mal'tzev [12]. They were called Mal'tzev algebras and satisfy the following two identities:

$$[x,y] = -[y,x],$$

$$[[[x,y],z],x] + [[[y,z],x],x] + [[[z,x],x],y] = [[x,y],[x,z]].$$

It is clear that the intersection of the varieties of Leibniz algebras and Mal'tzev algebras coincides exactly with the variety of Lie algebras.

All algebras considered in this paper are supposed to be defined over the field of complex numbers C.

2. Nilpotent Leibniz algebras

In this section we remind the reader to some observations on associative and Leibniz algebras.

If L is associative algebra without unit then the associative algebra A, obtained from L by the external adjoining of a unit is denoted by $A = L \oplus C\mathbf{1}$.

PROPOSITION 2.1. *Let L be a finite dimensional nilpotent associative algebra. Then the algebra $A = L \oplus C\mathbf{1}$ does not contain any nontrivial idempotents.*

Proof. Let a be an element of L and let $1 + a$ be an idempotent of $A = L \oplus C\mathbf{1}$. Let m be the index of nilpotency of a. It is evident that $1+a = (1+a)^n$ for any natural $n \geq 2$. On the other hand, one has $(1+a)^n = 1 + \sum_{k=1}^{n} C_n^k a^k$; this gives $a = na + \sum_{k=2}^{n} C_n^k a^k$, and multiplying both sides of this equality by a^{m-2} we get $(n-1)a = 0$, which shows that the idempotent is trivial. ∎

Let L be a complex Leibniz algebra. We put:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \in N.$$

So if L is a Leibniz algebra with $L^3 = 0$ then it is associative. For our purpose the following corollary of the Proposition 2.1 is important.

COROLLARY 2.2. *Let L be a finite dimensional Leibniz algebra with $L^3 = 0$. Then the algebra $A = L \oplus C\mathbf{1}$ has no nontrivial idempotents.*

The following proposition gives the another link between L and $A = L \oplus C\mathbf{1}$.

PROPOSITION 2.3. *Let L_1 and L_2 be finite dimensional associative algebras without unit. Then $L_1 \oplus C\mathbf{1} \cong L_2 \oplus C\mathbf{1}$ if only if $L_1 \cong L_2$.*

Another result which we will use in this paper is G. Mazzola's classification of 5-dimensional unitary associative algebras [10]. He gave the list of all isomorphism types of 5-dimensional unitary associative algebras, which is however too long to be given here since there are 59 isomorphism types. We assume the reader is familiar with this list.

The concept of nilpotency in the case of Leibniz algebras can be defined the similar manner as in the Lie algebras case.

DEFINITION 2.4. A Leibniz algebra L is said to be nilpotent if there exists a natural $s \in N$, such that $L^s = 0$.

DEFINITION 2.5. An n-dimensional Leibniz algebra L is said to be nulfiliform if $\dim L^i = n - i + 1$, where $2 \leq i \leq n + 1$.

DEFINITION 2.6. An n-dimensional Leibniz algebra L is said to be filiform if $\dim L^i = n - i$, where $2 \leq i \leq n$.

For a given $n-$dimensional nilpotent Leibniz algebra L we define the following isomorphism invariant:

$$\chi(L) = (\dim L^1, \dim L^2, \dots, \dim L^{n-1}, \dim L^n)$$

It is evident that

$$\dim L^1 > \dim L^2 > \cdots > \dim L^k > \cdots$$

PROPOSITION 2.7. *Let L be an n-dimensional nilpotent Leibniz algebra. If the first two coordinates of the invariant $\chi(L)$ are equal to n and $n - 1$, respectively, then L is a nulfiliform Leibniz algebra.*

Proof. From the assertion of the proposition we can easily conclude that L is one-generated algebra. Let $L =< x >$ for some $x \in L \backslash L^2$. Then using Leibniz identity we obtain that the following set of elements

$$\{x, [x, x], \dots, \underbrace{[[[x, x], x], \dots, x]}_{n-\text{times}}\}$$

forms the basis of the algebra L, such that $\dim L^i = n - i + 1$, where $2 \leq i \leq n + 1$. ∎

PROPOSITION 2.8. ([4]) *Up to isomorphism, there is only one n-dimensional non-Lie nulfiliform Leibniz algebra. It can be given by the following table of multiplications:*

$$[e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n - 1$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of L and omitted products are supposed to be zero.

In [4] the description of the set of all $(n+1)$-dimensional complex non-Lie filiform Leibniz algebras was given as a union of two disjoint classes. The motivation behind these classes can be explained by the existence of naturally graded filiform non-Lie Leibniz algebras. However, from the existence of naturally graded filiform Lie algebras one more class of $(n+1)$-dimensional complex non-Lie filiform Leibniz algebras appears. Combining these two facts we get the following proposition.

PROPOSITION 2.9. *Any $(n+1)$-dimensional complex non-Lie filiform Leibniz algebra can be included in one of the following three classes of non-Lie filiform Leibniz algebras:*

$$\mu_1^{\overline{\alpha},\theta} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1 \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + \cdots + \alpha_{n-1} e_{n-1} + \theta e_n, & \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + \cdots + \alpha_{n+1-j} e_n, & 1 \le j \le n-2 \end{cases}$$

$$\mu_2^{\overline{\beta},\gamma} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \le i \le n-1 \\ [e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + \cdots + \beta_n e_n, & \\ [e_1, e_1] = \gamma e_n, & \\ [e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + \cdots + \beta_{n+1-j} e_n, & 2 \le j \le n-2 \end{cases}$$

$$\mu_3^{\alpha,\beta,\gamma,\delta} = \begin{cases} [e_0, e_0] = \alpha e_n, & (\alpha,\beta,\gamma) \neq (0,0,0) \\ [e_1, e_1] = \beta e_n, & \\ [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1 \\ [e_0, e_1] = -e_2 + \gamma e_n, & \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le n-1 \\ [e_i, e_j] = -[e_j, e_i] \in lin < e_{i+j+1}, e_{i+j+2}, \dots, e_n >, & \\ \qquad 2 \le i \le j \le n-1-i, \quad i+j \neq n & \\ [e_{n-i}, e_i] = -[e_i, e_{n-i}] = (-1)^i \delta e_n, & 1 \le i \le n-1 \end{cases}$$

where $\{e_0, e_1, e_3, \dots, e_n\}$ is a basis, $\delta = 1$ for odd n and $\delta = 0$ for even n; omitted products are supposed to be zero.

In other words, the above proposition means that the set of all $(n+1)$-dimensional complex non-Lie filiform Leibniz algebras can be represented as a disjoint union of the above mentioned three subsets. The isomorphisms inside each class, however, are not known.

3. Description of 4-dimensional nilpotent complex Leibniz algebras

In this section we shall expose the list of all isomorphic types of four dimensional nilpotent complex Leibniz algebras. Here we will use the four-dimensional case of Proposition 2.9 and then define isomorphisms inside each of the classes. To find the other isomorphism types one uses the classification of unitary associative algebras of dimension five [10]. Namely, from the classification list of [10] we pick out according to some restrictions Leibniz algebras of our list.

First we derive the following subsidiary result:

Proposition 3.1. *Any 4-dimensional nilpotent complex Leibniz algebra L belongs to one of the following types:*
(i) *nulfiliform Leibniz algebras, that is* $\chi(L) = (4,3,2,1)$,
(ii) *filiform Leibniz algebras, that is* $\chi(L) = (4,2,1,0)$,
(iii) *associative algebras, with* $\chi(L) := (4,2,0,0)$ *or* $(4,1,0,0)$,
(iv) *abelian, that is* $\chi(L) = (4,0,0,0)$.

Proof. Let us consider all possible cases for $\chi(L)$. The following can occur.

a) $\chi(L) = (4,3,2,1)$, then in view of Proposition 2.8 L is a nulfiliform algebra.
b) $\chi(L) = (4,2,1,0)$, then in view of Proposition 2.9 L is a filiform algebra.
c) $\chi(L) = (4,2,0,0)$.
d) $\chi(L) = (4,1,0,0)$.
It is clear that in the last two cases the algebra L is associative.
e) $\chi(L) = (4,0,0,0)$, then we get an abelian algebra.

Note that Proposition 2.7 implies that the cases $\chi(L) := (4,3,2,0)$, $(4,3,1,0)$ and $(4,3,0,0)$ are impossible. ∎

From now on as a matter of convenience we assume that the undefined multiplications are zero; we also do not consider neither abelian algebras, nor Lie algebras or split Leibniz algebras, i.e., we do not consider Leibniz algebras which are direct sums of proper ideals.

So we are given a Leibniz algebra L with a basis $\{e_1, e_2, e_3, e_4\}$.

THEOREM 3.2. *The isomorphism classes of four-dimensional complex nilpotent Leibniz algebras are given by the following representatives.*

$$
\begin{aligned}
\Re_1 :\ & [e_1,e_1]=e_2,\ [e_2,e_1]=e_3,\ [e_3,e_1]=e_4;\\
\Re_2 :\ & [e_1,e_1]=e_3,\ [e_1,e_2]=e_4,\ [e_2,e_1]=e_3,\ [e_3,e_1]=e_4;\\
\Re_3 :\ & [e_1,e_1]=e_3,\ [e_2,e_1]=e_3,\ [e_3,e_1]=e_4;\\
\Re_4(\alpha) :\ & [e_1,e_1]=e_3,\ [e_1,e_2]=\alpha e_4,\ [e_2,e_1]=e_3,\ [e_2,e_2]=e_4,\\
& [e_3,e_1]=e_4, \quad \alpha\in\{0,1\};\\
\Re_5 :\ & [e_1,e_1]=e_3,\ [e_1,e_2]=e_4,\ [e_3,e_1]=e_4;\\
\Re_6 :\ & [e_1,e_1]=e_3,\ [e_2,e_2]=e_4,\ [e_3,e_1]=e_4;\\
\Re_7 :\ & [e_1,e_1]=e_4,\ [e_2,e_1]=e_3,\ [e_3,e_1]=e_4,\ [e_1,e_2]=-e_3,\\
& [e_1,e_3]=-e_4,\\
\Re_8 :\ & [e_1,e_1]=e_4,\ [e_2,e_1]=e_3,\ [e_3,e_1]=e_4,\ [e_1,e_2]=-e_3+e_4,\\
& [e_1,e_3]=-e_4,\\
\Re_9 :\ & [e_1,e_1]=e_4,\ [e_2,e_1]=e_3,\ [e_2,e_2]=e_4,\ [e_3,e_1]=e_4,\\
& [e_1,e_2]=-e_3+2e_4,\ [e_1,e_3]=-e_4,\\
\Re_{10} :\ & [e_1,e_1]=e_4,\ [e_2,e_1]=e_3,\ [e_2,e_2]=e_4,\ [e_3,e_1]=e_4,\\
& [e_1,e_2]=-e_3,\ [e_1,e_3]=-e_4,\\
\Re_{11} :\ & [e_1,e_1]=e_4,\ [e_1,e_2]=e_3,\ [e_2,e_1]=-e_3,\ [e_2,e_2]=-2e_3+e_4\\
\Re_{12} :\ & [e_1,e_2]=e_3,\ [e_2,e_1]=e_4,\ [e_2,e_2]=-e_3;\\
\Re_{13}(\alpha) :\ & [e_1,e_1]=e_3,\ [e_1,e_2]=e_4,\ [e_2,e_1]=-\alpha e_3,\ [e_2,e_2]=-e_4, \quad \alpha\in C;\\
\Re_{14}(\alpha) :\ & [e_1,e_1]=e_4,\ [e_1,e_2]=\alpha e_4,\ [e_2,e_1]=-\alpha e_4,\ [e_2,e_2]=e_4,\\
& [e_3,e_3]=e_4, \quad \alpha\in C;\\
\Re_{15} :\ & [e_1,e_2]=e_4,\ [e_1,e_3]=e_4,\ [e_2,e_1]=-e_4,\ [e_2,e_2]=e_4,\ [e_3,e_1]=e_4;\\
\Re_{16} :\ & [e_1,e_1]=e_4,\ [e_1,e_2]=e_4,\ [e_2,e_1]=-e_4,\ [e_3,e_3]=e_4;\\
\Re_{17} :\ & [e_1,e_2]=e_3,\ [e_2,e_1]=e_4;\\
\Re_{18} :\ & [e_1,e_2]=e_3,\ [e_2,e_1]=-e_3,\ [e_2,e_2]=e_4;\\
\Re_{19} :\ & [e_2,e_1]=e_4,\ [e_2,e_2]=e_3;\\
\Re_{20}(\alpha) :\ & [e_1,e_2]=e_4,\ [e_2,e_1]=\tfrac{1+\alpha}{1-\alpha}\,e_4,\ [e_2,e_2]=e_3, \quad \alpha\in C\backslash\{1\};\\
\Re_{21} :\ & [e_1,e_2]=e_4,\ [e_2,e_1]=-e_4,\ [e_3,e_3]=e_4.
\end{aligned}
$$

Proof. According to Proposition 3.1 any 4-dimensional nilpotent complex Leibniz algebra is either nulfiliform, filiform or associative. Let us consider each of these cases separately.

Let L be nulfiliform. Then in view of Proposition 2.8 there is only one nulfiliform Leibniz algebra and it can be given by the multiplication table:

$$[e_1, e_1] = e_2,\ [e_2, e_1] = e_3,\ [e_3, e_1] = e_4.$$

This is $\Re_1$.

Let now L be filiform. Then Proposition 2.9 implies that there are three classes of filiform Leibniz algebras, i.e., the following occurs:

$$\nabla(\alpha, \beta) : [e_1, e_1] = e_3,\ [e_1, e_2] = \alpha e_4,\ [e_2, e_1] = e_3, [e_2, e_2] = \beta e_4,\ [e_3, e_1] = e_4;$$

$$\Omega(\alpha, \beta) : [e_1, e_1] = e_3,\ [e_1, e_2] = \alpha e_4,\ [e_2, e_2] = \beta e_4,\ [e_3, e_1] = e_4;$$

and

$$\begin{aligned}\Phi(\alpha, \beta, \gamma) : [e_1, e_1] = \alpha e_4,\ [e_2, e_2] = \beta e_4,\ [e_2, e_1] = e_3,\ [e_3, e_1] = e_4,\\ [e_1, e_2] = -e_3 + \gamma e_4,\ [e_1, e_3] = -e_4,\end{aligned}$$

where in the latter case at least one of α, β, γ is not zero.

Let us consider the case of the algebras $\nabla(\alpha, \beta)$.

Case 1.1: $\beta = 0$ and $\alpha \neq 0$. Then the following change of the basis $\{e_1, e_2, e_3, e_4\}$ reduces $\nabla(\alpha, \beta)$ to $\nabla(1, 0)$:

$$e_1' = \alpha e_1,\ e_2' = \alpha e_2,\ e_3' = \alpha^2 e_3,\ e_4' = \alpha^3 e_4.$$

In this case we get $\Re_2$.

Case 1.2: $\beta = 0$ and $\alpha = 0$. Then $\nabla(0, 0)$ is $\Re_3$.

Case 2: $\beta \neq 0$. By changing the basis $\{e_1, e_2, e_3, e_4\}$ in the following way:

$$e_1' = \beta e_1,\ e_2' = \beta e_2,\ e_3' = \beta^2 e_3,\ e_4' = \beta^3 e_4$$

the algebra $\nabla(\alpha, \beta)$ can be reduced to $\nabla(\alpha, 1)$.

Thus we get the following table of multiplication for the algebras $\nabla(\alpha, 1)$:

$$[e_1, e_1] = e_3,\ [e_1, e_2] = \alpha e_4,\ [e_2, e_1] = e_3,\ [e_2, e_2] = e_4,\ [e_3, e_1] = e_4.$$

Next we define isomorphism types within the above family of algebras. It is not hard to see that the only following changes of the basis $\{e_1, e_2, e_3, e_4\}$ can occur:

$$\begin{aligned}&e_1' = ae_1 + be_2,\ e_2' = (a + b)e_2 + b(\alpha - 1)e_3,\\ &e_3' = a(a + b)e_3 + b(a\alpha + b)e_4,\ e_4' = a^2(a + b)e_4.\end{aligned}$$

Then for α, α', a, b we obtain the equalities $\alpha' = (a\alpha + b)/a^2$ and $b = a^2 - a$. Clearly at $\alpha = 1$ we get $\alpha' = 1$. But at $\alpha \neq 1$ setting $a = 1 - \alpha$ we obtain $\alpha' = 0$. Therefore, up to isomorphisms, there are only two algebras here:

$$\Re_4(1) : [e_1, e_1] = e_3,\ [e_2, e_1] = e_3,\ [e_2, e_2] = e_4,\ [e_3, e_1] = e_4,\ [e_1, e_2] = e_4;$$
$$\Re_4(0) : [e_1, e_1] = e_3,\ [e_2, e_1] = e_3,\ [e_2, e_2] = e_4,\ [e_3, e_1] = e_4.$$

We show that the displayed algebras are nonisomorphic. Note that the dimensions of maximal abelian subalgebras of the algebras $\Re_2$ and $\Re_4(\alpha)$ are different, and therefore $\Re_2$ is not isomorphic to the algebra $\Re_4(\alpha)$ for any $\alpha \in \{0, 1\}$. The algebra $\Re_3$ is not isomorphic to the algebras $\Re_2$ and $\Re_4(\alpha)$ for any $\alpha \in \{0, 1\}$, by dimensional reasons about the right annihilators.

Let us now consider the class $\Omega(\alpha, \beta)$. There are two possible cases.

Suppose $\beta = 0$ and $\alpha \neq 0$. Then the transformation

$$e_1' = e_1,\ e_2' = \alpha^{-1} e_2,\ e_3' = e_3,\ e_4' = e_4$$

brings us to $\Omega(1, 0)$ which is $\Re_5$.

Suppose now $\beta \neq 0$. It is easy then to see that the transformation

$$e_1' = \beta e_1,\ e_2' = \beta e_2,\ e_3' = \beta^2 e_3,\ e_4' = \beta^3 e_4$$

leads to the type

$$\Omega(\alpha, 1) : [e_1, e_1] = e_3,\ [e_1, e_2] = \alpha e_4,\ [e_2, e_2] = e_4,\ [e_3, e_1] = e_4.$$

In this case the following changes of the basis can occur:

$$e_1' = ae_1 + be_2,\ e_2' = ce_2 - abc^{-1} e_3,$$
$$e_3' = a^2 e_3 + b(a\alpha + b)e_4,\ e_4' = a^3 e_4.$$

It is not hard to notice, after some calculations, that for different values of the parameters α and α' we will get the equalities: $\alpha' = c(a\alpha + b)/a^3$ and $c^2 = a^3$. Now we set $b = -a\alpha$ to obtain $\alpha = 0$. Thus we have showed that the algebras $\Omega(\alpha, 1)$ are isomorphic to the algebra $\Re_6$ with the following multiplication table:

$$[e_1, e_1] = e_3,\ [e_2, e_2] = e_4,\ [e_3, e_1] = e_4.$$

It should be noted that the algebra $\Omega(0, 0)$ is split.

The algebra $\Re_5$ is not isomorphic to the algebra $\Re_6$, by dimensional reasons about the left annihilators.

Let us consider the last class of non-Lie filiform Leibniz algebras: $\Phi(\alpha, \beta, \gamma)$.

Since at least one of α, β, γ is not zero then not restricted of generality we can suppose that $\alpha \neq 0$. Taking the transformation

$$e_1' = e_1,\ e_2' = \alpha e_2,\ e_3' = \alpha e_3,\ e_4' = \alpha e_4$$

we obtain $\alpha = 1$.

To study the family $\Phi(1, \beta, \gamma)$ we consider the general change of the generator basic elements in the form:

$$e_1' = A_1 e_1 + A_2 e_2 + A_3 e_3,\ e_2' = B_1 e_1 + B_2 e_2 + B_3 e_3.$$

Then express the new basis $\{e_1', e_2', e_3', e_4'\}$ of the algebra $\Phi'(1, \beta', \gamma')$ with respect to the old basis $\{e_1, e_2, e_3, e_4\}$ and comparing the coefficients we obtain the following identities:

$$A_1^2 + A_1 A_2 \gamma + A_2^2 \beta = A_1^2 B_2, B_1 = 0, \quad A_1^2 B_2 \neq 0$$

$$\beta' = \frac{B_2 \beta}{A_1^2}, \quad \gamma' = \frac{A_1 \gamma + 2 A_2 \beta}{A_1^2}.$$

Note that the following identity is true

$$\gamma'^2 - 4\beta' = \frac{1}{A_1^2}(\gamma^2 - 4\beta).$$

Consider the case $\beta = 0$. Then $\beta' = 0$.

So, in this case if $\gamma = 0$, then $\gamma' = 0$ and we get $\Re_7$. But if $\gamma \neq 0$, then setting $A_1 = \gamma, B_2 = 1$ and $A_2 = 0$ we obtain $\gamma' = 1$, and we get $\Re_8$.

Consider the case $\beta \neq 0$. Then putting $B_2 = A_1^2/\beta$ we obtain $\beta' = 1$.

If $\gamma^2 - 4\beta = 0$, then taking $A_2 = (-2A_1\gamma + 4A_1^2)/\gamma^2$ and A_1 any non-zero number we obtain $\gamma' = 2$. Thus, in this case we obtain $\Re_9$.

If $\gamma^2 - 4\beta \neq 0$, then putting

$$A_1 = \sqrt{\frac{4\beta - \gamma^2}{4}}, \quad A_2 = -\frac{\gamma}{2\beta}\sqrt{\frac{4\beta - \gamma^2}{4}}$$

we obtain $\gamma' = 0$ and this is $\Re_{10}$.

Now we suppose that the algebra L has the type (iii) in Proposition 3.1, that is the same to say that L is a Leibniz algebra with $\chi(L) :=$ (4,2,0,0) or

(4,1,0,0), in particular L is associative. Then all above results on associative algebras are applicable to this case and so we deal with associative algebras.

We consider the associative algebra $A = L \oplus C$. It is 5-dimensional and unitary. Now using properties of L we delete from G. Mazzola's list [10] inappropriate algebras. The first condition that will be used is the number of central idempotents. In view of this condition A is not isomorphic to the algebras with numbers 1-15, 25, 26, 38, 39, 55. Then in view of Corollary 2.2 and of the fact that the image of an idempotent element under isomorphism is an idempotent element we decide that A is not isomorphic to each of the algebras 16-22, 27, 28, 29, 40, 41, 46, 47, 52, 58 in the list of G. Mazzola. Moreover, the condition $L^3 = 0$ implies that A is not isomorphic to the algebras 23, 24, 33, 34, 44. in the list of G. Mazzola. Then from the remaining algebras we pick out appropriate ones.

Consider N30. $A = C < x, y > /(xy + yx, xy - yx + y^2 - x^2) + (x, y)^3$. Choose the basis of A: $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = xy$, $e_4 = x^2$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ has the following composition law:

$$[e_1, e_1] = e_4,\ [e_1, e_2] = e_3,\ [e_2, e_1] = -e_3,\ [e_2, e_2] = -2e_3 + e_4.$$

This is algebra $\Re_{11}$.

Consider N31. $A = C < x, y > /(x^2, xy + y^2) + (x, y)^3$. Choose the basis of A: $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = xy$, $e_4 = yx$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ has the composition law:

$$[e_1, e_2] = e_3,\ [e_2, e_1] = e_4,\ [e_2, e_2] = -e_3.$$

This means that it coincides with $\Re_{12}$

Let now consider N32. $A = C < x, y > /(xy + y^2, \alpha x^2 + yx) + (x, y)^3$. Choose the basis for A: $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = x^2$, $e_4 = xy$. Consider the subalgebra $L =< e_1, e_2, e_3, e_4 >$. Then it is the algebra with the multiplication table:

$$[e_1, e_1] = e_3,\ [e_1, e_2] = e_4,\ [e_2, e_1] = -\alpha e_3,\ [e_2, e_2] = -e_4.$$

In view of Proposition 2.7 for distinct values of the parameter α these algebras are not isomorphic. In this case L is $\Re_{13}(\alpha)$.

Consider N35. $A = C < x, y, z > /(xz, yz, zx, zy, x^2 - y^2, x^2 - z^2, xy + yx, \alpha x^2 + yx)$, where $\alpha \neq 0$. Choose the basis for A: $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = z$, $e_4 = x^2$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ has the following composition law:

$$[e_1, e_1] = e_4,\ [e_1, e_2] = \alpha e_4,\ [e_2, e_1] = -\alpha e_4,\ [e_2, e_2] = e_4,\ [e_3, e_3] = e_4.$$

and it is $\Re_{14}(\alpha)$.

We remark that the algebras $\Re_{14}(\alpha_1)$ and $\Re_{14}(\alpha_2)$ $(\alpha_1 \neq \alpha_2)$ are not isomorphic except for the case where $\alpha_2 = -\alpha_1$, in the latter case they are indeed isomorphic [10].

Consider N36. $A = C < x, y, z > /(x^2, yz, zy, z^2, xy - xz, xy + yx, yx + y^2, yx + zx)$. As a basis here we choose $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = z, e_4 = xy$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ has the following law of composition

$$[e_1, e_2] = e_4,\ [e_1, e_3] = e_4,\ [e_2, e_1] = -e_4,\ [e_2, e_2] = e_4,\ [e_3, e_1] = e_4$$

and it is isomorphic to $\Re_{15}$ of the theorem.

Consider N37. $A = C < x, y, z > /(xz, y^2, yz, zx, zy, x^2 - z^2, x^2 - xy, x^2 + yx)$. As a basis we take $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = z$, $e_4 = xy$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ is isomorphic to the algebra:

$$\Re_{16} : [e_1, e_1] = e_4,\ [e_1, e_2] = e_4,\ [e_2, e_1] = -e_4,\ [e_3, e_3] = e_4.$$

Consider N42. $A = C < x, y > /(x^2, y^2) + (x, y)^3$. As a basis we take $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = xy$, $e_4 = yx$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ is the same as:

$$\Re_{17} : [e_1, e_2] = e_3,\ [e_2, e_1] = e_4.$$

Consider N43. $A = C[x, y]/(y^3, xy, x^3)$ and choose as a basis $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = x^2$, $e_4 = y^2$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ has the table of multiplication:

$$[e_1, e_1] = e_3,\ [e_2, e_2] = e_4.$$

But it is obvious that this algebra is decomposable and we can omit it.

Consider N48. $A = C < x, y > /(x^2, xy + yx) + (x, y)^3$. As a basis we choose $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = xy$, $e_4 = y^2$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ coincides with the algebra:

$$\Re_{18} : [e_1, e_2] = e_3,\ [e_2, e_1] = -e_3,\ [e_2, e_2] = e_4.$$

Consider N49 $(\alpha = 1)$. $A = C < x, y > /(x^2, xy) + (x, y)^3$. As a basis we take $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = y^2$, $e_4 = yx$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ is isomorphic to the algebra:

$$\Re_{19} : [e_2, e_1] = e_4,\ [e_2, e_2] = e_3.$$

Let now consider N49 $(\alpha \neq 1)$. $A = C < x, y > / \big(x^2, (1 + \alpha)xy + (1 - \alpha)yx\big) + (x, y)^3$. As a basis of A can be chosen vectors $e_0 = 1$, $e_1 = x$,

$e_2 = y$, $e_3 = y^2$, $e_4 = xy$, and the subalgebra $L =< e_1, e_2, e_3, e_4 >$ will be isomorphic to

$$\Re_{20}(\alpha) : [e_1, e_2] = e_4,\ [e_2, e_1] = \frac{1+\alpha}{1-\alpha} e_4,\ [e_2, e_2] = e_3.$$

By using Proposition 2.3 again we conclude that for different values of α we obtain non-isomorphic algebras.

Consider N50. $A = C < x, y, z > /(x^2, xz, y^2, yz, zx, zy, xy + yx, yx + z^2)$. As a basis of it we choose vectors $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = z$, $e_4 = xy$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ is isomorphic to the algebra:

$$\Re_{21} : [e_1, e_2] = e_4,\ [e_2, e_1] = -e_4,\ [e_3, e_3] = e_4.$$

We finally consider N51. $A = C < x, y, z > /(xz, yz, zx, zy, x^2 - y^2, x^2 - z^2, xy, yx)$. As a basis we choose $e_0 = 1$, $e_1 = x$, $e_2 = y$, $e_3 = z$, $e_4 = x^2$. Then the subalgebra $L =< e_1, e_2, e_3, e_4 >$ has the following table of multiplication:

$$[e_1, e_1] = e_4,\ [e_2, e_2] = e_4,\ [e_3, e_3] = e_4.$$

But this algebra can be included in to the family of algebras $\Re_{14}(\alpha)$ at $\alpha = 0$.

Using Proposition 2.3 it is easy to check that all obtained algebras are pairwise not isomorphic. ∎

Remark 3.3. All the other algebras of G. Mazzola's list are either Lie algebras or split Leibniz algebras.

It should be noted that there are a lot of Leibniz algebras in dimension four unlike the case of Lie algebras, where in dimension four there is only one non-split algebra.

Summarizing the classification of the above theorem and the classifications of complex nilpotent Lie algebras of dimension at most four and complex nilpotent Leibniz algebras of dimension at most three, we obtain the complete classification of complex nilpotent Leibniz algebras of dimension at most four.

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References

[1] Albeverio, S., Ayupov, Sh.A., Omirov, B.A., On nilpotent and simple Leibniz algebras, *Comm. Algebra,* **33** (1) (2005), 159 – 172.

[2] Albeverio, S., Omirov, B.A., Rakhimov, I.S., Varieties of nilpotent complex Leibniz algebras of dimension less than five, *Comm. Algebra,* **33** (5) (2005), 1575 – 1585.

[3] Ayupov, Sh.A., Omirov, B.A., On Leibniz algebras, in "Algebra and operators theory, Proceeding of the Colloquium in Tashkent 1997", Kluwer Academic Publishers, Dordrecht, 1998, 1 – 13.

[4] Ayupov, Sh.A., Omirov, B.A., On some classes of nilpotent Leibniz algebras, *Siberian Math. J.,* **42** (1) (2001), 15 – 24. (Russian: *Sibirsk. Mat. Zh.,* **42** (1) (2001), 18 – 29.)

[5] Casas, J.M., Pirashvili, T., Ten-term exact sequence of Leibniz homology, *J. Algebra,* **231** (2000), 258 – 264.

[6] Frabetti, A., Leibniz homology of dialgebras of matrices, *J. Pure Appl. Algebra,* **129** (2) (1998), 123 – 141.

[7] Goze, M., Khakimdjanov, Yu., "Nilpotent Lie Algebras", Mathematics and its Applications, 361, Kluwer Academic Publishers Group, Dordrecht, 1996.

[8] Loday, J.-L., Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseign. Math.* (2), **39** (3-4) (1993), 269 – 293.

[9] Loday, J.-L., Frabetti, A., Chapoton, F., Goichot, F., "Dialgebras and Related Operads", Lecture Notes in Mathematics, 1763, Springer-Verlag, Berlin, 2001.

[10] Mazzola, G., The algebraic and geometric classification of associative algebras of dimension five, *Manuscripta Math.,* **27** (1) (1979), 81 – 101.

[11] Pirashvili, T., On Leibniz homology, *Ann. Inst. Fourier (Grenoble),* **44** (2) (1994), 401 – 411.

[12] Sagle, A.A., Mal'tzev algebras, *Trans. Amer. Math. Soc.,* **101** (3), 1961, 426 – 458.