

Tensor Product and Local Interior G -Algebras

WENLIN HUANG

School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China
e-mail: wenlin@math.pku.edu.cn

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0. INTRODUCTION

G -algebra is a very important working object in the modern representation theory of finite groups, on which Puig set up his well-known theory of pointed groups, especially, many working objects in the representation theory of finite groups can be regarded as G -algebras, or more explicitly, as interior G -algebras (e.g. [11]).

Tensor product is a long story in mathematics, by which authors have discovered so many interesting results in many branches of mathematics.

In [10], Külshammer obtained some properties on the tensor product of some indecomposable modules, in [5] Harris generalized [10, Proposition 2.1] to the lattices over the complete discrete valuation ring. In [1], Aglhamdi and Khammash studied the tensor module of G -algebras and have achieved some important results on tensor product of Brauer homomorphism; moreover, Khammash analyzed the points, pointed groups and their defect groups in tensor algebra of G -algebras in [9].

In this paper, we devote to studying the (outer) tensor algebra of G -algebras (see §2) and the (inner) tensor algebra of G -algebras (see §3), especially, we concentrate on the defect groups of them and the relationship between block algebras and interior G -algebras under tensor product; additionally, we have discussed the centralizer of the subalgebra consisting of G -fixed elements in an interior G -algebra (see §4).

1. PRELIMINARIES

In this paper, F is always an algebraically closed field of characteristic p , where p is a prime integer. We assume all groups are finite, all algebras and modules are finitely dimensional ones. By a G -algebra (A, ϕ) we mean a F -algebra A with a group homomorphism $\phi : G \rightarrow \text{Aut}(A)$, where $\text{Aut}(A)$ denotes the group of F -algebra automorphisms of A , hence G acts naturally on A by ${}^g a := \phi(g)(a)$, $a \in A$. An interior G -algebra (A, ρ) is a F -algebra A with a F -algebra homomorphism $\rho : FG \rightarrow A$ such that $\rho(1_{FG}) = 1_A$, which becomes a G -algebra with the G -action defined by ${}^g a = \rho(g)a\rho(g)^{-1}$ for any $a \in A$, $g \in G$, and we also denote this G -algebra with (A, ρ) ; moreover, it becomes a FG -module by extending linearly this G -action, we denote this FG -module with ${}_G A$. Sometimes we also simply say A is an interior G -algebra instead of (A, ρ) whenever there exists no confusion; we do the same for G -algebras.

For any $H \leq G$, we always write A^H for the subalgebra consisting of H -fixed elements of G -algebra (A, ϕ) ; moreover, $\text{Tr}_H^G(A^H)$, also denoted by A_H^G , means the relative trace ideal of A^G , where $\text{Tr}_H^G(\cdot)$ is the well-known relative trace functor mapping a in A^H on $\sum_{g \in G/H} ({}^g a)$ (e.g. [4]). The same holds for FG -modules (e.g. [3]).

An interior G -algebra (A, ρ) is called an epimorphic one if ρ is epimorphic, and is called a local one (in [11], which is also called a primitive one) if A^G is a local algebra; in the latter case, we define the minimal subgroups H of G such that $1_A \in A_H^G$ as the defect groups of A , which are p -subgroups of G and unique in the sense of G -conjugation, hence we denote any one of them with $D(A)$. Particularly, every p -block algebra $B(= FGb)$, where b is a central primitive idempotent of FG , is an epimorphic local interior G -algebra in a canonical way.

Let V be a FG -module. For a p -subgroup P of G , the P -relative Brauer map of V is the natural map $\text{Br}_P^V : V^P \rightarrow V(P) := V^P / \sum_{Q \leq P} V_Q^P$, where $V_Q^P = \text{Tr}_Q^P(V^Q)$ (e.g. [3]). For some subgroup H of G , we say V being H -projective if V is a direct summand of $\text{Ind}_H^G \text{Res}_H^G V$, and then $V(P) = 0$ unless $P \leq_G H$ ([3, Proposition 1.3]); moreover, if V is indecomposable, we say the minimal subgroups H such that V is H -projective, which are mutually conjugate p -subgroups in G , the vertices of V , and we denote any one of them with $\text{Vtx}_G(V)$ ([11, Theorem 18.3]).

DEFINITION 1.1. Let (A, ρ) be an interior G -algebra, we call (A, ρ) belongs to some block $B(= FGb)$ of G if $\rho(b) = 1_A$.

Remark 1.2. In the case of local interior G -algebras, this definition is just the one in [8]. It is easy to see that if (A, ρ) belongs to some block of G , it belongs to a unique block, and every block algebra of group algebra FG belongs to itself, hence this definition is reasonable; moreover, this generalization makes sense (see §4, Proposition 4.2).

EXAMPLE 1.3. If M is a FG -module, then the endomorphism ring $E(M) := \text{End}_F(M)$ is an interior G -algebra by a natural way and hence a G -algebra defined by ${}^g f = g \cdot f \cdot g^{-1}$ for all $g \in G$ and $f \in E(M)$. Obviously, $E(M)$ is a local interior G -algebra if and only if M is an indecomposable FG -module, in this case $E(M)$ belongs to a block B of G if and only if M belongs to B , and by Higman's criterion $D(E(M)) = \text{Vtx}_G(M)$ under G -conjugation.

2. (OUTER) TENSOR G -ALGEBRA

(A_i, ϕ_i) is a G_i -algebra, $i = 1, 2$; in this paper, their (outer) tensor $G_1 \times G_2$ -algebra, which is denoted by $(A_1 \otimes A_2, \phi_1 \otimes \phi_2)$, is defined by the following way

$$(\phi_1 \otimes \phi_2)(g_1, g_2) = \phi_1(g_1) \otimes \phi_2(g_2)$$

for any $(g_1, g_2) \in G_1 \times G_2$, where and in the sequel, \otimes always means \otimes_F , that is, the tensor product of F -algebras or that of F -modules; more generally, we define similarly the (outer) tensor product of modules over group algebras. Obviously, the (outer) tensor algebra of interior G_i -algebra, $i = 1, 2$, is also an interior $G_1 \times G_2$ -algebra.

LEMMA 2.1. *Let i, j be idempotents in F -algebras A and B , respectively. Then $i \otimes j$ is an idempotent in $A \otimes B$; furthermore, $i \otimes j$ is primitive if and only if i and j are primitive in A and B , respectively.*

Proof. See [5, Corollary 3.3]. ■

LEMMA 2.2. *Let (A_i, ρ_i) be an interior G_i -algebra and B_i a block of G_i , where G_i is a finite group, $i = 1, 2$. Then $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ belongs to $B_1 \otimes B_2$ if and only if (A_i, ρ_i) belongs to B_i , $i = 1, 2$.*

Proof. First of all, we recall that $B_1 \otimes B_2$ is a block of $F(G_1 \times G_2) \cong FG_1 \otimes FG_2$. Let $B_i = FG_i b_i$, where b_i is the block idempotent of B_i , $i = 1, 2$, we have $B_1 \otimes B_2 = (FG_1 \otimes FG_2)(b_1 \otimes b_2)$ with $1_{B_1 \otimes B_2} = b_1 \otimes b_2$.

If (A_i, ρ_i) belongs to B_i , $i = 1, 2$, we obtain $\rho_i(b_i) = 1_{A_i}$, $i = 1, 2$, hence $(\rho_1 \otimes \rho_2)(b_1 \otimes b_2) = \rho(b_1) \otimes \rho(b_2) = 1_{A_1 \otimes A_2}$, that is, $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ belongs to $B_1 \otimes B_2$.

Conversely, Let $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ belongs to $B_1 \otimes B_2$, that is,

$$(\rho_1 \otimes \rho_2)(b_1 \otimes b_2) = 1_{A_1 \otimes A_2},$$

hence $\rho(b_1) \otimes \rho(b_2) = 1_{A_1} \otimes 1_{A_2}$. Let $1_{FG_i} = b_{i1} + b_{i2} + \dots + b_{i,j_i}$, a finite sum of block idempotents of FG_i with $b_{i1} = b_i$, $i = 1, 2$, we have

$$\rho(b_1) \otimes \rho(b_2) = 1_{A_1} \otimes 1_{A_2} = \rho_1(1_{FG_1}) \otimes \rho_2(1_{FG_2}) = \sum_{m=1}^{j_1} \sum_{n=1}^{j_2} \rho(b_{1m}) \otimes \rho(b_{2n}),$$

hence $\rho(b_{1m}) \otimes \rho(b_{2n}) = 0$ for all $m > 1$ or $n > 1$, and hence $\rho(b_{1m}) \otimes \rho(1_{FG_2}) = 0$ for all $m > 1$. Since $\rho(b_{1m}) \otimes a = (\rho(b_{1m}) \otimes a)(\rho(b_{1m}) \otimes \rho(1_{FG_2}))$ for all $a \in A_2$, we obtain $\rho(b_{1m}) \otimes A_2 = 0$ for all $m > 1$, and thus $\rho(b_1) = 1_{A_1}$, that is, (A_1, ρ_1) belongs to B_1 ; similarly, (A_2, ρ_2) belongs to B_2 . ■

THEOREM 2.3. *Let (A_i, ρ_i) be a local interior G_i -algebra with a defect group $D(A_i)$, and block $B_i(= FG_i b_i)$ of G_i with a defect group D_i , $i = 1, 2$. Then $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$, which is a local interior $G_1 \times G_2$ -algebra with a defect group $D(A_1) \times D(A_2)$, belongs to $B_1 \otimes B_2$ if and only if A_i belongs to B_i , $i = 1, 2$. Additionally, if A_i is an epimorphic local interior G_i -algebra belonging to B_i , $i = 1, 2$, there exists some simple $F(G_1 \times G_2)$ -module V such that*

$$Vtx_{G_1 \times G_2}(V) \leq D(A_1 \otimes A_2) \leq D_1 \times D_2,$$

under conjugation in $G_1 \times G_2$, and moreover, for any normal p -subgroup P_i of G_i , $i = 1, 2$, we have $P_1 \times P_2 \leq_{G_1 \times G_2} D(A_1 \otimes A_2)$.

Proof. Since (A_i, ρ_i) is a local interior G_i -algebra with a defect group $D(A_i)$, $i=1, 2$, we know that $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ is a local interior $G_1 \times G_2$ -algebra with a defect group $D(A_1) \times D(A_2)$ and *vice versa*, by [1, Lemma 2.1], Lemma 2.1 and [9, Corollary 4.4]; hence by Lemma 2.2 we have completed the first part of Theorem 2.3.

Let's go on in the case of the additional assumptions. By [8, Lemma 2.8], there are V_1 belonging to $Irr(FG_1)$, that is, the set of all irreducible FG_1 -modules, and V_2 belonging to $Irr(FG_2)$ such that $Vtx_{G_1}(V_1) \leq_{G_1} D(A_1) \leq_{G_1} D_1$ and $Vtx_{G_2}(V_2) \leq_{G_2} D(A_2) \leq_{G_2} D_2$; furthermore, by [7, Theorem 9.14] we

obtain $V_1 \otimes V_2$ belongs to $Irr(F(G_1 \times G_2))$ and by [10, Proposition 1.2] we have

$$Vtx_{G_1 \times G_2}(V_1 \otimes V_2) = {}_{G_1 \times G_2}Vtx_{G_1}(V_1) \times Vtx_{G_2}(V_2).$$

Then

$$P_1 \times P_2 \leq Vtx_{G_1 \times G_2}(V_1 \otimes V_2) \leq_{G_1 \times G_2} D(A_1 \otimes A_2) \leq_{G_1 \times G_2} D_1 \times D_2,$$

since it is well known that vertices of any simple FG -module contain any normal p -subgroup of G . Theorem 2.3 follows. ■

THEOREM 2.4. *Let (A_i, ρ_i) be a local interior G_i -algebra with a defect group D_i , $i=1, 2$. Then $D_1 \times D_2$ is the maximal of vertices of indecomposable summands of ${}_{G_1 \times G_2}A_1 \otimes A_2$, up to $G_1 \times G_2$ -conjugation.*

Proof. Let ${}_{G_i}A_i = A_{i1} \oplus A_{i2} \oplus \dots \oplus A_{i, n_i}$ be an indecomposable decomposition of A_i as FG_i -module, $i = 1, 2$. We have

$${}_{G_1 \times G_2}A_1 \otimes A_2 = \bigoplus_{s=1}^{n_1} \bigoplus_{t=1}^{n_2} (A_{1s} \otimes A_{2t})$$

be an indecomposable decomposition of $A_1 \otimes A_2$ as $F(G_1 \times G_2)$ -module, by [10, Proposition 1.1]. Hence

$$\begin{aligned} ({}_{G_1 \times G_2}A_1 \otimes A_2)(D_1 \times D_2) &= \bigoplus_{s=1}^{n_1} \bigoplus_{t=1}^{n_2} (A_{1s} \otimes A_{2t})(D_1 \times D_2) \\ &\simeq \bigoplus_{s=1}^{n_1} \bigoplus_{t=1}^{n_2} A_{1s}(D_1) \otimes A_{2t}(D_2), \end{aligned}$$

by [1, Theorem 2.6]. Since

$$({}_{G_1 \times G_2}A_1 \otimes A_2)(D_1 \times D_2) = (A_1 \otimes A_2)(D_1 \times D_2) \neq 0$$

by Theorem 2.3 and [11, Corollary 18.6], there are A_{1,s_0} and A_{2,t_0} such that $A_{1,s_0}(D_1) \otimes A_{2,t_0}(D_2) \neq 0$, for some s_0 and t_0 ; hence $A_{1,s_0}(D_1) \neq 0$ and $A_{2,t_0}(D_2) \neq 0$. Then, by [3, Proposition 1.3], we have $D_1 \leq_{G_1} Vtx_{G_1}(A_{1,s_0})$ and $D_2 \leq_{G_2} Vtx_{G_2}(A_{2,t_0})$. However, since A_1 is D_1 -projective by [6, Lemma 2.9], we have $A_{1,s}$ is also D_1 -projective, that is, $D_1 \geq_{G_1} Vtx_{G_1}(A_{1,s})$ for any s , hence $D_1 =_{G_1} Vtx_{G_1}(A_{1,s_0})$. Similarly, $D_2 =_{G_2} Vtx_{G_2}(A_{2,t_0})$, and then

$$D_1 \times D_2 =_{G_1 \times G_2} Vtx_{G_1}(A_{1,s_0}) \times Vtx_{G_2}(A_{2,t_0}) =_{G_1 \times G_2} Vtx_{G_1 \times G_2}(A_{1,s_0} \otimes A_{2,t_0}),$$

by [10, Proposition 1.2]. we have seen that, up to $G_1 \times G_2$ -conjugation, $D_1 \times D_2$ is the maximal of vertices of indecomposable summands of $A_1 \otimes A_2$ as $F(G_1 \times G_2)$ -module, by Krull-Schmidt Theorem. ■

3. (INNER) TENSOR G -ALGEBRA

(A_1, ϕ_1) and (A_2, ϕ_2) are two G -algebras, in this paper, their (inner) tensor G -algebra means a G -algebra $(A_1 \otimes A_2, \phi)$ with

$$\phi(g) := \phi_1(g) \otimes \phi_2(g)$$

for any $g \in G$; similarly, we define the (inner) tensor product of FG -modules. It is clear that the (inner) tensor G -algebra of two interior G -algebras remains to be an interior G -algebra.

Remark 3.1. Obviously, the tensor product of G -algebras in [11] is just the (inner) tensor G -algebra of G -algebras here; generally, not like the (outer) tensor G -algebra, the (inner) tensor G -algebra of two local interior G -algebras does not remain to be a local one, whereas the following Proposition 3.2 gives us a surprise.

(A, ϕ) is a G/N -algebra, where $N \trianglelefteq G$, the inflated G -algebra of (A, ϕ) is a G -algebra $(\text{inf}(A), \text{inf}(\phi))$, where $\text{inf}(A) = A$, $(\text{inf}(\phi))(g) = \phi(gN)$ for all $g \in G$. Obviously, if (A, ρ) is an interior G/N -algebra, $(\text{inf}(A), \text{inf}(\rho))$ is also an interior G -algebra, and moreover, if (A, ρ) is a local one, so is $(\text{inf}(A), \text{inf}(\rho))$.

PROPOSITION 3.2. *Let G be a finite group with a normal subgroup N and A be a G -algebra such that $\text{Res}_N^G(A)$ is a local N -algebra; C is a local G/N -algebra. Then the (inner) tensor G -algebra $A \otimes \text{inf}(C)$ is a local G -algebra.*

Proof. Since $(\text{Res}_N^G(A))^N$ is a local algebra, we have a decomposition

$$A^N = F \cdot 1_A \oplus J(A^N)$$

as F -module, hence as FG -module since $N \trianglelefteq G$; furthermore, since N acts trivially on $\text{inf}(C)$, we obtain

$$\begin{aligned} (A \otimes \text{inf}(C))^N &= (A \otimes \text{inf}(C))^{N \times N} && \text{(as (outer) tensor } G\text{-algebra)} \\ &= A^N \otimes (\text{inf}(C))^N && \text{(by [1, Lemma 2.1])} \\ &= A^N \otimes \text{inf}(C) \\ &= F \cdot 1_A \otimes \text{inf}(C) \oplus J(A^N) \otimes \text{inf}(C) && \text{(as } FG\text{-module)} \end{aligned}$$

and $J(A^N) \otimes \text{inf}(C)$ is a nilpotent ideal of $(A \otimes \text{inf}(C))^N$. On the other hand, since $F \cdot 1_A$ is a G -subalgebra of A , it is easy to see that

$$f : F \cdot 1_A \otimes \text{inf}(C) \rightarrow \text{inf}(C)$$

is an isomorphism of G -algebras by the following way

$$f(t \cdot 1_A \otimes c) = t \cdot c$$

for any $t \in F$ and any $c \in C$, hence

$$(F \cdot 1_A \otimes \text{inf}(C))^G = (F \cdot 1_A \otimes \text{inf}(C)) \cap (A \otimes \text{inf}(C))^G$$

is a local algebra. Then

$$(A \otimes \text{inf}(C))^G = (F \cdot 1_A \otimes \text{inf}(C))^G \oplus (J(A^N) \otimes \text{inf}(C))^G$$

has only one idempotent since

$$(J(A^N) \otimes \text{inf}(C))^G = (J(A^N) \otimes \text{inf}(C)) \cap (A \otimes \text{inf}(C))^G$$

is a nilpotent ideal in $(A \otimes \text{inf}(C))^G$ and $(F \cdot 1_A \otimes \text{inf}(C))^G$ is a local algebra, that is, we have known that $A \otimes \text{inf}(C)$ is a local G -algebra. ■

By Proposition 3.2 we see that for two F -algebras A and C , $A \otimes C$ is a local F -algebra if and only if both A and C are.

THEOREM 3.3. *In the case of Proposition 3.2, set D is a defect group of the local G -algebra $A \otimes \text{inf}(C)$, where D is some p -subgroup of G . Then DN/N is a defect group of C as a G/N -algebra.*

Proof. Since $A \otimes \text{inf}(C)$ is a local G -algebra with a defect group D by Proposition 3.2, let $1_{A \otimes \text{inf}(C)} = \text{Tr}_D^G(d)$, where $d \in (A \otimes \text{inf}(C))^D$, hence

$$\text{Tr}_D^{DN}(d) \in (A \otimes \text{inf}(C))^{DN} \subseteq (A \otimes \text{inf}(C))^N.$$

Since

$$(A \otimes \text{inf}(C))^N = A^N \otimes \text{inf}(C) = F \cdot 1_A \otimes \text{inf}(C) \oplus J(A^N) \otimes \text{inf}(C),$$

we have $\text{Tr}_D^{DN}(d) = 1_A \otimes i + j$ for some $i \in \text{inf}(C)$ and $j \in J(A^N) \otimes \text{inf}(C)$; moreover, since $N \trianglelefteq G$ and $F \cdot 1_A \otimes \text{inf}(C) \simeq \text{inf}(C)$ as G -algebras, it is easy to see that $i \in \text{inf}(C)^{DN}$ and $j \in (A \otimes \text{inf}(C))^{DN}$. Then,

$$1_{A \otimes \text{inf}(C)} = \text{Tr}_D^G(d) = 1_A \otimes \text{Tr}_{DN}^G(i) + \text{Tr}_{DN}^G(j),$$

where $Tr_{DN}^G(j) \in J((A \otimes inf(C))^G)$ since j is a nilpotent element; hence $1_A \otimes Tr_{DN}^G(i) \notin J((A \otimes inf(C))^G)$, and hence $1_A \otimes Tr_{DN}^G(i)$ is a unit in $(A \otimes inf(C))^G$ since $(A \otimes inf(C))^G$ is a local algebra. Then $Tr_{DN}^G(i)$ is a unit in $(inf(C))^G$, that is, $Tr_{DN/N}^{G/N}(i) = Tr_{DN}^G(i)$ is a unit in $C^{G/N}$, and it yields that the local G/N -algebra C is DN/N -projective; therefore $DN/N \geq H/N$, where $DN \geq H \geq N$ for some subgroup H of G such that H/N is a defect group of C as a G/N -algebra; we have that $inf(C)$ is H -projective, and $A \otimes inf(C)$ is also H -projective, by [11, Lemma 14.3]. It follows that H contains a defect group of $A \otimes inf(C)$, i.e. a conjugation of D , hence $H = DN$ and $H/N = DN/N$. we are done. ■

COROLLARY 3.4. *Let G be a finite group with a normal subgroup N and A be a G -algebra such that $Res_N^G(A)$ is a local N -algebra. Then, if D is a defect group of A , DN/N is a Sylow p -subgroup of G/N .*

Proof. In the case of Theorem 3.3, Let $C = F$, the trivial G/N -algebra. We have DN/N is a defect group of F as a trivial G/N -algebra, hence DN/N is a Sylow p -subgroup of G/N . ■

Remark 3.5. In the case of Example 1.3, Let M_1 be an indecomposable FG -module such that $Res_N^G(M_1)$ remains to be indecomposable, where N is a normal subgroup of G , and let M_2 be an indecomposable $F(G/N)$ -module. Then $Res_N^G E(M_1)$ is a local interior N -algebra and $E(M_2)$ is a local interior G/N -algebra. We see that $E(M_1 \otimes inf(M_2))$ remains to be a local interior G -algebra by Proposition 3.2, that is, $M_1 \otimes inf(M_2)$ is an indecomposable FG -module. Moreover, if D is a vertex of $M_1 \otimes inf(M_2)$ as FG -module, that is, a defect group of $E(M_1 \otimes inf(M_2))$, DN/N is a vertex of M_2 , by Theorem 3.3; especially, let $M_2 = F$, the trivial $F(G/N)$ -module, we have $Vtx(M_1)N/N$ is a Sylow p -subgroup of G/N , by Corollary 3.4. Hence, we have generalized [10, Proposition 2.1, Proposition 2.2].

4. CENTRALIZER OF THE G -FIXED ELEMENTS SUBALGEBRA

(A, ρ) is an interior G -algebra, the centralizer $C_A(A^G)$ of A^G in A is a subalgebra of A , with the same identity 1_A . It is easy to see that $\rho(FG) \subseteq C_A(A^G)$, hence we have an interior G -subalgebra $(C_A(A^G), \rho)$ of (A, ρ) by inheriting the structure of (A, ρ) , with the following property

$$(C_A(A^G))^G = Z(A^G) = Z(C_A(A^G)).$$

In this paper, we say that (A, ρ) is connected if $(C_A(A^G), \rho)$ is a local interior G -algebra, or equivalently, if $Z(A^G)$ is a local algebra; obviously, every local interior G -algebra is connected and if we restrict to the epimorphic interior G -algebras, the connected one is just the local one.

EXAMPLE 4.1. Under the notation of Example 1.3, let M be a FG -module, which can be regarded as an $E(M)$ -module. We have

$$\varepsilon(M) := \text{End}_{\text{End}_{FG}(M)}(M) = C_{E(M)}(E(M)^G),$$

which is an interior G -subalgebra of $E(M)$. In [2], L. Barker calls M a connected module if $\varepsilon(M)$ is a local interior G -algebra, that is, $E(M)$ is a connected interior G -algebra. Obviously every indecomposable FG -module is connected.

PROPOSITION 4.2. Every connected interior G -algebra (A, ρ) belongs to some block B of G . In this case, we have $D(C_A(A^G)) \leq_G D(B)$.

Proof. Let $B_i = FGb_i$, $i = 1, 2, \dots, n$, be block algebras of G such that $\sum_{i=1}^n b_i = 1_{FG}$, $n \geq 1$. It is easy to see that $\rho(b_i) \in C_A(A^G) \cap A^G = Z(A^G)$, and since $Z(A^G)$ has only one idempotent 1_{FG} , there is only one b_{i_0} such that $\rho(b_{i_0}) = 1_{FG}$ for some $1 \leq i_0 \leq n$ and $\rho(b_i) = 0$ for any other i , that is, (A, ρ) belongs to b_{i_0} . Since $\rho(B_{i_0}) \subseteq C_A(A^G)$, we have $D(C_A(A^G)) \leq_G D(\rho(B_{i_0}))$ by [1, Proposition 4.2], and since $(\rho(B_{i_0}), \rho)$ is an epimorphic local interior G -algebra belonging to B_{i_0} , we have $D(\rho(B_{i_0})) \leq_G D(B_{i_0})$ by [8, Lemma 2.8], hence $D(C_A(A^G)) \leq_G D(B_{i_0})$; we are done. ■

COROLLARY 4.3. Let (A_i, ρ_i) be a connected interior G_i -algebra and belong to the block algebra B_i of G_i , $i = 1, 2$. Then the (outer) tensor $G_1 \times G_2$ -algebra $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ is also a connected interior $G_1 \times G_2$ -algebra belonging to $B_1 \otimes B_2$. Moreover,

$$D(C_{A_1 \otimes A_2}((A_1 \otimes A_2)^{G_1 \times G_2})) =_{G_1 \times G_2} D(C_{A_1}(A_1^{G_1})) \times D(C_{A_2}(A_2^{G_2})).$$

Proof. Since

$$\begin{aligned} C_{A_1 \otimes A_2}((A_1 \otimes A_2)^{G_1 \times G_2}) &= C_{A_1 \otimes A_2}(A_1^{G_1} \otimes A_2^{G_2}) \\ &= C_{A_1}(A_1^{G_1}) \otimes C_{A_2}(A_2^{G_2}), \end{aligned}$$

we have

$$\begin{aligned} Z(C_{A_1 \otimes A_2}((A_1 \otimes A_2)^{G_1 \times G_2})) &= Z(C_{A_1}(A_1^{G_1}) \otimes C_{A_2}(A_2^{G_2})) \\ &= Z(C_{A_1}(A_1^{G_1})) \otimes Z(C_{A_2}(A_2^{G_2})), \end{aligned}$$

and since $Z(C_{A_1}(A_1^{G_1}))$ and $Z(C_{A_2}(A_2^{G_2}))$ are local algebras,

$$Z(C_{A_1 \otimes A_2}((A_1 \otimes A_2)^{G_1 \times G_2}))$$

is also a local algebra, by Lemma 2.1. Hence $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ is a connected interior $G_1 \times G_2$ -algebra belonging to some block of $G_1 \times G_2$ by Proposition 4.2, and moreover, it is clear that $(A_1 \otimes A_2, \rho_1 \otimes \rho_2)$ belongs to $B_1 \otimes B_2$, by Lemma 2.2. Then

$$\begin{aligned} D(C_{A_1 \otimes A_2}((A_1 \otimes A_2)^{G_1 \times G_2})) &= D(C_{A_1}(A_1^{G_1}) \otimes C_{A_2}(A_2^{G_2})) \\ &=_{G_1 \times G_2} D(C_{A_1}(A_1^{G_1})) \times D(C_{A_2}(A_2^{G_2})), \end{aligned}$$

by Theorem 2.3. ■

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