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Essential Descent Spectrum and Commuting Compact Perturbations

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1. Introduction

Let $\mathcal{L}(X)$ be the algebra of all bounded operators acting on an infinite-dimensional complex Banach space. For an operator $T \in \mathcal{L}(X)$, write $\sigma(T)$ for its spectrum and $\rho(T)$ for its resolvent. The range and the kernel of T are denoted respectively by $\mathrm{R}(T)$ and $\mathrm{N}(T)$. The operator T is called *upper semi-Fredholm* if $\dim \mathrm{N}(T)$ is finite and $\mathrm{R}(T)$ is closed, while T is called *lower semi-Fredholm* if $\operatorname{codim} \mathrm{R}(T)$ is finite, and in this case the closedness of the range follows immediately (see [2]). We shall simply say "semi-Fredholm" when the operator is either upper semi-Fredholm or lower semi-Fredholm. The index of such an operator T is defined by $\operatorname{ind}(T) = \dim \mathrm{N}(T) - \dim \mathrm{R}(T)$, and if it is finite then T is said to be *Fredholm*.

Let T be an operator acting on X, and consider the decreasing sequence $\mathrm{c}_n(T) := \dim(\mathrm{R}(T^n)/\mathrm{R}(T^{n+1}))$, $n \in \mathbb{N}$, see [4]. Following M. Mbekhta and M. Müller [14], we shall say that T has finite *essential descent* if $\mathrm{d_e}(T) := \inf\{n \geq 0 : \mathrm{c}_n(T) < \infty\}$, where the infimum over the empty set is taken to be infinite, is finite. Clearly, every lower semi-Fredholm operator has finite essential descent and we have $\mathrm{d_e}(T) = 0$. This class of operators contains also every operator of finite descent, i.e., every operator T such that the *descent*, $\mathrm{d}(T) = \inf\{n \geq 0 : \mathrm{c}_n(T) = 0\}$, is finite.

The notion of essential descent was studied in several article, for instance, we cite [4], [6], [5] and [14]. From [15] and [6], we mention the following useful characterizations:

$$\mathrm{d}(T) \text{ is finite} \iff \mathrm{R}(T) + \mathrm{N}(T^d) = X \text{ for some } d \geq 0\,, \tag{1.1}$$

and

$$\mathrm{d_e}(T) \text{ is finite} \iff \begin{array}{c}\mathrm{R}(T) + \mathrm{N}(T^d) \text{ has finite codimension} \\ \text{in } X \text{ for some } d \geq 0\,.\end{array} \tag{1.2}$$

Let T be a bounded operator on X, the *descent* and the *essential descent resolvent sets* are defined respectively by:

$$\rho_{\mathrm{des}}(T) := \{\lambda \in \mathbb{C} : \mathrm{d}(T-\lambda) \text{ is finite}\}\,,$$
$$\rho^{\mathrm{e}}_{\mathrm{des}}(T) := \{\lambda \in \mathbb{C} : \mathrm{d_e}(T-\lambda) \text{ is finite}\}\,.$$

The *descent* and the *essential descent* spectrum are respectively $\sigma_{\mathrm{des}}(T) := \mathbb{C} \setminus \rho_{\mathrm{des}}(T)$ and $\sigma^{\mathrm{e}}_{\mathrm{des}}(T) := \mathbb{C} \setminus \rho^{\mathrm{e}}_{\mathrm{des}}(T)$; evidently $\sigma^{\mathrm{e}}_{\mathrm{des}}(T) \subseteq \sigma_{\mathrm{des}}(T) \subseteq \sigma(T)$.

The paper is organized as follows. In section 2, we show that the essential descent spectrum is a compact subset of $\mathbb{C}$, and that it is empty precisely when the operator is algebraic. We shall also prove that the essential descent spectrum satisfies a holomorphic version of the Spectral Mapping Theorem. In [1], it was established that a power of an operator $F \in \mathcal{L}(X)$ has a finite-rank if and only if $\sigma_{\mathrm{des}}(T+F) = \sigma_{\mathrm{des}}(T)$ for every operator T commuting with F. In section 3, we give a similar characterization of such operators F in term of essential descent. In the final section we provide some sufficient conditions to obtain the closedness of the range of an operator with finite essential descent.

2. Characterization of the essential descent spectrum

For an operator T of finite essential descent, we associate $\mathrm{p}(T) = \inf\{n \geq 0 : c_p(T) = c_n(T) \text{ for all } p \geq n\}$. Clearly, $\mathrm{d_e}(T) \leq \mathrm{p}(T)$, and if $\mathrm{d}(T)$ is finite then we have $\mathrm{d}(T) = \mathrm{p}(T)$.

An operator $T \in \mathcal{L}(X)$ is called *semi-regular* if $\mathrm{R}(T)$ is closed and $\mathrm{N}(T^n) \subseteq \mathrm{R}(T)$ for all positive integer n. The *semi-regular resolvent set* is the open subset s-reg(T) of $\mathbb{C}$ formed by the complex numbers λ for which $T-\lambda$ is semi-regular, see [13].

We begin the statement of our results by the following theorem:

Theorem 2.1. *Let $T \in \mathcal{L}(X)$ be an operator for which $\mathrm{d_e}(T)$ is finite. Then there exists $\delta > 0$ such that for $0 < |\lambda| < \delta$ and $p := \mathrm{p}(T)$, we have the following assertions:*

(i) $T - \lambda$ *is semi regular;*

(ii) $\dim \mathrm{N}(T-\lambda)^n = n \dim \big(\mathrm{N}(T^{p+1})/\mathrm{N}(T^p)\big)$ *for all* $n \in \mathbb{N}$*;*

(iii) $\operatorname{codim} \mathrm{R}(T-\lambda)^n = n \dim \big(\mathrm{R}(T^p)/\mathrm{R}(T^{p+1})\big)$ *for all* $n \in \mathbb{N}$.

The proof of this theorem requires the following lemma.

Lemma 2.2. *If* $T \in \mathcal{L}(X)$ *is a semi-regular operator with finite codimensional range, then* $\operatorname{codim} \mathrm{R}(T^n) = n \operatorname{codim} \mathrm{R}(T)$ *for all positive integer* n.

Proof. Let $n \geq 2$ and $S : X \longmapsto X/\mathrm{R}(T^n)$ be the operator given by $Sx := T^{n-1}x + \mathrm{R}(T^n)$. Since T is semi-regular, we have $\mathrm{N}(S) = \mathrm{R}(T) + \mathrm{N}(T^{n-1}) = \mathrm{R}(T)$, and consequently $X/\mathrm{R}(T) \cong \mathrm{R}(T^{n-1})/\mathrm{R}(T^n)$. On the other hand, it is well-known that $X/\mathrm{R}(T^{n-1}) \times \mathrm{R}(T^{n-1})/\mathrm{R}(T^n) \cong X/\mathrm{R}(T^n)$. Therefore $X/\mathrm{R}(T^{n-1}) \times X/\mathrm{R}(T) \cong X/\mathrm{R}(T^n)$, and hence

$$\operatorname{codim} \mathrm{R}(T^n) = \operatorname{codim} \mathrm{R}(T^{n-1}) + \operatorname{codim} \mathrm{R}(T)\,.$$

Thus, a successive repetition of this argument leads to $\operatorname{codim} \mathrm{R}(T^n) = n \operatorname{codim} \mathrm{R}(T)$. ∎

In [11], it is shown that if $T \in \mathcal{L}(X)$ is a semi-regular operator such that its range possesses a closed complement subspace M in X, then $X = \mathrm{R}(T-\lambda) \oplus M$ for all λ in a small neighbourhood of 0 in $\mathbb{C}$. Therefore, we can add to the preceding lemma that $\operatorname{codim} \mathrm{R}(T-\lambda)^n = n \operatorname{codim} \mathrm{R}(T)$ for every $n \in \mathbb{N}$ and λ in the connect component of s-reg(T) that contains zero.

Proof of Theorem 2.1. Let T_{o} be the restriction of T to $\mathrm{R}(T^p)$, and define a new norm on $\mathrm{R}(T^p)$ by

$$|y| = \|y\| + \inf\{\|x\| \,:\, x \in X \text{ and } y = T^p x\}\,, \qquad \text{for all } y \in \mathrm{R}(T^p)\,.$$

It is a classical fact that $\mathrm{R}(T^p)$ equipped with this norm is a Banach space and that T_{o} is a bounded operator on $(\mathrm{R}(T^p), |\ |)$. Hence it follows that T_{o} is both semi-Fredholm and semi-regular. Indeed, T_{o} is semi-Fredholm because $\mathrm{R}(T_{\mathrm{o}}) = \mathrm{R}(T^{p+1})$ is of finite codimension in $\mathrm{R}(T^p)$. Moreover, since $\mathrm{d_e}(T)$ is finite, [4, Theorem 3.1] ensures that for all $n \in \mathbb{N}$, $\mathrm{N}(T) \cap \mathrm{R}(T^p) = \mathrm{N}(T) \cap \mathrm{R}(T^{p+n})$, and so

$$\mathrm{N}(T_{\mathrm{o}}) = \mathrm{N}(T) \cap \mathrm{R}(T^p) = \mathrm{N}(T) \cap \mathrm{R}(T^{p+n}) \subseteq \mathrm{R}(T^{p+n}) = \mathrm{R}(T_{\mathrm{o}}^n)\,.$$

Let $\delta > 0$ be such that $T_{\mathrm{o}} - \lambda$ is both semi-Fredholm and semi-regular for $|\lambda| < \delta$. We note that with no restriction on T, $X = \mathrm{R}(T-\lambda)^n + \mathrm{R}(T^p)$ for all positive integers p, n and non-zero complex number λ. In fact, consider the complex polynomials $p(z) =: (z-\lambda)^n$ and $q(z) = z^p$. Since p and q has no common divisors, there exists two complex polynomials u and v such that $1 = p(z)u(z) + q(z)v(z)$ for every $z \in \mathbb{C}$. Hence $I = p(T)u(T) + q(T)v(T)$, and thus $X = \mathrm{R}(T-\lambda)^n + \mathrm{R}(T^p)$. Consequently, for $0 < |\lambda| < \delta$, it follows by the preceding lemma that

$$\begin{aligned}
\operatorname{codim} \mathrm{R}(T-\lambda)^n &= \dim X/\mathrm{R}(T-\lambda)^n \\
&= \dim \big((\mathrm{R}(T^p) + \mathrm{R}(T-\lambda)^n)/\mathrm{R}(T-\lambda)^n\big) \\
&= \dim \big(\mathrm{R}(T^p)/\mathrm{R}(T^p) \cap \mathrm{R}(T-\lambda)^n\big) \\
&= \operatorname{codim} \mathrm{R}(T_\mathrm{o}-\lambda)^n = n \operatorname{codim} \mathrm{R}(T_\mathrm{o}) \\
&= n \dim \mathrm{R}(T^p)/\mathrm{R}(T^{p+1}) \,.
\end{aligned}$$

In particular, $T-\lambda$ is semi-Fredholm. Moreover, since $\mathrm{N}(T-\lambda) = \mathrm{R}(T^p) \cap \mathrm{N}(T-\lambda) = \mathrm{N}(T_\mathrm{o}-\lambda) \subseteq \mathrm{R}(T_\mathrm{o}-\lambda)^k \subseteq \mathrm{R}(T-\lambda)^k$ for all $k \in \mathbb{N}$, $T-\lambda$ is also semi-regular. For the second statement, we have

$$\begin{aligned}
\dim \mathrm{N}(T-\lambda)^n &= \dim \mathrm{N}(T_\mathrm{o}-\lambda) \\
&= \operatorname{ind}(T_\mathrm{o}-\lambda)^n + \operatorname{codim} \mathrm{R}(T_\mathrm{o}-\lambda)^n \\
&= n\big[\operatorname{ind}(T_\mathrm{o}-\lambda) + \operatorname{codim} \mathrm{R}(T_\mathrm{o}-\lambda)\big] \\
&= n\big[\operatorname{ind}(T_\mathrm{o}) + \operatorname{codim} \mathrm{R}(T_\mathrm{o})\big] \\
&= n \dim \mathrm{N}(T_\mathrm{o}) = n \dim \big(\mathrm{R}(T^p) \cap \mathrm{N}(T)\big) \,.
\end{aligned}$$

But, since T^p induces an isomorphism from $\mathrm{N}(T^{p+1})/\mathrm{N}(T^p)$ onto $\mathrm{R}(T^p) \cap \mathrm{N}(T)$, we obtain that

$$\dim \mathrm{N}(T-\lambda)^n = n \dim \big(\mathrm{R}(T^p) \cap \mathrm{N}(T)\big) = n \dim \big(\mathrm{N}(T^{p+1})/\mathrm{N}(T^p)\big) \,.$$

This completes the proof. ∎

Remark 2.3. It is interesting to note that if $T \in \mathcal{L}(X)$ has finite essential descent, then there exists a finite-dimensional subspace M of X such that $X = \mathrm{R}(T-\lambda) \oplus M$ for every λ in a sufficient small punctured neighbourhood of 0. Indeed, let T_o and p be as in the proof of Theorem 2.1. Since T_o is semi-regular with finite-codimensional range, there exists $\delta > 0$ and a finite dimensional subspace M such that $\mathrm{R}(T^p) = \mathrm{R}(T_\mathrm{o}-\lambda) \oplus M$ for $|\lambda| < \delta$. Hence, $X = \mathrm{R}(T-\lambda) + \mathrm{R}(T^p) = \mathrm{R}(T-\lambda) \oplus M$ for $0 < |\lambda| < \delta$.

In the following we recapture as corollary the Proposition 2.1 of [1].

Corollary 2.4. *Let $T \in \mathcal{L}(X)$ be an operator of finite descent d. Then there exists $\delta > 0$ such that the following assertions hold for $0 < |\lambda| < \delta$:*

(i) *$T - \lambda$ is onto;*

(ii) $\dim \mathrm{N}(T - \lambda) = \dim \mathrm{N}(T^{d+1})/\mathrm{N}(T^d)$.

Also as an immediate consequence of Theorem 2.1, we have:

Corollary 2.5. *If T is a bounded operator on X, then $\sigma^{\mathrm{e}}_{\mathrm{des}}(T)$ is a compact subset of $\mathbb{C}$.*

In [14], M. Mbekhta and V. Müller have established that the set $\{T \in \mathcal{L}(X) : \mathrm{d_e}(T) \text{ is finite}\}$ is a regularity in $\mathcal{L}(X)$; consequently, by [10, Theorem 1.4], the corresponding spectrum satisfies the spectral mapping theorem.

Theorem 2.6. *Let T be a bounded operator on X. If f is an analytic function on an open neighborhood of $\sigma(T)$, not identically constant on each connected component of its domain, then*

$$\sigma^{\mathrm{e}}_{\mathrm{des}}(f(T)) = f(\sigma^{\mathrm{e}}_{\mathrm{des}}(T))\,.$$

Recall that the *ascent* of an operator T is defined by $\mathrm{a}(T) = \inf\{n \geq 0 : \mathrm{N}(T^n) = \mathrm{N}(T^{n+1})\}$. It is familiar that T has finite ascent and descent if and only if 0 is a pole of the resolvent of T. The set of the poles of the resolvent of T will be denoted by $\mathrm{E}(T)$.

In the following theorem, we show that the operators whose essential descent spectrum is empty are exactly the algebraic operators, i.e, the operators that satisfy a non-trivial polynomial identity.

Theorem 2.7. *If T is a bounded operator on X, then*

$$\rho^{\mathrm{e}}_{\mathrm{des}}(T) \cap \partial\sigma(T) = \mathrm{E}(T)\,.$$

Moreover, $\sigma^{\mathrm{e}}_{\mathrm{des}}(T)$ is empty if and only if T is algebraic.

Before giving the proof of Theorem 2.7, we have to consider the following lemma:

Lemma 2.8. *Let T be a bounded operator on X. Then $\sigma_{\mathrm{des}}(T) \setminus \sigma^{\mathrm{e}}_{\mathrm{des}}(T)$ is an open subset of $\mathbb{C}$.*

Proof. Assume that $\lambda \in \sigma_{\mathrm{des}}(T) \setminus \sigma_{\mathrm{des}}^{\mathrm{e}}(T)$ and let $p := p(T-\lambda)$. Then by Theorem 2.1, there exists a deleted open neighborhood V of λ such that $V \cap \sigma_{\mathrm{des}}^{\mathrm{e}}(T) = \emptyset$ and for all $\mu \in V$ and $n \in \mathbb{N}$,

$$\operatorname{codim} \mathrm{R}(T-\mu)^n = n \dim \big(\mathrm{R}(T-\lambda)^p / \mathrm{R}(T-\lambda)^{p+1}\big) .$$

But, since $T-\lambda$ has infinite descent, $\dim \big(\mathrm{R}(T-\lambda)^p/\mathrm{R}(T-\lambda)^{p+1}\big)$ is non-zero, and hence $\{\operatorname{codim}(\mathrm{R}(T-\mu)^n)\}_n$ is a strictly increasing sequence for each $\mu \in V$. Thus $V \subseteq \sigma_{\mathrm{des}}(T)$, which completes the proof. ∎

Proof of Theorem 2.7. Let λ be in the boundary of $\sigma(T)$ and such that $\mathrm{d_e}(T-\lambda)$ is finite. It follows by theorem 2.1 that there exists a punctured neighborhood U of λ such that $\dim \mathrm{N}(T-\mu) = \dim \big(\mathrm{N}((T-\lambda)^{p+1})/\mathrm{N}((T-\lambda)^p)\big)$ and $\operatorname{codim} \mathrm{R}(T-\mu) = \dim \big(\mathrm{R}((T-\lambda)^p)/\mathrm{R}(T-\lambda)^{p+1})\big)$ for all $\mu \in U$, where $p := p(T-\lambda)$. Moreover, $U \setminus \sigma(T)$ is non-empty because $\lambda \in \partial\sigma(T)$. Therefore

$$\dim \big(\mathrm{N}((T-\lambda)^{p+1})/\mathrm{N}((T-\lambda)^p)\big) = \dim \big(\mathrm{R}((T-\lambda)^p)/\mathrm{R}((T-\lambda)^{p+1})\big) = 0 .$$

Thus $T-\lambda$ is of finite ascent and descent and so λ is a pole of the resolvent of T. The inverse inclusion is clair.

For the last statement, observe that $\sigma_{\mathrm{des}}^{\mathrm{e}}(T)$ is empty if and only if so is $\sigma_{\mathrm{des}}(T)$. In fact, suppose that $\sigma_{\mathrm{des}}^{\mathrm{e}}(T) = \emptyset$. Then, by the previous lemma, $\sigma_{\mathrm{des}}(T)$ is a clopen subset of $\mathbb{C}$, and hence it is empty. To complete the proof, we recall that by [1, Theorem 1.5], $\sigma_{\mathrm{des}}^{\mathrm{e}}(T) = \emptyset$ if and only if T is algebraic. ∎

Theorem 2.9. *Let T be a bounded operator on X. If Ω is a connected component of $\rho_{\mathrm{des}}^{\mathrm{e}}(T)$, then*

$$\Omega \subset \sigma(T) \qquad \text{or} \qquad \Omega \setminus \mathrm{E}(T) \subseteq \rho(T) .$$

Proof. Let Ω_r be the set of complex number $\lambda \in \Omega$ such that $T-\lambda$ is both semi-regular and semi Fredholm. Then, Theorem 2.1 implies that $\Omega \setminus \Omega_r$ is at most countable, and hence Ω_r is connected. Suppose that $\Omega \cap \rho(T)$ is non-empty, then so is $\Omega_r \cap \rho(T)$. Consequently, since $\operatorname{codim} \mathrm{R}(T-\lambda)$ is a constant function on Ω_r, we obtain that $\operatorname{codim} \mathrm{R}(T-\lambda) = 0$, and by the continuity of the index, we get that $\dim \mathrm{N}(T-\lambda) = 0$. Thus $\Omega_r \subseteq \rho(T)$. Now, $\Omega \setminus \Omega_r$ consists of an isolated points of the spectrum. Hence, by Lemma 2.7, $\Omega \setminus \Omega_r \subseteq \mathrm{E}(T)$, as required. ∎

Corollary 2.10. *If T is a bounded operator on X, then the following assertions are equivalent:*

(i) *$\sigma(T)$ is at most countable;*

(ii) *$\sigma_{\mathrm{des}}(T)$ is at most countable;*

(iii) *$\sigma^{\mathrm{e}}_{\mathrm{des}}(T)$ is at most countable; in this case, we have*

$$\sigma^{\mathrm{e}}_{\mathrm{des}}(T) = \sigma_{\mathrm{des}}(T) \text{ and } \sigma(T) = \sigma_{\mathrm{des}}(T) \cup \mathrm{E}(T)\,.$$

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (i) Suppose that $\sigma^{\mathrm{e}}_{\mathrm{des}}(T)$ is at most countable, then $\rho^{\mathrm{e}}_{\mathrm{des}}(T)$ is connected, and since $\rho(T) \subseteq \rho^{\mathrm{e}}_{\mathrm{des}}(T)$, Theorem 2.9 implies that $\rho^{\mathrm{e}}_{\mathrm{des}}(T) \setminus \mathrm{E}(T) = \rho(T)$. Therefore $\sigma(T) = \sigma^{\mathrm{e}}_{\mathrm{des}}(T) \cup \mathrm{E}(T)$ is at most countable.

For the last assertion, suppose that $\sigma(T)$ is at most countable. Then it follows by Lemma 2.8 that $\sigma_{\mathrm{des}}(T) \setminus \sigma^{\mathrm{e}}_{\mathrm{des}}(T)$ is a countable open set. Hence $\sigma_{\mathrm{des}}(T) = \sigma^{\mathrm{e}}_{\mathrm{des}}(T)$, as desired. ∎

3. Essential descent spectrum and perturbations

In [8], M. Kaashoek and D. Lay have shown that the descent spectrum is invariant under commuting perturbation F such that a power of F is of finite rank. Also they have conjectured that this perturbation property characterizes such operators F. Recently, M. Burgos, A. Kaidi, M. Mbekhta and M. Oudghiri provided in [1] an affirmative answer to this question. We generalizes these results as follows:

Theorem 3.1. *Let F be a bounded operator on X. Then the following assertions are equivalent:*

(i) *There exists a positive integer k for which F^k is of finite rank.*

(ii) *$\sigma^{\mathrm{e}}_{\mathrm{des}}(T+F) = \sigma^{\mathrm{e}}_{\mathrm{des}}(T)$ for every operator $T \in \mathcal{L}(X)$ commuting with F.*

Proof. (i) $\Rightarrow$ (ii) Suppose that F^k has finite-dimensional range. Then, by [8, lemma 2.1], we have

$$\dim\big(\mathrm{R}(T^{n+k-1})/\mathrm{R}(T+F)^n \cap \mathrm{R}(T^{n+k-1})\big) \le \dim \mathrm{R}(F^k) < \infty \tag{3.3}$$

for all positive integer n. Moreover, T has finite essential descent $d := \mathrm{d_e}(T)$, therefore $\dim\big(\mathrm{R}(T^d)/\mathrm{R}(T+F)^n \cap \mathrm{R}(T^{n+k-1})\big)$ is finite for $n \ge d$, and since

$$\mathrm{R}(T+F)^n \cap \mathrm{R}(T^{n+k-1}) \subseteq \mathrm{R}(T+F)^n \cap \mathrm{R}(T^d) \subseteq \mathrm{R}(T^d)\,,$$

we get that $\dim\big(\mathrm{R}(T^d)/\mathrm{R}(T+F)^n \cap \mathrm{R}(T^d)\big) < \infty$. Consequently,

$$\dim\big((\mathrm{R}(T^d)+\mathrm{R}(F^k))/\mathrm{R}(T+F)^n \cap \mathrm{R}(T^d)\big) < \infty \quad \text{for all } n \geq d\,. \tag{3.4}$$

On the other hand, by interchanging T and $T+F$ in (3.3), we obtain that

$$\dim\big(\mathrm{R}(T+F)^{n+k-1}/\mathrm{R}(T^n) \cap \mathrm{R}(T+F)^{n+k-1}\big) < \infty\,,$$

and so

$$\dim\big(\mathrm{R}(T+F)^{n+k-1}/\mathrm{R}(T^d) \cap \mathrm{R}(T+F)^{n+k-1}\big) < \infty \quad \text{for all } n \geq d\,. \tag{3.5}$$

Now by combining (3.4) and (3.5), it follows that

$$\dim\big((\mathrm{R}(T^d)+\mathrm{R}(F^k))/\mathrm{R}(T+F)^n\big) < \infty \quad \text{for all } n \geq d+k\,.$$

Thus $\dim\big(\mathrm{R}(T+F)^n/\mathrm{R}(T+F)^{n+1}\big)$ is finite for every $n \geq d+k$; which implies that $\mathrm{d_e}(T+F) \leq d+k$ as desired.

(ii) $\Rightarrow$ (i) First, if we take $T=0$, then we obtain that $\sigma^{\mathrm{e}}_{\mathrm{des}}(F)$ is empty, and hence F is algebraic with finite spectrum $\sigma(F)=\{\lambda_1,\lambda_2,\dots,\lambda_n\}$. Therefore, we have the following decomposition

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_n\,,$$

where X_i is a closed subspace and the restriction of $F-\lambda_i$ to this subspace is nilpotent.

We claim that if $\lambda_i \neq 0$, X_i is finite dimensional. Suppose to the contrary that $\lambda_i \neq 0$ and X_i is infinite dimensional. By [1, Proposition 4.3], there exists a non algebraic operator S_i on X_i commuting with the restriction F_i of F to this space. Let S denote the extension of S_i to X given by $S=0$ on each X_j such that $j \neq i$. Obviously $SF=FS$, and so $\sigma^{\mathrm{e}}_{\mathrm{des}}(S+F)=\sigma^{\mathrm{e}}_{\mathrm{des}}(S)$ by hypothesis. On the other hand, we have $\sigma^{\mathrm{e}}_{\mathrm{des}}(S)=\sigma^{\mathrm{e}}_{\mathrm{des}}(S_i)$ and $\sigma^{\mathrm{e}}_{\mathrm{des}}(S+F)=\sigma^{\mathrm{e}}_{\mathrm{des}}(S_i+F_i)$, and since $F_i-\lambda_i$ is nilpotent, the first implication ensures that $\sigma^{\mathrm{e}}_{\mathrm{des}}(S_i)=\sigma^{\mathrm{e}}_{\mathrm{des}}(S_i+F_i)=\sigma^{\mathrm{e}}_{\mathrm{des}}(S_i+\lambda_i)$. Now let α be an arbitrary complex number in $\sigma^{\mathrm{e}}_{\mathrm{des}}(S)\neq\emptyset$. Then it follows that $\alpha-n\lambda_i\in\sigma^{\mathrm{e}}_{\mathrm{des}}(S)$ for all $n\in\mathbb{N}$, which implies that $\lambda_i=0$, the desired contradiction. ∎

Notice that the preceding result can not be extended to compact perturbations. Indeed, consider the operator $T=0$ defined on the Hilbert space with an orthonormal basis $\{e_{i,j}\}_{i,j=1}^{\infty}$; clearly $\mathrm{d_e}(T)$ is finite. However, if we let K to be the operator defined by

$$Ke_{i,j}=\frac{1}{ij}e_{i,j+1}\,,$$

then we can see easily that K is a compact operator and that $e_{i,n+1} \in \mathrm{R}(K^n) \setminus \mathrm{R}(K^{n+1})$ for every $i \geq 1$ and every $n \in \mathbb{N}$. Thus $\mathrm{d_e}(K)$ is infinite.

We mention that when the operator F is assumed to be of finite-rank in the previous theorem, then the commutativity condition is redundant. In fact, M. Mbekhta and V. Müller have proved in [14] that if T is a bounded operator on X, then $\sigma_{\mathrm{des}}^{\mathrm{e}}(T+F) = \sigma_{\mathrm{des}}^{\mathrm{e}}(T)$ for every finite rank operator F on X. Hence, if we let $\mathcal{F}(X)$ denote the set of finite-rank operators on X, then we have:

$$\sigma_{\mathrm{des}}^{\mathrm{e}}(T) \subseteq \bigcap_{F \in \mathcal{F}(X)} \sigma_{\mathrm{des}}(T+F)\,. \tag{3.6}$$

Let $\operatorname{iso} K$ denote the set of isolated point of every subset K of $\mathbb{C}$, and $\operatorname{acc} K = K \setminus \operatorname{iso} K$ the set of its accumulation points. In the next result we show that the inclusion (3.6) becomes equality if we complete $\sigma_{\mathrm{des}}^{\mathrm{e}}(T)$ by the set, $\sigma_{\mathrm{sf}}^{+}(T)$, formed by the complex numbers λ such that $T - \lambda$ is not semi-Fredholm of positive index.

THEOREM 3.2. *If T is a bounded operator on X, then*

$$\sigma_{\mathrm{des}}^{\mathrm{e}}(T) \cup \operatorname{acc} \sigma_{\mathrm{sf}}^{+}(T) = \bigcap_{F \in F(X)} \sigma_{\mathrm{des}}(T+F)\,.$$

Proof. Suppose that λ is a complex number for which there exists a finite-rank operator such that $\mathrm{d}(T+F)$ is finite, then $\lambda \notin \sigma_{\mathrm{des}}^{\mathrm{e}}(T)$. Moreover, it follows from Corollary 2.4 that $T+F-\mu$ is a surjective operator, and hence $T-\mu$ is semi-Fredholm with positive index, when μ is in a small punctured neighbourhood of λ. This shows that $\lambda \notin \operatorname{acc} \sigma_{\mathrm{sf}}^{+}(T)$. For the converse, suppose $\lambda \notin \sigma_{\mathrm{des}}^{\mathrm{e}}(T) \cup \operatorname{acc} \sigma_{sf}^{+}(T)$. Then there exists $\delta > 0$ such that $T - \lambda$ is semi-Fredholm with positive index for $0 < |\lambda| < \delta$. Now, by [12, Theorem 2.1], there exists a finite-rank operator F such that $T+F-\lambda$ is onto for every $0 < |\lambda| < \delta$. Finally, since $\mathrm{d_e}(T)$, and so $\mathrm{d_e}(T+F)$, is finite, Theorem 2.1 implies that $\mathrm{d}(T+F)$ is finite. This completes the proof. ∎

4. ESSENTIAL DESCENT AND CLOSED RANGE

Let T be a bounded operator on X. A well-known result of T. Kato [9, Lemma 332] states that if $\mathrm{R}(T)$ has finite codimension then it is closed. A more general version of this result is done by S. Goldberg [3]: if $\mathrm{R}(T)$ has a closed complement M in X, then it is closed.

The closedness of the range can not follow for operators with finite essential descent. In fact, if we consider the operator T defined on the Hilbert space H with orthonormal basis $(e_n)_{n\in\mathbb{N}}$ by $Te_{2n} = \frac{1}{n}e_{2n-1}$ and $Te_{2n-1} = 0$. Then $\mathrm{R}(T)$ is not closed and $\mathrm{d}(T) = 2$.

Proposition 4.1. *Let $T \in \mathcal{L}(X)$ be an operator with finite-essential descent d and let k be a positive integer. If $\mathrm{N}(T^d) \cap \mathrm{R}(T^k)$ has a closed complement in $\mathrm{N}(T^d)$, then $\mathrm{R}(T^k)$ is closed.*

Proof. Let M be a closed subspace of $\mathrm{N}(T^d)$ such that $\mathrm{N}(T^d) = M \oplus \mathrm{N}(T^d) \cap \mathrm{R}(T^k)$. Since $d := \mathrm{d_e}(T)$ is finite, then so is $\mathrm{d_e}(T^k) \leq \mathrm{d_e}(T)$, and hence it follows by (1.2) that $\mathrm{codim}(\mathrm{R}(T^k)+\mathrm{N}(T^d))$ is finite. Thus there exists a finite dimensional subspace M_1 such that $X = [\mathrm{R}(T^k) + \mathrm{N}(T^d)] \oplus M_1 = \mathrm{R}(T^k) \oplus M \oplus M_1$, which shows that $\mathrm{R}(T^k)$ is closed. ∎

Note that if T is an operator with finite-essential descent and finite-dimensional kernel, then we obtain immediately from Proposition 4.1 that $\mathrm{R}(T)$ is closed. However, for such operator T, the range is of finite-codimension, i.e., $\mathrm{d_e}(T) = 0$. In fact, we have $\mathrm{codim}\,\mathrm{R}(T) = \mathrm{codim}(\mathrm{N}(T^d) + \mathrm{R}(T)) + \dim(\mathrm{N}(T^d) + \mathrm{R}(T))/\mathrm{R}(T)$. Clearly, $\mathrm{codim}(\mathrm{N}(T^d) + \mathrm{R}(T))$ is finite because $\mathrm{d_e}(T)$ is finite. Also, since $\dim \mathrm{N}(T)$ is finite, then so is $\dim \mathrm{N}(T^d)$, and hence $(\mathrm{N}(T^d) + \mathrm{R}(T))/\mathrm{R}(T)$ is finite-dimensional. Thus $\mathrm{codim}\,\mathrm{R}(T)$ is finite.

Corollary 4.2. *Let T be a bounded operator on X such that $\mathrm{d_e}(T) = 1$.*

(i) *If $\dim\big(\mathrm{N}(T) \cap \mathrm{R}(T)\big)$ is finite, then $\mathrm{R}(T)$ is closed.*

(ii) *If X is a Hilbert space, then $\mathrm{N}(T) \cap \mathrm{R}(T)$ is closed if and only if $\mathrm{R}(T)$ is closed.*

Also as consequence of Theorem 2.1 and Corollary 4.2 we derive the following proposition:

Proposition 4.3. *Let $T \in \mathcal{L}(X)$ and λ be a complex number such that $\mathrm{d_e}(T - \lambda) = 1$.*

(i) *If there exists a sequence of complex numbers $\{\lambda_n\}_n$ converging to λ and such that $\dim \mathrm{N}(T - \lambda_n)$ is finite for all $n \geq 1$ then $\mathrm{R}(T - \lambda)$ is closed.*

(ii) *If $\mathrm{R}(T - \lambda)$ is not closed, then λ is in the interior of the point spectrum and $\dim \mathrm{N}(T - \lambda) = \infty$ in a neighborhood of λ.*

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References

[1] Burgos, M., Kaidi, A., Mbekhta, M., Oudghiri, M., On the descent spectum, to appear in *J. Operator Theory.*

[2] Caradus, S.R., Pfaffenberger, W.E., Bertram, Y., "Calkin Algebras and Algebras of Operators on Banach Spaces", Marcel Dekker, New York, 1974.

[3] Goldberg, S., "Unbounded Linear Operators: Theory and Applications", McGraw-Hill, New York-Toronto, 1966.

[4] Grabiner, S., Uniform ascent and descent of bounded operators, *J. Math. Soc. Japan* **34** (1982) 317 – 337.

[5] Grabiner, S., Generalization of Fredholm perators, in "Banach Algebras '97" (Blaubeuren), de Gruyter, Berlin, 1998, 169 – 187.

[6] Grabiner, S., Zemánek, J., Ascent, descent and ergodic properties of linear operators, *J. Operator Theory* **48** (2002), 69 – 81.

[7] Kaashoek, M.A., Ascent, descent, nullity and defect: a note on a paper by A.E. Taylor, *Math. Ann.* **172** (1967), 105 – 115.

[8] Kaashoek, M.A., Lay, D.C., Ascent, descent, and commuting perturbations, *Trans. Amer. Math. Soc.* **169** (1972) 35 – 47.

[9] Kato, T., Perturbation theory for nullity, deficiency, and other quantities of linear operators, *J. Anal. Math.* **6** (1958), 261 – 322.

[10] Kordula, V., Müller, V., On the axiomatic theory of the spectrum, *Studia Math.* **119** (1996), 109 – 128.

[11] Mbekhta, M., On the Generalized Resolvent in Banach Spaces, *J. Math. Anal. Appl.* **189** (1995), 362 – 377.

[12] Mbekhta, M., Semi-Fredholm perturbations and commutators, *Math. Proc. Cambridge Philos. Soc.* **113** (1993), 173 – 177.

[13] Mbekhta, M., Ouahab, A., Opérateur s-régulier dans un espace de Banach et théorie spectrale, *Acta Sci. Math. (Szeged)* **59** (1994), 525 – 543.

[14] Mbekhta, M., Muller, V., On the axiomatic theory of spectrum II, *Studia Math.* **199** (1996) 129 – 147.

[15] Taylor, A.E., Lay, D.C., "Introduction to Functional Analysis", John Wiley & Sons, New York-Chichester-Brisbane, 1980.