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Projective Covers of Finitely Generated Banach Modules and the Structure of Some Banach Algebras

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Introduction

The investigation of the structure of biprojective Banach algebras with non-trivial radical [3] forces the author to suppose that the idea of projective cover, which is important in Ring Theory, can be effectively applied to Banach algebras and modules. But, in fact, the structural results on biprojectivity can be easier obtained without projective covers, so there are no references to this matter in [3]. Projective covers of Banach modules are considered in the present article. Except some assertions in Sections 1 and 6 we restrict our attention to the finitely generated case. The discussion concentrates on Banach algebras with conditions on the existence of projective covers.

Recall that for a unital ring $\mathcal{R}$, an epimorphism of $\mathcal{R}$-modules $\varepsilon\colon P \to X$ is called a projective cover (in another terminology, a projective envelope) if P is projective and a submodule Y coincides with P provided $\operatorname{Ker}\varepsilon + Y = P$. A unital ring $\mathcal{R}$ is called semi-perfect if every irreducible left $\mathcal{R}$-module admits a projective cover. This is equivalent to a stronger condition that every finitely generated module admits a projective cover. Bass' structural theorem asserts that *$\mathcal{R}$ is semi-perfect if and only if $\mathcal{R}/\operatorname{Rad}\mathcal{R}$ is classically semi-simple and every idempotent in $\mathcal{R}/\operatorname{Rad}\mathcal{R}$ can be lifted modulo $\operatorname{Rad}\mathcal{R}$.* (Here $\operatorname{Rad}\mathcal{R}$ is the Jacobson radical of $\mathcal{R}$.) A detailed discussion on semi-perfect rings is in [11, Chapter 11] and [6, Chapter 19].

Transferring the idea of projective cover into the Banach algebra context we have to be careful because of some peculiarities of Topological Homology.

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First, the main subject of Ring Theory is a unital ring. But for Banach algebras the existence of identity is too strong assumption to include many interesting examples. Second, we must distinguished two notions for Banach modules — relative projectivity and strict projectivity. (Usually 'relative' is omitted). The first notion is connected with cohomology groups but only the second condition, which is stronger, can be used as a base for a satisfactory structural theory. Of course, the difference with the pure algebraic case is of main interest.

In Section 1, two version of covers — projective and strictly projective — will be introduced. Their basic properties are considered and a description of the strictly projective cover of an irreducible module is given. Note that the dual notion of a strictly injective envelope of a Banach module was considered in [7], where it was shown that (as in the pure algebraic case) the strictly injective envelope always exists.

We say that a Banach algebra is *semi-perfect (strictly semi-perfect)* if every irreducible left Banach module admits a projective (strictly projective) cover. Strictly semi-perfect Banach algebras are considered in Section 2. The main result asserts that *a Banach algebra A is strictly semi-perfect if and only if $A/\operatorname{Rad} A$ is modular annihilator*. Note that this characterization is more transparent than in the pure algebraic case — we do not need the lifting idempotents assumption because by Dixon's theorem every idempotent in the quotient of a Banach algebra over the radical can be lifted. It will be shown in Section 3 that a Banach algebra is strictly semi-perfect if and only if every finitely generated left Banach module over it admits a strictly projective cover.

In the next sections we turn to projective covers. In the general situation, the existence of a projective cover is not sufficient to obtain structural results. Indeed, let $P \to X$ be a projective cover of an irreducible module X; then we need to solve the lifting problem

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ P & \longrightarrow & X. \end{array}$$

To do this it is sufficient to assume that the projection $A \to X$ is admissible or P is strictly projective. But we cannot guarantee neither the first nor the second assumption. The only known way to get over this difficulty is to use the approximation property. It will be shown in Section 4 that a semi-perfect Banach algebra such that the quotient by the radical has the approximation property is strictly semi-perfect. A stronger form of this theorem, which is valid for a quasi-biprojective Banach algebra, can be found in Section 6. As a preliminary result the existence of a projective cover for a wide class of modules over a quasi-biprojective Banach algebra will be shown.

In [15], Selivanov proved that a finitely generated, projective Banach module with the approximation property is strictly projective. Extending this approach in Section 5 we shall find some assumptions (in particular, on the approximation property) that imply the existence of a strictly projective cover for a finitely generated Banach module.

Finally, in Section 7, we consider some application to Harmonic Analysis. The purpose is to obtain a homological characterization of compactness that has the same form for groups and their non-commutative analogues. It was shown by Helemskii that a locally compact group G is compact if and only if the convolution Banach algebra $L^1(G)$ is biprojective [8]. Recently, non-commutative analogues of this result have appeared; see [1, 19] for Fourier algebras, [14] for algebras of nuclear operator with "the non-commutative convolution", and [2] for Kac algebras and locally compact quantum groups. But in order to get satisfactory results, Operator Space Theory is essentially involved in [1, 2, 19]. On the other hand, for algebras of nuclear operators, biprojectivity is equivalent to finiteness of G not only in the classical case as it is proved in [14] but in the operator space case also [2]. Thus biprojectivity is too strong to characterize compactness in this case. We shall show that both difficulties disappear when we use semi-perfect algebras instead of biprojective algebras.

1. Projectivity and covers

First, let us recall some standard homological definitions in the Banach algebra context [8, 9]. Let A be a Banach algebra. For simplicity we assume that A and all Banach A-modules are endowed with a multiplicative norm. An epimorphism $\nu : Y \to Z$ of (one-sided or two-sided) Banach A-modules is called *admissible* if there exists a bounded linear operator $\sigma : Z \to Y$ such that $\nu\sigma = 1$. Also, an epimorphism ν is called *strict* if the natural morphism $Y/\operatorname{Ker}\nu \to Z$ is an isomorphism. It is clear that each admissible epimorphism is strict. It is an obvious corollary of the open mapping theorem that an epimorphism ν is strict if and only if it is surjective. (Note that a morphism of Banach modules is an epimorphism if and only if its range is dense.)

A Banach A-module P is called *projective* if for every admissible epimorphism $\nu\colon Y \to Z$ and every morphism $\psi\colon P \to Z$ of Banach A-modules there exists a morphism $\varphi\colon P \to Y$ such that the diagram

$$\begin{tikzcd} & Y \arrow[d, "\nu"] \\ P \arrow[ur, dashed, "\varphi"] \arrow[r, "\psi"'] & Z \end{tikzcd} \tag{1.1}$$

commutes.

A Banach A-module P is called *strictly projective* if for every strict epimorphism $\nu : Y \to Z$ and every morphism $\psi : P \to Z$ of Banach A-modules there exists a morphism $\varphi\colon P \to Y$ such that Diagram (1.1) commutes.

Definition 1.1. A strict epimorphism $\varepsilon : P \to X$ of Banach A-modules is said to be a *cover* if for every strict epimorphism $\varphi : Y \to X$ and every morphism ψ such that $\varphi = \varepsilon\psi$ the morphism ψ is a strict epimorphism. If P is projective (strictly projective), then ε is a *projective (strictly projective) cover*. In this case, we say that X *admits a projective (strictly projective) cover*.

Proposition 1.2. *Suppose that $\varepsilon_1 : P_1 \to X$ and $\varepsilon_2 : P_2 \to X$ are strictly projective covers. Then there exists a topological isomorphism $\alpha : P_1 \to P_2$ such that the diagram*

$$\begin{tikzcd} P_1 \arrow[rr, dashed, "\alpha"] \arrow[dr, "\varepsilon_1"'] & & P_2 \arrow[dl, "\varepsilon_2"] \\ & X & \end{tikzcd} \tag{1.2}$$

commutes.

Proof. Since P_1 is strictly projective, there exists a morphism α such that Diagram (1.2) commutes. The epimorphism ε_2 is a cover, hence α is a strict epimorphism. Since P_2 is strictly projective, there exists a morphism β such that $\alpha\beta = 1$. Then $\varepsilon_1\beta = \varepsilon_2\alpha\beta = \varepsilon_2$. The epimorphism ε_1 is a cover; therefore β is a strict epimorphism. Since β is injective, it is an isomorphism and $\alpha = \beta^{-1}$. ∎

The unitization of a Banach algebra A is denoted by A_+. A left Banach A-module X is called n-generated for $n \in \mathbb{N}$ if there exists $x_1, \ldots, x_n \in X$ such that each $x \in X$ can represented as $x = \sum_{k=1} a_k \cdot x_k$ for some $a_1, \ldots, a_n \in A$. A cyclic module is just a 1-generated one.

Proposition 1.3. *Let X be an n-generated left Banach A-module for some $n \in \mathbb{N}$, and let $\varepsilon\colon P \to X$ be a strictly projective cover. Then P is topologically isomorphic to a closed complemented submodule in A_+^n containing in A^n.*

Proof. Let $x_1, \dots, x_n$ be generators of X. Consider the morphism of left Banach A-modules

$$\sigma_+\colon A_+^n \to X\colon (a_1, \dots a_n) \mapsto \sum_{k=1}^{n} a_k \cdot x_k \qquad (a_k \in A_+).$$

Since A_+^n is strictly projective, there exists a morphism χ such that the diagram

$$\begin{tikzcd} & P \arrow[d, "\varepsilon"] \\ A_+^n \arrow[ur, dashed, "\chi"] \arrow[r, "\sigma_+"'] & X \end{tikzcd}$$

commutes. Set $\chi' := \chi|_{A^n}$. It is obvious that $\varepsilon\chi' = \sigma_+|_{A^n}$. Since $\sigma_+|_{A^n}$ is a strict epimorphism and ε is a cover, χ' is a strict epimorphism. The module P is strictly projective, hence there exists a morphism $\varphi\colon P \to A^n$ such that $\chi'\varphi = 1$. It is clear that $\chi\varphi = 1$; therefore P is topologically isomorphic to a closed complemented submodule in A_+^n containing in A^n. ∎

Corollary 1.4. *Let X be a cyclic left Banach A-module, and let $\varepsilon\colon P \to X$ be a strictly projective cover. Then P is topologically isomorphic to Ap for some idempotent p in A.*

Proof. It follows from Proposition 1.3 (with $n = 1$) that P is topologically isomorphic to a closed complemented left ideal in A_+ containing in A. Hence there is an idempotent $p \in A_+$ such that $P \cong A_+p$. For every $a \in A_+$ the product ap is in A, in particular, $p = 1 \cdot p \in A$. Thus, $P \cong Ap$. ∎

Remarks 1.5. Suppose that A is a Banach algebra that does not contain a non-trivial idempotent. It follows from Corollary 1.4 that every cyclic Banach A-module does not admit a strictly projective cover.

Lemma 1.6. *A maximal submodule in a cyclic Banach module is closed.*

Proof. A cyclic Banach A-module has the form A/I, where I is a closed modular left ideal in A. Suppose that Y is a maximal submodule in A/I. By

σ denote the natural projection $A \to A/I$. It is easy to see that $\sigma^{-1}(Y)$ is a modular left ideal in A. Hence it is contained in a maximal modular left ideal J [9, Proposition 1.3.25]. Consequently, J/I is a proper submodule in A/I and $Y \subset J/I$. Since Y is maximal, $Y = J/I$. The ideal J is closed in A; therefore J/I is closed in A/I. ∎

PROPOSITION 1.7. *Let $\varepsilon : P \to X$ be a strictly projective cover of an irreducible Banach module X. Then* $\operatorname{Ker}\varepsilon$ *is the largest proper submodule in P.*

Proof. Assume the converse. Then there exists a submodule Z_0 such that $Z_0 \not\subset \operatorname{Ker}\varepsilon$. It follows from Corollary 1.4 that P is a cyclic A_+-module. It is well known that every submodule in a finitely generated module over a unital ring is contained in a maximal submodule [11, 2.3.11]. Then there is a maximal submodule Z that contains Z_0. Obviously, $Z \not\subset \operatorname{Ker}\varepsilon$. Note that $\operatorname{Ker}\varepsilon$ is maximal; therefore $\operatorname{Ker}\varepsilon + Z = P$. Lemma 1.6 implies that Z is closed. Hence the embedding $Z \to P$ is a morphism of Banach modules, and the composition $Z \to P \xrightarrow{\varepsilon} X$ is surjective. Since ε is a cover, $Z \to P$ is surjective also. Consequently, $Z = P$. This contradicts with the maximality of Z. ∎

LEMMA 1.8. *Suppose that a Banach module X contains a largest proper submodule Y. If, in addition, Y is closed, then $\varepsilon\colon X \to X/Y$ is a cover.*

Proof. Let $\psi\colon Z \to X$ be a morphism, let and $\varphi\colon Z \to X/Y$ be a strictly epimorphism of Banach modules such that $\varphi = \varepsilon\psi$. Then $\psi(Z) \subset Y$ or $\psi(Z) = X$. In the first case we get the impossible equality $\varphi = 0$. In the second case ψ is surjective. This concludes the proof. ∎

For a left Banach A-module X set

$$\varkappa_X\colon A \mathbin{\widehat{\otimes}_A} X \to X\colon a \otimes_A x \mapsto a \cdot x.$$

(Here $\widehat{\otimes}_A$ is the symbol for the projective tensor product of Banach A-modules.) If $\varkappa_X$ is surjective then by the open mapping theorem there exists a constant $C > 0$ such that for each $x \in X$ we can choose $\mu(x) \in A \mathbin{\widehat{\otimes}_A} X$ with $\varkappa_X\mu(x) = x$ and $\|\mu(x)\| \le C\|x\|$. (The map $\mu\colon X \to A \mathbin{\widehat{\otimes}_A} X$ may be non-linear.)

The next assertions will be used in Section 4.

Lemma 1.9. *Suppose that $\varkappa_X$ is surjective and fix a map μ as above. Then for each $y \in A \mathbin{\widehat{\otimes}}_A X$ and each decomposition $y = \sum_{i=1}^{\infty} a_i \otimes_A x_i$, where $(a_i) \subset A$, $(x_i) \subset X$, and $\sum_i \|a_i\| \, \|x_i\| < \infty$, we have*

$$y = \sum_i a_i \cdot \mu(x_i).$$

Proof. For every i fix a decomposition $\mu(x_i) = \sum_j b_{ij} \otimes_A t_{ij}$, where $(b_{ij}) \subset A$, $(t_{ij}) \subset X$, and $\sum_j \|b_{ij}\| \, \|t_{ij}\| < \infty$. Whence $x_i = \sum_j b_{ij} \cdot t_{ij}$. Finally,

$$\begin{aligned} \sum_i a_i \cdot \mu(x_i) &= \sum_i \sum_j a_i b_{ij} \otimes_A t_{ij} \\ &= \sum_i \sum_j a_i \otimes_A b_{ij} \cdot t_{ij} = \sum_i a_i \otimes_A x_i = y. \end{aligned}$$

∎

Corollary 1.10. *Let X be an n-generated left Banach A-module with generators $t_1, \dots, t_n$. Then $A \mathbin{\widehat{\otimes}}_A X$ is an n-generated left Banach A-module and every $y_1, \dots, y_n$ such that $\varkappa_X(y_i) = t_i$ for $i = 1, \dots, n$ are generators of $A \mathbin{\widehat{\otimes}}_A X$.*

Proof. It is sufficient to choose μ such that $\mu(t_i) = y_i$ and apply Lemma 1.9. ∎

Proposition 1.11. *Let X be a left Banach A-module. If the morphism $\varkappa_X : A \mathbin{\widehat{\otimes}}_A X \to X$ is surjective, then it is a cover.*

Proof. Suppose that $\varphi : Y \to X$ is a strict epimorphism and $\psi : Y \to A \mathbin{\widehat{\otimes}}_A X$ is a morphism such that $\varphi = \varkappa_X \psi$. Then there is a constant C and a map $\nu : X \to Y$ right inverse for φ and such that $\|\nu(x)\| \le C\|x\|$. Let us set $\mu = \psi\nu$. Lemma 1.9 implies that every $y \in A \mathbin{\widehat{\otimes}}_A X$ has the form $y = \sum_i a_i \cdot \mu(x_i)$, where $\sum \|a_i\| \, \|x_i\| < \infty$. Hence $y = \sum_i \psi(a_i \cdot \nu(x_i))$. This means that ψ is surjective, i.e. it is a strict epimorphism. ∎

I conclude this section with an example, which was the first motivation to consider projective covers in the Banach algebra context. Banach spaces F and E endowing with a non-trivial continuous bilinear functional $\langle \cdot, \cdot \rangle : F \times E \to \mathbb{C}$ form a *dual pair* of Banach spaces. The Banach space $E \mathbin{\widehat{\otimes}} F$ is a Banach algebra with respect to the multiplication

$$(e \otimes f) \cdot (e' \otimes f') := \langle f, e' \rangle (e \otimes f').$$

It is easy to see that E is a left Banach $E\widehat{\otimes}F$-module with the multiplication specified by $(e\otimes f)\cdot e' := \langle f, e'\rangle e$, and $F^\perp := \{e \in E : \langle F, e\rangle = 0\}$ is a closed submodule in E.

Proposition 1.12. *Let $(F, E, \langle\cdot,\cdot\rangle)$ be a dual pair of Banach spaces. Then the morphism of Banach $(E\widehat{\otimes}F)$-modules $E \to E/F^\perp$ is a strictly projective cover.*

Proof. Let Y be a submodule in E such that $Y \not\subset F^\perp$, i.e. there exists $e_0 \in Y \setminus F^\perp$. Then there is $f_0 \in F$ such that $\langle f_0, e_0\rangle = 1$. Therefore for every $e \in E$ we have $(e\otimes f_0)\cdot e_0 = e$. Hence $Y = E$. This yields that $F^\perp$ is the largest proper submodule in E. It follows from Lemma 1.8 that $E \to E/F^\perp$ is a cover.

Now fix $e \in E$ and $f \in F$ such that $\langle f, e\rangle = 1$. Then E is isometrically isomorphic to $(E\widehat{\otimes}F)(e\otimes f) = (E\widehat{\otimes}F)_+(e\otimes f)$. The latter module is strictly projective. This completes the proof. ∎

2. Strictly semi-perfect Banach algebras

Definition 2.1. We say that a Banach algebra A is

(A) *semi-perfect* if every irreducible left Banach A-module admits a projective cover;

(B) *strictly semi-perfect* if every irreducible left Banach A-module admits a strictly projective cover.

It will be shown later on (Corollary 2.7) that Condition (B) is equivalent to the similar condition for right modules, but the following question is open.

Question 2.2. Is it true that a Banach algebra is semi-perfect if and only if every irreducible right Banach module over it admits a projective cover?

The discussion on semi-perfect Banach algebras is in Sections 4 and 6. In this section and the next one we concentrate our attention on strictly semi-perfect Banach algebras.

Recall that a complex associative algebra is called *modular annihilator* if it is semi-prime and the right annihilator of every maximal modular left ideal is not trivial. For example, the algebras of compact operators, approximable operators, and nuclear operators in a Banach space are modular annihilator. The reader can find a detailed treatment on modular annihilator algebras in [13, Section 8.4].

Proposition 2.3. *If a semi-simple Banach algebra is strictly semi-perfect, then it is modular annihilator.*

Proof. Let A be a semi-simple, strictly semi-perfect Banach algebra, and let I be a maximal modular left ideal in A. Then $X := A/I$ is an irreducible left Banach module. Therefore there exists a strictly projective cover $\varepsilon\colon P \to X$.

(1) We claim that ε is an isomorphism. First, Corollary 1.4 implies that $P \oplus Q \cong A$ for some left Banach module Q. Let J be an arbitrary maximal modular left ideal in A. By Proposition 1.7, $\operatorname{Ker}\varepsilon$ is the largest proper submodule in P. Then $P \subset J$ or $J = \operatorname{Ker}\varepsilon \oplus Q$. Hence $\operatorname{Ker}\varepsilon \subset J$. It follows that $J \subset \operatorname{Rad} A$. Since $\operatorname{Rad} A = 0$, the strict epimorphism ε is injective; therefore it is an isomorphism.

(2) We have seen that X is strictly projective. Consequently, there exists a morphism $\varphi\colon X \to A$ right inverse for the projection $A \to X$. Let $b := \varphi(u+I)$, where u is a right modular identity for I. Since φ is injective, $b \neq 0$. Therefore for every $a \in I$

$$ab = \varphi(a \cdot (u + I)) = \varphi(au + I) = \varphi(a + I) = 0,$$

i.e., b is in the right annihilator of I. Hence A is modular annihilator. ∎

Corollary 2.4. *A unital, semi-simple, strictly semi-perfect Banach algebra is classically semi-simple.*

Proof. Proposition 2.3 says that a semi-simple, strictly semi-perfect Banach algebra is modular annihilator. By [13, Theorem 8.4.14] a unital, modular annihilator, Banach algebra is finite-dimensional. Finally, a semi-simple, finite-dimensional algebra is classically semi-simple. ∎

Proposition 2.5. (sf. [6, Lemma 27.16B]) *If I is a two-sided closed ideal in a strictly semi-perfect Banach algebra A, then A/I is strictly semi-perfect.*

Proof. Set $B := A/I$ and let X be an irreducible left Banach B-module. Then X is an irreducible left Banach A-module with respect to the natural multiplication. Since A is strictly semi-perfect, there exists a strictly projective cover $\varepsilon\colon P \to X$. Note that $I \cdot X = 0$; therefore

$$\varepsilon'\colon P/\overline{I \cdot P} \to X : y + \overline{I \cdot P} \mapsto \varepsilon(y),$$

is a well-defined morphism of Banach B-modules.

(1) We claim that $P/\overline{I\cdot P}$ is strictly projective. Suppose that

$$\psi\colon P/\overline{I\cdot P}\to Z$$

is a morphism and $\nu\colon Y\to Z$ is a strict epimorphism of Banach B-modules. Both morphisms can be considered as morphisms of A-modules. Since P is a strictly projective A-module, there exists a morphism of Banach A-modules χ such that the big triangle in the diagram

$$\begin{tikzcd} & & Y \arrow[d, "\nu"] \\ P \arrow[r] \arrow[urr, dashed, "\chi"] & P/\overline{I\cdot P} \arrow[r, "\psi"'] \arrow[ur, dashed, "\varphi"'] & Z \end{tikzcd} \tag{2.1}$$

is commutative. It follows from $I\cdot Y=0$ that there is a morphism of Banach B-modules $\varphi : P/\overline{I\cdot P}\to Y$ lifting ν. This means that $P/\overline{I\cdot P}$ is strictly projective.

(2) It remains to show that ε' is a cover. Suppose that $\varphi : Y\to X$ is a strict epimorphism and ψ is a morphism of Banach B-module such that the diagram

$$\begin{tikzcd} & P/\overline{I\cdot P} \arrow[d, "\varepsilon'"] \\ Y \arrow[ur, "\psi"] \arrow[r, "\varphi"'] & X \end{tikzcd}$$

commutes. By M denote the inverse image of $\operatorname{Im}\psi$ in P. By Proposition 1.7, $M\subset \operatorname{Ker}\varepsilon$ or $M=P$. The first case impossible because it implies $\psi=0$. So $M=P$; hence $\operatorname{Im}\psi=P/\overline{I\cdot P}$. This proves that ε' is a cover. ∎

Theorem 2.6. *A Banach algebra A is strictly semi-perfect if and only if $A/\operatorname{Rad}A$ is modular annihilator.*

Proof. ($\Rightarrow$) It follows immediately from Propositions 2.5 and 2.3.

($\Leftarrow$) Let X be an irreducible left Banach A-module. Denote the projection $A\to A/\operatorname{Rad}A$ by σ and set $B:=A/\operatorname{Rad}A$. It is clear that X is an irreducible left Banach B-module. Since B is modular annihilator, X is topologically isomorphic (as a B-module, hence, as an A-module) to Bq for some non-trivial idempotent $q\in B$ [13, Theorem 8.4.5(e)]. Every idempotent in a Banach algebra can be lifted modulo the radical [13, Proposition 4.3.12(d)].

Hence there is an idempotent $p \in A$ such that $\sigma(p) = q$. Obviously, Ap is a strictly projective left Banach A-module.

We claim that the restriction of σ to Ap considered as a map onto Bq is a cover. By Lemma 1.8, it suffices to show that Ap contains the largest proper submodule and this submodule is closed. Since Ap is a direct summand in A_+, it contains a maximal submodule, say Y. Then $Y \oplus A(1-p)$ is a maximal left ideal in A. This implies $\operatorname{Rad} A \subset Y \oplus A(1-p)$. Hence $Ap \cap \operatorname{Rad} A \subset Y$.

Now suppose towards the contradiction that $\sigma(Y) \neq 0$. Since Bq is irreducible, $\sigma(Y) = Bq$. Therefore there is $y \in Y$ such that $y - p \in \operatorname{Rad} A$. But $y - p \in Ap$; hence $y - p \in Y$. Finally, $p \in Y$; thus, $Y = Ap$. The contradiction with the maximality of Y shows that $Y \subset \operatorname{Ker} \sigma = \operatorname{Rad} A$. Consequently, $Y = \operatorname{Rad} A \cap Ap$. Thus, Ap contains a unique maximal left submodule and this submodule is closed. ∎

Corollary 2.7. *A Banach algebra is strictly semi-perfect if and only if every irreducible right Banach module over it admits a strictly projective cover.*

Proof. It follows from [13, Theorem 8.4.5] that a Banach algebra is modular annihilator if and only if the opposite Banach algebra is modular annihilator. So the statement follows immediately from Theorem 2.6. ∎

3. Finitely generated modules over strictly semi-perfect Banach algebras

Let A be a complex associative algebra (possibly, non-unital), and let X be a finitely generated left A-module with generators $t_1, \dots, t_n$. Consider the morphism of left Banach A-modules

$$\sigma\colon A^n \to X\colon (a_1, \dots, a_n) \mapsto \sum_{k=1}^{n} a_k \cdot t_k \qquad (a_k \in A)$$

and denote by K the kernel of σ. Consider the set $\mathrm{M}_n(A)$ of all $n \times n$-matrices with entries from A as an algebra with respect to the usual matrix multiplication.

Proposition 3.1. *Let K' be the subset in $\mathrm{M}_n(A)$ such that for every $(a_{jk}) \in K'$ the element $(a_{j1}, \dots, a_{jn}) \in K$ for $j = 1, \dots, n$. Then K' is a modular left ideal in $\mathrm{M}_n(A)$.*

Proof. Suppose that $(a_{jk}) \in K'$. Then $\sum_k a_{jk} \cdot t_k = 0$ for each $j = 1, \dots, n$. Therefore for every $(b_{ij}) \in \mathrm{M}_n(A)$ and every $i = 1, \dots, n$

$$\sigma\big(\sum_j b_{ij} a_{jk}\big) = \sum_{j,k} b_{ij} a_{jk} \cdot t_k = 0.$$

This means that the matrix $(b_{ij})(a_{jk}) = (\sum_j b_{ij} a_{jk})$ belongs to K'. Thus, K' is a left ideal.

Fix $u_{jk} \in A$ $(j, k = 1, \dots, n)$ such that $t_j = \sum_k u_{jk} \cdot t_k$ for every j. We claim that the matrix (u_{jk}) is a right modular identity for K'. Indeed, if $(b_{ij}) \in \mathrm{M}_n(A)$, then for each $i = 1, \dots, n$

$$\begin{aligned} \sigma\Big(\sum_j b_{ij} u_{j1} - b_{i1}, \, \dots, \, \sum_j b_{ij} u_{jn} - b_{in}\Big) &= \sum_k \Big(\sum_j b_{ij} u_{jk} - b_{ik}\Big) \cdot t_k \\ &= \sum_j b_{ij} \Big(\sum_k u_{jk} \cdot t_k\Big) - \sum_k b_{ik} \cdot t_k = 0, \end{aligned}$$

i.e., $(b_{ij})(u_{jk}) - (b_{ik}) \in K'$. ∎

We need a simple lemma.

Lemma 3.2. *A finite sum of minimal left ideals in a semi-prime Banach algebra is closed.*

Proof. It is easy to check by induction on the cardinality of a generating set that a finite sum of minimal left ideals a semi-prime Banach algebra A has the form Ap for some idempotent p. ∎

Proposition 3.3. *Let A be a semi-simple Banach algebra with dense socle, and let X be a finitely generated left Banach A-module. Then X is topologically isomorphic to $\bigoplus_{i=1}^m Ap_i$ for some minimal idempotents $p_1, \dots, p_m$ in A.*

Proof. Fix a submodule in A^n such that $X \cong A^n/K$. The set K' defined in Proposition 3.1 is a modular left ideal in $\mathrm{M}_n(A)$. We can treat $\mathrm{M}_n(A)$ as a Banach algebra with some multiplicative norm. For example, $\mathrm{M}_n(A)$ can be identified with $A \mathbin{\widehat{\otimes}} \mathrm{M}_n(\mathbb{C})$, where $\mathrm{M}_n(\mathbb{C})$ is endowed with the operator norm.

Denote by S the socle of A. Then $\mathrm{M}_n(S)$ is dense in $\mathrm{M}_n(A)$. It is not hard to see that a right modular identity $u = (u_{jk})$ for K' can be chosen in $\mathrm{M}_n(S)$

[3, Lemma 3.4]. The set $\{u_{jk}\colon j,k=1,\dots,n\}$ is contained in a finite sum of minimal left ideals in A. Denote this sum by I. Since

$$\Big(\sum_j a_j u_{j1}-a_1,\dots,\sum_j a_j u_{jn}-a_n\Big)\in K,\quad (a_j\in A^n),$$

we have $A^n=I^n+K$. By Lemma 3.2 I^n is a Banach A-module. Obviously, $X\cong A^n/K\cong I^n/(I^n\cap K)$. The module I^n is semi-simple (i.e. a sum of simple modules); hence all its quotient modules are semi-simple. Therefore X is topologically isomorphic to a semi-simple, finitely generated Banach module. Since the radical of A is trivial and the socle is dense, A is modular annihilator [13, Proposition 8.7.2(c)]. Hence for every simple summand in X there is an algebraic isomorphism onto Ap_i for some minimal idempotent p_i $(i=1,\dots,m)$ [13, Theorem 8.4.5(e)]. Since Ap_i is closed, this isomorphism is topological [9, Corollary 6.2.10]. Thus, $X\cong\bigoplus_{i=1}^m Ap_i$ topologically. ∎

In the following proposition we need only algebraic properties of modular annihilator algebras — the existence of any norm on A is not assumed.

Proposition 3.4. *Let A be a modular annihilator algebra, and let S be the socle of A. Then every finitely generated left A-module is finitely generated as a left S-module.*

Proof. Suppose that X is a finitely generated left A-module. Then $X/(S\cdot X)$ is finitely generated also. Further, $X/(S\cdot X)$ is finitely generated as an A/S-module. Since A is a modular annihilator, the algebra A/S is radical [13, Theorem 8.4.5(c)]. It easy to see that each finitely generated module over a radical algebra is trivial. Therefore $X=S\cdot X$. This proves that X is a finitely generated S-module. ∎

Lemma 3.5. *Let I be a left ideal in an algebra A, and let X is a left A-module finitely generated as an I-module. Then every morphism of I-modules from X to every A-module is a morphism of A-modules.*

The proof is straightforward.

In the proof of the following theorem we need the notions of a small submodule and the radical of a module. A submodule X_0 in a module X is called *small* if for a submodule Y in X the equality $X_0+Y=X$ implies $Y=X$. The *radical* of a module X is the intersection of all its maximal submodule or, equivalently, the sum of all its small submodules [11, 9.1.1, 9.1.2]. The notation is $\operatorname{rad}X$.

THEOREM 3.6. *Let A be a Banach algebra such that $A/\operatorname{Rad} A$ is modular annihilator. Then every finitely generated left (right) Banach A-module admits a strictly projective cover.*

Proof. Suppose that X is a finitely generated left Banach A-module. Let $R := \operatorname{Rad} A$. Then $X/\overline{R\cdot X}$ is a finitely generated left Banach A/R-module. Denote the socle of A/R by S. By Proposition 3.4, $X/\overline{R\cdot X}$ a finitely generated S-module; therefore it is a finitely generated Banach $\overline{S}$-module. Since $\overline{S}$ is a two-sided ideal in the semi-simple algebra A/R, it is also semi-simple. It follows from Proposition 3.3 that there are minimal idempotents $q_1,\dots,q_m$ in $\overline{S}$ such that $X/\overline{R\cdot X}$ and $\bigoplus_{i=1}^m \overline{S}q_i$ are topologically isomorphic as $\overline{S}$-modules. It is obvious that $\overline{S}q_i = (A/R)q_i$ for every i. Lemma 3.5 implies that $X/\overline{R\cdot X}$ and $\bigoplus_{i=1}^m (A/R)q_i$ are isomorphic as Banach A/R-modules. Denote by φ the composition of the natural map

$$\bigoplus_{i=1}^m Ap_i \to \bigoplus_{i=1}^m (A/R)q_i$$

and the isomorphism between the latter module and $X/\overline{R\cdot X}$.

By [13, Proposition 4.3.12(d)], there exist idempotents $p_i \in A$ such that $q_i = p_i + R$. The Banach A-module $P := \bigoplus_{i=1}^m Ap_i$ is strictly projective. Hence there is a morphism ψ of left Banach modules such that the diagram

$$\begin{tikzcd} & X \arrow[d] \\ P \arrow[ur, "\psi"] \arrow[r, "\varphi"'] & X/\overline{R\cdot X} \end{tikzcd}$$

commutes. Since Ap_i is finitely generated, $Rp_i = R\cdot(Ap_i)$ is a small submodule in Ap_i [11, Theorem 9.2.1(d)]. Therefore

$$\operatorname{Ker}\varphi = \bigoplus_{i=1}^m Rp_i \subset \operatorname{rad} P.$$

Since P is finitely generated, $\operatorname{rad} P$ is a small submodule in P [11, Theorem 9.2.1(c)]. Obviously, $\operatorname{Ker}\psi \subset \operatorname{Ker}\varphi$. Thus, $\operatorname{Ker}\psi$ is contained in a small submodule; hence it is small itself. Therefore ψ is a cover in the category of pure A-modules; whence it is a cover in the category of Banach A-modules.

The proof of the right case is similar. ∎

Combining Theorems 2.6 and 3.6, and using the obvious fact that an irreducible module is finitely generated, we obtain the following result.

THEOREM 3.7. *The following are equivalent for a Banach algebra A:*

(i) *A is strictly semi-perfect;*
(ii) *every cyclic left (right) Banach A-module admits a strictly projective cover;*
(iii) *every finitely generated left (right) Banach A-module admits a strictly projective cover.*

4. SEMI-PERFECT BANACH ALGEBRAS AND THE APPROXIMATION PROPERTY

Remind that a Banach space E has the *approximation property* if for every compact subset K in E and every $\varepsilon > 0$ there exists a bounded linear operator $\varphi\colon E \to E$ of finite rank such that

$$\|\varphi(x) - x\| < \varepsilon \quad \text{for all } x \in K.$$

PROPOSITION 4.1. *If a semi-simple Banach algebra with the approximation property is semi-perfect, then it is modular annihilator.*

Proof. Let A be a semi-simple, semi-perfect Banach algebra with the approximation property, and let X be a irreducible left Banach A-module. Then there exists a projective cover $\varepsilon\colon P \to X$. Assume that $\operatorname{Ker}\varepsilon \neq 0$ and fix some non-trivial $y \in \operatorname{Ker}\varepsilon$. Since A has the approximation property, and P is projective, there exists $\chi \in {}_A\mathrm{h}(P, A_+)$ such that $\chi(y) \neq 0$ [8, Corollary IV.4.5]. By Proposition 1.7, $\operatorname{Ker}\varepsilon = \operatorname{rad} P$. On the other hand, $\chi(\operatorname{rad} P) \subset \operatorname{Rad} A_+$ [11, 9.1.4(a)]. But by assumption $\operatorname{Rad} A_+ = \operatorname{Rad} A = 0$; therefore $\chi(y) = 0$. This contradiction implies that ε is an isomorphism. So we see that every irreducible left Banach A-module is projective. Since A has the approximation property it follows from [16, Corollary 4.39] that A is modular annihilator. ∎

Note that the second part of the proof of Proposition 2.3 is just a simple form of the argument from [16, Corollary 4.39] cited in the previous proof.

PROPOSITION 4.2. *If I is a two-sided closed ideal in a semi-perfect Banach algebra A, then A/I is semi-perfect.*

Proof. The argument is exactly as in the proof of Proposition 2.5. We need only to assume that the epimorphism ν in Diagram (2.1) is admissible. ∎

THEOREM 4.3. *Let A be a semi-perfect Banach algebra such that $A/\operatorname{Rad} A$ has the approximation property. Then A is strictly semi-perfect.*

Proof. It follows from Propositions 4.2 and 4.1 that $A/\operatorname{Rad} A$ is modular annihilator. Theorem 2.6 implies that A is strictly semi-perfect. ∎

So we get a new proof of Selivanov's theorem: *if A is a Banach algebra such that $A/\operatorname{Rad} A$ has the approximation property and every left Banach A-module is projective, then A is classically semi-simple.* Indeed, since A_+/A is projective, A is unital. Thus, Theorem 4.3 and Corollary 2.4 can be applied to $A/\operatorname{Rad} A$. It follows that A/Rad is finite-dimensional; whence $A \to A/\operatorname{Rad} A$ is admissible. Since the left module $A/\operatorname{Rad} A$ is projective, $\operatorname{Rad} A$ has the right identity; therefore $\operatorname{Rad} A = 0$.

QUESTION 4.4. Does there exist a semi-perfect Banach algebra that is not strictly semi-perfect?

This question can be considered as a general form of an old problem in Topological Homology [10, Problem 6]: *Does exist a semi-simple Banach algebra that is not classically semi-simple and such every left Banach module (or every irreducible left Banach module) over it is projective?*

5. PROJECTIVE COVERS OF FINITELY GENERATED MODULES WITH THE APPROXIMATION PROPERTY

The following assertion is [15, Lemma 1].

THEOREM 5.1. *A Banach space E has the approximation property if and only if for every Banach space E_1, every compact subset K in $E_1 \mathbin{\widehat{\otimes}} E$, and $\varepsilon > 0$ there exist a bounded linear operator $\varphi\colon E \to E$ of finite rank such that $\|(1 \otimes \varphi)(u) - u\| < \varepsilon$ for all $u \in K$.*

This result is the basis for the following approximation theorem: *If X is a projective left Banach A-module with the approximation property, then 1_X can be approximated uniformly on compact subsets of X by morphisms of the form*

$$y \mapsto \sum_{i=1}^{k} \chi_i(y) \cdot x_i, \quad \textit{where } \chi_i \in {}_A\mathrm{h}(X, A_+) \textit{ and } x_i \in X,$$

[15, Theorem 1]. (Here ${}_A\mathrm{h}(\cdot,\cdot)$ is the set of morphisms of left Banach A-modules.) Applied to finite subsets, this statement, in turn, gives a very useful theorem: *If X is a finitely generated projective left Banach A-module with the approximation property, then X is strictly projective* [15, Theorem 2].

We need a modification of these results. For a left Banach module X consider the morphism of left Banach modules

$$\hat\pi_X\colon A\mathbin{\widehat{\otimes}}X\to A\mathbin{\widehat{\otimes}}_A X\colon a\otimes x\mapsto a\otimes_A x.$$

THEOREM 5.2. *Let X be a left Banach A-module with the approximation property. Suppose that there exists a morphism of left Banach modules*

$$\hat\rho_X\colon A\mathbin{\widehat{\otimes}}_A X\to A\mathbin{\widehat{\otimes}}X$$

such that $\hat\pi_X\hat\rho_X=1$. Then for every compact subset K in $A\mathbin{\widehat{\otimes}}_A X$ and every $\varepsilon>0$ there are $\chi_1,\dots,\chi_n\in {}_A\mathrm{h}(A\mathbin{\widehat{\otimes}}_A X,A)$ and $x_1,\dots,x_n\in X$ such that

$$\Bigl\|y-\sum_{i=1}^n\chi_i(y)\otimes_A x_i\Bigr\|<\varepsilon\quad\text{for all } y\in K.$$

Proof. Denote by K_1 the compact subset $\hat\rho_X(K)$ in $A\mathbin{\widehat{\otimes}}X$. By Theorem 5.1, there exist a bounded linear operator $\varphi\colon X\to X$ of finite rank such that $\|u-(1\otimes\varphi)(u)\|<\varepsilon$ for all $u\in K_1$. Since $\|\hat\pi_X\|\le 1$, for every $y\in K$ we have

$$\|y-\hat\pi_X(1\otimes\varphi)\hat\rho_X(y)\|\le\|\hat\rho_X(y)-(1\otimes\varphi)\hat\rho_X(y)\|<\varepsilon. \tag{5.1}$$

The operator φ has the form $\varphi(x)=\sum_{i=1}^n f_i(x)x_i$, where $f_i\in X^*$ and $x_i\in X$. Whence,

$$\hat\pi_X(1\otimes\varphi)\hat\rho_X(y)=\sum_{i=1}^n\chi_i(y)\otimes_A x_i,$$

where $\chi_i=(1\otimes f_i)\hat\rho_X$. To conclude the proof, it remains to apply (5.1). ∎

THEOREM 5.3. *Let X be a finitely generated left Banach A-module with the approximation property. Suppose that there exists a morphism of left Banach A-modules $\hat\rho_X\colon A\mathbin{\widehat{\otimes}}_A X\to A\mathbin{\widehat{\otimes}}X$ such that $\hat\pi_X\hat\rho_X=1$. Then $A\mathbin{\widehat{\otimes}}_A X$ is strictly projective.*

Proof. Let $t_1,\dots,t_m$ be generators of X. Choose $y_1,\dots,y_m\in A\mathbin{\widehat{\otimes}}_A X$ such that $\varkappa_X(y_j)=t_j$ for all i. Consider the morphism of Banach modules

$$\sigma\colon A^m\to A\mathbin{\widehat{\otimes}}_A X\colon(a_1,\dots,a_m)\mapsto\sum_{j=1}^m a_j\cdot y_j. \tag{5.2}$$

It follows from Corollary 1.10 that σ is surjective. The open mapping theorem implies that there is a constant $C>0$ such that for every $y\in A\mathbin{\widehat{\otimes}}_A X$ there exist $a_1,\dots,a_m\in A$ with $y=\sum_j a_j\cdot y_j$ and $\sum_j\|a_j\|\le C\|y\|$. By Theorem 5.2 we can choose $\chi_1,\dots,\chi_n\in {}_A\mathrm{h}(A\mathbin{\widehat{\otimes}}_A X,A)$ and $x_1,\dots,x_n\in X$ such that

$$\Big\|y_j-\sum_{i=1}^n\chi_i(y_j)\otimes_A x_i\Big\|<1/C$$

for all $j=1,\dots,m$.

Take an arbitrary element y in $A\mathbin{\widehat{\otimes}}_A X$ and write it as $y=\sum_j a_j\cdot y_j$ with $\sum_j\|a_j\|\le C\|y\|$. Then

$$\begin{aligned}\Big\|y-\sum_{i=1}^n\chi_i(y)\otimes_A x_i\Big\|&\le\sum_{j=1}^m\|a_j\|\,\Big\|y_j-\sum_{i=1}^n\chi_i(y_j)\otimes_A x_i\Big\|\\&<\frac{1}{C}\sum_{j=1}^m\|a_j\|\le\Big\|\sum_{j=1}^m a_j\cdot y_j\Big\|=\|y\|.\end{aligned}\tag{5.3}$$

Let ψ takes y to $\sum_{i=1}^n\chi_i(y)\otimes_A x_i$. Since (5.3) holds for every y, we have $\|1-\psi\|<1$. Consequently, ψ is an invertible element in the Banach algebra ${}_A\mathrm{h}(A\mathbin{\widehat{\otimes}}_A X,A\mathbin{\widehat{\otimes}}_A X)$ [9, Proposition 1.2.39].

Now consider the morphisms of Banach modules

$$\begin{aligned}&\tau\colon A\mathbin{\widehat{\otimes}}_A X\to A^n\colon y\mapsto(\chi_1\psi^{-1}(y),\dots,\chi_n\psi^{-1}(y)),\\&\sigma'\colon A^n\to A\mathbin{\widehat{\otimes}}_A X\colon(b_1,\dots,b_n)\mapsto\sum_{i=1}^n b_i\otimes_A x_i.\end{aligned}$$

Note that

$$\sigma'\tau(y)=\sum_{i=1}^n\chi_i\psi^{-1}(y)\otimes_A x_i=\psi\psi^{-1}(y)=y.$$

Thus, the left Banach A-module $A\mathbin{\widehat{\otimes}}_A X$ is topologically isomorphic to a direct summand in A^n.

Further, write $x_i=\sum_j c_{ij}\cdot t_j$ for each $i=1,\dots m$, and set

$$\theta\colon A^n\to A^m\colon(b_1,\dots,b_n)\mapsto\Big(\sum_i b_ic_{i1}\,,\ \dots\ ,\ \sum_i b_ic_{im}\Big).$$

Since $c \cdot y = c \otimes_A \varkappa_X(y)$ for every $c \in A$ and $y \in A \mathbin{\widehat{\otimes}}_A X$, we obtain

$$\begin{aligned}\sigma\theta(b_1, \dots, b_n) &= \sum_{j,i} b_i c_{ij} \cdot y_j = \sum_{j,i,k} b_i c_{ij} \otimes_A \varkappa_X(y_j) \\ &= \sum_{j,i} b_i c_{ij} \otimes_A t_j = \sum_{j,i} b_i \otimes_A c_{ij} \cdot t_j \\ &= \sum_{i} b_i \otimes_A x_i = \sigma'(b_1, \dots, b_n).\end{aligned}$$

Therefore $\theta\tau$ is a right inverse morphism for σ. Since (5.2) defines a morphism from A_+^m to $A \mathbin{\widehat{\otimes}}_A X$ that extends σ, the latter module is topologically isomorphic to a direct summand in the strictly projective module A_+^m. Thus, $A \mathbin{\widehat{\otimes}}_A X$ is strictly projective.

Corollary 5.4. *Under the conditions of Theorem* 5.3, *we have that* $\varkappa_X \colon A \mathbin{\widehat{\otimes}}_A X \to X$ *is a strictly projective cover.*

The proof is immediate from Proposition 1.11 and Theorem 5.3.

6. Modules over quasi-biprojective Banach algebras

A Banach algebra A is called *quasi-biprojective* if $\overline{A^2} = A$ and there exists a morphism of Banach A-bimodules $\hat{\rho}$ that is right inverse for

$$\hat{\pi} \colon A \mathbin{\widehat{\otimes}} A \to A \mathbin{\widehat{\otimes}}_A A \colon a \otimes b \mapsto a \otimes_A b$$

[17, 18]. It is shown in [18, Theorem 3.13] that the following Banach algebras are quasi-biprojective:

- ℓ_p for $p \geq 1$;
- the Shatten ideals S_p for $p \in [1, 2]$;
- $L^p(G)$ for $p \geq 1$, the continuous function space $C(G)$, and the Fourier algebra $A(G)$, where G is a compact group, all with respect to the convolution product.

Now set

$$\varkappa \colon A \mathbin{\widehat{\otimes}}_A A \to A \colon a \otimes_A b \mapsto ab.$$

Proposition 6.1. *Let A a Banach algebra such that $\overline{A^2} = A$, and let X be a left Banach A-module such that $\varkappa_X \colon A \mathbin{\widehat{\otimes}}_A X \to X$ is surjective. Then*

$$\varkappa \otimes_A 1 \colon A \mathbin{\widehat{\otimes}}_A A \mathbin{\widehat{\otimes}}_A X \to A \mathbin{\widehat{\otimes}}_A X$$

is a topological isomorphism.

Proof. First, note that $\varkappa \otimes_A 1$ is surjective. This is just the projectivity property of $\widehat{\otimes}_A$: since $\varkappa_X$ is a strict epimorphism, then $\varkappa \otimes_A 1 = 1 \otimes_A \varkappa_X$ is a strict epimorphism.

Second, to prove that $\varkappa \otimes_A 1$ is injective it is sufficient to show that $(\varkappa \otimes_A 1)^*$ is surjective. We can identify $(A \mathbin{\widehat{\otimes}_A} X)^*$ and $(A \mathbin{\widehat{\otimes}_A} A \mathbin{\widehat{\otimes}_A} X)^*$ with the spaces of corresponding multilinear functionals. So we need to check that for every continuous trilinear functional $f \colon A \times A \times X \to \mathbb{C}$ that is balanced, i.e.

$$f(ab, c, x) = f(a, bc, x) = f(a, b, c \cdot x) \quad \text{for all } a, b, c \in A \text{ and } x \in X,$$

there exists a continuous balanced bilinear functional $g \colon A \times X \to \mathbb{C}$ such that

$$f(a, b, x) = g(ab, x) \quad \text{for all } a, b \in A \text{ and } x \in X. \tag{6.1}$$

Since $\varkappa_X$ is surjective, it follows from [8, Propostition II.3.6] that for $x \in X$ there are sequences $(c_j) \subset A$ and $(x_j) \subset X$ such that $x = \sum_{j=1}^{\infty} c_j \cdot x_j$ and $\sum_i \|c_j\| \, \|x_j\| < \infty$. If $x = \sum_{j=1}^{m} c'_j \cdot x'_j$ is another such decomposition, then

$$\sum_j f(ab, c_j, x_j) = f(a, b, x) = \sum_j f(ab, c'_j, x'_j)$$

for all $a, b \in A$. Since $\overline{A^2} = A$, the sum $\sum_j f(a, c_j, x_j)$ does not depend on the decomposition of x for every $a \in A$. So

$$g(a, x) := \sum_j f(a, c_j, x_j), \qquad (a \in A,\ x \in X)$$

is well defined. It is easy to see that $g \colon A \times X \to \mathbb{C}$ is bilinear, balanced (i.e. $g(ab, x) = g(a, b \cdot x)$ for all $a, b \in A$ and $x \in X$), and (6.1) is satisfied. By the open mapping theorem, there exists a constant $C > 0$ such that for every $\varepsilon > 0$ the decomposition of x can be chosen with $\sum_i \|c_j\| \, \|x_j\| \le C\|x\| + \varepsilon$. This implies that the functional g is continuous. ∎

THEOREM 6.2. *Let A be a quasi-biprojective Banach algebra, and let X be a left Banach A-module such that $\varkappa_X \colon A \mathbin{\widehat{\otimes}_A} X \to X$ is surjective. Then*
(A) *the left Banach A-module $A \mathbin{\widehat{\otimes}_A} X$ is projective;*
(B) *the morphism*

$$\hat{\pi}_X \colon A \mathbin{\widehat{\otimes}} X \to A \mathbin{\widehat{\otimes}_A} X \colon a \otimes x \mapsto a \otimes_A x$$

admits a right inverse morphism of left Banach A-modules.

Proof. (A) Let $\hat{\rho}$ be the morphism from the definition of a quasi-biprojective algebra. By Proposition 6.1, $\varkappa \otimes_A 1$ is a topological isomorphism. Therefore

$$\rho' := (\hat{\rho} \otimes_A 1)(\varkappa \otimes_A 1)^{-1} \tag{6.2}$$

is a right inverse morphism of left Banach modules for

$$(\pi \otimes_A 1)\colon A \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}}_A X \to A \mathbin{\widehat{\otimes}}_A X,$$

where π is the morphism associated with the multiplication. Therefore the module $A \mathbin{\widehat{\otimes}}_A X$ is projective.

(B) Let ρ' be a right inverse for $\pi_A \otimes_A 1$; for example, it can be defined by (6.2). Since $\pi_A \otimes_A 1 \cong \hat{\pi}_X(1 \mathbin{\widehat{\otimes}} \varkappa_X)$, the morphism $(1 \mathbin{\widehat{\otimes}} \varkappa_X)\rho'$ is right inverse for $\hat{\pi}_X$. ∎

Now we can get the main result of this section.

THEOREM 6.3. *Suppose that A is a quasi-biprojective Banach algebra and X is a left Banach A-module such that*

$$\varkappa_X \colon A \mathbin{\widehat{\otimes}}_A X \to X \colon a \otimes_A x \mapsto a \cdot x$$

is surjective. Then $\varkappa_X$ is a projective cover.

Proof. It follows immediately from Theorem 6.2(A) and Proposition 1.11. ∎

COROLLARY 6.4. *A quasi-biprojective Banach algebra is semi-perfect.*

Proof. It suffices to note that for a finitely generated Banach A-module X the morphism $\varkappa_X$ is surjective. ∎

Theorems 5.3 and 6.2 allow us to get a modification of Theorem 4.3 in the case where algebra is quasi-biprojective. For the proof we need the following assertions.

THEOREM 6.5. [3, Theorem 5.3(A)] *Let A be a Banach algebra such that $\overline{A^2} = A$. Suppose that there exists a closed two-sided ideal R_0 contained in* $\operatorname{Rad} A$, *containing the prime radical, and such that A/R_0 has the approximation property. If X is an irreducible left Banach A-module such that $A \mathbin{\widehat{\otimes}}_A X$ is projective, then X has the approximation property.*

THEOREM 6.6. *Suppose that the conditions of Theorem 6.5 are satisfied for the algbera A. If X is an irreducible left Banach A-module such that there exist a morphism of left Banach A-modules $\hat{\rho}_X : A\widehat{\otimes}_A X \to A\widehat{\otimes} X$ right inverse for $\hat{\pi}_X$, then $\varkappa_X$ is a strictly projective cover.*

Proof. By Theorem 6.5, X has the approximation property. So we can apply Corollary 5.4. ∎

Now we can prove the last result of this section.

THEOREM 6.7. *Let A be a quasi-biprojective Banach algebra. Suppose that there exists a closed two-sided ideal R_0 contained in* $\operatorname{Rad} A$, *containing the prime radical, and such that A/R_0 has the approximation property; then A is strictly semi-perfect.*

Proof. It follows from Theorem 6.2(B) that for each irreducible left Banach A-module X there exists a morphism of left Banach A-modules $\hat{\rho}_X$ such that $\hat{\pi}_X\hat{\rho}_X = 1$. So we can apply Theorem 6.6, which states that $\varkappa_X : A\widehat{\otimes}_A X \to X$ is a strictly projective cover. ∎

In view of this theorem and Corollary 6.4 the following question is natural.

QUESTION 6.8. Is every quasi-biprojective Banach algebra strictly semi-perfect?

7. Examples from Harmonic Analysis

A *Kac algebra* is a quadruple $(M, \Delta, \varkappa, \varphi)$, where M is a von Neumann algebra, Δ is a co-associative normal $*$-homomorphism from M to its spatial tensor square, $\varkappa : M \to M$ is a co-involution, and φ is a weight on M, that satisfies to certain axioms [5, 2.2.5]. It is well known that the predual space of M is a Banach algebra with respect to the multiplication $(\omega * \theta)(x) := (\omega \otimes \theta)\Delta(x)$ $(\omega, \theta \in M_*,\ x \in M)$. Before to give a necessary and sufficient condition for M_* to be strictly semi-perfect, we need the following observation.

LEMMA 7.1. *Let $(M, \Delta, \varkappa, \varphi)$ be a Kac algebra. Then the algebra M_* is semi-simple.*

Proof. It follows from the definition of $\varkappa$ [5, 2.2.5] that $\varkappa_* : M_* \to M_*$ is an isometric involution. To prove that M_* is semi-simple it suffices to show that there exists an injective $*$-representation of M_* [9, Theorems 4.5.20, 4.5.35]. But it is easy to see that the Fourier representation of M_* (as defined in [5, 2.5.3]) is injective. ∎

A Kac algebra is said to be *of compact type* if there is a normal state F on M that is left invariant in the sense that

$$(\omega \otimes F)\Delta(x) = \omega(1)F(x)$$

for all $\omega \in M_*$ and $x \in M$; in this case, F coincides with φ up to a scalar; for the details and the discussion see [2].

THEOREM 7.2. *A Kac algebra $(M, \Delta, \varkappa, \varphi)$ is of compact type if and only if the Banach algebra M_* is strictly semi-perfect.*

Proof. ($\Rightarrow$) If $(M, \Delta, \varkappa, \varphi)$ is of compact type, then all irreducible $*$-representations of M_* is finite-dimensional. For an irreducible $*$-representation α denote its dimension by d_α. Let $\{u_{ij}^\alpha\} \subset M$ be the set of coefficients of irreducible $*$-representations. (Here α runs over all equivalence classes of irreducible $*$-representations of M_* and $i, j = 1, \ldots, d_\alpha$.) Then

$$\Delta(u_{ij}^\alpha) = \sum_k u_{ik}^\alpha \otimes u_{kj}^\alpha. \tag{7.1}$$

The functional $x \mapsto \varphi(x u_{ij}^\alpha)$ belongs to M_*; denote it by $\hat{u}_{ij}^\alpha$. Let S be the linear span of $\{\hat{u}_{ij}^\alpha\}$.

Let $x \in M$ and $\omega \in M_*$. By the definition of a Kac algebra [5, 2.2.1],

$$(1 \otimes \varphi)\big(\Delta(x)(1 \otimes u_{ij}^\alpha)\big) = (\varkappa \otimes \varphi)\big((1 \otimes x)\Delta(u_{ij}^\alpha)\big).$$

Combining this with (7.1), we have

$$\begin{aligned}(\omega * \hat{u}_{ij}^\alpha)(x) &= (\omega \otimes \varphi)\big(\Delta(x)(1 \otimes u_{ij}^\alpha)\big) \\ &= (\omega\varkappa \otimes \varphi)\big((1 \otimes x)\Delta(u_{ij}^\alpha)\big) = \sum_k \omega\varkappa(u_{ik}^\alpha)\hat{u}_{kj}^\alpha(x).\end{aligned}$$

Thus, S is a left ideal in M_*.

Since S is dense in M_* and $\hat{u}_{ij}^\alpha * \hat{u}_{kl}^\beta = d_\alpha^{-1}\delta_{\alpha\beta}\delta_{jk}\hat{u}_{il}^\alpha$ (see, for example, [4, Proposition 2.2]), the socle of M_* is dense (in fact, S is the socle). By [13, Proposition 8.7.2(c)], M_* is modular annihilator. Lemma 7.1 and Theorem 2.6 imply that M_* is strictly semi-perfect.

($\Leftarrow$) Denote by I_0 the kernel of $\tau : M_* \to \mathbb{C} : \omega \mapsto \omega(1)$. It is easy to see that τ is an algebra homomorphism. Therefore I_0 is a modular ideal. If M_* is strictly semi-perfect, then, by Lemma 7.1 and Proposition 2.3, it is modular annihilator. It follows from [13, Proposition 8.4.3(b)] that there is a minimal idempotent $p \in M_*$ such that $I_0 p = 0$. Since M_*/I_0 is one-dimensional, $M_* = I_0 \oplus M_* p$. Whence I_0 is a complemented left ideal in M_*. By [2, Proposition 1.2(1), Theorem 2.3] $(M, \Delta, \varkappa, \varphi)$ is of compact type. ∎

Since the convolution algebra $L^1(G)$ and the Fourier algebra $A(G)$ of a locally compact group are preduals of Kac algebras, we get the following result.

Corollary 7.3. *Let G be a locally compact group.*

(A) *The Banach algebra $L^1(G)$ is strictly semi-perfect if and only if G is compact.*

(B) *The Banach algebra $A(G)$ is strictly semi-perfect if and only if G is discrete.*

Both statements in the corollary can be proved without a reference to Kac algebras; but the argument is the same as in Theorem 7.2.

Let G be a locally compact group, and let $p \in (1, \infty)$. Denote the Banach space of nuclear operators on $L^p(G)$ by $\mathcal{N}^p(G)$. In [12], Neufang introduced a "non-commutative convolution" on $\mathcal{N}^p(G)$ and proved that $\mathcal{N}^p(G)$ is a Banach algebra with respect to this convolution. Some homological property of $\mathcal{N}^p(G)$ are studied in [14].

Theorem 7.4. *The Banach algebra $\mathcal{N}^p(G)$ is strictly semi-perfect if and only if G is compact.*

Proof. ($\Rightarrow$) It follows from [12, Satz 5.3.4] that there is a closed two-sided ideal I in $\mathcal{N}^p(G)$ such that $\mathcal{N}^p(G)/I \cong L^1(G)$. By Proposition 2.5, if $\mathcal{N}^p(G)$ is strictly semi-perfect, then $L^1(G)$ is strictly semi-perfect. Corollary 7.3(A) implies that G is compact.

($\Leftarrow$) By [12, Satz 5.3.4], $\mathcal{N}^p(G)\, I = 0$. Hence I is a radical algebra. Since $L^1(G)$ is semi-simple, $I = \operatorname{Rad} \mathcal{N}^p(G)$ [13, Theorem 4.3.2(c)]. If G is compact, then, by Corollary 7.3(A), $L^1(G)$ is strictly semi-perfect. Applying Theorem 2.6 and Proposition 2.3 we obtain that $\mathcal{N}^p(G)$ is strictly semi-perfect. ∎

Remark 7.5. In all results of this section "strictly semi-perfect" can be replaced to "semi-perfect". We just have to make a little change in the proof of sufficiency of Theorem 7.2 and use Proposition 4.2 instead of Proposition 2.5 in the proof of necessity in Theorem 7.4.

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