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On Generalized d'Alembert Functional Equation

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Let G be a locally compact group. Let σ be a continuous involution of G and let μ be a complex bounded measure. In this paper we study the generalized d'Alembert functional equation

$$\mathrm{D}(\mu) \qquad \int_G f(xty)\mathrm{d}\mu(t) + \int_G f(xt\sigma(y))\mathrm{d}\mu(t) = 2f(x)f(y)\,, \qquad x, y \in G\,,$$

where $f : G \to \mathbb{C}$ to be determined is a measurable and essentially bounded function.

We give some conditions under which all solutions are of the form

$$\frac{\prec \pi(x)\xi, \zeta \succ + \prec \pi(\sigma(x))\xi, \zeta \succ}{2}\,,$$

where $(\pi, \mathcal{H})$ is a continuous unitary representation of G such that $\pi(\mu)$ is of rank one and $\xi, \zeta \in \mathcal{H}$. Furthermore, we also consider the case when f is an integrable solution. In the particular case where G is a connected Lie group, we reduce the solution of $\mathrm{D}(\mu)$ to a certain problem in operator theory. We prove that the solutions of $\mathrm{D}(\mu)$ are exactly the common eigenfunctions of some operators associated to a left invariant differential operators on G.

Key words: Functional equation, Gelfand measure, μ-spherical function, positive denite function, representation theory, Lie group, invariant differential operator.

1. Introduction

The classical d'Alembert's functional equation has the form

$$f(xy) + f(xy^{-1}) = 2f(x)f(y)\,, \tag{1}$$

where x and y run over a group G and $f : G \to \mathbb{C}$ is the unknown function.

At present the theory of d'Alembert functional equation is extensively developed. The monographs by Aczél and Dhombres [1] have references and detailed discussions.

The basic result for the study of equation (1) is a result obtained by Kannappan [9]. It says that every nonzero continuous solution f of d'Alembert's functional equation (1) which satisfies the condition $f(xyz) = f(yxz)$ for all $x, y \in G$ has the form

$$f(x) = \frac{\chi(x) + \chi(x^{-1})}{2}\,, \qquad x \in G\,, \tag{2}$$

where $\chi : G \to \mathbb{C}$ is a nonzero continuous multiplicative function of G. In particular, all nonzero bounded and continuous solution of equation (1) on a locally compact abelian group G are represented by the formula $f(x) = \mathrm{Re}(\chi(x))$ with some continuous character χ of the group G (see Ger [7]).

As a continuation of these investigations, in the present paper we are going to study the following functional equation

$$\int_G f(xty)\mathrm{d}\mu(t) + \int_G f(xt\sigma(y))\mathrm{d}\mu(t) = 2f(x)f(y)\,, \qquad x, y \in G\,, \tag{3}$$

where μ is a complex bounded measure and σ is an involution of G. The equation (3) is our generalization of d'Alembert's functional equation (1) in which $\mu = \delta_e$: the Dirac measure concentrated at the identity element of G.

2. Notation and preliminaries

Throughout this paper, G will be a Hausdorff topological locally compact group; $M(G)$ denotes the Banach algebra of complex bounded measures, it is the dual of $C_0(G)$ the Banach space of continuous functions vanishing at infinity; $C(G)$ (resp. $C_b(G)$) designates the space of continuous (resp. continuous and bounded) complex valued functions; σ denotes a continuous involution of G (i.e., $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$).

Given $\mu \in M(G)$, we say that μ is σ-invariant, and we write $\mu = \sigma(\mu)$, if

$$\prec \mu, f \circ \sigma \succ = \prec \mu, f \succ \qquad \text{for all } f \in C_b(G)\,,$$

where

$$\prec \mu, f \succ = \int_G f(x)\mathrm{d}\mu(x)\,.$$

For all $\mu, \nu \in M(G)$, we recall that the convolution $\mu * \nu$ is the measure given by

$$\prec \mu * \nu, f \succ := \int_G \int_G f(ts)\mathrm{d}\mu(t)\mathrm{d}\nu(s)\,, \qquad f \in C_b(G)\,,$$

and the involution is defined in $M(G)$ by

$$\mu^* = \check{\bar{\mu}}\,,$$

where $\prec \bar{\mu}, f \succ = \overline{\prec \mu, \bar{f} \succ}$, $\prec \check{\mu}, f \succ = \prec \mu, \check{f} \succ$, with $\check{f}(x) = f(x^{-1})$ and $\bar{f}(x) = \overline{f(x)}$ for all $x \in G$.

For every $\mu \in M(G)$ and every continuous and bounded function $f : G \to \mathbb{C}$ we set

$$f_\mu(x) = \int_G \int_G f(txs)\mathrm{d}\mu(t)\mathrm{d}\mu(s)\,,$$

and we say that f is μ-biinvariant if $f_\mu = f$. Noting that if $\mu * \mu = \mu$, then f is μ-biinvariant if and only if f is right μ-invariant $\left(\int_G f(xt)\mathrm{d}\mu(t) = f(x)\right)$ and f is left μ-invariant $\left(\int_G f(tx)\mathrm{d}\mu(t) = f(x)\right)$.

For every $x \in G$, δ_x designates the Dirac measure concentrated at x. If $f \in C_b(G)$ we say that f satisfies the condition $\mathrm{K}(\mu)$ if

$$\int_G \int_G f(ysxtz)\mathrm{d}\mu(s)\mathrm{d}\mu(t) = \int_G \int_G f(xsytz)\mathrm{d}\mu(s)\mathrm{d}\mu(t) \quad \text{for all } x, y, z \in G\,.$$

Definition 2.1. Let $\mu \in M(G)$; μ is called a generalized Gelfand measure if $\mu * \mu = \mu$ and the Banach algebra $\mu * M(G) * \mu$ is commutative (under the convolution).

For the notion of Gelfand measure see [3].

Definition 2.2. Let $\mu \in M(G)$. A non zero function $\Phi \in C_b(G)$ is a μ-spherical function if it satisfies the functional equation $\int_G \Phi(xty)\mathrm{d}\mu(t) = \Phi(x)\Phi(y)$ for all $x, y \in G$.

In a previous paper see [6], the continuous and bounded solutions of (3) are completely determined under the condition that f satisfies the Kannappan

type condition K(μ) and in the particular case where μ is a generalized Gelfand measure. In both cases the solutions are expressed in the formula

$$f(x) = \frac{\Phi(x) + \Phi(\sigma(x))}{2}, \tag{4}$$

where Φ is a μ-spherical function.

In the first part of this paper (Section 3) we are going to study the general properties of (3). In Theorem 3.1 we give necessary and sufficient conditions for measurable and essentially bounded function f to satisfy equation (3). One of these condition is

$$\check{\mu} * h * f + (\check{\mu} * h * f) \circ \sigma = 2 \prec h, \check{f} \succ f \qquad \text{for all } h \in L_1(G), \tag{5}$$

which explains why we restrict our selves to solutions $f \in C_b(G)$.

In Theorem 3.2 and Theorem 3.3 we prove that if $f \in C_b(G)$ and μ is a generalized Gelfand measure on G which is σ-invariant, then the map $h \mapsto \int_G h(x)f(x)\mathrm{d}x$ is a character of the commutative Banach subalgebra $\big(P(L_1(G))\big)^\mu$ if and only if f is a solution of the functional equation (3), where $P(h)(x) = \frac{h(x)+h(\sigma(x))}{2}$ and $\big(P(L_1(G))\big)^\mu = \mu * P(L_1(G)) * \mu$.

The purpose of Section 4 is to give some conditions under which all solutions of the functional equation (3) are of the form

$$\frac{\prec \pi(x)\xi, \zeta \succ + \prec \pi(\sigma(x))\xi, \zeta \succ}{2},$$

where $(\pi, \mathcal{H})$ is a continuous unitary representation of G such that $\pi(\mu)$ is of rank one and $\xi, \zeta \in \mathcal{H}$.

We prove in Theorem 4.1 that if $\mu = \bar{\mu} = \check{\mu}$ is a generalized Gelfand measure, then the positive definite solution of (3) with $\sigma(x) = x^{-1}$ are of the form

$$\frac{\prec \pi(x)\xi, \xi \succ + \overline{\prec \pi(x)\xi, \xi \succ}}{2},$$

where $(\pi, \mathcal{H})$ is an irreducible representation of G such that $\pi(\mu)$ is of rank one. In Theorem 4.2 we give the some similar general result under the condition that μ is σ-invariant and the solutions satisfies the condition K(μ). We treat also the case of the integrable solutions. As a consequence we obtain a characterization of the solutions which satisfies the Kannappan type condition K(μ) in compact groups.

In Section 5 we consider the case when G is a connected Lie group. We prove that the solutions are the common eigenfunctions of some operators associated to a left invariant differential operators on G. The result is given in Theorem 5.1.

3. General theory

In Theorem 3.1 below we present the necessary and sufficient conditions for a measurable and essentially bounded function to be a solution of equation (3).

Theorem 3.1. *Let f be a measurable and essentially bounded function on G. Then the following statements are equivalent:*

(1) $\int_G f(xty)\mathrm{d}\mu(t)+\int_G f(xt\sigma(y))\mathrm{d}\mu(t) = 2f(x)f(y)$ *for almost all* $x, y \in G$;

(2) $\check{\mu} * h * f + (\check{\mu} * h * f) \circ \sigma = 2 \prec h, \check{f} \succ f$ *for all* $h \in L_1(G)$;

(3) $\check{\mu} * \vartheta * f + (\check{\mu} * \vartheta * f) \circ \sigma = 2 \prec \vartheta, \check{f} \succ f$ *for all* $\vartheta \in M(G)$;

(4) $\check{\mu} * \delta_x * f + (\check{\mu} * \delta_x * f) \circ \sigma = 2\check{f}(x)f$ *for all* $x \in G$;

(5) $\prec \vartheta' * \mu * \vartheta, f \succ + \prec \vartheta' * \mu * \sigma(\vartheta), f \succ = 2 \prec \vartheta, f \succ \prec \vartheta', f \succ$ *for all* $\vartheta, \vartheta' \in M(G)$.

Proof. (1) $\Rightarrow$(2) For all $h \in \mathcal{K}(G)$ (functions with compact support) and for almost all $y \in G$, we get

$$\begin{aligned} 2 \prec h, \check{f} \succ f(y) &= \int_G 2f(x^{-1})f(y)h(x)\mathrm{d}x = \int_G\int_G f(x^{-1}ty)h(x)\mathrm{d}\mu(t)\mathrm{d}x \\ &\quad + \int_G\int_G f(x^{-1}t\sigma(y))h(x)\mathrm{d}\mu(t)\mathrm{d}x \\ &= \int_G (h*f)(t^{-1}y)\mathrm{d}\check{\mu}(t) + \int_G (h*f)(t^{-1}\sigma(y))\mathrm{d}\check{\mu}(t) \\ &= (\check{\mu} * h * f)(y) + (\check{\mu} * h * f)(\sigma(y))\,, \end{aligned}$$

which proves (2).

(2) $\Rightarrow$(3) Follows immediately from the fact that $L_1(G)$ is weakly dense in $M(G)$.

(4) $\Rightarrow$(1) First note that $(\check{\mu}*\delta_x*f)(y) = \int_G f(x^{-1}ty)\mathrm{d}\mu(t)$, hence we have that

$$\int_G f(x^{-1}ty)\mathrm{d}\mu(t) + \int_G f(x^{-1}t\sigma(y))\mathrm{d}\mu(t) = 2\check{f}(x)f(y) \qquad \text{for all } x, y \in G\,.$$

This proves (1).

(3) $\Rightarrow$(5) If $\vartheta, \vartheta' \in M(G)$, then $\prec \vartheta' * \mu * \vartheta \succ + \prec \vartheta' * \mu * \sigma(\vartheta), f \succ = \prec \vartheta, \check{\mu} * \check{\vartheta}' * f \succ + \prec \vartheta, (\check{\mu} * \check{\vartheta}' * f) \circ \sigma \succ = \prec \vartheta, 2 \prec \check{\vartheta}', \check{f} \succ f \succ =$

$2 \prec \vartheta', f \succ \prec \vartheta, f \succ$. By a small computation we proves the other point of the theorem. ∎

Consequently we shall assume throughout the paper that the solutions of equation (3) are bounded and continuous functions on G.

The connection between continuous characters of the commutative Banach algebra $\big(P(L_1(G))\big)^\mu = P(L_1(G)^\mu)$ is illustrated by the following two theorems.

THEOREM 3.2. *Let G be unimodular, let μ be a generalized Gelfand measure on G which is σ-invariant. Let $f \in C_b(G)$ be a solution of equation* (3). *Then the mapping $h \mapsto \prec h, f \succ = \int_G h(x)f(x)\mathrm{d}x$ is a continuous character of the commutative Banach algebra $\big(P(L_1(G))\big)^\mu$.*

Proof. Assume that $f \in C_b(G)$ is a solution of (3). Then, in view of [6, Lemma 2.3], f is μ-biinvariant and $f(\sigma(x)) = f(x)$ for all $x \in G$. Therefore, for all $h, g \in L_1(G)$ we have

$$\begin{aligned}\prec\left(\frac{g+g\circ\sigma}{2}\right)^\mu * \left(\frac{h+h\circ\sigma}{2}\right)^\mu, f\succ &= \frac{1}{4}\int_G\int_G\Big[(g(x)h(y)\\ &\quad + g(x)(h\circ\sigma)(y) + (g\circ\sigma(x))h(y)\\ &\quad + (g\circ\sigma)(x)(h\circ\sigma((y))\int_G f_\mu(xty)\mathrm{d}(\mu*\mu)(t)\Big]\mathrm{d}x\mathrm{d}y\,.\end{aligned}$$

Since $\mu*\mu=\mu$, $f_\mu = f$, $f\circ\sigma = f$ and G is unimodular, then we get

$$\begin{aligned}&\prec\left(\frac{g+g\circ\sigma}{2}\right)^\mu * \left(\frac{h+h\circ\sigma}{2}\right)^\mu, f\succ\\ &\quad= \frac{1}{2}\int_G\int_G g(x)h(y)\Big[\int_G f(xty)\mathrm{d}\mu(t) + \int_G f(xt\sigma(y))\mathrm{d}\mu(t)\Big]\mathrm{d}x\mathrm{d}y\\ &\quad= \int_G f(x)g(x)\mathrm{d}x\int_G f(y)h(y)\mathrm{d}y\\ &\quad= \int_G f(x)\frac{g(x)+g(\sigma(x))}{2}\mathrm{d}x\int_G f(y)\frac{h(y)+h(\sigma(y))}{2}\mathrm{d}y\\ &\quad= \prec\left(\frac{g+g\circ\sigma}{2}\right)^\mu, f\succ \prec\left(\frac{h+h\circ\sigma}{2}\right)^\mu, f\succ,\end{aligned}$$

which concludes the proof of the theorem. ∎

THEOREM 3.3. *Let G be unimodular. Let μ be a generalized Gelfand measure which is σ-invariant on G. If $\chi : P(L_1(G))^\mu \to \mathbb{C}^*$ is a continuous character of $\big(P(L_1(G))\big)^\mu$, then there exists $f \in C_b(G)$ solution of the functional equation* (3) *such that $\chi(g) = \prec g, f \succ$ for all $g \in \big(P(L_1(G))\big)^\mu$.*

Proof. Let χ be a nonzero continuous character of the Banach algebra $\big(P(L_1(G))\big)^\mu$. The map $L_1(G) \to \mathbb{C}$, $g \mapsto \chi((\frac{g+g\circ\sigma}{2})^\mu)$, is continuous and linear. Consequently, there exists $f \in \pounds_\infty(G)$ such that $\chi((\frac{g+g\circ\sigma}{2})^\mu) = \prec g, f \succ$. In addition, f may be chosen continuous: let $f_1 \in \mathcal{K}(G)$ such that $f_1 = f_1 \circ (\sigma)$ and $\chi((P(f_1))^\mu) = 1$; for all $h \in \mathcal{K}(G)$ we have

$$\begin{aligned} < h, f > &= \chi\big((P(f_1))^\mu\big)\chi\big((P(h))^\mu\big) \\ &= \chi\big((P(h))^\mu * (P(f_1))^\mu\big) = \chi\big((P(P(h) * f_1^\mu))^\mu\big) \\ &= < P(h) * f_1^\mu, f > = < P(h), f * (\check{f}_1^\mu) > \\ &= < h, P(f * (\check{f}_1^\mu)) > . \end{aligned}$$

Consequently $f = P(f * (\check{f}_1^\mu))$ and hence f is a continuous function.

On the other hand

$$\begin{aligned} \chi\left(\left(\frac{g + g\circ\sigma}{2}\right)^\mu\right) = \chi\left(\frac{g^\mu + g^\mu\circ\sigma}{2}\right) &= \chi\left(\left(\frac{g^\mu + g^\mu\circ\sigma}{2}\right)^\mu\right) \\ &= \prec g^\mu, f \succ = \prec g, f_\mu \succ . \end{aligned}$$

It follows that f is μ-biinvariant.

On the other hand

$$\begin{aligned} \chi\left(\frac{g^\mu + g^\mu\circ\sigma}{2}\right) &= \chi\left(\left(\frac{\frac{g^\mu+g^\mu\circ\sigma}{2} + \frac{g^\mu+g^\mu\circ\sigma}{2}}{2}\right)^\mu\right) \\ &= \prec \frac{g^\mu + g^\mu\circ\sigma}{2}, f \succ = \prec g, \frac{f_\mu + f_\mu\circ\sigma}{2} \succ \\ &= \prec g, \frac{f + f\circ\sigma}{2} \succ . \end{aligned}$$

Then we get $f = \frac{f+f\circ\sigma}{2}$, i.e., $f\circ\sigma = f$. Now, for all $g, h \in L_1(G)$ we have

$$\begin{aligned}
&\prec \left(\frac{g+g\circ\sigma}{2}\right)^{\mu} * \left(\frac{h+h\circ\sigma}{2}\right)^{\mu}, f \succ \\
&\qquad = \frac{1}{2}\int_G\int_G g(x)h(y)\left[\int_G f(xty)\mathrm{d}\mu(t) + \int_G f(xt\sigma(y))\mathrm{d}\mu(t)\right]\mathrm{d}x\mathrm{d}y \\
&\qquad = \prec \left(\frac{g+g\circ\sigma}{2}\right)^{\mu}, f \succ \prec \left(\frac{h+h\circ\sigma}{2}\right)^{\mu}, f \succ \\
&\qquad = \int_G\int_G g(x)h(y)f(x)f(y)\mathrm{d}x\mathrm{d}y\,,
\end{aligned}$$

hence it follows that

$$\int_G f(xty)\mathrm{d}\mu(t) + \int_G f(xt\sigma(y))\mathrm{d}\mu(t) = 2f(x)f(y) \qquad \text{for all } x, y \in G\,,$$

which proves our theorem. ∎

4. Solutions of equation (3) and representations of G

An ambitious project is to obtain the general solution f of the functional equation (3), where $\mu \in M(G)$. In this section we produce the explicit solution formulas for the functional equation (3) in question by means of coefficients of irreducibles and continuous unitary representation of G.

We start by the following theorem which extend some results which have been obtained in [5].

Theorem 4.1. *Let $\mu = \bar{\mu} = \check{\mu}$ be a generalized Gelfand measure. Let $f \in C_b(G)$ be a positive definite function satisfying equation (3) with $\sigma(x) = x^{-1}$. Then, there exists an irreducible, continuous and unitary representation $(\pi, \mathcal{H})$ of G and $\xi \in \mathcal{H}$, such that $\pi(\mu)$ is of rank one and*

$$f(x) = \frac{\prec \pi(x)\xi, \xi \succ + \overline{\prec \pi(x)\xi, \xi \succ}}{2} \qquad \textit{for all } x \in G\,.$$

Noting that in this case $\Phi(x) = \prec \pi(x)\xi, \xi \succ$ is a positive definite μ-spherical function on G.

Proof. It is elementary to check that the function

$$\frac{\prec \pi(x)\xi, \xi \succ + \overline{\prec \pi(x)\xi, \xi \succ}}{2}$$

is a solution of equation (3) with $\sigma(x) = x^{-1}$, where $(\pi, \mathcal{H})$ is a continuous and unitary representation of G such that $\pi(\mu)(\eta) = \prec \eta, \xi \succ \xi$ for all $\eta \in \mathcal{H}$. Thus what is left is to show that each positive definite, continuous and bounded solution f occurs in the formula of Theorem 4.1.

Let $f \in C_b(G)$ be a nonzero positive definite solution of equation

$$\int_G f(xty)\mathrm{d}\mu(t) + \int_G f(xty^{-1})\mathrm{d}\mu(t) = 2f(x)f(y)\,. \tag{6}$$

By [6, Lemma 2.3], f is μ-invariant; hence, in view of [3, Theorem 5.3], there exists a bounded positive Radon measure ϱ_f on Ω_μ: the set of positive definite μ-spherical functions on G such that

$$f(x) = \int_{\Omega_\mu} \omega(x)\mathrm{d}\varrho_f(\omega) \qquad \text{for all } x \in G\,. \tag{7}$$

Using this expression for f and the fact that f satisfies equation (6) gives us

$$\int_{\Omega_\mu} \omega(x)\Big[\int_{\Omega_\mu} \varpi(y)\mathrm{d}\varrho_f - \mathrm{Re}(\omega)(y)\Big]\mathrm{d}\varrho_f(\omega) = 0\,.$$

It follows from the injection map $M(\Omega_\mu) \to C_b(G)$, $i(\vartheta)(x) = \int_{\Omega_\mu} \omega(x)\mathrm{d}\vartheta(\omega)$ (see [2]), that

$$\mathrm{Re}(\omega)(y) = \int_{\Omega_\mu} \varpi(y)\mathrm{d}\varrho_f(\varpi) = f(y)$$

on the support of ϱ_f.

Using the linear independence of the μ-spherical functions (see [6, Lemma 2.2]), we get

$$\mathrm{Supp}(\varrho_f) = \{\omega,\ \bar{\omega}\}\,, \qquad \omega \in \Omega_\mu\,,$$

and

$$\varrho_f = a\delta_\omega + b\delta_{\bar{\omega}} \qquad \text{where } a, b \in \mathbb{R}^+.$$

This implies that

$$f(x) = (a+b)\,\mathrm{Re}(\omega)(x) + i(a-b)\,\mathrm{Im}(\omega)(x)\,,$$

since $f(e) = 1$ and $f(x) = f(x^{-1})$, hence there exists $\omega \in \Omega_\mu$ such that $f(x) = \mathrm{Re}(\omega)(x)$ for all $x \in G$.

By [3, Theorem 4.5.2], there exits an irreducible, continuous and unitary representation $(\pi, \mathcal{H})$ of G and $\xi \in \mathcal{H}$, such that $\pi(\mu)$ is of rank one and $\omega(x) = \prec \pi(x)\xi, \xi \succ$. This completes the proof of Theorem 4.1. ∎

Now we are going to give some conditions under which the solutions of equation (3) are expressed in terms of coefficients of irreducible, continuous and unitary representations of G, we also consider the integrable solutions, in particular if G is compact the complete solutions formulas are determined under the condition that f satisfies $\mathrm{K}(\mu)$.

THEOREM 4.2. *Let μ be a σ-invariant measure. Let $f \in C_b(G)$ be a solution of the functional equation* (3) *satisfying* $\mathrm{K}(\mu)$.

(1) *There exists a μ-spherical function Φ such that $f(x) = \frac{\Phi(x)+\Phi(\sigma(x))}{2}$ for all $x \in G$.*

(2) *If $\Phi(x) = \prec \pi(x)\xi, \zeta \succ$ and*

$$f(x) = \frac{\prec \pi(x)\xi, \zeta \succ + \prec \pi(\sigma(x))\xi, \zeta \succ}{2},$$

then π admits an irreducible subrepresentation $\pi^{'}$ such that $\pi^{'}(\mu)$ is of rank one,

$$f(x) = \frac{\prec \pi^{'}(x)\xi^{'}, \zeta \succ + \prec \pi^{'}(\sigma(x))\xi^{'}, \zeta \succ}{2}$$

for all $x \in G$, and $\pi^{'}(\mu)\eta = \prec \eta, \zeta \succ \xi^{'}$.

(3) *If Φ is a positive definite function and $f(x) = \frac{\Phi(x)+\Phi(\sigma(x))}{2}$, then there exists an irreducible, continuous and unitary representation $(\pi, \mathcal{H})$ of G such that $\pi(\mu)$ is of rank one and*

$$f(x) = \frac{\mathrm{tr}\big(\pi(x)\pi(\mu)\big) + \mathrm{tr}\big(\pi(\sigma(x))\pi(\mu)\big)}{2}.$$

(4) *If f is integrable, then there exists an irreducible, unitary and integrable representation $(\pi, \mathcal{H})$ of G such that $\pi(\mu)$ is of rank one and*

$$f(x) = \frac{\prec \pi(x)\xi, \zeta \succ + \prec \pi(\sigma(x))\xi, \zeta \succ}{2}$$

for all $x \in G$, where $\xi, \zeta \in \mathcal{H}$ are such that $\pi(\mu)(\eta) = \prec \eta, \zeta \succ \xi$.

(5) *If G is compact then, there exists an irreducible, continuous and unitary representation $(\pi, \mathcal{H})$ of G such that $\pi(\mu)$ is of rank one and*

$$f(x) = \frac{\prec \pi(x)\xi, \zeta \succ + \prec \pi(\sigma(x))\xi, \zeta \succ}{2}$$

for all $x \in G$.

Proof. See [6, Theorem 2.2] for (1), and [4, Theorem 2.2] for (2) and (3).

(4) According to the proof of [6, Theorem 2.1], the μ-spherical functions Φ in (1) are written in the form

$$\Phi(x) = f(x) + k\left[\int_G f(xta)\mathrm{d}\mu(t) - \int_G f(xt\sigma(a))\mathrm{d}\mu(t)\right] \qquad \text{for all } x \in G$$

for some $k \in \mathbb{C}$ and $a \in G$ such that

$$\int_G f(ata)\mathrm{d}\mu(t) - \int_G f(at\sigma(a))\mathrm{d}\mu(t) \neq 0 \qquad \text{or} \qquad \Phi \equiv f\,.$$

It follows directly from these formulas that f is integrable if and only Φ is integrable. Now, by applying [4, Theorem 2.5] we derive the rest of the proof. ∎

5. Generalized d'Alembert functional equation on Lie groups

In the present section G stands for connected Lie group and σ designate a continuous automorphism of G which satisfies $\sigma \circ \sigma = I$.

A form of the continuous solutions of equation (3) we shall characterize in terms of eigenfunctions of some operators. At this place we recall some definitions used in the sequel.

For each fixed $a \in G$, we define the translation operator as follows: if $g \in C(G)$ then $(L_a g)(x) = g(a^{-1}x)$. We will say that a operator $T : C(G) \to C(G)$ is *left-invariant* if $(L_a T)(g) = T(L_a g)$ for all $g \in C(G)$.

The following result will be used later.

Proposition 5.1. *For any operator $T : C(G) \to C(G)$, the operator T_μ defined by*

$$T_\mu(g)(x) = \frac{1}{2}T\left\{(L_{x^{-1}}g)_\mu + (L_{x^{-1}}g)_\mu \circ \sigma\right\}(e)\,, \qquad g \in C_b(G)\,,\ x \in G\,,$$

satisfies the following properties:

(i) *T_μ is left invariant;*

(ii) *$T_\mu(g)(e) = \frac{1}{2}\,T\{g_\mu + g_\mu \circ \sigma\}(e)$; in particular, if $g_\mu = g$ and $g \circ \sigma = g$ we have $T_\mu(g)(e) = T(g)(e)$;*

(iii) *if f is a right μ-invariant solution of equation (3), then f is a common eigenfunction of the operator T_μ; more precisely $T_\mu(f) = T(f)(e)f$.*

Proof. (i) Let $g \in C_b(G)$ and let $a \in G$; for all $x \in G$ we have

$$\begin{aligned} L_a(T_\mu g)(x) = T_\mu g(a^{-1}x) &= \frac{1}{2}\, T\big\{(L_{x^{-1}a}g)_\mu + (L_{x^{-1}a}g)_\mu \circ \sigma\big\}(e) \\ &= \frac{1}{2}\, T\big\{(L_{x^{-1}}(L_a g))_\mu + (L_{x^{-1}}(L_a g))_\mu \circ \sigma\big\}(e) = T_\mu(L_a g)(x)\,, \end{aligned}$$

which proves (i).

(ii) Is evident.

(iii) Let f be a right μ-invariant solution of equation (3); this means that

$$\int_G f(xty)\mathrm{d}\mu(t) + \int_G f(xt\sigma(y))\mathrm{d}\mu(t) = 2f(x)f(y)$$

and

$$\int_G f(xt)\mathrm{d}\mu(t) = f(x)$$

for all $x, y \in G$. We conclude that

$$\begin{aligned} (L_{y^{-1}}f)_\mu(x) + (L_{y^{-1}}f)_\mu(\sigma(x)) \;&=\; \int_G\int_G (L_{y^{-1}}f)(sxt)\mathrm{d}\mu(s)\mathrm{d}\mu(t) \\ &\quad + \int_G\int_G (L_{y^{-1}}f)(s\sigma(x)t)\mathrm{d}\mu(s)\mathrm{d}\mu(t) \\ &= \int_G\int_G f(ysxt)\mathrm{d}\mu(s)\mathrm{d}\mu(t) + \int_G\int_G f(ys\sigma(x)t)\mathrm{d}\mu(s)\mathrm{d}\mu(t) \\ &= \int_G f(ysx)\mathrm{d}\mu(s) + \int_G f(ys\sigma(x))\mathrm{d}\mu(s) \;=\; 2f(x)f(y)\,. \end{aligned}$$

For $x = e$, we get

$$T_\mu(f)(y) \;=\; \frac{1}{2}\big\{(L_{y^{-1}}f)_\mu + (L_{y^{-1}}f)_\mu \circ \sigma\big\}(e) \;=\; T(f)(e)f(y)\,.$$

Hence it follows that $T_\mu(f) = T(f)(e)f$. This completes the proof. ∎

Let ${C_\mu}^\infty(G) = \check{\mu} * C^\infty * \triangle\check{\mu}$ denote the space of C^∞ and μ-biinvariant functions on G, where $\triangle$ denotes the modular function on G. The subspace of ${C_\mu}^\infty(G)$ of functions g which satisfies $g \circ \sigma = g$, will be denoted by ${\mathcal{C}_\mu}^\infty(G)$. Finally $\mathbb{D}(G)$ denote the algebra of left invariant differential operators on G.

PROPOSITION 5.2. *Let μ be a σ-invariant measure with compact support and let $T \in \mathbb{D}(G)$. For all $g \in \mathcal{C}_\mu{}^\infty(G)$ we have*

$$T_\mu g = \frac{1}{2}\big\{Tg * \triangle\check{\mu} + (Tg * \triangle\check{\mu}) \circ \sigma\big\}\,.$$

Furthermore $T_\mu g \in \mathcal{C}_\mu{}^\infty(G)$.

Proof. Let $T \in \mathbb{D}(G)$ and let $g \in \mathcal{C}_\mu{}^\infty(G)$. For all $x, y \in G$ we have

$$(L_{x^{-1}}g)_\mu(y) = \int_G \int_G g(xtys)\mathrm{d}\mu(t)\mathrm{d}\mu(s)\,.$$

Since g is right μ-invariant, then we get

$$(L_{x^{-1}}g)_\mu(y) = \int_G g(xty)\mathrm{d}\mu(t) = \int_G (L_{(xt)^{-1}}g)(y)\mathrm{d}\mu(t)\,,$$

it follows that

$$\begin{aligned} T(L_{x^{-1}}g)_\mu(e) &= \int_G (TL_{(xt)^{-1}}g)(e)\mathrm{d}\mu(t) \\ &= \int_G (L_{(xt)^{-1}}Tg)(e)\mathrm{d}\mu(t) = \int_G (Tg)(xt)\mathrm{d}\mu(t) \;=\; Tg * \triangle\check{\mu}\,. \end{aligned}$$

On the other hand

$$(L_{x^{-1}}g)_\mu(\sigma(y)) = \int_G \int_G g(xt\sigma(y)s)\mathrm{d}\mu(t)\mathrm{d}\mu(s) = \int_G g(xt\sigma(y))\mathrm{d}\mu(t)\,.$$

Since $g \circ \sigma = g$ and $\sigma(\mu) = \mu$, then we have

$$(L_{x^{-1}}g)_\mu(\sigma(y)) = \int_G g(\sigma(x)ty)\mathrm{d}\mu(t) = \int_G (L_{(\sigma(x)t)^{-1}}g)(y)\mathrm{d}\mu(t)\,.$$

This implies that

$$\begin{aligned} T(L_{x^{-1}}g)_\mu(e) &= \int_G (TL_{(\sigma(x)t)^{-1}}g)(e)\mathrm{d}\mu(t) \;=\; \int_G (L_{(\sigma(x)t)^{-1}}Tg)(e)\mathrm{d}\mu(t) \\ &= \int_G (Tg)(\sigma(x)t)\mathrm{d}\mu(t) \;=\; (Tg * \triangle\check{\mu})(\sigma(x))\,. \end{aligned}$$

Consequently

$$(T_\mu g)(x) = \frac{1}{2}\{Tg * \triangle\check{\mu})(x) + (Tg * \triangle\check{\mu})(\sigma(x)\}\,.$$

This proves the first point of Proposition 5.2.

It's clear that $(T_\mu g)(\sigma(x)) = (T_\mu g)(x)$ for all $x \in G$.

Now we are going to prove that $T_\mu g$ is μ-biinvariant. In virtue of the fact that $g \in C_b(G)$ is μ-biinvariant (with $\sigma(\mu) = \mu$) if and only if $g \circ \sigma$ is μ-biinvarint and according to expression of $T_\mu g$, it's sufficient to prove that $T_\mu g * \triangle\check{\mu}$ is μ-biinvariant. So

$$\check{\mu} * (Tg * \triangle\check{\mu}) * \triangle\mu \;=\; \check{\mu} * Tg * \triangle\check{\mu} \;=\; T(\check{\mu} * g) * \triangle\check{\mu} \;=\; Tg * \triangle\check{\mu}\,.$$

This shows that $T_\mu g$ is μ-biinvariant.

On the other hand, by using the fact that μ is a compactly supported measure, it's easy to prove that $T_\mu g \in C^\infty(G)$. This completes the proof. ∎

The main result of the present section is the following theorem.

THEOREM 5.3. *Let G be a connected Lie group. Let μ be a complex measure with compact support such that $\sigma(\mu) = \mu = \mu * \mu$. Let $f \in C(G)$. Then, the following statements are equivalent:*

(1) *f is a solution of the functional equation* (3)*;*

(2) (i) *f is μ-biinvariant, $f \circ \sigma = f$,*
  (ii) *f is analytic, and*
  (iii) *f is the common eigenfunction of the operators T_μ for all $T \in \mathbb{D}(G)$.*

Proof. (1)⇒(2) Follows directly from [6, Proposition 5.1, Lemma 2.3].

(2)⇒(1) Suppose that (2) holds, with $T_\mu f = \lambda(T) f$ for any operator $T \in \mathbb{D}(G)$. By Proposition 5.1 (4), $\lambda(T) = T(f)(e)$. For a fixed element x in G, we define the new function

$$g(y) = \frac{1}{2}\left\{ \int_G f(xty)\,\mathrm{d}\mu(t) + \int_G f(xt\sigma(y))\,\mathrm{d}\mu(t) \right\}, \qquad y \in G\,.$$

Since $\mu * \mu = \mu$, $\sigma(\mu) = \mu$ and f is μ-biinvariant, then also g is μ-biinvariant. In virtue of $\sigma(\mu) = \mu$ and $f \circ \sigma = f$ we get

$$g(y) = \frac{1}{2}\left\{ \int_G (L_{(xt)^{-1}} f)(y)\,\mathrm{d}\mu(t) + \int_G (L_{(\sigma(x)t)^{-1}} f)(y)\,\mathrm{d}\mu(t) \right\}.$$

Consequently for all $T \in \mathbb{D}(G)$ we have

$$(T_\mu(g))(y) = \frac{1}{2}\left\{ \int_G T_\mu(L_{(xt)^{-1}} f)(y)\,\mathrm{d}\mu(t) + \int_G T_\mu(L_{(\sigma(x)t)^{-1}} f)(y)\,\mathrm{d}\mu(t) \right\}.$$

Since T_μ is left invariant (see Proposition 5.1), then we obtain

$$\begin{aligned}(T_\mu g)(y) &= \frac{1}{2}\left\{\int_G T_\mu f(xty)\mathrm{d}\mu(t) + \int_G T_\mu f(\sigma(x)ty)\mathrm{d}\mu(t)\right\}\\ &= T(f)(e)\frac{1}{2}\left\{\int_G f(xty)\mathrm{d}\mu(t) + \int_G f(\sigma(x)ty)\mathrm{d}\mu(t)\right\}\\ &= T(f)(e)\frac{1}{2}\left\{\int_G f(xty)\mathrm{d}\mu(t) + \int_G f(\sigma(y)tx)\mathrm{d}\mu(t)\right\}\\ &= T(f)(e)g(y)\,.\end{aligned}$$

In particular

$$(T_\mu g)(e) = T(f)(e)g(e)\,.$$

By Proposition 5.1 (ii) we have

$$(T_\mu g)(e) = T(g)(e)\,,$$

and hence it follows that

$$T(g - g(e)f)(e) = 0 \qquad \text{for all } T \in \mathbb{D}(G)$$

since $g - g(e)f$ is analytic function on the connected Lie group G. Then, in view of [8, Chapter II] we obtain

$$g - g(e)f \equiv 0 \qquad \text{on } G\,.$$

We conclude that

$$\begin{aligned}\frac{1}{2}&\left\{\int_G f(xty)\mathrm{d}\mu(t) + \int_G f(xt\sigma(y))\mathrm{d}\mu(t)\right\}\\ &\qquad = \int_G f(xt)\mathrm{d}\mu(t)f(y) = f(x)f(y)\end{aligned}$$

for all $x, y \in G$. This ends the proof of Theorem 5.3. ∎

Corollary 5.4. *Let G be a connected Lie group, let μ be a complex measure with compact support such that $\sigma(\mu) = \mu = \mu * \mu$, and let $f \in C(G)$. If f is a solution of the functional equation (3), then f is the unique solution of $\{g \in \mathcal{C}_\mu{}^\infty(G) : T_\mu(g) = T(f)(e)g$ for all $T \in \mathbb{D}(G)\}$.*

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