EXTRACTA MATHEMATICAE Vol. **20**, Núm. 1, 51 – 70 (2005)

Some Open Problems on Functional Analysis and Function Theory

V. K. Maslyuchenko[1], A. M. Plichko[2] (Editors)

[1] *Department of Mathematics, Chernivtsi National University str. Kotsjubyn'skogo* 2, *Chernivtsi*, 58012 *Ukraine*
[2] *Instytut Matematyki, Politechnika Krakowska, ul. Warszawska* 24, *Kraków, Poland*

e-mail: popov@chv.ukrpack.net, aplichko@usk.pk.edu.pl

AMS *Subject Class.* (2000): 46– 06 *Received October* 28, 2004

This collection is dedicated to the memory of Hans Hahn.

1. Introduction

The name of Hans Hahn (1879–1934), an Austrian mathematician, a Professor of Chernivtsi (1909–1916), Bonn (1916–1921) and Vienna (1921–1934) Universities is well known among mathematicians mainly due to the famous Hahn-Banach Theorem on extensions of linear functionals. Much less known is the fact that H. Hahn independently of S. Banach proved another basic principle of Functional Analysis - the uniform boundedness principle. Some other well-known results due to H. Hahn are: the Hahn decomposition theorem, the Vitali-Hahn-Saks theorem in Measure Theory, the Hahn-Mazurkiewicz theorem on continuous images of the unit segment in Topology, the Hahn embedding theorem in the Theory of Partially Ordered Sets. The notions of local connectivity and reflexivity introduced by Hahn also play an important role in modern mathematics. H. Hahn was a very versatile mathematician. His scientific heritage contains papers in Calculus of Variations, Real Functions Theory, Functional Analysis, Topology, History and Philosophy of Mathematics.

In honour of the memory of Hans Hahn, mathematicians from Chernivtsi National University (Ukraine) organized regular conferences, beginning in 1984. The first and the second conferences dedicated to the memory of H. Hahn were held in Chernivtsi in 1984 and 1994, respectively.

Around 120 mathematicians from different countries participated in the 3-rd Conference. For the first time a Problem Section was organized during which a number of problems in Functional Analysis and Function Theory were posed. Under some correction by the Editors, these problems were placed into the base of the note.

This Problem section is divided into independent parts, each of which has its own authors.

2. On the extension of z-linear maps

J. M. F. Castillo

Departamento de Matemáticas, Universidad de Extremadura,
06071 *Badajoz, Spain, e-mail:* castillo@unex.es

In the spirit of the Hahn-Banach extension theorem for linear continuous functionals on Banach spaces, let us consider the problem of the extension of z-linear functionals on Banach spaces. Recall that a functional $f : X \curvearrowright \mathbb{R}$ (this notation is to stress the fact that these are, in general, non-linear maps) is said to be z-linear if there is a constant C such that for all finite families $x_1, \dots, x_n \in X$ one has

$$\Big\| \sum_{i=1}^{n} f(x_i) - f\big(\sum_{i=1}^{n} x_i\big)\Big\| \leq C \sum_{i=1}^{n} \|x_i\| \tag{1}$$

The infimum of the constant C above is called the z-linearity constant of f and denoted $Z(f)$. Observe that a z-linear functional need not be either bounded, linear or continuous. It may sound surprising but every z-linear functional f defined on a subspace X of a Banach space X_1 can be extended to a z-linear functional $\hat{f}$ on the whole space, although it is not clear that $Z(\hat{f}) = Z(f)$ can also be reached.

The connection with the Hahn-Banach theorem appears after realizing that to get the extension result for z-linear maps one relies on the following result: *Every z-linear map $f : X \curvearrowright \mathbb{R}$ admits a linear map $\ell : X \to \mathbb{R}$ such that $\|f - \ell\| \leq Z(f)$.* This is nothing different from a rewording of the Hahn-Banach theorem (called "nonlinear Hahn-Banach theorem" in [1], since it admits an independent proof) as the following final remainder pieces of the puzzle show: z-linear maps between two Banach spaces $f : X \curvearrowright Y$ correspond with exact sequences $0 \to Y \to E \to X \to 0$ (i.e., with Banach spaces E such that $E/Y = X$). And z-linear maps admitting a linear map $\ell : X \to Y$ such that $\|f - \ell\| < +\infty$ correspond with exact sequences $0 \to Y \to E \to X \to 0$ that split (i.e., with Banach spaces E such that Y is complemented in E and $E/Y = X$). Since the Hahn-Banach theorem says that every exact sequence $0 \to \mathbb{R} \to E \to X \to 0$ in which E is a Banach space splits, every z-linear map $fX \curvearrowright \mathbb{R}$ has a linear map at finite distance, and can therefore be written as $f = \ell + b$ where b is a bounded map in the sense that $\|b(x)\| \leq M\|x\|$ for some constant M.

The extension result follows now taking a bounded projection $m : X_1 \to X$ and then a linear projection $L : X_1 \to X$ and setting $F = bm + \ell L$. This is a z-linear map $F : X \curvearrowright \mathbb{R}$ that extends f. The problem of such rude way of extension is that $Z(F)$ can be much larger than $Z(f)$.

The balance between the properties is quite delicate: everything can fail if $\mathbb{R}$ is replaced by another (infinite dimensional) Banach space or f is asked to be simply quasi-linear, which means that (1) just holds for couples of points (instead of finite families). As an example of the former assertion, the Kalton-Peck map [3] $F_2 : l_2 \curvearrowright l_2$ -which is perfectly z-linear- cannot be extended to $L_1[0, 1]$; as an example of the latter, Ribe's map [6] $R : l_1 \curvearrowright \mathbb{R}$ -which is just quasi-linear but not z-linear- cannot be extended to $C[0, 1]$.

Problem 2.1. Why?

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3. A type of bases

V. M. Kadets

Department of Mathematics, Kharkov National University, Univ. Sq., 1, *Kharkov, Ukraine, e-mail:* anna.m.vishnyakova@univer.kharkov.ua

DEFINITION 3.1. A biorthogonal system $\{e_n, e_n^*\}_{n=1}^{\infty}$ for a Banach space X is said to be an *almost basis* if the identity operator is a limiting point for the sequence of partial sum operators $S_n(x) = \sum_1^n e_k^*(x)e_k$ in the topology of pointwise convergence.

PROBLEM 3.2. Does every separable Banach space have an almost basis?

4. An isomorphism problem for spaces of analytic functions

S.V. Kislyakov

POMI, Fontanka str. 27, *St. Petersburg, Russia, e-mail:* skis@pdmi.ras.ru

The problem I want to present is not new. I formulate it just to recall that in the Banach space theory some basic objects have not yet been distinguished isomorphically. (By the way, a challenging couple of such objects is formed by the spaces of one time continuously differentiable functions on the square and on the 3-cube.)

Consider a compact subset K of the complex plane. Various sup-norm spaces of analytic functions can be associated with it. For instance, we may consider the space $C_A(K)$ of all functions continuous on K and analytic on the interior of K, the closure $P(K)$ of the analytic polynomials (i.e. polynomials of a complex variable) in the norm of $C(K)$, or the similar closure $R(K)$ of all rational functions with poles off K. Let X be any of these spaces. It is known that X is not linearly homeomorphic to any $C(S)$-space unless $X = C(K)$ (note that sometimes $R(K) = C(K)$); see [1].

PROBLEM 4.1. Does there exist a compact set $K \subset \mathbb{C}$ such that some of the above spaces X is a proper subset of $C(K)$ and is not isomorphic to the disc-algebra C_A?

We remind the reader that the disc-algebra is the space $C_A(\{z : |z| \leq 1\})$. The result of Wojtaszczyk saying that the disc-algebra is isomorphic to the c_0-sum of countably many copies of it (see [2]) and conformal mapping theory suggest that such a K (if it exists) must be of rather sophisticated structure.

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5. On the extensions of Hölder-Lipschitz maps

G. Lancien [1], **B. Randrianantoanina** [2]

[1] *Département de Mathématiques, Université de Franche-Comté,* 16 *Route de Gray,* 25030 *Besançon, France, e-mail:* glancien@math.univ-fcomte.fr

[2] *Department of Mathematics and Statistics, Miami University, Oxford, OH* 45056*, USA, e-mail:* randrib@muohio.edu

If (X, d) and (Y, ρ) are metric spaces, $\alpha \in (0, 1]$ and $K > 0$, we say that a map $f : X \to Y$ is α-Hölder with constant K (or in short (K, α)-Hölder) if

$$\forall x, y \in X, \quad \rho(f(x), f(y)) \le K d(x, y)^\alpha.$$

We refer to [2] for background and more information about Hölder maps.

In [12] and [9] the following notation was introduced: for $C \ge 1$, $\mathcal{B}_C(X, Y)$ denotes the set of all $\alpha \in (0, 1]$ such that any (K, α)-Hölder function f from a subset of X into Y can be extended to a (CK, α)-Hölder function from X into Y. If $C = 1$, such an extension is called an isometric extension. When $C > 1$, it is called an isomorphic extension. If a (CK, α)-Hölder extension exists for all $C > 1$, we say that f can be almost isometrically extended. Thus the following sets are defined:

$$\mathcal{A}(X, Y) = \mathcal{B}_1(X, Y), \quad \mathcal{B}(X, Y) = \bigcup_{C \ge 1} \mathcal{B}_C(X, Y), \quad \widetilde{\mathcal{A}}(X, Y) = \bigcap_{C > 1} \mathcal{B}_C(X, Y).$$

The study of these sets goes back to a classical result of Kirszbraun [8] asserting that if H is a Hilbert space, then $1 \in \mathcal{A}(H, H)$. This was extended by Grünbaum and Zarantonello [4] who showed that $\mathcal{A}(H, H) = (0, 1]$. Then the complete description of $\mathcal{A}(L^p, L^q)$ for $1 < p, q < \infty$ relies on works by Minty [11] and Hayden, Wells and Williams [5] (see also the book of Wells and Williams [14] for a very nice exposition of the subject). More recently, K. Ball [1] introduced a very important notion of non linear type or cotype and used it to prove a general extension theorem for Lipschitz maps. Building on this work, Naor [12] and Naor, Peres, Schramm and Sheffield [13] described completely the sets $\mathcal{B}(L^p, L^q)$ for $1 < p, q < \infty$.

In [9] we studied $\mathcal{A}(X,Y)$ and $\widetilde{\mathcal{A}}(X,Y)$, when X is a Banach space and Y is a space of continuous functions on a compact space equipped with the supremum norm. (This can also be viewed as a non linear generalization of the results of Lindenstrauss and Pełczyński [10] and of Johnson and Zippin [6, 7] on the extension of linear operators with values in $C(K)$ spaces.) We showed that for any finite dimensional space X, $\widetilde{\mathcal{A}}(X, C(K)) = (0,1]$ and $\mathcal{A}(X, C(K))$ is either $(0,1]$ or $(0,1)$ and we gave examples of both occurrences. To our knowledge, this is the first example of Banach spaces X and Y such that $\mathcal{A}(X,Y)$ is not closed in $(0,1]$ and also such that $\mathcal{A}(X,Y) \neq \widetilde{\mathcal{A}}(X,Y)$.

This leads us to a number of questions concerning the above defined sets:

Problem 5.1. Is $\widetilde{\mathcal{A}}(X,Y)$ always closed? If yes, is $\widetilde{\mathcal{A}}(X,Y) = \overline{\mathcal{A}(X,Y)}$?

Problem 5.2. Is $\mathcal{B}(X,Y)$ always closed? Is $\widetilde{\mathcal{B}}_C(X,Y) \stackrel{\text{def}}{=} \bigcap_{\varepsilon>0} \mathcal{B}_{C+\varepsilon}(X,Y)$ always closed? If yes, is $\widetilde{\mathcal{B}}_C(X,Y) = \overline{\mathcal{B}_C(X,Y)}$? Or, more generally, is $\widetilde{\mathcal{B}}_C(X,Y) \subseteq \overline{\mathcal{B}_C(X,Y)}$?

Problem 5.3. Is the collection of sets $\mathcal{B}_C(X,Y)$ continuous with respect to C?

Problem 5.4. Does there always exist $C > 0$ so that $\mathcal{B}(X,Y) = \mathcal{B}_C(X,Y)$? (It is so in the examples that we know.)

Brudnyi and Shvartsman [3] proved that if Y is a Banach space then the set $\mathcal{B}(X,Y)$ is always a subinterval of $(0,1]$ with the left endpoint equal to 0 (see also Naor [12]). Naor asked whether the same is true for the set $\mathcal{A}(X,Y)$. It is also natural to ask

Problem 5.5. Do the sets $\mathcal{B}_C(X,Y)$ or $\widetilde{\mathcal{B}}_C(X,Y)$ have to be intervals? If yes, does the left endpoint have to be 0?

We note that all the above questions make sense in the setting when X and Y are assumed to be either metric spaces or Banach spaces, and the answers may differ in these two settings.

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6. Metric approximation properties

Å. Lima [1], **E. Oja** [2]

[1] *Department of Mathematics, Agder University College, Serviceboks* 422, 4604 *Kristiansand, Norway, e-mail:* Asvald.Lima@hia.no

[2] *Faculty of Mathematics and Computer Science, Tartu University, J. Liivi* 2, *EE*-50409 *Tartu, Estonia, e-mail:* eve.oja@ut.ee

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the Banach space of bounded linear operators from X to Y, and by $\mathcal{F}(X,Y)$ and $\mathcal{K}(X,Y)$ its subspaces of finite rank operators and compact operators.

Recall that a Banach space X is said to have the *metric approximation property* (MAP) if for every compact set K in X and every $\varepsilon > 0$, there is an operator $T \in B_{\mathcal{F}(X,X)}$ (the closed unit ball) such that $\|Tx - x\| \leq \varepsilon$ for all $x \in K$. Recall that X is said to have the *metric compact approximation property* (MCAP) if for every compact set K in X and every $\varepsilon > 0$, there is an operator $T \in B_{\mathcal{K}(X,X)}$ such that $\|Tx - x\| \le \varepsilon$ for all $x \in K$.

Since finite rank operators are compact, the MCAP is formally weaker than the MAP. It really is weaker: Willis [5] has constructed a separable reflexive Banach space with the MCAP but without the MAP.

Let us consider the trace mapping V from the projective tensor product $X^*\hat{\otimes}_\pi X$ to $\mathcal{F}(X,X)^*$, the dual space of $\mathcal{F}(X,X)$, defined by

$$(Vu)(T) = \mathrm{trace}(Tu), \quad u \in X^*\hat{\otimes}_\pi X, \quad T \in \mathcal{F}(X,X),$$

that is, if $u = \sum_{n=1}^{\infty} x_n^* \otimes x_n$, then $(Vu)(T) = \sum_{n=1}^{\infty} x_n^*(Tx_n)$. The following well-known criterion of the MAP is due to Grothendieck [1].

Theorem 6.1. (Grothendieck) *A Banach space X has the MAP if and only if the trace mapping $V : X^*\hat{\otimes}_\pi X \to \mathcal{F}(X,X)^*$ is isometric.*

It is not known whether the similar result holds for the MCAP.

Problem 6.2. Does X have the MCAP if and only if the trace mapping $V : X^*\hat{\otimes}_\pi X \to \mathcal{K}(X,X)^*$ is isometric?

The following criterion holds for the general version of the metric approximation property defined by any operator ideal $\mathcal{A}$ (in the sense of Pietsch), studied, for instance, by Reinov [4] and Grønbæk and Willis [2]. A Banach space X is said to have the *metric $\mathcal{A}$-approximation property* (M-$\mathcal{A}$-AP) if for every compact set K in X and every $\varepsilon > 0$, there is an operator $T \in B_{\mathcal{A}(X,X)}$ such that $\|Tx - x\| \le \varepsilon$ for all $x \in K$. Clearly, the MAP coincides with the M-$\mathcal{F}$-AP and the MCAP coincides with the M-$\mathcal{K}$-AP.

Below, $\mathcal{A}(X,X)$ is always equipped with the norm topology from $\mathcal{L}(X,X)$. Thus the trace mapping $V : X^*\hat{\otimes}_\pi X \to \mathcal{A}(X,X)^*$ has norm one. We regard X as a subspace of X^{**}. Thus the identity operator I_X on X is also considered as the embedding.

Theorem 6.3. (see [3]) *Let $\mathcal{A}$ be an operator ideal. A Banach space X has the M-$\mathcal{A}$-AP if and only if $I_X \in V^*(B_{\mathcal{A}(X,X)^{**}})$ for the trace mapping $V : X^*\hat{\otimes}_\pi X \to \mathcal{A}(X,X)^*$.*

Theorem 6.3 indicates a simple reason why the "if" part in Theorem 6.1 works for the M-$\mathcal{A}$-AP for all operator ideals $\mathcal{A}$.

Corollary 6.4. (see [3]) *Let X be a Banach space and let $\mathcal{A}$ be an operator ideal. If the trace mapping $V : X^*\hat{\otimes}_\pi X \to \mathcal{A}(X,X)^*$ is isometric, then X has the M-$\mathcal{A}$-AP.*

Proof. Since $V^* : \mathcal{A}(X,X)^{**} \to \mathcal{L}(X,X^{**})$ is the conjugate of an into isometry, for every $T \in (X,X^{**})$, in particular for $T = I_X$, there exists $\varphi \in \mathcal{A}(X,X)^{**}$ satisfying $V^*\varphi = T$ and $\|\varphi\| = \|T\|$. Hence, $I_X \in V^*(B_{\mathcal{A}(X,X)^{**}})$, meaning that X has the M-$\mathcal{A}$-AP.

PROBLEM 6.5. Let $\mathcal{A}$ be an operator ideal containing $\mathcal{K}$ and let a Banach space X have the M-$\mathcal{A}$-AP. Is then the trace mapping $V : X^*\hat{\otimes}_\pi X \to \mathcal{A}(X,X)^*$ isometric?

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7. CONNECTIONS BETWEEN JOINT AND SEPARATE PROPERTIES OF FUNCTIONS OF SEVERAL VARIABLES

V.K. Maslyuchenko

Department of Mathematics, Chernivtsi National University, str. Kotsiubyns'kogo 2*, Chernivtsi,* 58012*, Ukraine, e-mail:* vmasl@chnu.cv.ua

1. We start with probably the most known problem in this concern which is due to M. Talagrand [5] in 1985. By the author's view, this problem is quite difficult.

By $C(f)$ we denote the set of all points of continuity of a function f.

PROBLEM 7.1. Does there exist a Baire space X, a compact Y and a separately continuous function $f : X \times Y \to \mathbb{R}$ for which $C(f) = \emptyset$?

Comment. In [2] it was constructed a completely regular Baire space X, a countably compact space Y and a separately continuous everywhere discontinuous function $f : X \times Y \to \mathbb{R}$ which takes values in the two-point set $\{0,1\}$. Moreover, the space X in this construction is α-favourable in the Choquet's game and the space Y can be chosen to be τ-compact, i.e. for which for an arbitrary open covering of Y of cardinality $\leq \tau$ there exists a finite sub-covering. Besides, in [2] it is found a topology $\mathcal{T}$ on $[0,1]$ such that for the topological space $X([0,1],\mathcal{T})$ and a compact $Y = \beta\mathbb{N} \setminus \mathbb{N}$ there exists a separately continuous everywhere discontinuous function $f : X \times Y \to \mathbb{R}$. A question whether the space X from this construction is Baire remains open.

2. Recall that a topological space Y is called *co-Namioka* if for each Baire space X and each separately continuous function $f : X \times Y \to \mathbb{R}$ there exists a dense G_δ-set A in X such that $A \times Y \subseteq C(f)$. Metrizable, Eberlein, Corson and (more general) Valdivia compacts, and also a Tikhonov cube of an arbitrary weight are co-Namioka. A function $f : X \times Y \to Z$ is called *horizontally almost separately continuous* if it is continuous with respect to the second variable and the set $\{y \in Y : f_y = f(\cdot, y)$ is continuous$\}$ is dense in Y. We say that a topological space Y is a *Hahn space* with respect to a topological space Z if for each topological space X and each horizontally almost separately continuous function $f : X \times Y \to Z$ the set $C_Y(f) = \{x \in X : \{x\} \times Y \subseteq C(f)\}$ is residual in X. A topological space Y is called a *Hahn space* if it is so with respect to any metrizable space Z. Metrizable compacts and, more general the second countable spaces are Hahn spaces but the Tikhonov cube $[0,1]^{[0,1]}$ is not a Hahn space even with respect to $\mathbb{R}$ (see [3]). A. Bouziad [1] showed that Eberlein, Corson and Valdivia compacts are Hahn spaces.

Problem 7.2. Characterize those co-Namioka compacts which are Hahn spaces (with respect to $\mathbb{R}$).

3. Upon some additional conditions on spaces X, Y and Z every separately continuous function $f : X \times Y \to Z$ is quasi-continuous and point-wise discontinuous (i.e. $C(f)$ is dense in $X \times Y$). To this concern, it would be interesting to solve the following problems.

Problem 7.3. Do there exist topological spaces X and Y such that every separately continuous function $f : X \times Y \to \mathbb{R}$ is point-wise discontinuous and simultaneously there is a separately continuous function $f_0 : X \times Y \to \mathbb{R}$ which is not quasi-continuous?

Problem 7.4. Do there exist topological spaces X and Y such that every separately continuous function $f : X \times Y \to \mathbb{R}$ is quasi-continuous and simultaneously there is a separately continuous function $f_0 : X \times Y \to \mathbb{R}$ which is not point-wise discontinuous?

4. The following problem which is due to V.V. Mykhaylyuk and O.V. Sobchuk [4] turned out quite complicated and up today is unsolved, in spite of attempts of mathematicians from Chernivtsi, Lviv and Paris.

Problem 7.5. Does every function $f : [0,1]^2 \to \mathbb{R}$ for which all vertical sections $f^x = f(x,\cdot)$ are continuous and all horizontal sections $f_y = f(\cdot,y)$ belong to the first Baire class is a point-wise limit of a sequence of separately continuous functions $f_n : [0,1]^2 \to \mathbb{R}$?

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8. Extensions of operators

Y. Moreno Salguero [1], A.M. Plichko[2]

[1] *Departamento de Matemáticas, Universidad de Extremadura,* 06071 *Badajoz, Spain, e-mail:* ymoreno@unex.es

[2] *Instytut Matematyki, Politechnika Krakowska, ul. Warszawska* 24, *Kraków, Poland, e-mail:* aplichko@usk.pk.edu.pl

Throughout this section X, Y and E_n are assumed to be finite dimensional normed spaces.

Definition 8.1. Let us say that an X has *b-extension property*, $1 \le b < \infty$, if for each subspace Y of X and each $T \in \mathcal{L}(Y,X)$ there exists an extension $\tilde{T} \in \mathcal{L}(X)$ with $\|\tilde{T}\| \le b\,\|T\|$.

For example, ℓ_2^n and c_0^n have 1-extension property for each $n \in \mathbb{N}$. Our first problem is whether these examples are unique in the following sense.

PROBLEM 8.2. Let E_n has b-extension property and $\dim E_n = n$ for each $n \in \mathbb{N}$. Whether

$$\sup_n \min\Big\{ \mathrm{d}(E_n, \ell_2^n),\ \mathrm{d}(E_n, c_0^n) \Big\} < \infty?$$

DEFINITION 8.3. We say that an X has *b-automorphic property*, $1 \le b < \infty$, if for each subspace Y of X and each injective $T \in \mathcal{L}(Y, X)$ there exists an injective extension $\tilde{T} \in \mathcal{L}(X)$ with $\|\tilde{T}\|\|\tilde{T}^{-1}\| \le b\,\|T\|\|T^{-1}\|$.

Of course, ℓ_2^n has 1-automorphic property for each $n \in \mathbb{N}$.

PROBLEM 8.4. Whether c_0^n has b-automorphic property for each $n \in \mathbb{N}$ and some $b \ge 1$?

PROBLEM 8.5. Let E_n has b-automorphic property and $\dim E_n = n$ for each $n \in \mathbb{N}$. Whether

$$\sup_n \mathrm{d}(E_n, \ell_2^n) < \infty?$$

9. On almost isometric Banach spaces which are not isometric

E. Odell

Department of Mathematics, The University of Texas at Austin, 1 *University Station* C1200*, Austin, TX, USA., e-mail:* odell@math.utexas.edu

We use the following notations for Banach spaces: $X \sim Y$ means that X and Y are isomorphic and $X \cong Y$ means that X and Y are isometric. For a Banach space X we set $I(X) = \{Y : \ Y \sim X\}$.

Recall that the Banach-Mazur distance between Banach spaces X and Y is defined as

$$\mathrm{d}(X, Y) = \inf\{\|T\|\|T^{-1}\| : \ T : X \to Y \quad \text{is an isomorphism}\}.$$

Obviously, if $X \cong Y$ then $\mathrm{d}(X, Y) = 1$. It is a well known fact that the converse is not true in general.

PROBLEM 9.1. Does for every separable infinite dimensional Banach space X there exist Banach spaces $Y, Z \in I(X)$ such that $\mathrm{d}(Y, Z) = 1$ and $Y \ncong Z$? This is true for $X = c_0$ or $X = \ell_p$, $1 \le p < \infty$, for example.

PROBLEM 9.2. Describe the separable infinite dimensional Banach spaces X having the following property: for each Banach space Y the condition $\mathrm{d}(X,Y)=1$ implies $Y \cong X$.

G. Godefroy pointed out to us that c_0 is such a space. Evidently the spaces ℓ_p and L_p also have this property.

10. On a property of bases

E. Odell

Department of Mathematics, The University of Texas at Austin, 1 *University Station* C1200*, Austin, TX, USA, e-mail:* odell@math.utexas.edu

OBSERVATION. Let $(e_i)_1^\infty$ be a normalized monotone basis of a Banach space X having the following property: for each $C \geq 1$ there is a constant $D \geq 1$ such that for every normalized monotone basis (x_i) of any Banach space, if $i_1 < \cdots < i_n$ and $(x_{i_j})_{j=1}^n$ is C-equivalent to $(e_i)_1^n$ then

$$\Big\|\sum_{j=1}^n a_{i_j} x_{i_j}\Big\| \leq D \Big\|\sum_{j=1}^\infty a_j x_j\Big\|$$

for any scalars $(a_i)_{i=1}^\infty$. Then (e_i) is equivalent to the unit vector basis of c_0.

PROBLEM 10.1. Let $(e_i)_1^\infty$ be a normalized monotone subsymmetric basis of a Banach space X having the following property: for each $C \geq 1$ there is a constant $D \geq 1$ such that every normalized weakly null sequence (x_i) in any Banach space admits a subsequence $(y_i)_1^\infty \subseteq (x_i)_1^\infty$ such that if $i_1 < \cdots < i_n$ and $(y_{i_j})_{j=1}^n$ is C-equivalent to $(e_i)_1^n$ then

$$\Big\|\sum_{j=1}^n a_{i_j} y_{i_j}\Big\| \leq D \Big\|\sum_{j=1}^\infty a_j y_j\Big\|$$

for any scalars $(a_i)_i^\infty$. Is (e_i) equivalent to the unit vector basis of c_0 or ℓ_1? If not then characterize these bases.

The fact that ℓ_1 has this property is due to Argyros, Mercourakis and Tsarpalias [1]. ℓ_p fails this property for $1 < p < \infty$.

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11. Uniqueness of Hahn-Banach extensions of functionals on compact operators

E. Oja[1], **M. Põldvere**[2]

[1]*Faculty of Mathematics and Computer Science, Tartu University, J. Liivi 2,* EE-50409 *Tartu, Estonia, e-mail:* eve.oja@ut.ee

[2]*Institute of Pure Mathematics, Tartu University, J. Liivi* 2, 50904 *Tartu, Estonia, e-mail:* mart.poldvere@ut.ee

Let X be a Banach space and let $Y \subset X$ be a closed subspace.

Definition 11.1. (R. R. Phelps [4], 1960) The subspace Y has *property* U in X if every functional $g \in Y^*$ has a unique Hahn-Banch extension $f \in X^*$ (i.e. there is exactly one $f \in X^*$ satisfying $f|_Y = g$ and $\|f\| = \|g\|$).

Definition 11.2. (G. Godefroy, N.J. Kalton, P.D. Saphar [1], 1993) The subspace Y is an *ideal* in X if there exists a bounded linear projection P on X^* such that $\|P\| = 1$ and $\ker P = Y^{\perp} := \{f \in X^* \colon\ f|_Y = 0\}$.

Definition 11.3. (E.M. Alfsen, E.G. Effros, 1972, see [2]) The subspace Y is an M*-ideal* in X if Y is an ideal in X with respect to an ideal projection P such that, for all $f \in X^*$,

$$\|f\| = \|Pf\| + \|f - Pf\|.$$

Definition 11.4. (Å. Lima, 1977, see [2]) The subspace Y is a *semi* M*-ideal* in X if whenever $g \in X^*$ is such that $\|g|_Y\| = \|g\|$, and $h \in Y^{\perp}$, then

$$\|g + h\| = \|g\| + \|h\|.$$

Clearly, Y is an M-ideal in X if and only if it is both an ideal and a semi M-ideal in X; and if Y is a semi M-ideal in X, then it has property U in X.

The following problems concern property U for the subspace $K(X)$ of compact operators in the Banach space $L(X)$ of all bounded linear operators on X. (Property U for $K(X)$ in $L(X)$ has been studied e.g. in [3] and [5].)

Problem 11.5. Does there exist a Banach space X such that $K(X)$ has property U in $L(X)$ without being an ideal in $L(X)$?

The same question is open for semi M-ideals.

Problem 11.6. Does there exist a Banach space X such that $K(X)$ is a semi M-ideal in $L(X)$ without being an M-ideal in $L(X)$?

It is known that $K(X)$ is an M-ideal in $L(X)$ if and only if $K(X)$ is an M-ideal in $\text{span}\big(K(X)\cup\{I_X\}\big)$, the linear span of $K(X)$ and I_X, the identity operator on X (see e.g. [2]).

PROBLEM 11.7. Does $K(X)$ have property U in $L(X)$ if $K(X)$ has property U in $\text{span}\big(K(X)\cup\{I_X\}\big)$?

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12. NORM-ATTAINING OPERATORS

M.I. Ostrovskii

Department of Mathematics, The Catholic University of America, Washington, D.C. 20064*, U.S.A., e-mail:* ostrovskii@cua.edu

Let X be a Banach space, $L(X)$ the space of all bounded linear operators on X. We say that $A\in L(X)$ *attains its norm* if there exists $x\in X$ such that $||x||=1$ and $||Ax||=||A||$. The following problem sounds as a classical one. Nevertheless, to the best of my knowledge, it is still open.

PROBLEM 12.1. Does there exist an infinite dimensional Banach space X such that each $A\in L(X)$ attains its norm?

RELATED RESULTS AND OBSERVATIONS:

(1) R. C. James's characterization of reflexivity implies that if X is such that each $A\in L(X)$ attains its norm, then X and $L(X)$ are reflexive spaces. (To show that $L(X)$ is reflexive, we use the identification $L(X)=(X\hat{\otimes}_\pi X^*)^*$.

Recall that this identification is valid for any reflexive X without any approximation property assumptions.)

(2) J. R. Holub [2] proved that if X has the approximation property, then the reflexivity of $L(X)$ implies that X is finite dimensional.

(3) N. J. Kalton [3, Theorem 2] proved that $L(X)$ cannot be reflexive for nonseparable X.

(4) Hence the only possible candidates for X in the problem are separable reflexive spaces without 1-complemented infinite-dimensional subspaces having the approximation property.

(5) Some related results and observations can be found in [1] (see p. 693), [4], and [5].

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13. Weak embeddings of L_1

M.M. Popov

Department of Mathematics, Chernivtsi National University, str. Kotsiubyns'kogo 2, *Chernivtsi,* 58012*, Ukraine,*
e-mail: popov@chv.ukrpack.net

Recall that an operator $T \in \mathcal{L}(X,Y)$ between Banach spaces X and Y is said to be an (isomorphic) embedding provided $\|Tx\| \geq \delta\|x\|$ for some $\delta > 0$ and each $x \in X$. We shall discuss the following three weaker notions.

Definition 13.1. (H.P. Lotz, N.T. Peck, H. Porta, 1979) An injective operator $T \in \mathcal{L}(X,Y)$ is called a semi-embedding if TB_X is closed.

Definition 13.2. (J. Bourgain, H.P. Rosenthal, 1983) An injective operator $T \in \mathcal{L}(X,Y)$ is called a G_δ-embedding if TK is a G_δ-set for each closed bounded $K \subset X$.

Definition 13.3. (H.P. Rosenthal, 1981) An injective operator $T \in \mathcal{L}(L_1, X)$ is called a sign-embedding if $\|Tx\| \geq \delta\|x\|$ for some $\delta > 0$ and every sign $x \in L_1$ (sign means that x takes values from $\{-1, 0, 1\}$).

The most interesting case is when the domain space is L_1 and we shall consider only it in the sequel. It is said that L_1 semi-embeds (or sign-embeds, or "other type"-embeds) in a Banach space X provided there exists a semi-embedding (sign-embedding, or respectively "other type" embedding) $T \in \mathcal{L}(L_1, X)$.

The connections between these notions can be described as follows.

(i) Every semi-embedding is automatically a G_δ-embedding [1] and no other implication is true [2]. Hence, if L_1 semi-embeds in X then L_1 G_δ-embeds in X.

(ii) If L_1 G_δ-embeds in X then L_1 sign-embeds in X [3]. Hence, if L_1 semi-embeds in X then L_1 sign-embeds in X.

(iii) There exists a Banach space X which contains no subspace isomorphic to L_1 such that L_1 semi-embeds in X [4]. Hence, all three notions of weak embedding are in fact weaker than the notion of isomorphic embedding even passing to subspaces.

Problem 13.4. A) Suppose that L_1 sign-embeds in X. Does L_1 G_δ-embed in X?

B) Suppose that L_1 G_δ-embeds in X. Does L_1 semi-embed in X?

C) Suppose that L_1 sign-embeds in X. Does L_1 semi-embed in X?

For more details we refer to [2].

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14. Greedy approximation bases

P. Wojtaszczyk

Instytut Matematyki Stosowanej i Mechaniki, Uniwersytet Warszawski, 02-097 *Warszawa, ul. Banacha* 2, *e-mail:* pwojt@mimuw.edu.pl

Let $(x_n)_1^\infty$ be a normalized basis for a Banach space X and $(x_n^*)_1^\infty$ be its biorthogonal functionals. Given $x \in X$ and $m \in \mathbb{N}$, we define *greedy approximation m-th partial sum* by putting

$$\mathcal{G}_m(x) = \sum_{n \in \Lambda} x_n^*(x) x_n$$

where Λ is a subset of the integers with $|\Lambda| = m$ such that $|x_n^*(x)| \geq |x_s^*(x)|$ for each $n \in \Lambda$ and $s \notin \Lambda$. Sometimes the set Λ is not uniquely defined but then we take any Λ.

Definition 14.1. A normalized basis $(x_n)_1^\infty$ for a Banach space X is called to be *C-greedy* $(1 \leq C < \infty)$ if

$$\|x - \mathcal{G}_m(x)\| \leq C \inf\Big\{ \Big\| x - \sum_{j \in \Lambda} \alpha_j x_j \Big\| : \ |\Lambda| = m, \ \alpha_j \in \mathbb{R} \Big\}$$

for each $x \in X$.

It can be shown that this definition does not depend on the choice of Λ in case there is any ambiguity (see [2]). A basis which is C-greedy for some C is called greedy.

Problem 14.2. Find an example of a 1-greedy non-symmetric basis.

It is known that the Haar system is a greedy basis for L_p where $1 < p < \infty$ (see [1]). Recently I have shown (unpublished) that L_p are the only such rearrangement invariant spaces, that is we have

Theorem 14.3. *Let X be a rearrangement invariant function space on $[0,1]$ such that the Haar system is a greedy basis in X. Then $X = L_p[0,1]$ for some $1 < p < \infty$ (maybe with equivalent norm).*

It is easy to show that symmetric basis is greedy. It is known that natural rearrangement invariant spaces (different from Hilbert space) do not have symmetric basis. This suggest the following

Problem 14.4. Does the Lorentz space $L_{p,q}$ have a greedy basis if $p \neq q$?

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15. Subspaces of biduals

D. Yost

School of I.T. & Mathematical Sciences, University of Ballarat, Australia 3353, *e-mail:* dyost@ballarat.edu.au

Problem 15.1. If X is an infinite-dimensional Banach space, does X^{**} contain an infinite-dimensional reflexive subspace?

Clearly not every Banach space contains an infinite-dimensional reflexive subspace; c_0 is an obvious counterexample. Nor does every dual space; ℓ_1 is a splendid counterexample. But for biduals, who knows?

This is the case for every example of a Banach space known today, but there is no clear reason why it should always be true. This may be considered the latest in a sequence of progressively weaker questions, for each of which counterexamples have been found.

Perhaps it all began with the question of whether every infinite-dimensional Banach space contains an isomorphic copy of either c_0 or ℓ_p for some finite value of p. This was eventually disproved by Tsirelson. His example was reflexive, even super-reflexive.

A weaker question is whether every infinite-dimensional Banach space contains an isomorphic copy of either c_0, ℓ_1 or an infinite-dimensional reflexive subspace. Recall the result of James, that this is so in every Banach space with an unconditional basis. For general Banach spaces, a counterexample was eventually found by Gowers.

Note that c_0^{**} and ℓ_1^{**} both contain infinite dimensional Hilbert spaces. Thus our question is a weakening of the preceding question. The space of Gowers may also turn to be a counterexample for our question but it is unclear at this stage.

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16. On m-convex compacts in $\mathbb{R}^n$

Ju. B. Zelins'kyj

Institute of Mathematics of National Acad. Sci. Ukraine, Kyiv, Ukraine, e-mail: zel@imath.kiev.ua

Definition 16.1. A compact $K \subset \mathbb{R}^n$ is called to be *m-convex* $(m < n)$ if for each $x \in \mathbb{R}^n \setminus K$ there exists an m-plane $T(x)$ such that $x \in T(x)$ and $T(x) \cap K = \emptyset$.

Problem 16.2. Does there exist a homeomorphic embedding of $K = S_2$ to $\mathbb{R}^4$ with 2-convex range?

Problem 16.3. Let $K \subset \mathbb{R}^n$ be a compact. Suppose that for each hyperplane $T \subset \mathbb{R}^n$ the intersection $T \cap K$ is $(m-1)$-convex. Find a condition on K which together with the above one would be sufficient for K to be m-convex.