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Equivariant Unfoldings of G-Stratified Pseudomanifolds

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To Rodolfo Ricabarra, in memoriam.

Introduction

Let X be a Thom-Mather stratified space with depth $d(X) = n$. The De Rham Intersection Cohomology of X with differential forms was defined in [3] by means of an auxiliar construction called *unfolding*, which is a continuous map $\mathcal{L} : \widetilde{X} \to X$ where $\widetilde{X}$ is a smooth manifold obtained trough a finite composition

$$\mathcal{L} : \widetilde{X} = X_n \xrightarrow{\mathcal{L}_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{\mathcal{L}_1} X_0 = X$$

of topological operations $X_i \xrightarrow{\mathcal{L}_i} X_{i-1}$, called *elementary unfoldings*. This iterative construction is possible because the stratification of X is controlled by the existence of a family of conical fiber bundles over the singular strata. Later in [11] we find a more abstract definition of unfoldings, which impose some conditions of transversality over the singular strata. For instance, if the depth of X is 1 then the first elementary unfolding of X is an unfolding in the new sense.

Now let G be a compact Lie group. We introduce the definition of a G-stratified pseudomanifold in the category of stratified pseudomanifolds. Our definition is related to a previous one given in [10]. A G-stratified pseudomanifold is a stratified pseudomanifold in the usual sense together with a continuous action preserving the strata, and whose local model near each singular strata is given by a conical slice. We also give a definition of equivariant unfoldings, which is a suitable adaptation of the usual definition of unfolding to the family of G-stratified pseudomanifolds. We give a sufficient condition for the existence of equivariant unfoldings, which is related to the choice of a good family of tubular neighborhoods and a sequence of equivariant elementary unfoldings of X. Additionally, if G is abelian, then each elementary unfolding induces an elementary unfolding on each orbit space together with a factorization diagram.

The content of this paper is the following. In Sections 1 and 2 we recall the usual definition of stratified pseudomanifolds and give an equivariant version for stratified actions of a compact Lie Group G. In Section 3 we present the G-stratified fiber bundles and show some of their main properties. Section 4 is devoted to the study equivariant tubular neighborhoods, which are equivariant versions of the usual ones. In Section 5 we pass to the orbit space of a stratified action and provide some useful factorization theorems. Finally, in Sections 6 and 7 we develop the construction of equivariant unfoldings trough a finite composition of elementary desingularizations (see, for instance, [1]).

1. Stratified pseudomanifolds

In this section we review the usual definitions of stratified spaces, stratified morphisms and stratified pseudomanifolds. For a more detailed introduction see [7], [9].

1.1. Stratified spaces Let X be a Hausdorff, locally compact and second countable space. A *stratification* of X is a locally finite partition $\mathcal{S}_X$ satisfying:

(i) Each element $S \in \mathcal{S}_X$ is a connected manifold with the induced topology, which a *stratum* of X.

(ii) If $S' \cap \overline{S} \neq \emptyset$ then $S' \subset \overline{S}$ for any two strata S, $S' \in \mathcal{S}_X$. In this case we write $S' \leq S$ and we say that S *incides* on S'.

We say that $(X, \mathcal{S}_X)$ is a *stratified space* whenever $\mathcal{S}_X$ is a stratification of X.

With the above conditions, the incidence relationship is a partial order on $\mathcal{S}_X$. More over, since $\mathcal{S}_X$ is locally finite, any strictly ordered chain

$$S_0 < S_1 < \cdots < S_m$$

in $\mathcal{S}_X$ is finite. The *depth* of X is by definition the supreme (possibly infinite) of the integers m such that there is a strictly ordered chain as above. We write this as $d(X)$.

The maximal (resp. minimal) strata in X are open (resp. closed) in X. A *singular* stratum is a non-maximal stratum in X. The union of the singular strata is the *singular part* of X, denoted by $\Sigma \subset X$, which is closed in X. Its complement $X - \Sigma$ is open and dense in X. The family of minimal strata will often be denoted by $\mathcal{S}_X^{min}$, while the union of minimal strata will be denoted by Σ^{min}, which we call the *minimal part* of X.

1.2. Examples (1) For any manifold M the trivial stratification of M is the family

$$\mathcal{S}_M = \{C : C \text{ is a connected component of } M\}.$$

(2) For any connected manifold M, the space $M \times X$ is a stratified space, with the stratification

$$\mathcal{S}_{M\times X} = \{M \times S : S \in \mathcal{S}_X\}.$$

Notice that $d(M \times X) = d(X)$.

(3) The *cone* of a compact stratified space L is the quotient space

$$c(L) = L \times [0, \infty)/L \times \{0\}.$$

We write $[p, r]$ for the equivalence class of $(p, r) \in L \times [0, \infty)$. The symbol $*$ will be used for the equivalence class of $L \times \{0\}$, this is the *vertex* of the cone. The family

$$\mathcal{S}_{c(L)} = \{*\} \cup \{S \times (0, \infty) : S \in \mathcal{S}_L\}$$

is the canonical stratification of $c(L)$. Notice that $d(c(L)) = d(L) + 1$.

1.3. Stratified subspaces and morphisms Let $(X, \mathcal{S}_X)$ be a stratified space. For each subset $Z \subset X$ the *induced partition* is the family

$$\mathcal{S}_{Z/Y} = \{C : C \text{ is a connected component of } Z \cap S,\ S \in \mathcal{S}_X\}.$$

We will say that Z is a *stratified subspace* of X, whenever the induced partition on Z is a stratification of Z.

Now let $(Y, \mathcal{S}_Y)$ be another stratified space. A *morphism* (resp. *isomorphism*) is a continuous map $f : X \to Y$ (resp. homeomorphism) which smoothly (resp. diffeomorphicaly) sends strata into strata. In particular, f is a *embedding* if $f(X)$ is a stratified subspace of Y and $f : X \to f(X)$ is an isomorphism.

Henceforth, we will write $\mathrm{Iso}(X, \mathcal{S}_X)$ for the group of isomorphisms of a stratified space X. The following statement will be used later, we leave the proof to the reader.

Lemma 1.4. *Let $(X, \mathcal{S}_X)$ be a stratified space, and $\mathfrak{F} \subset \mathcal{S}_X$ a subfamily of equidimensional strata. The connected components of $M = \cup_{S \in \mathfrak{F}} S$ are the strata in $\mathfrak{F}$.*

Stratified pseudomanifolds were used by Goresky and MacPherson in order to introduce the Intersection Homology and extend the Poincaré duality to the family of stratified spaces [5], [6]; see also [7] for a brief introduction.

1.5. Stratified pseudomanifolds The definition of a stratified pseudomanifold is made by induction on the depth of the space. More precisely:

(1) A stratified pseudomanifold of depth 0 is a manifold with the trivial stratification.

(2) An arbitrary stratified space $(X, \mathcal{S}_X)$ is a *stratified pseudomanifold* if, for any singular stratum $S \in \mathcal{S}_X$, there is a compact stratified pseudomanifold L_S depending on S (called a *link* of S) such that each point $x \in S$ has a coordinate neighborhood $U \subset S$ and an embedding onto an open subset of X.

$$\varphi : U \times c(L_S) \to X$$

such that $x \in \mathrm{Im}(\varphi)$. The pair (U, φ) is a *chart* of x *modeled* on L_S.

1.6. Examples Here are some examples of stratified pseudomanifolds.

(1) If X is a stratified pseudomanifold, then any open subset $A \subset X$ is also a stratified pseudomanifold. Also the product $M \times X$ (with the canonical stratification) is a stratified pseudomanifold, for any manifold M.

(2) If L is a compact stratified pseudomanifold, then $c(L)$ is a stratified pseudomanifold.

2. G-stratified pseudomanifolds

From now on, we fix an abelian compact Lie group G. We will study the family of actions of G which preserve the strata. Our definition is strongly related to the previous one given in [10]. Also some easy proofs in this section can be seen in [8].

Given a stratified space $(X, \mathcal{S}_X)$ and a effective action $\Phi : G \times X \to X$; we write $\Phi(g, x) = gx$ for any $g \in G$, $x \in X$. We denoted X/K by the K-orbit space for every K closed subgroup of G, and by $\pi : X \to X/K$ the orbit map. The group of G-equivariant isomorphisms of X will be denoted by $\mathrm{Iso}_G(X, \mathcal{S}_X)$.

2.1. G-STRATIFIED SPACES We say that X is *G-stratified* whenever:

(1) For each stratum $S \in \mathcal{S}_X$ the points of S all have the same isotropy group, denoted by G_S.

(2) Each $g \in G$ induces an isomorphism $\Phi_g : X \to X \in \mathrm{Iso}_G(X, \mathcal{S}_X)$.

2.2. EXAMPLES Here are some examples of G-stratified spaces:

(1) Each G-manifold M has a natural structure of G-stratified space, when M is endowed with the stratification given by orbit types.

(2) If X is a G-stratified space, then $M \times X$ is a G-stratified space with the action $g(m, x) = (m, gx)$; for any manifold M.

(3) If L is a compact G-stratified space then $c(L)$, with the action $g[x, r] = [gx, r]$, is a G-stratified space.

Now we introduce the definition of a G-stratified pseudomanifold.

2.3. G-STRATIFIED PSEUDOMANIFOLDS A G-stratified pseudomanifold is a stratified pseudomanifold in the usual sense, endowed with a structure of G-stratified space (i.e. G acts by isomorphisms) and whose local model is described through conical slices. Conical slices were introduced in [10] in order to state a sufficient condition on any continuous action of a compact Lie group (abelian or not) on stratified pseudomanifold so that the corresponding orbit space would remain in the same class of spaces.

Let $(X, \mathcal{S}_X)$ be a G-stratified space. Take a singular stratum $S \in \mathcal{S}_X$ a point $x \in S$. A *conical slice* of x in X is a slice S_x in the usual sense of [2], with a conical part transverse to the stratum S. In other words:

(1) S_x is an invariant G_S-space containing x.

(2) For any $g \in G$, if $gS_x \cap S_x \neq \emptyset$ then $g \in G_S$.

(3) GS_x is open in X. And

(4) There is a G_S-equivalence $\beta : \mathbb{R}^i \times c(L) \to S_x$ where $i \geq 0$ and L is a compact G_S-stratified space. Here the action of G_S on $\mathbb{R}^i$ is trivial (notice that β induces on S_x a structure of G_S-stratified space).

The definition of a *G-stratified pseudomanifold* is made by induction on the depth of the space. A G-stratified pseudomanifold with depth 0 is a manifold with a smooth free action of G. In general, we will say that X is a G-stratified pseudomanifold if, for each singular stratum $S \in \mathcal{S}_X$, there is a compact G_S-stratified pseudomanifold L_S such that each point $x \in S$ has a conical slice

$$\beta : \mathbb{R}^i \times c(L_S) \to S_x$$

and the usual map on the twisted product

$$\alpha : G \times_{G_S} S_x \to X \qquad \alpha([g, y]) = gy$$

is an equivariant (stratified) embedding on an open subset of X. We say that the triple (S_x, β, L_S) is a *distinguished slice* of x.

2.4. Examples Here there are some examples of G-stratified pseudomanifolds.

(1) Take a smooth effective action $\Phi : G \times M \to M$ with fixed points on a manifold M endowed with the stratification by orbit types. By the Equivariant Slice Theorem, M is a G-stratified pseudomanifold.

(2) If X is a G-stratified pseudomanifold then $M \times X$ is a G-stratified pseudomanifold with the obvious action.

(3) If L is a compact G-stratified pseudomanifold, then $c(L)$ is a G-stratified pseudomanifold with the obvious action.

(4) Any invariant open subspace of a G-stratified pseudomanifold is itself a G-stratified pseudomanifold.

2.5. Remark *Each G-stratified pseudomanifold is a stratified pseudomanifold in the previous sense.*

To see this, proceed by induction on the depth. Take a G-stratified pseudomanifold X. For $d(X) = 0$ the statement is trivial. Assume the inductive hypothesis and suppose that $d(X) > 0$. Take a singular stratum $S \in \mathcal{S}_X$, a point $x \in S$ and a distinguished slice (S_x, β, L_S) of x. The isotropy subgroup G_S acts on G by the restriction of the group operation. We fix a slice S_e of the identity element $e \in G$ with respect to this action. Since $G_S S_e$ is open in G, the composition

$$\begin{aligned}(S_e \times \mathbb{R}^i) \times c(L_S) \to S_e \times (\mathbb{R}^i \times cL_S) \to S_e \times S_x \to \\ \to S_e \times (G_S \times_{G_S} S_x) \to (G_S S_e) \times_{G_S} S_x \to X\end{aligned}$$

is an embedding. Notice that by induction L_S is a stratified pseudomanifold, since $S_e \times \mathbb{R}^i \simeq S_e G_S (S \cap S_x)$ is open in S. We have obtained a chart of x modeled on L_S.

2.6. Remark *If X is a G-stratified pseudomanifold and K is any closed subgroup of G, then X is also a K-stratified pseudomanifold.*

It is straightforward that X is a K-stratified space. For any singular stratum S and any $x \in S$, in order to choose a distinguished slice in x we proceed as follows: Take a distinguished slice $\beta : \mathbb{R}^i \times c(L_S) \to S_x$ in x with respect to the action of G. Take also a slice V_e of the identity element $e \in G$ with respect to the action of $G_S K$ in G. Then $\imath \times \beta : (V \times \mathbb{R}^i) \times c(L_S) \to V S_x$ is a distinguished slice of x with respect to the action of K.

3. G-stratified fiber bundles

Henceforth we fix a compact, abelian Lie group G. In this section we introduce the notion of a G-stratified fiber bundle. This is a previous step in order to study the family of equivariant tubular neighborhoods. The reader will find in [12] a detailed introduction to the fiber bundles, while [13] provides the usual definition of a tubular neighborhood in the stratified context (see also [2] for the smooth case).

3.1. G-stratified fiber bundles Let $\xi = (E, p, B, F)$ be a locally trivial fiber bundle with (maximal) trivializing atlas $\mathcal{A}$. We will say that ξ is a *G-stratified fiber bundle* whenever:

(1) The total space E is a G-stratified space.

(2) The base space B is a manifold, endowed with a smooth action $\Psi : G \times B \to B$ and with constant isotropy $H \subset G$ at all its points.

(3) The fiber F is a H-stratified space.

(4) The projection $p : E \to B$ is G-equivariant.

(5) The group G acts by isomorphisms. In other words, each chart

$$\varphi : U \times F \to p^{-1}(U) \in \mathcal{A}$$

is H-equivariant; and for any two charts $(U, \varphi), (U', \varphi') \in \mathcal{A}$ such that $U' \cap g^{-1} U \neq \emptyset$ for some $g \in G$, there is a map

$$g_{\varphi,\varphi'} : U' \cap g^{-1} U \to \mathrm{Iso}_H(F, \mathcal{S}_F)$$

such that

$$\varphi^{-1} g \varphi'(b, z) = (gb, g_{\varphi,\varphi'}(b) z).$$

Lemma 3.2. *Let $\xi = (E, p, B, F)$ be a G-stratified fiber bundle, H the isotropy of B. If F is an H-stratified pseudomanifold, then E is a G-stratified pseudomanifold.*

Proof. Fix a singular stratum S in E and a point $x \in S$. We must prove the existence of a link L_S depending only on S and, a distinguished slice (S_x, β, L_S) in x. For this purpose, let's take a trivializing chart

$$\varphi : U \times F \to p^{-1}(U) \in \mathcal{A}$$

such that $x \in p^{-1}(U)$. Take $z = p(x)$ and a G-slice V_z in B. Since V_z is contractible, we assume that $V_z \cong \mathbb{R}^k$ and V_z is contained in U.

Write $\varphi^{-1}(x) = (z, y) \in V_z \times F$ and take S' the stratum in F containing y. Since F is an H-stratified pseudomanifold, we can choose a distinguished slice S_y in y; say

$$\beta_0 : S_y \to \mathbb{R}^i \times c(L_{S'}).$$

Consider the following composition

$$\varphi(V_z \times S_y) \overset{\varphi^{-1}}{\to} V_z \times S_y \overset{\imath \times \beta_0}{\to} V_z \times \mathbb{R}^i \times c(L_{S'}) \cong R^{i+k} \times c(L_{S'}).$$

We will show that

$$(S_x, \beta, L_S) = (\varphi(V_z \times S_y), (\imath \times \beta_0) \circ \varphi^{-1}, L_{S'})$$

is a distinguished slice in x. We proceed in three steps.

• *L_S only depends on S*: If $(U', \psi) \in \mathcal{A}$ is another trivializing chart covering x, $\psi^{-1}(x) = (z, y') \in V_z \times F$ and $\beta_0' : S_{y'} \to \mathbb{R}^i \times c(L_{S''})$ is a distinguished slice in y'; then the composition $\beta' \beta^{-1}$ induces an H-isomorphism $L_S \overset{\cong}{\to} L_{S''}$.

• *S_x is a conical slice*: We verify the conditions (1) to (4) of § 2.3.
(1) Since V_z is a slice of $z \in B$, we have $gp(x) = p(gx) = p(x) \in V_z$ for any $g \in G_S$. So $G_S = H \cap G_S = H_S$, but φ is H-equivariant, hence $G_S = H_S = H_{S'}$. Again, since φ is H-equivariant and S_y is $H_{S'} = G_S$ invariant, we obtain that S_x is G_S-invariant.
(2) Take $g \in G$, $x' \in S_x$ such that $gx' \in S_x$. Then $gp(x') = p(gx') \in V_z$, so $g \in H$ and $gp(x') = p(x')$. Since φ is H-equivariant, if $x' = \varphi(p(x'), y)$ then $g.x' = \varphi(p(x'), gy)$, and $gy \in S_y$; hence $g \in H_{S'} = G_S$.
(3) Take a slice S_e of the identity element $e \in G$ with respect to the action of H. Since S_e is contractible, we can assume that $S_e V_z \subset U$. Notice that $S_e H$

is open in G. Since $GS_x = \bigcup_{g \in G} g(S_e H)S_x$, we only have to show that $(S_e H)S_x$ is open in X. But φ is H-equivariant and the action of H on V_z is trivial, so we get the following equality

$$(S_e H)S_x = S_e\left(H\varphi(V_z \times S_y)\right) = S_e\varphi(V_z \times HS_y).$$

Since HS_y is open in F we deduce that $S_e\varphi(V_z HS_y)$ is open in $S_e\varphi(V_z \times F)$. Finally we show that $S_e\varphi(V_z \times F) = S_e p^{-1}(V_z)$ is open in X: Since p is equivariant and $S_e V_z$ is open in U the set $S_e p^{-1}(V_z) = p^{-1}(S_e V_z) = p^{-1}(S_e H V_z)$ is open in $p^{-1}(U)$ (and so in X).

(4) It is straightforward that the map β is a G_S-equivalence.

- *S_x is a distinguished slice*: We will show that usual the map

$$\alpha : G \times_{G_S} S_x \to X$$

is a (stratified) embedding.

(a) *α preserves the strata*: Take a stratum S^0 in S_x. We will prove that $G'S^0$ is an open subset in some stratum of X, for any connected component $G' \subset G$. It is enough to prove this for the connected component G_0 of the identity element $e \in G$. Let H_0 be the connected component of the identity element $e \in H$. The set $S_e H_0$ is a connected open subset in $S_e H$, so is also connected and open in G_0. Since $G_0 S^0$ is connected, we need to prove that $S_e H_0 S^0$ is open in some stratum of X. But $S_e H S_x$ is contained in $p^{-1}(S_e V_z)$ and φ is a stratified embedding, and so we only have to show that $\varphi^{-1}(S_e H_0 S^0)$ is open in some stratum of $(S_e V_z) \times F$. Consider the map

$$\begin{array}{rccl} f: & S_e H \times V_z \times S_y & \to & (S_e V_z) \times F \\ & (gh, b, l) & \mapsto & \big(ghb, (gh)_{\varphi\varphi}(b)(z)\big) = \big(gb, g_{\varphi\varphi}(b)(hz)\big). \end{array}$$

Let S^1 be the stratum of S_y such that $S^0 = \varphi(V_z \times S^1)$. By hypothesis S_y is a distinguished slice of y in F, and there is a stratum S^2 in F such that $H_0 S^1$ is open in S^2. Notice that

$$\varphi^{-1}(S_e H_0 S^0) = f(S_e H_0 \times V_z \times S^1) = f(S_e \times V_z \times H_0 S^1).$$

Also, since φ is H-equivariant, we have

$$p\big(\varphi^{-1}(S_e H_0 S^0)\big) = S_e V_z.$$

Hence the projection $pr_2 : U \times F \to F$ sends $\varphi^{-1}(S_e H_0 S^0)$ on some open subset of S^2. Notice that $\varphi^{-1}(S_e H_0 S^0)$ is connected, so

$$pr_2\big(\varphi^{-1}(S_e H_0 S^0)\big) = \bigcup_{(g,b)\in S_e\times V_z} g_{\varphi\varphi}(b)(H_0 S^1)$$

is a connected subset of F. Each $g_{\varphi\varphi}(b)$ is an H-equivariant stratified isomorphism; hence $g_{\varphi\varphi}(b)(H_0 S^1)$ is open is some stratum of F with the same dimension of S^2. Since $e_{\varphi\varphi}(b)(H_0 S^1) = H_0 S^1 \subset S^2$, by § 1.4 the set

$$\bigcup_{(g,b)\in S_e\times V_z} g_{\varphi\varphi}(b)(H_0 S^1)$$

is contained in S^2.

(b) *α is smooth on each stratum*: Since $G\times_{G_x} S_x$ has the quotient stratification induced on $G \times S_x$ by the action of H, the stratification of S_x is induced by X and the action of G is smooth on each stratum of $G \times X$. We conclude that the restriction of α to each stratum is smooth. ∎

4. Equivariant tubular neighborhoods

In this section we will study the family of equivariant tubular neighborhoods, which are equivariant version of the usual ones. We fix, as before, a compact Lie Group G and a G-stratified pseudomanifold X. Given a singular stratum S in X, a tubular neighborhood is just a locally trivial fiber bundle over a S whose fiber is $c(L_S)$, the cone of the link of S, and whose structure group is $\mathrm{Iso}_{G_S}(L_S, \mathcal{S}_{L_S})$.

4.1. Equivariant tubular neighborhoods An equivariant tubular neighborhood is a conical locally trivial fiber bundle. For a detailed introduction the reader can see [9], [13]. In [1], the tubular neighborhoods are used in order to show the existence of an unfolding for any manifold endowed with a Thom-Mather structure. We will provide an equivariant version of this fact for any G-stratified pseudomanifold.

Let X be a G-stratified pseudomanifold with $d(X) > 0$. Let's take a singular stratum S in X. An *equivariant tubular neighborhood* of S is a G-stratified fiber bundle $(T_S, \tau_S, S, c(L_S))$ with (maximal) trivializing atlas $\mathcal{A}$, verifying

(1) T_S is an open invariant neighborhood of S and the inclusion $S \to T_S$ is a section of $\tau_S : T_S \to S$.

(2) G preserves the conical radium: For any two charts $(U,\varphi),(U',\varphi^{'}) \in \mathcal{A}$ such that $U' \cap g^{-1}U \neq \emptyset$ for some $g \in G$, there is a map

$$g_{\varphi,\varphi'} : U' \cap g^{-1}U \to \mathrm{Iso}_{G_S}(L_S, \mathcal{S}_{L_S})$$

such that

$$\varphi^{-1} g \varphi' (b, [l,r]) = \Big(gb, [g_{\varphi,\varphi'}(b)l, r]\Big) .$$

This allows us to define a (global) *radium* on T_S, as the map $\rho_S : T_S \to [0,\infty)$ satisfying

$$\rho_S(\varphi(z,[l,r])) = r \quad \forall\, (z,[l,r]) \in U \times c(L_S);\ (U,\varphi) \in \mathcal{A}.$$

We also define the *radial action* $\delta_S : \mathbb{R}^+ \times T_S \to T_S$ as follows

$$\delta_S(r,x) = \varphi(z,[l,rt])$$
$$\forall\, (z,[l,t]) \in U \times c(L_S);\ (U,\varphi) \in \mathcal{A} \ \text{ (for } x = \varphi(z,[l,t])).$$

We will write rx instead of $\delta_S(r,x)$ in the future. These functions satisfy

(a) $\rho_S(rx) = r\rho_S(x)$ and $\rho_S(gx) = \rho_S(x)$ for any $r \in \mathbb{R}^+$, $x \in T_S$, $g \in G$.
(b) $S \cap \rho_S^{-1}(0,\infty) = \emptyset$
(c) The radial action commutes with the action of G on T_S.

4.2. Thom-Mather spaces (see [14], [15]) A *Thom-Mather G-stratified pseudomanifold* is a pair $(X,\mathcal{T})$ where X is a G-stratified pseudomanifold and $\mathcal{T} = \{T_S : S \in \mathcal{S}_X^{sing}\}$ is a family of equivariant tubular neighborhoods satisfying the following condition:

$$T_S \cap T_R \neq \emptyset \ \Leftrightarrow\ R \leq S \ \text{ or } \ S \leq R$$

for any two singular strata R, S in X. We will usually omit the family $\mathcal{T}$ if there is no possible confusion.

4.3. Examples Here are some examples of G-stratified tubular neighborhoods.

(1) Following [2, p. 306], for any manifold M endowed with a smooth action $\Phi : G \times M \to M$ there is a Riemannian metric μ such that G acts by μ-isometries. By the local properties of the exponential map, each singular stratum S of M has a smooth G-equivariant tubular neighborhood which

can be realized as the normal fiber bundle $N_\mu(S)$ over S with respect to μ. The cocycles of this bundle are orthogonal actions. Hence, this tubular neighborhood is actually a G-stratified tubular neighborhood.

(2) If L is a compact G-stratified pseudomanifold, the map $c(L) \to \{\star\}$ is a G-stratified tubular neighborhood of the vertex.

(3) If $\xi = (T_S, \tau_S, S, c(L_S))$ is a G-stratified tubular neighborhood of S in X, then $(M \times T_S, \imath_M \times \tau_S, M \times S, c(L_S))$ is a G-stratified tubular neighborhood of $M \times S$ in $M \times X$; for any connected manifold M.

(4) If $f : Y \to X$ is a G-equivariant isomorphism, then for any G-stratified tubular neighborhood $\xi = (T_S, S, \tau_S, c(L_S))$ of a stratum S in X; the pull-back $f^*(\xi) = (f^{-1}(T_S), f^{-1}\tau_S f, f^{-1}(S), c(L_S))$ is a G-stratified tubular neighborhood of $f^{-1}(S)$ in Y.

5. Orbit Spaces

In this section we will expose some factorization theorems, concerning G-stratified pseudomanifolds and the equivariant tubular neighborhoods. This is made in order to get a consistent theory when passing to the orbit spaces, we do it for any compact *abelian* Lie group G. For similar results in the non-abelian context the reader can see [10].

In the sequel, we fix a G-stratified space X and a closed subgroup $K \subset G$. Write $\pi : X \to X/K$ for the usual orbit map. The orbit space X/K inherits a *canonical stratification* given by the family

$$\mathcal{S}_{X/K} = \{\pi(S) : S \in \mathcal{S}_X\}.$$

Notice also that $d(X) = d(X/K)$.

Lemma 5.1. *The orbit space X/K is a G/K-stratified space.*

Proof. Write $\overline{g} \in G/K$ for the equivalence class of $g \in G$. Consider the quotient action

$$\overline{\Phi} : G/K \times X/K \to X/K, \qquad \overline{g} \cdot \pi(x) = \pi(gx).$$

This action is well defined because G is abelian. So:

• *The isotropy groups are constant over the strata of X/K*:
This is straightforward, since for each stratum $S \in \mathcal{S}_X$ we have

$$(G/K)_{\pi(S)} = KG_S/K.$$

Hence $\pi(S)$ has constant isotropy.

• *Each* $\overline{g}$ *induces an isomorphism* $\overline{\Phi}_g \in G/K \in \mathrm{Iso}(X/K, \mathcal{S}_{X/K})$: For each $g \in G$ we have a K-equivariant isomorphism $\Phi_g \in \mathrm{Iso}(X, \mathcal{S}_X)$. Passing to the quotients we obtain an isomorphism $\overline{\Phi}_g \in G/K \in \mathrm{Iso}(X/K, \mathcal{S}_{X/K})$. The differentiability of this map on $\pi(S)$ is immediate from the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\Phi_g} & gS \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \pi(S) & \xrightarrow{\overline{\Phi}_g} & \pi(gS) \end{array}$$

∎

Now we pass to the conical context.

PROPOSITION 5.2. *Assume that X is a G-stratified pseudomanifold. Then X/K is a G/K-stratified pseudomanifold.*

Proof. Proceed by induction on $l = d(X)$. For $l = 0$ it is straightforward, since $d(X/K) = d(X) = 0$. Assume the inductive hypothesis and suppose that $d(X) > 0$. By § 5.1, X/K is a G/K-stratified space, so we must verify the existence of conical slices.

Take a singular stratum $S \in \mathcal{S}_X$, fix a point $x \in S$ and a distinguished slice (S_x, β, L_S) of a x. The G_S-equivariant isomorphism $\beta : \mathbb{R}^i \times c(L_S) \to S_x$ induces an isomorphism on the orbit spaces

$$\overline{\beta} : \mathbb{R}^i \times c(L_S/G_S \cap K) \to \pi(S_x), \qquad \overline{\beta}(b, [\overline{w}, r]) = \pi(\beta(b, [w, r])).$$

Now we will show that the triple $(\pi(S_x), \overline{\beta}, L_S/G_S \cap K)$ is a distinguished slice of $\pi(x) \in X/K$. We do it in three steps.

• $\pi(S_x)$ *is a slice of* $\pi(x)$: This is straightforward, since $(G/K)_{\pi(x)} = KG_S/K$, the quotient $\pi(S_x)$ is a $(G/K)_{\pi(x)}$-space with the quotient action and the orbit map π is an open map.

• $\overline{\beta}$ *is a KG_S/K-equivalence*: This is immediate, since β is an H-equivalence. Notice that, by induction on the depth, $L_S/G_S \cap K$ is a KG_S/K-stratified pseudomanifold.

• *The induced map* $\overline{\alpha} : (G/K) \times_{(G_S/G_S \cap K)} \pi(S_x) \to X/K$ *is an embedding*: This $\overline{\alpha}$ is given by the rule $\overline{\alpha}([\overline{g}, \pi(z)]) = \overline{g}.\pi(z)$, and is a homeomorphism.

We consider the following commutative diagram

$$\begin{array}{ccc} G\times_{G_S} S_x & \xrightarrow{\alpha} & X \\ {\scriptstyle\overline{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ (G/K)\times_{(G_S/G_S\cap K)} \pi(S_x) & \xrightarrow{\overline{\alpha}} & X/K \end{array}$$

Since the vertical arrows are submersions, and α is an embedding, we obtain that $\overline{\alpha}$ is an embedding. ∎

The following result provides a factorization theorem for tubular neighborhoods.

Proposition 5.3. *Let S be a singular stratum in X, $\xi = (T_S, \tau_S, S, c(L_S))$ be an equivariant tubular neighborhood of S in X and write*

$$\overline{\tau_S} : \pi(T_S) \to \pi(S)$$

for the induced quotient map. Then $\xi/K = (\pi(T_S), \overline{\tau_S}, \pi(S), c(L_S/G_S \cap K))$ is an equivariant tubular neighborhood of $\pi(S)$ in X/K.

Proof. Since π is an open map, $\pi(T_S)$ is an open neighborhood of $\pi(S)$ in X/K. Also the inclusion $\pi(S) \to \pi(T_S)$ is a section of $\overline{\tau_S} : \pi(T_S) \to \pi(S)$. In order to prove that ξ/K is a G-stratified tubular neighborhood we should first verify that it is a G-stratified fiber bundle, but the conditions § 3.1-(1) to (4) are straightforward.

Now we will prove § 4.1-(2), which implies § 3.1-(5). We will show that the trivializing atlas $\mathcal{A} = \{(U, \varphi)\}$ of ξ induces a trivializing atlas $\mathcal{A}/K = \{(V, \psi)\}$ of ξ/K. Write $\pi' : L_S \to L_S/G_S \cap K$ for the orbit map induced by the action of $G_S \cap K$ in L_S.

• *Trivializing charts*: Take a chart $(U, \varphi) \in \mathcal{A}$ and a point $x \in U$. Take also a K-slice V of x in S, we assume that $V \subset U$. Since G_S acts trivially on V and KV is open in S we deduce that

$$V = V/G_S \cap K = \pi(KV)$$

is open in $\pi(S)$. Since φ is G_S-equivariant, the function

$$(1) \qquad \psi : V \times c(L_S/G_S \cap K) \to \pi(T_S), \qquad \psi(b, [\pi'(l), r]) = \pi(\varphi(b, [l, r])).$$

is well defined. Moreover, ψ is injective because G acts by isomorphisms and V is a K-slice in S. Notice that $W = KV \cap U$ is open in U; since G also preserves the radium in T_S,

$$\mathrm{Im}(\psi) = \pi(\varphi(W \times c(L_S))).$$

Hence $\mathrm{Im}(\psi)$ is open in X/K. It is straightforward that ψ sends smoothly strata onto strata, so actually ψ is an embedding.

• *Atlas and cocycles*: We consider the family $\mathcal{A}/K = \{V, \psi)\}$ of all the pairs (V, ψ) as in (1). We will show that $\mathcal{A}/K$ is a trivializing atlas of ξ/K. Take two charts $(V, \psi); (V', \psi') \in \mathcal{A}/K$ respectively induced by $(U, \varphi); (U', \varphi') \in \mathcal{A}$. Assume that there is some $\overline{g_0} \in G/K$ such that $\overline{g_0}^{-1} V \cap V' \neq \phi$; so $g^{-1} U \cap U' \neq \phi$ for some $g \in g_0 K$. By § 4.1-(2), there is a map

$$g_{\varphi\varphi'} : g^{-1} U \cap U' \to \mathrm{Iso}_{G_S}(L_S, \mathcal{S}_{L_S})$$

satisfying

$$g\varphi^{'}(b, [l, r]) = \varphi(gb, [g_{\varphi\varphi'}(b)(l), r]), \qquad (b, [l, r]) \in (g^{-1} U \cap U') \times c(L_S).$$

Passing to the orbit space $L_S/G_S \cap K$ we obtain the induced map

$$\overline{g_0}_{\psi\psi'} : \overline{g_0}^{-1} V \cap V' \to \mathrm{Iso}_{(G_S/G_S \cap K)}(L_S/G_S \cap K, \mathcal{S}_{L_S/G_S \cap K})$$

satisfying

$$\overline{g_0}\psi'\big(b, [\pi'(l), r]\big) = \psi\big(\overline{g_0} b, [\overline{g_0}_{\psi\psi'}(b)(\pi'(l)), r]\big);$$
$$(b, [\pi'(l), r]) \in (\overline{g_0}^{-1} V \cap V') \times c(L_S/G_S \cap K).$$

Notice that, by definition, G/K preserves the radium of $\pi(T_S)$. ∎

6. Elementary unfoldings

An unfolding of a stratified pseudomanifold is an auxiliar construction which allows us to define the intersection cohomology from the point of view of differential forms [1], [3]. In the rest of this paper we will find conditions for the existence of equivariant unfoldings. For this, we will introduce the elementary unfoldings. The main idea is that, from a finite number of elementary unfoldings one can get an equivariant unfolding in the usual sense.

6.1. ELEMENTARY UNFOLDING OF A G-STRATIFIED PSEUDOMANIFOLD The elementary unfolding of a Thom-Mather space is essentially the resolution of singularities given in [4] for the smooth case. This topological operation can be done because the stratification is controlled through a family of tubular neighborhoods. Under certain conditions, after the iterated composition of finitely many elementary unfoldings, one obtains an equivariant unfolding as defined above. We follow the exposition of [1].

Henceforth we fix a Thom-Mather G-stratified pseudomanifold X, a closed (hence minimal) stratum S in X and an equivariant tubular neighborhood $(T_S, \tau_S, S, c(L_S))$ of S. Define the *unitary sub-bundle* as the set $E_S = \rho_S^{-1}(1)$; this is by construction a G-invariant stratified subspace of X. The restriction $\tau_S : E_S \to S$ is a G-stratified fiber bundle with fiber L_S. Consider the map

$$\mathcal{L}_{T_S} : E_S \times \mathbb{R} \to T_S, \qquad \mathcal{L}_{T_S}(x,t) = \begin{cases} |t| * x & \text{if } t \neq 0, \\ \tau_S(x) & \text{if } t = 0. \end{cases} \tag{2}$$

Each chart (U, φ) in the trivializing atlas provides a local description of $\mathcal{L}_{T_S}$ through the following commutative square

$$\begin{array}{ccc} U \times L_S \times \mathbb{R} & \xrightarrow{\widehat{\varphi}} & E_S \times \mathbb{R} \\ {\scriptstyle 1_U \times \mathcal{L}_C} \downarrow & & \downarrow {\scriptstyle \mathcal{L}_{T_S}} \\ U \times cL_S & \xrightarrow{\varphi} & T_S \end{array}$$

where $\widehat{\varphi}(x,l,t) = (\varphi(x,[l,1],t))$ and $\mathcal{L}_C(l,t) = [l,|t|]$. We also obtain the following properties:
(a) The map $\widehat{\varphi}$ is a G_S-equivariant embedding.
(b) The composition $\tau_S \circ \mathcal{L}_{T_S} : E_S \times \mathbb{R} \to S$ is a locally trivial fiber bundle with fiber $L_S \times \mathbb{R}$ and structure group $\mathrm{Iso}_{G_S}(L_S, \mathcal{S}_{L_S})$.
(c) $d(E_S \times \mathbb{R}) = d(E_S) = d(T_S) - 1$.

Now take a disjoint family of equivariant tubular neighborhoods $\{T_S : S \in \mathcal{S}_X^{min}\}$ of the minimal strata. The *elementary unfolding* of X with respect to the family $\{T_S : S \in \mathcal{S}_X^{min}\}$ is the pair $(\widehat{X}, \mathcal{L})$ constructed as follows: First $\widehat{X}$ is the amalgamated sum

$$\widehat{X} = [\bigsqcup_{S \text{ minimal}} E_S \times \mathbb{R}] \bigcup_{\theta} [(X - \Sigma^{min}) \times \{\pm 1\}], \tag{3}$$

where S runs over the family of minimal strata and, for each $S \in \mathcal{S}_X^{min}$, the map θ restricted to E_S is given by

$$\theta : E_S \times \mathbb{R}^* \to [X - \Sigma^{min}] \times \{\pm 1\}, \qquad \theta(x,t) = (|t| * x, |t|^{-1} t). \tag{4}$$

Second, $\mathcal{L}$ is the continuous map given by the rule

$$(5) \qquad \mathcal{L} : \widehat{X} \to X, \quad \mathcal{L}(x) = \begin{cases} \mathcal{L}_{T_S}(x)\,, & x \in E_S \times \mathbb{R}\,, \\ y\,, & x = (y,j) \in (X - \Sigma^{min}) \times \{\pm 1\}\,. \end{cases}$$

Here there are some properties of the elementary unfoldings.

Proposition 6.2. *Let $\mathcal{L} : \widehat{X} \to X$ be the elementary unfolding of a Thom-Mather G-stratified pseudomanifold X. Then*
(1) $\widehat{X}$ is a G-stratified pseudomanifold, whose stratification is the family $\mathcal{S}_{\widehat{X}}$ consisting of all the following sets

$$\widehat{R} = [\bigsqcup_{S \text{ minimal}} (E_S \cap R) \times \mathbb{R}] \bigsqcup_{\theta} (R \times \{\pm 1\})\,,$$

where R runs over the non closed strata in X. Moreover, $\widehat{X}$ satisfies the Thom-Mather condition.
(2) The map $\mathcal{L}$ is a G-equivariant morphism. The restriction

$$\mathcal{L} : \mathcal{L}^{-1}(X - \Sigma^{min}) \to X - \Sigma^{min}$$

is a (trivial) double covering.
(3) $d(\widehat{X}) = d(X) - 1$. In particular, if $d(X) = 1$ then $\mathcal{L} : \widehat{X} \to X$ is an equivariant unfolding (see § 7.1).
(4) If X is compact, then so is $\widehat{X}$.
(5) If G is abelian then, for any closed subgroup $K \subset G$, the induced map

$$\overline{\mathcal{L}} : \widehat{X}/K \to X/K$$

is an elementary unfolding.

Proof. (1) The stratification of $\widehat{X}$ can be seen in [1]. Since each equivariant tubular neighborhood is a G-stratified pseudomanifold (because they are invariant open subsets of X); so are the unitary sub-bundles (see § 3.2), and hence $\widehat{X}$ is a G-stratified pseudomanifold. Now we verify the Thom-Mather condition: Take a family $\{T_S : S \in \mathcal{S}_X\}$ of equivariant tubular neighborhoods in X. Take also a stratum $\widehat{R}$ in $\widehat{X}$ induced by a non closed stratum R in X. Define

$$T_{\widehat{R}} = \bigsqcup_{S \text{ minimal}} (E_S \cap T_R) \times \mathbb{R} \cup (T_R \times \{\pm 1\}) = \mathcal{L}^{-1}(T_R)$$

where θ is the map given in the equation (4) of § 6.1. This $T_{\widehat{R}}$ is an equivariant tubular neighborhood of $\widehat{R}$ in $\widehat{X}$; we leave the details to the reader.
(2) and (3) are straightforward, see again [1] for more details. The last observation of (3) is a consequence of Definition § 7.1.
(4) Since X is compact, $\mathcal{S}_X^{min}$ is finite. But $\widehat{X}$ is the quotient of the finite family of compact spaces $\bigsqcup_{S \text{ min.}} (E_S \times [-1,1])$ and $[X - \bigsqcup_{S \text{ min.}} \rho_S^{-1}[0,1/2)] \times \{-1,1\}$. Then we get the result.
(5) This is a consequence of § 5.3. ∎

6.3. Remark *With tubular neighborhood of* 4.3-3, $\widehat{M \times X} = M \times \widehat{X}$, *for any manifold* M.

7. Equivariant unfoldings

For a detailed introduction to unfoldings, the reader can see [4], [11]. In this section we introduce equivariant unfoldings, these are a suitable adaptation of the usual unfoldings to the equivariant category. As an example, we show how, for any compact Lie group G and any smooth G-manifold M, there is always an equivariant unfolding. When G is abelian this construction passes well to the orbit space M/K for any closed subgroup $K \subset G$.

7.1. Equivariant unfoldings Broadly speaking, an un unfolding of a stratified pseudomanifold X is a manifold $\widetilde{X}$ and a surjective continuous map $\mathcal{L} : \widetilde{X} \to X$ such that $\mathcal{L}^{-1}(X - \Sigma)$ is a union of finitely many disjoint copies of $X - \Sigma$, and which smoothly unfolds the singular part so that the restriction $\mathcal{L} : \mathcal{L}^{-1}(S) \to S$ is a submersion, for any singular stratum S.

As for the usual unfoldings, the definition of an equivariant unfolding is made by induction on the depth. Fix a compact abelian Lie group G. Let X be a G-stratified pseudomanifold. An *equivariant unfolding* of X is a manifold $\widetilde{X}$ together with a smooth free action $\widetilde{\Phi} : G \times \widetilde{X} \to \widetilde{X}$; a surjective continuous equivariant map

$$\mathcal{L} : \widetilde{X} \to X$$

and a family of equivariant unfoldings $\{\mathcal{L}_{L_S} : \widetilde{L_S} \to L_S\}_S$ where S runs on the singular strata of X; satisfying:

(1) The restriction $\mathcal{L} : \mathcal{L}^{-1}(X - \Sigma) \to X - \Sigma$ is a smooth finite trivial covering.

(2) For each singular stratum S and each $x \in S$, there is a *liftable modeled*

chart, i.e.; a commutative square

$$\begin{array}{ccc} U \times \widetilde{L_S} \times \mathbb{R} & \xrightarrow{\widetilde{\varphi}} & \widetilde{X} \\ {\scriptstyle \mathcal{L}_c}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{L}} \\ U \times c(L_S) & \xrightarrow{\varphi} & X \end{array}$$

such that

(a) (U, φ) is a G_S-equivariant chart of x modeled on L_S.
(b) $\widetilde{\varphi}$ is a G_S-equivariant smooth embedding on an open subset of $\widetilde{X}$.
(c) The map $\mathcal{L}_c$ is given by the rule $\mathcal{L}_c(u, z, t) = (u, [\mathcal{L}_{L_S}(z), |t|])$.

A G-stratified pseudomanifold X is said to be *unfoldable* whenever it has an equivariant unfolding.

7.2. Examples Here are some examples of equivariant unfoldings.
(1) For any free smooth action $\Phi : G \times M \to M$ the identity $\imath : M \to M$ is an equivariant unfolding.
(2) If $\mathcal{L} : \widetilde{X} \to X$ is an equivariant unfolding, then for any manifold M the product $\imath : M \times \widetilde{X} \to M \times X$ is also an equivariant unfolding.
(3) For any equivariant unfolding $\mathcal{L} : \widetilde{L} \to L$ over a compact G-stratified pseudomanifold L, the map $\mathcal{L}_c : \widetilde{L} \times \mathbb{R} \to c(L)$ defined above is also an equivariant unfolding.

7.3. Iteration of elementary unfoldings From now on, we fix a Thom-Mather G-stratified pseudomanifold X. Our main goal is to prove that, from a finite composition of elementary unfoldings, one gets an equivariant unfolding. This is not surprising since, as we have already seen, for any elementary unfolding $\mathcal{L} : \widehat{X} \to X$, the space $\widehat{X}$ is again a Thom-Mather G-stratified pseudomanifold and satisfies $d(\widehat{X}) = d(X) - 1$. This allows us to ask for the behavior of a chain

$$X_l \xrightarrow{\mathcal{L}_l} X_{l-1} \xrightarrow{\mathcal{L}_{l-1}} \dots \xrightarrow{\mathcal{L}_2} X_1 \xrightarrow{\mathcal{L}_1} X \tag{6}$$

of elementary unfoldings, where $l = d(X)$. As we shall see, under certain conditions on the tubular neighborhoods, this iterative process leads us to an equivariant unfolding

$$\mathcal{L} : \widetilde{X} \to X \,,$$

where $\widetilde{X} = X_l$ and $\mathcal{L} = \mathcal{L}_1 \dots \mathcal{L}_l$.

Recall the definition of a *saturated subspace* [1]. Let $Y \subset X$ be a stratified subspace of X. We say that Y is *saturated* whenever

$$Y \cap T_S = \tau_S^{-1}(Y \cap S), \qquad \forall S \in \mathcal{S}_X.$$

For instance, if S is a singular stratum and $U \subset S$ is open, then $Y = \tau_S^{-1}(U)$ is a saturated. Also the unitary sub-bundle $Y = E_S$ is saturated.

7.4. Transverse morphisms Now we introduce the family of transverse morphisms, whose main feature is the preservation of the tubular neighborhoods. Let $H \subset G$ be a closed subgroup, Y a Thom-Mather H-stratified pseudomanifold and M be a connected manifold. A morphism

$$\psi : M \times Y \to X$$

is *transverse* whenever:

(1) $\mathrm{Im}(\psi)$ is a saturated open subspace of X.

(2) If $\psi(M \times S) \subset R$ then $\psi^{-1}(T_R) = M \times T_S$, for any $R \in \mathcal{S}_X, S \in \mathcal{S}_Y$.

Now let $\psi : M \times Y \to X$ be a transverse morphism. The *lifting* of ψ is, by definition, the map

$$\widehat{\psi} : M \times \widehat{Y} \to \widehat{X}\,, \quad \widehat{\psi}(m,z,t) = \begin{cases} (\psi(m,z),t), & (m,z,t) \in M \times E_S \times \mathbb{R}\,, \\ (\psi(m,z),t), & (m,z,t) \in M \times (Y - \Sigma^{min}) \times \{\pm 1\}\,. \end{cases}$$

This is the unique morphism such that the diagram

$$\begin{array}{ccc} M \times \widehat{Y} & \xrightarrow{\widehat{\psi}} & \widehat{X} \\ {\scriptstyle \imath_M \times \mathcal{L}_Y}\downarrow & & \downarrow{\scriptstyle \mathcal{L}_X} \\ M \times Y & \xrightarrow{\psi} & X \end{array}$$

commutes.

7.5. Examples For any smooth effective action of G, the charts of the tubular neighborhoods are transverse morphisms: Take a smooth action $\Phi : G \times M \to M$ and an invariant metric μ in M. Then M has a structure of Thom-Mather G-stratified pseudomanifold, where $\mathcal{S}_M$ is the stratification induced by the orbit types of the action. For any singular stratum S with codimension $\mathrm{codim}(S) = q + 1 > 0$, the equivariant tubular neighborhood

$T_S = N_\mu(S)$ is the normal fiber bundle over S induced by μ (see § 4.3). Then any a trivializing chart

$$\varphi : U \times c(\mathbb{S}^q) \to \tau_S^{-1}(U)$$

is transverse: Notice that $\mathrm{Im}(\varphi)$ is a saturated open subspace in M, so we only have to verify § 7.4-(2). Let S' be a stratum in $c(\mathbb{S}^q)$, R a stratum in M. Suppose that $\varphi(U \times S') \subset R$. There are two cases:

- $S' = \{\star\}$ *is the vertex*: It is straightforward, since $R = S$ and $T_{S'} = c(\mathbb{S}^q)$.
- $S' = S'' \times \mathbb{R}^+$ *for some stratum* S'' *in* $\mathbb{S}^q$: Then $S < R$. We consider in T_S the following decomposition of the metric:

$$\mu \mid_{T_S} = \mu_H + \mu_V$$

corresponding to the orthogonal decomposition of the tangent $T(T_S)$ in the horizontal and vertical sub-bundles. Hence

$$\varphi^{-1}(T_R) = \varphi^*(N_\mu(R)) = N_{\varphi^*(\mu)}(U \times S') = U \times N_{\mu_V}(S') = U \times T_{S'} .$$

Now we show two easy properties of the transverse morphisms.

PROPOSITION 7.6. *Let K, H a closed subgroups of G, L a Thom-Mather H-stratified pseudomanifold, $\psi : M \times L \to X$ a transverse morphism. Then*
(1) The lifting $\widehat{\psi} : M \times \widehat{L} \to \widehat{X}$ is transverse.
(2) If additionaly G is abelian, then the induced quotient map $\overline{\psi} : M \times (L/H \cap K) \to X/K$ is transverse.

Proof. (1) is straightforward from Definition § 7.4. (2) is a consequence of § 5.3. ∎

Finally, we give a sufficient condition for the existence of equivariant unfoldings.

THEOREM 7.7. *Let X be a Thom-Mather G-stratified pseudomanifold. Suppose that for any singular stratum S, each trivializing chart*

$$\varphi : U \times c(L_S) \to T_S$$

is transverse. Then
(1) The composition of the l elementary unfoldings of starting at X induces an equivariant unfolding $\mathcal{L} : \widetilde{X} \to X$ where $\widetilde{X}$ is the last (non trivial) elementary unfolding and $\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2 \dots \mathcal{L}_l$ (see eq. (6) at the beginning of this section).
(2) If G is abelian then, for any closed subgroup $K \subset G$, the induced map $\overline{\mathcal{L}} : \widetilde{X}/K \to X/K$ is an unfolding.

Proof. (1) Take a family of equivariant tubular neighborhoods in X with transverse trivializing charts. Let

$$X_l \xrightarrow{\mathcal{L}_l} X_{l-1} \xrightarrow{\mathcal{L}_{l-1}} \dots \xrightarrow{\mathcal{L}_2} X_1 \xrightarrow{\mathcal{L}_1} X$$

be the chain of elementary unfoldings induced by the tubular neighborhoods. Proceed by induction on $l = d(X)$; for $l = 1$ it is straightforward. For $l > 1$ we assume the inductive hypothesis, so $\mathcal{L}' : \widetilde{X} \to X_1$ is an equivariant unfolding, for $\widetilde{X} = X_l$ and $\mathcal{L}' = \mathcal{L}_2 \dots \mathcal{L}_l$. Take a closed stratum S and a transverse trivializing chart

$$\varphi : U \times c(L_S) \to \tau_S^{-1}(U) \subset T_S\,.$$

Apply the chain of elementary unfoldings and use § 7.6; you will get the following commutative diagram:

$$\begin{array}{ccc}
U \times \widetilde{L_S} \times \mathbb{R} & \xrightarrow{\widetilde{\psi_1}} & \widetilde{X} \\
{\scriptstyle \imath_U \times \mathcal{L}_{L_S} \times \imath_{\mathbb{R}}} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathcal{L}'} \\
U \times L_S \times \mathbb{R} & \xrightarrow{\psi_1} & X_1 \\
{\scriptstyle \imath_U \times \mathcal{L}_S} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathcal{L}_1} \\
U \times c(L_S) & \xrightarrow{\psi} & X
\end{array}$$

We conclude that $\mathcal{L} = \mathcal{L}_1 \mathcal{L}' : \widetilde{X} \to X$ is an equivariant unfolding.
(2) This is a consequence of § 6.2-(5). ∎

Corollary 7.8. (Unfolding of a G-manifold) *Let M be a manifold, $\Phi : G \times M \to M$ a smooth effective action, possibly with fixed points. Endow M with the stratification induced by the orbit types and the usual structure of a Thom-Mather G-stratified pseudomanifold. Then there is an equivariant unfolding $\mathcal{L} : \widetilde{M} \to M$.*

Proof. Apply the above theorem to the transverse charts obtained in § 7.5. ∎

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