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# On Continuous Surjections from Cantor Set

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It is a famous result of Alexandroff and Urysohn [1] that every compact metric space is a continuous image of Cantor set $\Delta$. In this short note we complement this result by showing that a certain "uniqueness" property holds.

Given topological spaces $X$ and $Y$, let $C(X,Y)$ denote the collection of all continuous mappings from $X$ to $Y$. If $Y$ is metrized by $d$, one can endow $C(X,Y)$ with the uniform metric

$$\operatorname{dist}(f,g) = \sup_{x\in X} d(f(x),g(x)).$$

It is clear that if $\varphi$ is a homeomorphism of $X$, the map $f \mapsto \varphi \circ f$ is an isometry of $C(X,Y)$.

Our result is the following.

Theorem. *Let $K$ be a compact metric space and let $f$ and $g$ be two continuous mappings from $\Delta$ onto $K$. For every $\varepsilon > 0$ there exists a homeomorphism $\varphi$ of $\Delta$ such that* $\operatorname{dist}(g, f\circ\varphi) < \varepsilon$.

Before going into the proof, let us fix some notations. We regard the points of $\Delta = \{0,1\}^{\mathbb{N}}$ as functions $x : \mathbb{N} \to \{0,1\}$ and we write $\Delta^{(n)}$ for the set of all two-valued functions $y : \{1,2,\dots,n\} \to \{0,1\}$. Given $y \in \Delta^{(n)}$, put

$$\Delta_y = \{x \in \Delta : \ x(k) = y(k),\ 1 \le k \le n\}.$$

Obviously $\Delta_y$ is homeomorphic to $\Delta$. The $n$-th standard decomposition of $\Delta$ is the partition of $\Delta$ into the $2^n$ clopen sets

$$\Delta = \bigoplus_{y\in\Delta^{(n)}} \Delta_y.$$

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LEMMA. *Let $S$ be a finite set and let $f$ and $g$ be be two surjections $\Delta \to S$ which are constant on the sets of some standard decomposition of $\Delta$. Then there is a homemorphism $\varphi$ of $\Delta$ such that $g = f \circ \varphi$.*

*Proof.* Since $\Delta \approx \Delta \oplus \Delta$ we see that for every $s \in S$ there is a homeomorphism $\varphi_s : g^{-1}(s) \to f^{-1}(s)$. In fact,

$$h^{-1}(s) \approx \Delta \oplus \overset{k_h}{\ldots} \oplus \Delta \approx \Delta \qquad (h = f,\, g).$$

The required homeomorphism is then given by

$$\varphi = \left( \bigoplus_{s \in S} \varphi_s \right) : \Delta = \bigoplus_{s \in S} g^{-1}(s) \to \bigoplus_{s \in S} f^{-1}(s) = \Delta.$$

∎

*Proof of the Theorem.* Fix $\varepsilon > 0$ and take $n$ such that if $\Delta = \oplus_y \Delta_y$ is the $n$-th standard decomposition, then the sets

$$h(\Delta_y) \qquad (h = f, g)$$

have all diameter at most $\varepsilon$. Also, for $y \in \Delta^{(n)}$, let $y^+ = \min \Delta_y$ and define $h_1 : \Delta \to K$ taking $h_1(x) = h(y^+)$ for $x \in \Delta_y$. Clearly, $\operatorname{dist}(h, h_1) \leq \varepsilon$ for $h = f, g$ and so the sets $h_1(\Delta)$ are $\varepsilon$-nets on $K$ having at most $2^n$ points.

Let $K_1$ be the range of $f_1$ and define $g_2 : \Delta \to K_1$ taking $g_2(\Delta_y)$ as the point $k \in K_1$ minimizing the distance to $g_1(\Delta_y) = g(y^+)$. Clearly, $\operatorname{dist}(g_2, g_1) \leq \varepsilon$. Of course, $g_2$ need not be onto $K_1$, but its range, say $S$ is a $2\varepsilon$-net in $K$ and so there exists a projection of $K_1$ onto its subset $S$ such that $\operatorname{dist}(\pi, \mathrm{Id}_S) \leq 2\varepsilon$. Taking $f_2 = \pi \circ f_1$, we see that $\operatorname{dist}(f_2, f_1) \leq 2\varepsilon$.

Finally, we can apply the Lemma to the pair $f_2, g_2$ to get a homeomorphism $\varphi$ of $\Delta$ such that $g_2 = f_2 \circ \varphi$. Therefore

$$\begin{aligned} \operatorname{dist}(g, f \circ \varphi) &\leq \operatorname{dist}(g, g_2) + \operatorname{dist}(g_2, f \circ \varphi) \\ &\leq \operatorname{dist}(g, g_2) + \operatorname{dist}(g_2, f_2 \circ \varphi) + \operatorname{dist}(f_2 \circ \varphi, f \circ \varphi) \leq 5\varepsilon, \end{aligned}$$

which completes the proof. ∎

*Remarks.* Every non-empty clopen subset of $\Delta$ is homeomorphic to $\Delta$ (this follows from Hausdorff characterization of $\Delta$ as the only totally disconnected perfect compact metric space [3]). So, the Lemma holds without any restriction on $f$ and $g$.

The only property of $\Delta$ needed to get the conclusion of the Theorem is that given $\varepsilon > 0$ there exists a decomposition $\Delta = C_1 \oplus \cdots \oplus C_n$ into clopen sets of diameter less that $\varepsilon$ with each $C_i$ homeomorphic to $\Delta$.

(See [2] for unexplained terms.) The above Theorem makes transparent that the set of continuous surjections $\Delta \to K$ that admit (or do not admit) a regular averaging operator is uniformly dense in the set of all continuous surjections $\Delta \to K$ whenever it is not empty; see [2, Theorem 3].

## References

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