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Invariantes diferenciales en presencia de una conexión lineal

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Resumen

Invariantes diferenciales en presencia de una conexión lineal

por Raúl MARTÍNEZ BOHÓRQUEZ

En este trabajo se estudian los invariantes diferenciales asociados a varias estructuras geométricas, las cuales comparten un nexo común: la presencia de una conexión lineal. Se presenta un marco para el estudio de los invariantes diferenciales: las estructuras de espacio anillado, fibrado natural y haz natural. Se obtienen descripciones de estos espacios de invariantes en términos de ciertas representaciones lineales de grupos clásicos. Además, se muestran ejemplos de aplicaciones de estas descripciones, como teoremas de unicidad de los operadores torsión y curvatura asociados a una conexión lineal y el cálculo de las identidades dimensionales de la curvatura en la geometría Fedosov.

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Abstract

Differential invariants in presence of a linear connection

by Raúl MARTÍNEZ BOHÓRQUEZ

In this work we study differential invariants associated to various geometric structures, which share a common link: the presence of a linear connection. A framework for the study of differential invariants is presented: the structures of ringed space, natural bundle and natural sheaf. We obtain descriptions for these spaces of invariants in terms of certain linear representations of classical groups. Moreover, we show examples of applications of these descriptions, such as theorems of uniqueness of the torsion and curvature operators associated to a linear connection and the computation of dimensional curvature identities in Fedosov geometry.

Contents

Introduction

The purpose of this work is to present a method of computing differential invariants associated to some geometric structures which involve linear connections. Differential invariants are geometric objects which are built in a 'natural' way, independent on the system of coordinates of choice.

These invariants are more precisely understood as follows: let X be a smooth manifold of dimension n, $\mathcal{G}$ a sheaf of any geometric structure over X and $\mathcal{T}$ a sheaf of tensors over X. Then, we define differential invariants (which we will usually call natural tensors) as natural morphisms of sheaves $\mathcal{G} \to \mathcal{T}$. This means that, for any open subset $U \subseteq X$, we get a morphism $\mathcal{G}(U) \to \mathcal{T}(U)$ commuting with restrictions to open subsets of U and the action of local diffeomorphisms $\tau : U \to V$.

Natural operations have long been a fundamental concept in the field of differential geometry and its applications, an example being its relevance in general relativity ([29]). The most significant classical results were by Gilkey, particularly on the characterization of the Pontryagin forms on Riemannian manifolds ([12]) and on the uniqueness of the Chern–Gauss–Bonnet formula ([14]), both during the mid-70s. Later on, characterizations of notorious differential operations were obtained, such as the exterior differential ([38]), the Lie bracket ([26]) or the improvement of the description characteristic classes in Riemannian geometry by Atiyah-Bott-Patodi ([2]), utilised later in the proof of the index theorem for elliptic operators. Notorious applications have also been recently developed in various fields, such as contact geometry ([3]), homotopy theory ([10, 37]), Riemannian and Kähler geometry ([15, 16, 31, 44]), general relativity ([34]), or quantum field theory ([23, 24]) .

In 1993, Kolář-Michor-Slovák published what has become the standard reference in this subject ([25]), a monograph that summarises and enhances the main results and techniques known to that point. However, this book can be complex and difficult to follow at times to the non-specialist, due to its level of generality and its functorial language, which is why there appeared important references rewriting its leading results ([10, 24, 37]).

Let us briefly describe the contents of this memoir.

We devote Chapter 1 to introduce what we call Main Theorems, which describe differential invariants in terms of certain linear representations of classical groups.

We begin by recalling the definition of natural bundles proposed by J. Sancho Guimerá, which is equivalent to the usual definition (see [25] for a monograph on the usual definition and [33] for a proof of the equivalence with our definition) and, in our opinion, is easier to understand and to work with.

Of even greater importance for us will be the sheaves of smooth sections associated to a natural bundle, as we define differential invariants at sheaf level. In fact, in this work we propose a generalization of such sheaves, called natural sheaves, which provide a more suitable framework for this work, as it covers geometries given by sheaves which are not the sheaf of smooth sections of any natural bundle, such as the sheaf of Fedosov structures. Being able to cover such geometries vindicates moving the main focus from bundles to sheaves.

Our next step is to define natural morphisms between natural sheaves, and in particular natural tensors, which will be our main interest. We culminate the first Chapter with the statements of the aforementioned Main Theorems, along with a brief historical exposition of which related results appeared before, which appeared later as improvements of what was done, and which are novel to this work.

In Chapter 2, the more technical concepts that appear in this theory are laid out. We start by defining the category of ringed spaces: a generalization of smooth manifolds with nice categorical properties. In particular, they admit inverse limits (which will allow us to consider 'infinite dimensional' spaces) and quotients by a group action. This will later be the structure endowed to the space of ∞-jets of a fibre bundle, a space that can be informally thought of as the set of Taylor expansions of smooth sections of the fibre bundle at any point.

The concepts defined up until now are intertwined, as the renowned Peetre-Slovák Theorem states:

Peetre-Slovák Theorem ([25], Sect. 19.7). *Let X be a smooth manifold. Let $E \to X$ and $E' \to X$ be natural bundles over X, and let $\mathcal{E}$ and $\mathcal{E}'$ be their respective sheaves of smooth sections over X.*

The choice of a point $x_0 \in X$ allows the definition of a bijection:

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \phi\colon \mathcal{E} \longrightarrow \mathcal{E}' \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_{x_0}\textit{-equivariant smooth maps} \\ J^\infty_{x_0} E \longrightarrow E'_{x_0} \end{array} \right\},$$

where Diff_{x_0} *stands for the group of germs of diffeomorphisms τ between open sets of X such that $\tau(x_0) = x_0$.*

We conclude Chapter 2 proving that differential invariants can be computed locally and that we may fix a section whenever diffeomorphisms act transitively (Proposition 2.19 and 2.21). These facts will become of particular interest for Fedosov structures, as well as geometries involving an orientation. In that regard, two modified versions of the Peetre-Slovak Theorem are stated at the end of the section, involving symplectic forms and orientations, respectively.

Chapter 3 is the centrepiece of this memoir, and contains the full proof of our Main Theorems. The sections of Chapter 3 each go over the proof of the Main Theorems of the various geometries considered in this thesis, which correspond to the classical groups:

1. Linear connections (the general linear group Gl).
2. Linear connections and orientations (the special linear group Sl).
3. Riemannian metrics (the orthogonal group O).
4. Riemannian metrics and orientations (the special orthogonal group SO).
5. Fedosov structures (the symplectic group Sp).

All of the proofs are similar in their structure. They start by considering what is usually referenced in the literature as 'normal extensions' or 'normal tensors' (the term of our preference, as it stresses their tensorial character) of the relevant geometric object. These tensors are naturally constructed from the geometric object, utilizing the normal coordinates associated to a linear connection (hence the name), and recover much of their information.

In fact, for each geometric structure we prove a corresponding Reduction Theorem, which expresses the space of ∞-jets of the geometry, obtained by applying the Peetre-Slovak Theorem, in terms of a product of the spaces of normal tensors.

These theorems already would, by themselves, describe the natural tensors associated to the geometry as smooth maps coming from a product of vector spaces, equivariant by the action of a classical group. To improve the results further, the Homogeneous Function Theorem (Appendix A) assures the dependence on a finite amount of variables.

In the cases where the Homogeneous Function Theorem cannot be applied directly, an homogeneity condition can be added. Such a condition has physical meaning: it corresponds to the notion of unit of measure, and thus makes it a reasonable condition to consider. In fact, it has been frequently used in the literature, some examples being the characterization of the Levi-Civita connection by Epstein ([5]) and the characterization of the Pontryagin forms by Gilkey ([13]).

The process described above to compute the differential invariants associated to a geometrical structure does not work as smoothly in Fedosov geometry, as some deep technical problems arise. These problems will be introduced in Chapter 1, tackled in Chapter 3 and expanded upon in Chapter 6.

Moving on to Chapter 4, applications of the various Main Theorems are given. Firstly, uniqueness theorems for the torsion and curvature operators associated to linear connections are obtained as an application of the Main Theorems for linear connections and for linear connections and orientations. In the same vein, we characterise the Chern forms as the only natural differential forms associated to linear connections.

Further on Chapter 4, we discuss various applications of the Main Theorem for Fedosov structures, and compare what is obtained to the known results in Riemannian geometry. In particular, we prove that in Fedosov geometry there are no objects equivalent to the scalar curvature or the Laplacian of Riemannian geometry.

To finalise the applications of the Main Theorem for Fedosov structures, we compute dimensional curvature identities in Fedosov geometry. Let us introduce this concept.

Our Main Theorems compute differential invariants which are local in nature, and thus the choice of base manifold is irrelevant as long as the dimension is maintained. In contrast, many differential invariants of great relevance can be defined in any dimension, such as the curvature tensor. We call these invariants 'universal'.

Now, we can consider whether there are universal invariants which become null in certain dimensions – this is what we call dimensional curvature identities. The 'curvature' part of the name comes from the fact that these invariants can be understood as identities that the curvature operator satisfies in certain dimensions.

A study of these identities in Riemannian geometry firstly appeared in general relativity ([27]), and a modern exposition can be checked at ([15]) or ([30]). There exists some results in other settings, such as Kähler geometry ([16]).

In our work, we add the Fedosov setting to the list above, obtaining that the most elementary dimensional curvature identities are constructed with the symplectic form and the Pontryagin forms of the symplectic connection.

We dedicate the last chapter to a brief dissertation on open problems in this theory. In particular, we expose to greater detail the technical problems commented above for Fedosov structures, which appear in the study of many other geometries defined by natural PDEs on a smooth manifold, such as Einstein or Kähler geometries. The discussion is centred around the generalization of the Peetre-Slovak Theorem to the setting of natural sheaves.

To finish the memoir, we state and prove the Homogeneous Function Theorem mentioned above. We have also included a recollection of results in classical invariant theory which are utilised in the applications. In particular, the First Fundamental Theorem of the various classical groups are stated, which are of the utmost importance in the computation of the differential invariants, as they describe all maps invariant by the group action. In the case of the symplectic group Sp, the Second Fundamental Theorem is also stated, which exposes the linear relations between the maps described in the First Fundamental Theorem. A brief introduction to Fedosov manifolds has been added for the sake of completion; the interested reader is encouraged to check the work of Gelfand-Retakh-Shubin ([11]).

The MATLAB code developed during this thesis and employed in some of the applications have been added at the end. The version of software utilised was MATLAB R2021a (9.10.0.1602886).

Chapter 1

Presentation of the Main Theorems

Let us begin this first Chapter by introducing the concepts required to state our Main Theorems: natural bundles and sheaves, natural morphisms and natural tensors.

Let X be a smooth manifold of dimension n. Let Diff (X) denote the set of local diffeomorphisms between open subsets of X.

Throughout this memoir, the term *diffeomorphism* will refer to a local diffeomorphism between two open subsets of a smooth manifold, unless explicitly stated.

The following definition of natural bundles is due to J. B. Sancho Guimerá, and it is equivalent[1] to the usual one (which can be checked at [25]) :

Definition 1.1. Let $\pi\colon E \to X$ be a fibre bundle over X. A natural bundle over X is a bundle $E \to X$ together with a map

$$\begin{aligned} \mathrm{Diff}\,(X) &\longrightarrow \mathrm{Diff}\,(E) \\ \tau &\longmapsto \tau_* \end{aligned} \quad ,$$

called lifting of diffeomorphisms, satisfying the following properties:

- If $\tau\colon U \to V$ is a diffeomorphism between open subsets of X, then $\tau_*\colon E_U \to E_V$ is a diffeomorphism covering τ; meaning that it makes the following square commutative

$$\begin{tikzcd} E_U \arrow[r, "\tau_*", "\sim"'] \arrow[d, "\pi"'] & E_V \arrow[d, "\pi"] \\ U \arrow[r, "\tau", "\sim"'] & V \end{tikzcd} ,$$

  where $E_U := \pi^{-1}(U)$ and $E_V := \pi^{-1}(V)$.

- Functoriality: $\mathrm{Id}_* = \mathrm{Id}$ and $(\tau \circ \tau')_* = (\tau)_* \circ (\tau')_*$.

- Locality: for any diffeomorphism $\tau : U \to V$ and any open subset $U' \subset U$, $(\tau_{|U'})_* = (\tau_*)_{|E_{U'}}$.

[1] The interested reader can check [33] for a proof of this equivalence of definitions.

- Regularity[2]: for any smooth family of diffeomorphisms $\{\tau_t : U_t \to V_t\}_{t\in T}$, the family $\{\tau_{t*} : E_{U_t} \to E_{V_t}\}_{t\in T}$ is also smooth.

Examples:

1. Let Y be a smooth manifold. Then, the trivial bundle $E = X \times Y \to X$ is a natural bundle: the lifting of a diffeomorphism τ is $\tau_* = (\tau, \mathrm{Id})$.

2. The tangent bundle $TX \to X$ is a natural bundle, the lifting of a diffeomorphism τ being its tangent linear map τ_*. Similarly, the cotangent bundle $T^*X \to X$ is a natural bundle, with the lifting being the inverse of the cotangent linear map.

3. In the same vein, any tensor bundle and the orientation bundle are also natural bundles.

4. The bundle of linear connections $\mathrm{Conn} \to X$ is a natural bundle. The lifting is the following:

$$\begin{aligned} \mathrm{Conn}_U &\longrightarrow \mathrm{Conn}_V \\ \nabla &\longmapsto (\tau_*\nabla)_D D' := \tau_*^{-1}(\nabla_{\tau_* D}\tau_* D'), \end{aligned}$$

   where the τ_* at the right hand side is the tangent linear map.

Definition 1.2. A natural sheaf $\mathcal{E}$ over X is a subsheaf of the sheaf of smooth sections of a natural bundle $E \to X$ over X such that, for any diffeomorphism $\tau : U \to V$, the morphism

$$\begin{aligned} \tau_* \colon \mathcal{E}(U) &\longrightarrow \mathcal{E}(V) \\ s &\longmapsto \tau_* \circ s \circ \tau^{-1} \end{aligned}$$

is well defined[3].

Examples:

1. Let $E \to X$ be a natural bundle. Then, the sheaf of smooth sections of E is a natural sheaf, because the lift τ_* covers the diffeomorphism τ. As such, the following sheaves are natural sheaves:
   - The sheaf $\mathcal{T}_p^q$ of (p,q)-tensors over X.
   - The sheaf $\mathcal{C}$ of linear connections over X.
   - The sheaf Or of orientations over X.

[2] Epstein and Thurston prove in [6] that this property can be derived from the rest. However, we have chosen to include it in this definition, as it will appear in the generalization of natural bundles given in later chapters.

[3] Observe that we are committing an abuse of notation: we are denoting by τ_* both the lifting of τ to E and the 'action' of τ on $\mathcal{E}$.

- The sheaf $\mathcal{M}_{(s_+,s_-)}$ of pseudo-riemannian metrics of fixed signature (s_+, s_-) over X.

2. Let $U \subseteq X$ be an open subset and let $\mathcal{E}$ a natural sheaf over X. The restriction of $\mathcal{E}$ to U, defined as the inverse image of $\mathcal{E}$ by the inclusion map $U \hookrightarrow X$ (or directly on any open subset $U' \subseteq U$ as $\mathcal{E}_U(U') := \mathcal{E}(U')$) is trivially a natural sheaf over U.

3. The Fedosov sheaf, defined on any open subset $U \subseteq X$ as

$$\mathcal{F}(U) := \{(\omega, \nabla) \in (\Omega \times \mathcal{C}^{\mathrm{sym}})(U) : \nabla\omega = 0\} ,$$

is a natural sheaf, where Ω denotes the sheaf of non-singular 2-forms on X and $\mathcal{C}^{\mathrm{sym}}$ denotes the sheaf of symmetric linear connections on X. Observe that the condition $\nabla\omega = 0$ is natural: if $(\omega, \nabla) \in \mathcal{F}(U)$, then $(\tau_*\nabla)(\tau_*\omega) = 0$, and so $\tau_* \circ (\nabla, \omega) \circ \tau^{-1} \in \mathcal{F}(V)$ for any $(\omega, \nabla) \in \mathcal{F}(U)$.

4. The sheaf of Einstein metrics, defined on any open subset $U \subseteq X$ as

$$\mathcal{E}(U) := \{g \in \mathcal{O}(U) : \mathrm{Ric}(g) = kg\} ,$$

is a natural sheaf, where $\mathcal{O}$ is the sheaf of pseudo-Riemannian metrics on X. Again, the condition $\mathrm{Ric}(g) = kg$ is natural.

Definition 1.3. Let $\mathcal{E}$ and $\mathcal{E}'$ be natural sheaves over X. A morphism of sheaves $\phi : \mathcal{E} \to \mathcal{E}'$ is natural if it is regular[4] and commutes with the action of diffeomorphisms on sections; that is to say, if for any diffeomorphism $\tau : U \to V$, the following square commutes:

$$\begin{tikzcd} \mathcal{E}(U) \arrow[r, "\phi"] \arrow[d, "\tau_*"'] & \mathcal{E}'(U) \arrow[d, "\tau_*"] \\ \mathcal{E}(V) \arrow[r, "\phi"] & \mathcal{E}'(V) . \end{tikzcd} \tag{1.1}$$

Definition 1.4. Let $\mathcal{E}$ be a natural sheaf and let $\mathcal{T}$ be the sheaf of (p, q)-tensors on X. A natural tensor (of type (p, q) associated to sections of $\mathcal{E}$) is a natural morphism of sheaves $\mathcal{E} \to \mathcal{T}$.

Examples:

- The torsion tensor of a linear connection can be understood as a natural $(2, 1)$-tensor $\mathrm{Tor}\colon \mathcal{C} \to \mathcal{T}_2^1$, whose value on a linear connection ∇ is the following $(2, 1)$-tensor:

$$\mathrm{Tor}_\nabla(D_1, D_2, \omega) := \omega(\nabla_{D_1} D_2 - \nabla_{D_2} D_1 - [D_1, D_2]) .$$

[4]The regularity condition is technical in nature, and as such it will be properly defined in Chapter 2, Definition 2.13.

- Similarly, the curvature tensor of a linear connection can be understood as a natural $(3,1)$-tensor $R\colon \mathcal{C} \to \mathcal{T}_3^1$, whose value on a linear connection ∇ is the following $(3,1)$-tensor:

$$R_\nabla(D_1, D_2, D_3, \omega) := \omega(\nabla_{D_1}\nabla_{D_2}D_3 - \nabla_{D_2}\nabla_{D_1}D_3 - \nabla_{[D_1,D_2]}D_3)\,.$$

Let us expose the statements of the Main Theorems, that is, the description of natural tensors in terms of certain linear representations of classical groups. The first of such descriptions in the literature appears in the context of Riemannian geometry, by P. Gilkey ([13]), and it was later improved by Atiyah-Bott-Patodi ([2]), which utilised the invariant theory of the orthogonal group in order to greatly simplify the proof.

Briefly after the work of Atiyah-Bott-Patodi, P. Stredder ([41]) included the use of normal coordinates to simplify computations. Then, J. Slovak ([40]) introduced the concept of natural bundles into the equation, pointing towards the translation of these results to other geometrical structures. The exact enunciate that we are about to present was given by J. Navarro-Garmendia ([33]), which clarified the results of J. Slovak by employing the language of sheaves, ringed spaces and the aforementioned equivalent definition of natural bundles:

Theorem 1.5. *Let X be a smooth manifold of dimension n, and let $\mathcal{M}_{(s_+,s_-)}$ denote the sheaf of pseudo-riemannian metrics of fixed signature (s_+, s_-). Let $\mathcal{T}_p$ be the sheaf of p-covariant tensors over X.*

Fixing a point $x_0 \in X$ and a pseudo-riemannian metric g_{x_0} of signature (s_+, s_-) at x_0 produces a $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural tensors} \\ \mathcal{M}_{(s_+,s_-)} \longrightarrow \mathcal{T}_p \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{O}(s_+,s_-)\textit{-equivariant smooth maps} \\ \prod_{i=2}^{\infty} N_i \longrightarrow \mathcal{T}_p \end{array} \right\}$$

where $d_2, \ldots, d_r$ run over the non-negative integer solutions of the equation

$$2d_2 + \ldots + rd_r = p - \delta\,,$$

and where $\mathrm{O}(s_+, s_-)$ denotes the generalised orthogonal group.

Here, the spaces N_m (called spaces of normal tensors) denote vector spaces made of tensors which recover the symmetries of the metric tensor in normal coordinates at the point x_0. They will be rigorously defined during Chapter 3.

In order to produce a more tangible description, a condition of homogeneity is considered:

Definition 1.6. Let $\delta \in \mathbb{R}$. We say that a natural tensor $T : \mathcal{M} \to \mathcal{T}$ is homogeneous of weight δ if, for all non-zero $\lambda \in \mathbb{R}$, it holds that:

$$T(\lambda^2 g) = \lambda^\delta T(g)\,,$$

and that a natural tensor $T : \mathcal{M} \times Or \to \mathcal{T}$ is homogeneous of weight δ if, for all non-zero $\lambda \in \mathbb{R}$, it holds that:

$$T(\lambda^2 g, \Omega) = \lambda^\delta T(g, \Omega) .$$

Examples:

- The metric itself can be understood as a natural $(2,0)$-tensor homogeneous of weight 2.
- The Riemann-Christoffel curvature operator, defined as a natural $(4,0)$-tensor whose value on a pseudo-riemannian metric g defined on an open set $U \subset X$ is:

  $$R_g(D_1, D_2, D_3, D_4) := g(\nabla_{D_1} \nabla_{D_2} D_3 - \nabla_{D_2} \nabla_{D_1} D_3 - \nabla_{[D_1, D_2]} D_3, D_4) ,$$

  where ∇ denotes the Levi-Civita connection of g. It is an homogeneous tensor of weight 2, whereas the usual $(3,1)$ curvature operator (defined again by the Levi-Civita connection of g) is an homogeneous natural tensor of weight 0.
- The scalar curvature, defined as a natural $(0,0)$-tensor (that is, a natural function) whose value on a pseudo-riemannian metric g defined on an open set $U \subset X$ is:

  $$r_g := \langle R_g, g \otimes g \rangle ,$$

  where $\langle \cdot, \cdot \rangle$ denotes the inner product induced by g over $\bigotimes^4 T^*U$. It is homogeneous of weight -2.
- In dimension 4, we may define a natural function associated to metrics and orientations whose value on a pseudo-riemannian metric g and an orientation Ω defined on an open set $U \subset X$ is:

  $$\langle R_g, \Omega \rangle ,$$

  where $\langle \cdot, \cdot \rangle$ is defined as above. It is homogeneous of weight -6.

This condition allows the use of the Homogeneous Function Theorem (Appendix A), reducing the infinite product of normal spaces to a finite one:

Theorem 1.7. *Let X be a smooth manifold of dimension n, and let $\mathcal{M}_{(s_+, s_-)}$ denote the sheaf of pseudo-riemannian metrics of fixed signature (s_+, s_-). Let $\mathcal{T}_p$ be the sheaf of p-covariant tensors over X. Let $\delta \in \mathbb{Z}$.*

Fixing a point $x_0 \in X$ and a pseudo-riemannian metric g_{x_0} of signature (s_+, s_-) at x_0 produces a $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural tensors} \\ \mathcal{M}_{(s_+,s_-)} \longrightarrow \mathcal{T}_p \\ \textit{homogeneous of weight } \delta \end{array} \right\} = \bigoplus_{d_2,\ldots,d_r} \mathrm{Hom}_{\mathrm{O}(s_+,s_-)}(S^{d_2}N_2 \otimes \ldots \otimes S^{d_r}N_r, T_{p,x_0}) \,,$$

where $d_2, \ldots, d_r$ run over the non-negative integer solutions of the equation

$$2d_2 + \ldots + rd_r = p - \delta \,,$$

and where $\mathrm{O}(s_+, s_-)$ denotes the generalised orthogonal group.

As commented before, the more general context provided by the theory of natural bundles pointed towards the possibility of describing natural operations associated to other geometrical structures. Specifically, the differential invariants associated to a linear connection were described in first instance by J. Slovak ([40]), and rewritten in a language much closer to ours by A. Gordillo-Merino and J. Navarro-Garmendia ([18]), although in the context of moduli spaces, and by D.A. Timashev ([44]), who rewrote the characterization of the differential classes associated to a linear connection.

Nevertheless, the statement below was given for the first time in full form by Gordillo-Merino, Martínez-Bohórquez and Navarro-Garmendia in [19], see Appendix D:

Theorem 1.8. *Let X be a smooth manifold and let $\mathcal{C}$ denote the sheaves of linear connections on X. Let $\mathcal{T}$ be the sheaf of (p, q)-tensors over X.*

If we fix a point $x_0 \in X$, there exists an $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural tensors} \\ \mathcal{C} \longrightarrow \mathcal{T} \end{array} \right\} = \bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Gl}}(S^{d_0}N_0 \otimes \ldots \otimes S^{d_k}N_k, T^q_{p,x_0}X) \,,$$

where $d_0, \ldots, d_k$ run over the non-negative integer solutions of the equation

$$d_0 + \ldots + (k+1)d_k = p - q$$

and $\mathrm{Gl} = \mathrm{Gl}(n, \mathbb{R})$ denotes the general linear group of invertible nxn matrices with real coefficients.

In contrast to the Riemannian case, a condition of homogeneity is not necessary here due to the equivariance by homotheties, as we will prove in Chapter 4.

This description was utilised, in the same work ([19], see Appendix D), in order to produce characterizations of the torsion and curvature operators associated to linear connections:

Theorem 1.9. *The only vector-valued 2-form naturally associated to linear connections satisfying the first Bianchi identity is the torsion tensor.*

Theorem 1.10. *For any smooth $n-$manifold (with $n \geq 3$), the constant multiples of the curvature are the only endomorphism-valued natural 2-forms (associated to symmetric linear connections) that satisfy both the first and second Bianchi identities.*

These results will be proven in Chapter 5.

A similar description of the natural operations associated to linear connections and orientations was given by Gordillo-Merino, Martínez-Bohórquez and Navarro-Garmendia in [20] (Appendix E), rewriting results of Kolář-Michor-Slovák in [25] much in the same spirit as before:

Theorem 1.11. *Let X be a smooth manifold and let $\mathcal{C}$ and Or_X denote the sheaves of linear connections and orientations on X, respectively.*

Let T be a natural sub-bundle of the bundle of (p,q)-tensors T^q_p and let $\mathcal{T}$ be its sheaf of smooth sections.

If we fix a point $x_0 \in X$ and an orientation Ω at an open neighbourhood of x_0, there exists an $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{C} \times Or_X \longrightarrow \mathcal{T} \end{array} \right\} = \bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Sl}}(S^{d_0}N_0 \otimes \ldots \otimes S^{d_k}N_k, T_{x_0}) \ ,$$

where $d_0, \ldots, d_k$ run over the non-negative integer solutions of the equation

$$d_0 + \ldots + (k+1)d_k = p - q \ .$$

Similar characterizations of the torsion and curvature operators to the ones obtained in [19] were given in [20], now considering them as natural operators associated to linear connections and orientations.

Although minor at first sight, this work included the reduction of differential invariants associated to the sheaf $\mathcal{C} \times Or$ of linear connections and orientations by the transitive action of diffeomorphisms in the sheaf Or of orientations. This tool would later prove to be of great usefulness in other geometrical settings, such as Fedosov structures, as we will now expose.

Even though the monograph by Kolář-Michor-Slovák ([25]) is very thorough in the description of differential invariants associated to several geometries, which relate to the classical groups exposed above (the orthogonal group O, the general linear group Gl and the special linear group Sl), it lacks a description of differential invariants associated to Fedosov structures, which correspond to the symplectic group Sp: pairs (ω, ∇) of a symplectic form ω and a symplectic connection ∇ (i.e. a linear connection compatible with ω, that is, that $\nabla\omega = 0$). It is the skew-symmetric version of Riemannian geometry, and as we will see later it possesses a strong relation to the symplectic group Sp.

The work of Gelfand-Retakh-Shubin ([11]) provided a description of the invariants associated to Fedosov structures, but without the generality provided by the languages of natural bundles and sheaves. They essentially started at jet level, that is, Taylor expansions of Fedosov structures at a point.

The main roadblock, which we will overcome in this memoir, is that the sheaf $\mathcal{F}$ of Fedosov structures is not the sheaf of smooth sections of any natural bundle, due to the compatibility condition $\nabla\omega = 0$. This essentially is a necessary condition in the general results provided in [25].

Utilizing the transitive action of diffeomorphisms in symplectic forms (due to the existence of Darboux coordinates), natural tensors associated to the sheaf $\mathcal{F}$ are reduced to natural tensors associated to the sheaf $\mathcal{C}_\omega$ of symplectic connections compatible with a fixed symplectic form ω, which is the sheaf of smooth sections of a natural bundle (generalizing the notion of natural bundle to include the action by just a pseudogroup of diffeomorphisms).

This development allows the use of the general machinery in order to proceed. However, additional problems appear: normal coordinates do not work as well if the symplectic form is fixed. There exists some coordinates that are apparently better suited to our problem, defined by B. Fedosov ([7, 8]), but we could not find natural ideas behind their construction.

Our approach was to 'take a step back', unfixing the symplectic form. This route posed the following technical question: what *is* the space of jets associated to Fedosov structures? The usual definition of the jet space defines jets as equivalence classes of smooth sections of fibre bundles (see [25], for example), and so the case of the Fedosov sheaf is not covered, as it is not the sheaf of smooth sections of any natural bundle.

In this memoir, we solve this problem by defining spaces of jets associated to 'natural sheaves': we generalise the notion of sheaves of smooth sections of a natural bundle to a setting where both the notions of naturalness and jet theory still make sense.

Some problems arise with this definition, which in the Fedosov case are solved by Theorem 3.10. It is not clear to us how to generalise this solution to other natural sheaves, as the proof of Theorem 3.10 requires very specific properties of symplectic forms (namely, a formal version of the Poincaré Lemma). This topic will be discussed further in Chapter 6.

With these considerations, we arrive to the description of natural operations associated to Fedosov structures ([21], see Appendix F):

Theorem 1.12. *Let X be a smooth manifold of dimension $2n$, and let $\mathcal{F}$ denote the sheaves of Fedosov structures. Let $\mathcal{T}$ be the sheaf of smooth sections of a natural subbundle $T \to X$ of the bundle p-covariant tensors on X.*

Fixing a point $x_0 \in X$ and a non-singular 2-form η_{x_0} at x_0 produces a $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Sp}(2n,\mathbb{R})\textit{-equivariant smooth maps} \\ \prod_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\} .$$

where $\mathrm{Sp}(2n,\mathbb{R}) := \{\mathrm{d}_{x_0}\tau \colon \tau \in \mathrm{Aut}(\eta)_{x_0}\}$.

As it happened in Riemannian geometry, a condition of homogeneity can be added to assure the finiteness in the dependence of variables:

Theorem 1.13. *Let X be a smooth manifold of dimension $2n$, and let $\mathcal{F}$ denote the sheaf of Fedosov structures. Let $\mathcal{T}$ be the sheaf of p-covariant tensors over X. Let $\delta \in \mathbb{Z}$.*

Fixing a point $x_0 \in X$ and a non-singular 2-form η at x_0 produces a $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \\ \textit{homogeneous of weight } \delta \end{array} \right\} = \bigoplus_{d_1,\ldots,d_r} \mathrm{Hom}_{\mathrm{Sp}}(S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r, T_{p,x_0}) \, ,$$

where $d_1, \ldots, d_r$ run over the non-negative integer solutions of the equation

$$2d_1 + \ldots + (r+1)d_r = p - \delta \, ,$$

and where $\mathrm{Sp} = \mathrm{Sp}(2n,\mathbb{R})$ *denotes the symplectic group.*

This result is later employed in Chapter 5 to characterise some scalar differential invariants, as a testament of the potency of the result above, as well as to produce the description of the most relevant dimensional curvature identities in Fedosov geometry, much in the spirit of the descriptions of such identities in Riemannian geometry given by Gilkey-Park-Sekigawa ([15]), which where later rewritten by Navarro-Navarro ([30]), as well as in Kähler geometry by Gilkey-Park-Sekigawa ([16]).

Chapter 2

The Peetre-Slovak Theorem

In this Chapter, we will begin by introducing the category of ringed spaces, followed by the concept of G-natural bundles and G-natural sheaves that we have developed. They generalise the concepts of natural bundles and sheaves of smooth sections of natural bundles, respectively, covering geometrical structures such as the symplectic connections compatible with a fixed symplectic form. Deeper results about them, involving a transitive action of diffeomorphisms, are then stated.

2.1 The Category of Ringed Spaces

Definition 2.1. A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sub-algebra of the sheaf of real-valued continuous functions on X.

A morphism of ringed spaces[1] $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $\varphi\colon X \to Y$ such that composition with φ induces a morphism of sheaves $\varphi^*\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$; that is, such that for any open set $V \subset Y$ and any function $f \in \mathcal{O}_Y(V)$, the composition $f \circ \varphi$ lies in $\mathcal{O}_X(\varphi^{-1}V)$.

Ringed spaces are a generalization of smooth manifolds, as any smooth manifold X can be understood as the ringed space $(X, \mathcal{C}^\infty_X)$, where $\mathcal{C}^\infty_X$ is the sheaf of smooth real-valued functions. If X and Y are smooth manifolds, a morphism of ringed spaces $X \to Y$ becomes a smooth map.

As we are thinking of ringed spaces as a generalization of smooth manifolds, it will be frequent to employ the same terminology: on any ringed space $(X, \mathcal{O}_X)$, the sheaf $\mathcal{O}_X$ will be called the sheaf of smooth functions, and a morphism of ringed spaces $X \to Y$ will be called a smooth map.

The category of ringed spaces has nicer (categorical) properties than the category of smooth manifolds. Many operations one would like to make in smooth manifolds do not exist in general smooth manifolds, such as the inverse limit of a sequence of smooth manifolds and the quotient by the action of a group. In contrast, both of

[1]By similarity with the category of smooth manifolds, we will often call morphisms of ringed spaces as *smooth morphisms*.

these operations can be made in the context of ringed spaces, which we will require throughout this work. Let us being with the inverse limit:

Definition 2.2. The inverse limit of a sequence of smooth manifolds and smooth maps between them

$$\ldots \to X_{k+1} \xrightarrow{\varphi_{k+1}} X_k \xrightarrow{\varphi_k} X_{k-1} \to \ldots$$

is the ringed space $(X_\infty, \mathcal{O}_\infty)$, which is defined as follows:

- The underlying topological space is the inverse limit of the topological spaces X_k, i.e., the set
$$X_\infty := \lim_{\leftarrow} X_k$$
is endowed with the minimum topology for which the canonical projections $\pi_k \colon X_\infty \to X_k$ are continuous.
- Its sheaf of smooth functions is the direct limit $\mathcal{O}_\infty := \lim\limits_{\rightarrow} \pi_k^* \mathcal{O}_{X_k}$.

This last condition means that, for any open set $U \subseteq X_\infty$, a continuous map $f \colon U \to \mathbb{R}$ lies in $\mathcal{O}_\infty(U)$ if and only if, for any point $x \in U$, there exist $k \in \mathbb{N}$, an open neighborhood $\pi_k(x) \in V_k \subseteq X_k$, and a smooth map $f_k \colon V_k \to \mathbb{R}$ such that the following triangle commutes:

$$\begin{tikzcd} \pi_k^{-1}(V_k) \arrow[rr, "f"] \arrow[dr, "\pi_k"'] & & \mathbb{R} \\ & V_k \arrow[ur, "f_k"'] & \end{tikzcd} .$$

Examples:

- Let $E \to X$ be a fibre bundle over a smooth manifold X. The bundle of ∞-jets of sections of $E \to X$ is defined as the inverse limit of the sequence of k-jets fiber bundles:
$$\ldots \to J^k E \to J^{k-1} E \to \ldots \to E \to X \, .$$
- Let $N_0, N_1, N_2, \ldots$ be a countable family of finite-dimensional $\mathbb{R}$-vector spaces. The vector space $\prod\limits_{i=0}^{\infty} N_i$ is defined as the inverse limit of the projections:
$$\ldots \to \prod_{i=0}^{k+1} N_i \to \prod_{i=0}^{k} N_i \to \ldots \to N_1 \times N_0 \to N_0 \, .$$

Universal property of the inverse limit: *For any smooth manifold Y, the projections $\pi_k \colon X_\infty \to X_k$ induce a bijection that is functorial on Y* [2]*,*

$$\begin{aligned} \mathcal{C}^\infty(Y, X_\infty) &= \lim_{\leftarrow} \mathcal{C}^\infty(Y, X_k) \\ \varphi &\longmapsto (\pi_k \circ \varphi)_k \,, \end{aligned}$$

where $\mathcal{C}^\infty(\_,\_)$ denotes the set of morphisms of ringed spaces.

Proof: Let $\varphi \in \mathcal{C}^\infty(Y, X_\infty)$. As the projections π_k are smooth maps, the composition $\pi_k \circ \varphi$ is smooth. The sequence $(\pi_k \circ \varphi)_k$ is an element of the inverse limit: if we denote by $f_{ij} : X_j \to X_i$ the transition morphisms of $\{X_k\}_k$ (for $i \leq j$), then the transition morphisms of the system $\{\mathcal{C}(Y, X_k)\}$ are $f_{ij}\circ$, and $f_{ij} \circ \pi_j \circ \varphi = \pi_i \circ \varphi$.

As for the other inclusion, let $\{\varphi_k\}_k \in \lim\limits_{\leftarrow} \mathcal{C}^\infty(Y, X_k)$. Define the map $\varphi(y) := (\varphi_k(y))_k$, continuous as the maps $\varphi_k = \pi_k \circ \varphi$ are continuous. It is smooth, as locally for any smooth function $f : X_\infty \to \mathbb{R}$ there exists $k \in \mathbb{N}$ and $f : X_k \to \mathbb{R}$ such that $f = f_k \circ \pi_k$, and so

$$f \circ \varphi = f_k \circ \pi_k \circ \varphi = f_k \circ \varphi_k \,,$$

which is smooth because φ_k is smooth. □

Proposition 2.3. *Let Z be a smooth manifold. A continuous map $\varphi \colon X_\infty \to Z$ is smooth if and only if it locally factors through a smooth map defined on some X_k.*

Proof: Let $\varphi \colon X_\infty \to Z$ be a smooth map; let $x \in X_\infty$ be a point and let $(U, z_1, \ldots, z_n)$ be a coordinate chart around $\varphi(x)$ in Z. Each of the functions $z_1 \circ \varphi, \ldots, z_n \circ \varphi \in \mathcal{O}_\infty(\varphi^{-1}U)$ locally factors through some X_j; as they are a finite number, there exists $k \in \mathbb{N}$ and an open neighborhood V of x such that all of them, when restricted to V, factor through X_k. Hence, $\varphi_{|V} = (\varphi_k \circ \pi_k)_{|V}$, where $\varphi_k = (z_1 \circ \varphi, \ldots, z_n \circ \varphi)$.

The converse is obvious because the composition of morphisms of ringed spaces is a morphism of ringed spaces. □

Let us now move onto quotients by the action of a group. Let G be a group acting on a ringed space X. Let us denote by X/G the quotient topological space and by $\pi : X \to X/G$ the quotient map.

Definition 2.4. The quotient ringed space $(X/G, \mathcal{O}_{X/G})$ is the ringed space whose underlying topological space is the quotient topological space X/G and whose sheaf of smooth functions is defined, on any open set $U \subseteq X/G$ as:

$$\mathcal{O}_{X/G}(U) := \{f \in \mathcal{C}(U, \mathbb{R}) : f \circ \pi \in \mathcal{O}_X(\pi^{-1}(U))\} = \mathcal{O}_X(\pi^{-1}(U))^G \,,$$

where $\mathcal{O}_X(\pi^{-1}(U))^G$ stands for the set of maps $f \in \mathcal{O}_X^\infty(\pi^{-1}(U))$ such that $f(g \cdot p) = f(p)$ for any $g \in G$ and $p \in \pi^{-1}(U)$.

[2]This means that the bijection defines a functor isomorphism $\mathcal{C}^\infty(\_, X_\infty) = \lim_\leftarrow \mathcal{C}^\infty(\_, X_k)$, considered as functors from the category of smooth manifolds to the category of sets.

It is then routine to check that the quotient map $\pi : X \to X/G$ is a morphism of ringed spaces that satisfies the following property:

Universal property of the quotient: *For any ringed space Y, the quotient map $\pi\colon X \to X/G$ induces a bijection that is functorial on Y:*

$$\left\{ \begin{array}{c} \text{Morphisms of ringed spaces } X \to Y \\ \text{constant along the orbits of } G \end{array} \right\} = \left\{ \begin{array}{c} \text{Morphisms of ringed spaces} \\ X/G \longrightarrow Y \end{array} \right\} .$$

Corollary 2.5 (Orbit reduction)**.** *Let G be a group acting on a ringed space X, and let $f : X \to Y$ be a surjective morphism of ringed spaces that, locally on Y, admits smooth sections passing through any point of X.*

If the orbits of G coincide with the fibers of f, then the corresponding map $\bar{f} : X/G \to Y$ is an isomorphism of ringed spaces.

Proof: The hypothesis on the fibers assures that the induced morphism $\bar{f} : X/G \to Y$ is bijective. The inverse map $\bar{f}^{-1}$ is also a morphism of ringed spaces because it locally coincides with the projection into the quotient of any smooth section of f. $\square$

Corollary 2.6. *Let G be a group acting on two ringed spaces X and Y, and let $H \subseteq G$ be a subgroup that acts trivially on Y.*

The universal property of the quotient applied to H restricts to a bijection:

$$\left\{ \begin{array}{c} \textit{G-equivariant morphisms} \\ \textit{of ringed spaces } X \to Y \end{array} \right\} = \left\{ \begin{array}{c} G/H\textit{-equivariant morphisms} \\ \textit{of ringed spaces } X/H \longrightarrow Y \end{array} \right\} .$$

2.2 G-natural bundles and G-natural sheaves

In the last chapter, we recalled the notion of natural bundle and considered what we have called natural sheaves: a generalization of sheaves of smooth sections of a natural bundle that included geometries defined by natural equations, such as Fedosov geometry. In the same vein, we now take it a step further by 'weakening' the naturalness condition, allowing us to cover geometric structures that are preserved by some pseudogroup of diffeomorphisms.

Following the notations of Chapter 1, let us begin the section by recalling the notion of pseudogroup of diffeomorphisms ([43]):

Definition 2.7. A pseudogroup of diffeomorphisms on X is a subset $G \subseteq \mathrm{Diff}\,(X)$ that satisfies the following:

1. The domains of the elements $g \in G$ cover X.
2. Let $g : U \to V$ be an element of G, and let $U' \subset U$. Then, $g_{|U'} : U' \to g(U')$ is also in G.

3. If $g : U \to V$ and $f : V \to W$ are both in G, then $f \circ g \in G$.
4. If $g : U \to V$ is in G, then $g^{-1} : V \to U$ is also in G.
5. The property of being in G is local: for any open subsets $\{U_i\}_{i\in I}$ of X, if $g : \bigcup_{i\in I} U_i \to V$ is a diffeomorphism such that $g_{|U_i} \in G$ for all $i \in I$, then $g \in G$.
6. For any $x, y \in X$ there exists a diffeomorphism $\tau : U \to V$ in G between open neighbourhoods $x \in U \subseteq X$ and $y \in V \subseteq X$ such that $\tau(x) = y$.

Although this last axiom is not usually included in the literature, we have chosen to include it as part of the definition, as all pseudogroups considered during this work verify such property and it is required in the proof of the Peetre-Slovak Theorem.

Examples:

1. Trivially, the set of all diffeomorphisms $\mathrm{Diff}\,(X)$ is a pseudogroup of diffeomorphisms.
2. Let (X, Ω) be an oriented manifold. Then,

$$\mathrm{Aut}(\Omega) = \{\tau : U \to V : \tau \in \mathrm{Diff}\,(X), \tau^*(\Omega_{|V}) = \Omega_{|U}\}$$

   is a pseudogroup of diffeomorphisms.
3. In a similar manner, let (X, ω) be a symplectic manifold. Then,

$$\mathrm{Aut}(\omega) = \{\tau : U \to V : \tau \in \mathrm{Diff}\,(X), \tau^*(\omega_{|V}) = \omega_{|U}\}$$

   is a pseudogroup of diffeomorphisms.

Definition 2.8. Let T be a smooth manifold and let $U_t, V_t \subseteq X$ be open subsets of X for all $t \in T$. A family of diffeomorphisms $\{\tau_t : U_t \to V_t\}_{t\in T}$ is said to be smooth if the sets $U := \bigsqcup_{t\in T} U_t$ and $V := \bigsqcup_{t\in T} V_t$ are open subsets of $T \times X$ and the map

$$\begin{aligned} \tau : U &\longrightarrow V \\ (t, x) &\longmapsto \tau(t, x) := \tau_t(x) \end{aligned}$$

is smooth.

Definition 2.9. Let $\pi \colon E \to X$ be a (fibre) bundle over X and $G \subseteq \mathrm{Diff}\,(X)$ be a pseudogroup. A G-natural bundle over X is a bundle $E \to X$ together with a map

$$\begin{aligned} G &\longrightarrow \mathrm{Diff}\,(E) \\ \tau &\longmapsto \tau_* \end{aligned} \quad ,$$

called lifting of diffeomorphisms, satisfying the following properties:

- Lifting: if $\tau\colon U \to V$ is a diffeomorphism in G, then $\tau_*\colon E_U \to E_V$ is a diffeomorphism covering τ; meaning that it makes the following square commutative
$$\begin{array}{ccc} E_U & \xrightarrow[\sim]{\tau_*} & E_V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ U & \xrightarrow[\sim]{\tau} & V\,, \end{array}$$
where $E_U := \pi^{-1}(U)$ and $E_V := \pi^{-1}(V)$.

- Functoriality: $\mathrm{Id}_* = \mathrm{Id}$ and $(\tau \circ \tau')_* = (\tau)_* \circ (\tau')_*$ for any diffeomorphism τ in G.

- Locality: for any diffeomorphism $\tau : U \to V$ in G and any open subset $U' \subset U$, $(\tau_{|U'})_* = (\tau_*)_{|E_{U'}}$.

- Regularity: for any smooth family of diffeomorphisms $\{\tau_t : U_t \to V_t\}_{t\in T}$ such that $\tau_t \in G$ for all $t \in T$, the family $\{\tau_{t*} : E_{U_t} \to E_{V_t}\}_{t\in T}$ is also smooth.

A morphism of G-natural bundles is a morphism of bundles $\varphi : E \to E'$ between G-natural bundles that commutes with the lifting of diffeomorphisms; that is, such that for any diffeomorphism $\tau : U \to V$ in G, the following square commutes[3]:

$$\begin{array}{ccc} E_U & \xrightarrow{\varphi} & E'_U \\ {\scriptstyle\tau_*}\downarrow & & \downarrow{\scriptstyle\tau_*} \\ E_V & \xrightarrow{\varphi} & E'_V. \end{array}$$

A subbundle $F \to X$ of a G-natural bundle $E \to X$ is said to be a G-natural subbundle of E if it is a G-natural bundle such that the inclusion $i : F \hookrightarrow E$ is a morphism of G-natural bundles.

G-natural bundles generalise the notion of natural bundle given in Chapter 1: a bundle $E \to X$ is natural if it is $\mathrm{Diff}\,(X)$-natural, and a morphism of bundles $\varphi : E \to E'$ is natural if it is $\mathrm{Diff}\,(X)$-natural.

Remark 2.10. Observe that any G-natural bundle has diffeomorphic fibres: given two points $x, y \in X$, let $\tau : U \to V$ be a diffeomorphism in G between open neighbourhoods U and V of the points x and y, respectively (such diffeomorphism exists, due to axiom 6 of Definition). Then, the diffeomorphism between the fibres at x and y is given by the restriction to the fibre at x of τ_*.

Examples:

1. Any natural bundle is a $\mathrm{Diff}\,(X)$-bundle, as mentioned above.

[3] Abusing notation, we will denote by τ_* the lifting of τ to both E and E'. The context will clarify which lifting we are referring to.

2. Let $G \subseteq G'$ be two pseudogroups of diffeomorphisms on X. Then, any G'-natural bundle is a G-natural bundle.

3. Let G be a pseudogroup of diffeomorphisms on X, and let $E_1 \to X$ and $E_2 \to X$ be G-natural bundles over X. Then, the fibre product $E_1 \times_X E_2 \to X$ is a G-natural bundle, with the lifting $\tau_* = (\tau_{*_1}, \tau_{*_2})$, where τ_{*_1} is the lifting of τ to E_1 and τ_{*_2} is the lifting of τ to E_2.

4. The k-jet prolongation of a G-natural bundle is a G-natural bundle: if we denote by $\tau_* : E_U \to E_V$ the lifting of a diffeomorphism $\tau : U \to V$ in G to a G-natural bundle $E \to X$, then the lifting of τ to $J^r E$ is

$$\begin{aligned} J^r E\colon J^r E_U &\longrightarrow J^r E_V \\ j^r_{x_0} s &\longmapsto j^r_{\tau(x_0)}(\tau_* \circ s \circ \tau^{-1}). \end{aligned}$$

5. Let (X, ω) be a symplectic manifold. The fibre bundle $\mathrm{Conn}_\omega \to X$ of symplectic connections compatible with ω (that is, symmetric linear connections ∇ such that $\nabla\omega = 0$) is an $\mathrm{Aut}(\omega)$-natural bundle. The definition of the lifting is the same as for the bundle of all linear connections.

Definition 2.11. Let G be a pseudogroup of diffeomorphisms. A G-natural sheaf $\mathcal{E}$ over X is a subsheaf of the sheaf of smooth sections of a natural bundle $E \to X$ over X such that, for any diffeomorphism $\tau : U \to V$ in G, the morphism

$$\begin{aligned} \tau_*\colon \mathcal{E}(U) &\longrightarrow \mathcal{E}(V) \\ s &\longmapsto \tau_* \circ s \circ \tau^{-1} \end{aligned}$$

is well defined[4].

Examples:

1. Any natural sheaf, as defined in Chapter 1, is a $\mathrm{Diff}\,(X)$-natural sheaf.

2. Let (X, ω) be a symplectic manifold. The sheaf $\mathcal{C}_\omega$ of symplectic connections compatible with ω, defined on any open subset $U \subseteq X$ as

$$\mathcal{C}_\omega(U) := \{\nabla \in \mathcal{C}(U) : \nabla(\omega_{|U}) = 0\}\,,$$

is an $\mathrm{Aut}(\omega)$-natural sheaf.

Definition 2.12. Let $E \to X$ be a fibre bundle over X, let $U \subseteq X$ be an open subset and let T be a smooth manifold. A family of sections $\{s_t : U \to E\}_{t\in T}$ is said to be

[4]Observe that we are committing an abuse of notation: we are denoting by τ_* both the lifting of τ to E and the 'action' of τ on $\mathcal{E}$. However, the context will make clarify which morphism we are working with.

smooth if the map

$$\begin{aligned} s : T \times U &\longrightarrow E \\ (t, x) &\longmapsto s(t, x) := s_t(x) \end{aligned}$$

is smooth.

Definition 2.13. Let $\mathcal{E}$ and $\mathcal{E}'$ be (sub)sheaves of the sheaves of smooth sections of the fibre bundles $E \to X$ and $E' \to X$, and let T be a smooth manifold. A morphism of sheaves $\phi : \mathcal{E} \to \mathcal{E}'$ is said to be regular if, for any smooth family of sections $\{s_t : U \to E\}_{t \in T}$ such that $U \simeq \mathbb{R}^n$ and $s_t \in \mathcal{E}(U)$ for all $t \in T$, the family $\{\phi(s_t) : U \to E'\}_{t \in T}$ is also smooth.

Definition 2.14. Let $G \subseteq \operatorname{Diff}(X)$ be a pseudogroup. Let $\mathcal{E}$ and $\mathcal{E}'$ be G-natural sheaves over X. A morphism of sheaves $\phi : \mathcal{E} \to \mathcal{E}'$ is G-natural if it is regular and commutes with the action of G on sections; that is to say, if for any diffeomorphism $\tau : U \to V$ in G, the following square commutes:

$$\begin{tikzcd} \mathcal{E}(U) \arrow[r, "\phi"] \arrow[d, "\tau_*"'] & \mathcal{E}'(U) \arrow[d, "\tau_*"] \\ \mathcal{E}(V) \arrow[r, "\phi"] & \mathcal{E}'(V) . \end{tikzcd} \tag{2.1}$$

where $\tau_* : \mathcal{E}(U) \to \mathcal{E}(V)$ is defined as in Definition 2.11.

As before, we will say that a sheaf is natural if it is $\operatorname{Diff}(X)$-natural and that a morphism of sheaves over X is natural if it is $\operatorname{Diff}(X)$-natural.

The following result is a particular case of the Peetre-Slovák Theorem, and is the reason why we have added the condition of regularity to the definition of natural morphism of sheaves:

Theorem 2.15 (Peetre-Slovák). *Let X be a smooth manifold. Let $E \to X$ and $E' \to X$ be fibre bundles over X, and let $\mathcal{E}$ and $\mathcal{E}'$ be their respective sheaves of smooth sections over X. Then,*

$$\left\{ \begin{array}{c} \textit{Regular morphisms of sheaves} \\ \phi \colon \mathcal{E} \longrightarrow \mathcal{E}' \end{array} \right\} = \left\{ \begin{array}{c} \textit{Smooth maps} \\ J^\infty E \longrightarrow E' \end{array} \right\} .$$

The full proof of this result can be checked at ([33]). Although the notion of regularity defined in this work is not exactly the same, it is close enough so that the same proof holds, as all statements are proven locally.

By adding a naturalness condition to this bijection, we can fix a point at the right hand side of the bijection, obtaining the following result:

Theorem 2.16 (Peetre-Slovák). *Let X be a smooth manifold and $G \subseteq \mathrm{Diff}(X)$ a pseudogroup. Let $E \to X$ and $E' \to X$ be G-natural bundles over X, and let $\mathcal{E}$ and $\mathcal{E}'$ be their respective sheaves of smooth sections over X.*

The choice of a point $x_0 \in X$ allows the definition of a bijection:

$$\left\{ \begin{array}{c} \textit{G-natural morphisms of sheaves} \\ \phi\colon \mathcal{E} \longrightarrow \mathcal{E}' \end{array} \right\} = \left\{ \begin{array}{c} G_{x_0}\textit{-equivariant smooth maps} \\ J^\infty_{x_0} E \longrightarrow E'_{x_0} \end{array} \right\},$$

where G_{x_0} stands for the group of germs of diffeomorphisms τ in G such that $\tau(x_0) = x_0$.

Proof: Let us first prove the bijection

$$\left\{ \begin{array}{c} G\text{-natural morphisms of sheaves} \\ \phi\colon \mathcal{E} \longrightarrow \mathcal{E}' \end{array} \right\} = \left\{ \begin{array}{c} G\text{-natural smooth maps} \\ J^\infty E \longrightarrow E' \end{array} \right\},$$

where a smooth map $P : J^\infty E \longrightarrow E'$ is said to be G-natural if it commutes with the lifting of diffeomorphisms in G[5]. Let $\tau : U \to V$ be a diffeomorphism in G and, abusing the notation, let τ_* denote the lifting of τ to either E or E', indistinctively. Let $\phi\colon \mathcal{E} \to \mathcal{E}'$ be a G-natural morphism of sheaves, and let $P_\phi : J^\infty E \to E'$ be the associated smooth map: $P_\phi(j^\infty_x s) := \phi(s)(x)$. Then,

$$P_\phi(j^\infty_{\tau(x)} \tau_* \circ s \circ \tau^{-1}) = \phi(\tau_* \circ s \circ \tau^{-1})(\tau(x)) = \tau_* \circ \phi(s) \circ \tau^{-1}(\tau(x)) = \tau_*(P_\phi(j^\infty_x s)) \,.$$

Reciprocally, let $P : J^\infty E \to E'$ be a G-natural smooth map, and let $\phi_P : \mathcal{E} \to \mathcal{E}'$ be the associated regular morphism of sheaves. Then,

$$\phi_P(\tau_* \circ s \circ \tau^{-1})(\tau(x)) = P(j^\infty_{\tau(x)} \tau_* \circ s \circ \tau^{-1}) = \tau_*(P(j^\infty_x s)) = \tau_*(\phi_P(s)(x)) = \tau_* \circ \phi_P(s) \circ \tau^{-1}(\tau(x)) \,.$$

Due to property 6 of Definition 2.7, the bijection

$$\left\{ \begin{array}{c} G\text{-natural smooth maps} \\ J^\infty E \longrightarrow E' \end{array} \right\} = \left\{ \begin{array}{c} G_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0} E \longrightarrow E'_{x_0} \end{array} \right\},$$

is proven by the standard arguments (see [20], Appendix E). □

This theorem is usually written in the base case $G = \mathrm{Diff}(X)$ (see [19] (Appendix D), [20] (Appendix E)). However, this generalization will be of particular interest for this work, as we obtain the following two statements:

Theorem 2.17. *Let X be an orientable manifold and let Ω be an orientation on X. Let $E \to X$ and $E' \to X$ be $\mathrm{Aut}(\Omega)$-natural bundles over X, and let $\mathcal{E}$ and $\mathcal{E}'$ be their respective sheaves of smooth sections over X.*

[5] Observe that we cannot say that it is a morphism of G-natural bundles because $J^\infty E$ is not a bundle (its dimension is not necessarily finite). Otherwise, both notions are the same.

The choice of a point $x_0 \in X$ defines a bijection

$$\left\{ \begin{array}{c} \mathrm{Aut}(\Omega)\text{-}\textit{natural morphisms of sheaves} \\ \phi\colon \mathcal{E} \longrightarrow \mathcal{E}' \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\Omega)_{x_0}\text{-}\textit{equivariant smooth maps} \\ J^\infty_{x_0}E \longrightarrow E'_{x_0} \end{array} \right\},$$

where $\mathrm{Aut}(\Omega)$ is the pseudogroup of diffeomorphisms $\tau : U \to V$ such that $\tau_*(\Omega_{|U}) = \Omega_{|V}$ and $\mathrm{Aut}(\Omega)_{x_0}$ denotes the group of germs of diffeomorphisms τ of X such that $\tau(x_0) = x_0$ and $\tau_*\Omega = \Omega$.

Theorem 2.18. *Let X be a symplectic manifold and let ω be a symplectic form on X. Let $E \to X$ and $E' \to X$ be $\mathrm{Aut}(\omega)$-natural bundles over X, and let $\mathcal{E}$ and $\mathcal{E}'$ be their respective sheaves of smooth sections over X.*

The choice of a point $x_0 \in X$ defines a bijection

$$\left\{ \begin{array}{c} \mathrm{Aut}(\omega)\text{-}\textit{natural morphisms of sheaves} \\ \phi\colon \mathcal{E} \longrightarrow \mathcal{E}' \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\omega)_{x_0}\text{-}\textit{equivariant smooth maps} \\ J^\infty_{x_0}E \longrightarrow E'_{x_0} \end{array} \right\},$$

where $\mathrm{Aut}(\omega)$ is the pseudogroup of diffeomorphisms $\tau : U \to V$ such that $\tau_*(\omega_{|U}) = \omega_{|V}$ and $\mathrm{Aut}(\omega)_{x_0}$ denotes the group of germs of diffeomorphisms τ of X such that $\tau(x_0) = x_0$ and $\tau_*\omega = \omega$.

2.3 Reduction by a transitive action

Proposition 2.19. *Let X be a smooth manifold of dimension n, and let $\mathcal{E}$ and $\mathcal{E}'$ be natural sheaves over X. For any chart $U \simeq \mathbb{R}^n$ on X, the restriction to U produces a bijection:*

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \phi : \mathcal{E} \longrightarrow \mathcal{E}' \end{array} \right\} = \left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \phi_U : \mathcal{E}_U \longrightarrow \mathcal{E}'_U \end{array} \right\},$$

where $\mathcal{E}_U$ and $\mathcal{E}'_U$ denote the restrictions of $\mathcal{E}$ and $\mathcal{E}'$ to the open subset U, respectively.

Proof: If ϕ is a natural morphism of sheaves $\phi : \mathcal{E} \to \mathcal{E}'$, its restriction to U (that is, $\phi_U(s) := \phi(s)$) is trivially a natural morphism of sheaves.

Reciprocally, for any natural morphism of sheaves $f : \mathcal{E}_U \to \mathcal{E}'_U$, let us construct the corresponding natural morphism of sheaves $\phi_f : \mathcal{E} \to \mathcal{E}'$: for any $V \subseteq X$, $s \in \mathcal{E}(V)$ and $x \in V$, we must define $\phi_f(s)(x)$.

As $\phi_f(s)(x) = \phi_f(s_{|W})(x)$ for any $W \subseteq V$ containing x, we may suppose that V is also a chart, thus obtaining a diffeomorphism $\tau : V \to U$, and so we may define:

$$\phi_f(s) = \tau_*^{-1}(f(\tau_* s))$$

at a neighbourhood of x.

Let us check that this morphism is well defined, natural, regular and the inverse of the map $\phi \to \phi_U$:

- ϕ_f is well defined: let $x \in V' \subseteq X$ and let $\sigma : V' \to U$ be another diffeomorphism. Taking the intersection $V \cap V'$ (non-empty, as $x \in V \cap V'$), we may set $V' = V$. As $\tau \circ \sigma^{-1} : U \to U$ is a diffeomorphism between open subsets of U and f is natural, we obtain

$$\sigma_*^{-1}(f(\sigma_* s)) = \tau_*^{-1}\tau_*\sigma_*^{-1}(f(\sigma_* s)) = \tau_*^{-1}(f(\tau_*\sigma_*^{-1}\sigma_* s)) = \tau_*^{-1}(f(\tau_* s)) .$$

- ϕ_f is natural: let $V, W \subseteq X$, $s \in \mathcal{E}(V)$ and let $\sigma : V \to W$ be a diffeomorphism. Let $\tau : W \to U$ be any diffeomorphism. Then,

$$\phi_f(\sigma_* s) = \tau_*^{-1}(f(\tau_*\sigma_* s)) = \sigma_*\sigma_*^{-1}\tau_*^{-1}(f(\tau_*\sigma_* s)) = \sigma_*\phi_f(s) ,$$

  as $\tau \circ \sigma : V \to U$ is a diffeomorphism and ϕ_f does not depend on the choice of the diffeomorphism.

- ϕ_f is regular: let T be a smooth manifold, let $E \to X$ be a bundle such that $\mathcal{E}$ is a subsheaf of the sheaf of smooth sections of E and let $(s_t)_{t\in T}$ be a smooth family of sections on a chart $V \subseteq X$ such that $s_t \in \mathcal{E}(V)$ for all $t \in T$. Let $\tau : V \to U$ be any diffeomorphism. Then, the map

$$\phi_f(s)(t, x) := \phi_f(s_t)(x) = \tau_*^{-1}(f(\tau_* s_t))(x)$$

  is smooth.

- The map $f \to \phi_f$ is the inverse of the map $\phi \to \phi_{|U}$:

$$\phi_{\phi_U}(s) = \tau_*^{-1}(\phi_U(\tau_* s)) = \tau_*^{-1}(\phi(\tau_* s)) = \tau_*^{-1}\tau_*(\phi(s)) = \phi(s) ,$$

$$(\phi_f)_U(s) = \phi_f(s) = \mathrm{Id}_*^{-1}(f(\mathrm{Id}_* s)) = f(s) .$$

□

Definition 2.20. Let G be a pseudogroup on X and let $\mathcal{E}$ be a natural sheaf over X. We say that G acts transitively on $\mathcal{E}$ if for any $s \in \mathcal{E}(U)$, $s' \in \mathcal{E}(V)$ such that $U, V \simeq \mathbb{R}^n$, there exists a diffeomorphism $\tau : U \to V$ such that $\tau_* s = s'$.

Example. The pseudogroup $G = \mathrm{Diff}(X)$ acts transitively on the sheaf $\mathcal{S}$ of symplectic forms over X: given $U, V \simeq \mathbb{R}^n$ open subsets of X and symplectic forms ω_1 and ω_2 on U and V respectively, let us fix a symplectic form ω_0 with constant coefficients on $\mathbb{R}^n$ and consider diffeomorphisms $\tau_1 : U \to \mathbb{R}^n$ and $\tau_2 : V \to \mathbb{R}^n$ such that $\tau_{1,*}\omega_1 = \omega_0$ and $\tau_{2,*}\omega_2 = \omega_0$. Then, $\tau := \tau_2^{-1} \circ \tau_1$ is a diffeomorphism that verifies $\tau_*\omega_1 = \tau_{2,*}^{-1}\tau_{1,*}\omega_1 = \omega_2$.

Proposition 2.21. *Let X be a smooth manifold of dimension n. Let $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$ be natural sheaves over X such that $\mathcal{E}'(X) \neq \emptyset$ and Diff(X) acts transitively on $\mathcal{E}'$. Let $\mathcal{G} \subset \mathcal{E} \times \mathcal{E}'$ be a natural sheaf over X.*

The choice of a global section $s_0 \in \mathcal{E}'(X)$ *defines a bijection:*

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{G} \longrightarrow \mathcal{E}'' \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(s_0)\textit{-natural morphisms of sheaves} \\ \mathcal{E}_{s_0} \longrightarrow \mathcal{E}'' \end{array} \right\},$$

where $\mathcal{E}_{s_0} := \pi_2^{-1}(s_0)$ *is the sheaf defined as*

$$\mathcal{E}_{s_0}(U) = \{s \in \mathcal{E}(U) : (s, s_{0|U}) \in \mathcal{G}(U)\}$$

and $\mathrm{Aut}(s_0)$ *is the pseudogroup of diffeomorphisms* $\tau : U \to V$ *between open sets of* X *such that* $\tau_*(s_{0|U}) = s_{0|V}$.

Proof: Given a natural morphism of sheaves $\phi : \mathcal{G} \to \mathcal{E}''$, the corresponding morphism of sheaves $\hat{\phi} : \mathcal{E}_{s_0} \to \mathcal{E}''$ is given, at any open subset $U \subseteq \mathbb{R}^n$, by

$$\hat{\phi}_U(s) := \phi_U(s, s_{0|U}),$$

which is trivially an $\mathrm{Aut}(s_0)$-natural morphism of sheaves.

Let us give the inverse map, that is, to define a natural morphism of sheaves $\tilde{\varphi} : \mathcal{G} \to \mathcal{E}''$ from an $\mathrm{Aut}(s_0)$-natural morphism of sheaves $\varphi : \mathcal{E}_{s_0} \to \mathcal{E}''$. Let $(s, s') \in \mathcal{G}(U)$ and $x \in U$, and let us define the value of $\tilde{\varphi}(s, s')(x)$. By hypothesis, there exists an open subset $V \subseteq U$ and a diffeomorphism $\tau : V \to V$ such that $x \in V$ and $\tau_*(s'_{|V}) = s_{0|V}$. As the value at x of $\tilde{\varphi}(s, s')$ does not depend on the chosen neighbourhood of x, we may assume that $V = U$. Then, at U we can set

$$\tilde{\varphi}(s, s') := \tau_*^{-1}(\varphi(\tau_* s)).$$

Let us prove that this morphism is well defined, natural, regular and the inverse of the map $\phi \to \hat{\phi}$:

- $\tilde{\varphi}$ is well defined: without loss of generality, let $\sigma : U \to U$ be another diffeomorphism such that $\sigma_*(s') = s_0$. As $\tau \circ \sigma^{-1} : U \to U$ is a diffeomorphism such that $(\tau \circ \sigma^{-1})_* s_0 = s_0$ (at U) and φ is $\mathrm{Aut}(s_0)$-natural, we obtain

$$\sigma_*^{-1}(\varphi(\sigma_* s)) = \tau_*^{-1} \tau_* \sigma_*^{-1}(\varphi(\sigma_* s)) = \tau_*^{-1}(\varphi(\tau_* \sigma_*^{-1} \sigma_* s)) = \tau_*^{-1}(\varphi(\tau_* s)).$$

- $\tilde{\varphi}$ is natural: let $U, V \subseteq X$, $(s, s') \in \mathcal{G}(U)$ and let $\sigma : U \to V$ be a diffeomorphism. Let $\tau : V \to U$ be a diffeomorphism such that $\tau_*(\sigma_*(s')) = s_0$ at U. Then,

$$\tilde{\varphi}(\sigma_*(s, s')) = \tilde{\varphi}(\sigma_* s, \sigma_* s') = \tau_*^{-1}(\varphi(\tau_* \sigma_* s)) = \sigma_* \sigma_*^{-1} \tau_*^{-1}(\varphi(\tau_* \sigma_* s)) = \sigma_* \tilde{\varphi}(s, s'),$$

as $\tau \circ \sigma : U \to U$ is a diffeomorphism such that $(\tau \circ \sigma)_*(s') = s_0$ at U and $\tilde{\varphi}$ does not depend on the choice of such diffeomorphism.

- $\tilde{\varphi}$ is regular: let T be a smooth manifold, let $E \to X$ be a bundle such that $\mathcal{G}$ is a subsheaf of the sheaf of smooth sections of E and let $(s_t, s'_t)_{t\in T}$ be a smooth family of sections on a chart $U \subseteq X$ such that $(s_t, s'_t) \in \mathcal{G}(U)$ for all $t \in T$. Let $\tau_t : U \to U$ be a smooth family of diffeomorphisms such that $\tau_{t,*}(s'_t) = s_0$. Then, the map

$$\tilde{\varphi}(s,s')(t,x) := \tilde{\varphi}(s_t, s'_t)(x) = \tau_{t,*}^{-1}(\varphi(\tau_{t,*}s_t))(x)$$

is smooth.

- The map $\varphi \to \tilde{\varphi}$ is the inverse of the map $\phi \to \hat{\phi}$: with the previous notation,

$$\tilde{\hat{\phi}}(s,s') = \tau_*^{-1}(\hat{\phi}(\tau_* s)) = \tau_*^{-1}(\phi(\tau_* s, s_0)) = \tau_*^{-1}\tau_*(\phi(s, \tau_*^{-1}s_0)) = \phi(s,s') \,,$$

$$\hat{\tilde{\varphi}}(s) = \tilde{\varphi}(s, s_0) = \mathrm{Id}_*^{-1}(\varphi(\mathrm{Id}_* s)) = \varphi(s) \,.$$

□

Proposition 2.22. *Let $E \to X$ be a natural bundle, and let $\mathcal{E}$ be the sheaf of smooth sections associated to E.*

The choices of a point $x_0 \in X$ and an orientation Ω at an open neighbourhood of x_0 define a bijection

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{E} \times Or \longrightarrow \mathcal{T} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\Omega)_{x_0}\textit{-equivariant smooth maps} \\ J^\infty_{x_0} E \longrightarrow T_{x_0} \end{array} \right\} ,$$

where $\mathrm{Aut}(\Omega)_{x_0}$ denotes the group of germs of diffeomorphisms τ of X such that $\tau(x_0) = x_0$ and $\tau_*\Omega = \Omega$.

Proof: Let us fix $x_0 \in X$ and an orientation Ω at an open subset $x_0 \in V \subseteq X$. Applying Proposition 2.19 to a chart $U \subseteq V$ around x_0 and considering that $U \simeq \mathbb{R}^n$, we get a bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{E} \times Or \longrightarrow \mathcal{T} \end{array} \right\} = \left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{E}_U \times Or_U \longrightarrow \mathcal{T}_U \end{array} \right\} .$$

As $(U, \Omega_{|U})$ is an oriented manifold and diffeomorphisms act transitively on the sheaf Or_U, we can apply Proposition 2.21:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{E}_U \times Or_U \longrightarrow \mathcal{T}_U \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\Omega)\text{-natural} \\ \text{morphisms of sheaves} \\ \mathcal{E}_U \longrightarrow \mathcal{T}_U \end{array} \right\} .$$

Finally, Theorem 2.17 applied to the oriented smooth manifold U gives:

$$\left\{ \begin{array}{c} \mathrm{Aut}(\Omega)\text{-natural} \\ \text{morphisms of sheaves} \\ \mathcal{E}_U \longrightarrow \mathcal{T}_U \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\Omega)_{x_0}\text{-equivariant} \\ \text{smooth maps} \\ J^\infty_{x_0} E \longrightarrow T_{x_0} \end{array} \right\}.$$

□

Remark 2.23. Observe that one could have applied the Peetre-Slovak theorem to, for example, differential invariants associated to linear connections and orientations, as the sheaf $\mathcal{C} \times Or$ is indeed the sheaf of smooth sections of a natural fibre bundle (the direct product of the bundle of linear connections and the bundle of orientations, both of which are natural). The transitive action of diffeomorphisms would then by considered in the space of jets at a point of this geometric structure (this line of reasoning can be checked at [20], Appendix E).

However, the same argument cannot be replicated for the space of differential invariants associated to Fedosov structures, as that sheaf is not the sheaf of smooth sections of any natural bundle, which is why we have chosen to make the reduction by the transitive action of diffeomorphism at sheaf level.

Proposition 2.24. *Let X be a smooth manifold of dimension $2n$. Let $\mathcal{F}$ be the sheaf of Fedosov structures over X. The choices of a point $x_0 \in X$ and a symplectic form ω at an open neighbourhood of x_0 define a bijection*

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\omega)_{x_0}\textit{-equivariant smooth maps} \\ J^\infty_{x_0} Conn_{\,\omega} \longrightarrow T_{x_0} \end{array} \right\},$$

where $\mathrm{Aut}(\omega)_{x_0}$ denotes the group of germs of diffeomorphisms τ of X such that $\tau(x_0) = x_0$ and $\tau_* \omega = \omega$.

Proof: The proof of this result is similar to that of Proposition 2.22: let us fix a point $x_0 \in X$, a chart U around x_0 and apply Proposition 2.19, producing a bijection

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \end{array} \right\} = \left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F}_U \longrightarrow \mathcal{T}_U \end{array} \right\}.$$

Now, fix a symplectic form ω on U and recall that $\mathcal{F} \subset \mathcal{C} \times \mathcal{S}$, where $\mathcal{C}$ is the sheaf of linear connections over X and $\mathcal{S}$ is the sheaf of symplectic forms over X. As $\mathrm{Diff}\,(X)$ acts transitively on $\mathcal{S}$ (as seen in the example after Definition 2.20), Proposition 2.21 gives the following bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F}_U \longrightarrow \mathcal{T}_U \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\omega)\text{-natural} \\ \text{morphisms of sheaves} \\ \mathcal{C}_\omega \longrightarrow \mathcal{T}_U \end{array} \right\},$$

where $\mathcal{C}_\omega$ is the sheaf of symplectic connections compatible with ω over U.

All that is left is to invoke Theorem 2.18, finishing the proof:

$$\left\{ \begin{array}{c} \mathrm{Aut}(\omega)\text{-natural} \\ \text{morphisms of sheaves} \\ \mathcal{C}_\omega \longrightarrow \mathcal{T}_U \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\omega)_{x_0}\text{-equivariant} \\ \text{smooth maps} \\ J^\infty_{x_0}\mathrm{Conn}_\omega \longrightarrow T_{x_0} \end{array} \right\}.$$

□

Chapter 3

Jets of geometric structures involving a linear connection

The Peetre-Slovak Theorem 2.16 stated in the previous Chapter enables a nice description of the spaces of natural tensors, which will be achieved by taking a closer look at the structure of the spaces of jets at a fixed point.

As natural tensors do not depend on the choice of coordinates, we shall choose a system of coordinates in which our geometrical objects become simpler. In the presence of a linear connection, such coordinates always exist: they are known as normal coordinates, and they are the fundamental tool of our Reduction Theorems, which describe the spaces of jets in terms of vector spaces (called spaces of normal tensors).

3.1 Normal extensions of linear connections

Let X be a smooth manifold of dimension n. Let ∇ be the germ of a linear connection at a point $x_0 \in X$, and let $\bar{\nabla}$ be the germ of the flat connection at $x_0 \in X$ corresponding, via the exponential map, to the flat connection of $T_{x_0}X$.

Let $\mathbb{T} := \nabla - \bar{\nabla}$ be the difference tensor, that is, the following $(2,1)$-tensor:

$$\mathbb{T}(\omega, D_1, D_2) := \omega(D_1^{\nabla} D_2 - D_1^{\bar{\nabla}} D_2) .$$

Definition 3.1. For any integer $m \geq 0$, the m-th normal tensor of ∇ at x_0 is $\bar{\nabla}^m_{x_0}\mathbb{T}$.

In a system of normal coordinates $(x_1, \ldots, x_n)$ for ∇ at x_0, the tensor $\bar{\nabla}^m_{x_0}\mathbb{T}$ takes the expression below:

$$\bar{\nabla}^m_{x_0}\mathbb{T} = \sum_{i,j,k,a_1,\ldots,a_m} \Gamma^k_{ij,a_1\ldots a_m} \cdot \left(\frac{\partial}{\partial x_k}\right)_{x_0} \otimes \mathrm{d}_{x_0}x_i \otimes \mathrm{d}_{x_0}x_j \otimes \mathrm{d}_{x_0}x_{a_1} \otimes \ldots \otimes \mathrm{d}_{x_0}x_{a_m} ,$$

or, in index notation,

$$(\bar{\nabla}^m_{x_0}\mathbb{T})^k_{ij,a_1\ldots a_m} := \Gamma^k_{ij,a_1\ldots a_m} ,$$

where $\Gamma^k_{ij,a_1\dots a_m} := \frac{\partial^m \Gamma^k_{ij}}{\partial x_{a_1}\dots\partial x_{a_m}}(x_0)$ and Γ^k_{ij} are the germs at x_0 of the Christoffel symbols of ∇ in the coordinates $(x_1, \dots, x_n)$.

Definition 3.2. Let $m \geq 0$ be an integer. The space N_m of normal tensors of order m at x_0 is the vector subspace of $(m+2,1)$-tensors T at x_0 satisfying the following symmetries:

1. they are symmetric in the last m covariant indices:

$$T^l_{ijk_1\dots k_m} = T^l_{ijk_{\sigma(1)}\dots k_{\sigma(m)}}\,, \quad \forall\, \sigma \in S_m\,; \tag{3.1}$$

2. the symmetrization of the $m+2$ covariant indices is zero:

$$\sum_{\sigma\in S_{m+2}} T^l_{\sigma(i)\sigma(j)\sigma(k_1)\dots\sigma(k_m)} = 0. \tag{3.2}$$

Example. The m-th normal tensor $\bar{\nabla}^m_{x_0}\mathbb{T}$ of a linear connection ∇ lies in N_m: it trivially verifies Equation 3.1 and Equation 3.2 holds due to Gauss's Lemma (see [18], for reference).

These spaces are closely related to the space of jets of linear connections at a point. In particular, for any $m \geq 0$ we get a map (well defined, due to the example above):

$$\begin{aligned} \phi_m\colon J^m_{x_0}\mathrm{Conn} &\longrightarrow N_0 \times N_1 \times \dots \times N_m \\ j^m_{x_0}\nabla &\longmapsto (\mathbb{T}_{x_0}, \bar{\nabla}^1_{x_0}\mathbb{T}, \dots, \bar{\nabla}^m_{x_0}\mathbb{T})\,. \end{aligned}$$

These maps are compatible, meaning that the diagram

$$\begin{tikzcd} J^{m+1}_{x_0}\mathrm{Conn} \arrow[r, "\phi_{m+1}"] \arrow[d] & N_0 \times \dots \times N_{m+1} \arrow[d] \\ J^m_{x_0}\mathrm{Conn} \arrow[r, "\phi_m"] & N_0 \times \dots \times N_m, \end{tikzcd}$$

commutes for any m. This, in turn, defines a morphism of ringed spaces between the corresponding inverse limits, using the universal property of the inverse limit $\prod_{i=0}^{\infty} N_i$ and applying Proposition 2.3 to smooth morphisms $J^\infty_{x_0}\mathrm{Conn} \to N_m$:

$$\begin{aligned} \phi_\infty\colon J^\infty_{x_0}\mathrm{Conn} &\longrightarrow \prod_{i=0}^{\infty} N_i \\ j^\infty_{x_0}\nabla &\longmapsto (\bar{\nabla}^1_{x_0}\mathbb{T}, \bar{\nabla}^2_{x_0}\mathbb{T}, \dots). \end{aligned}$$

For any $m \geq 1$, let us consider the Lie groups $\mathrm{Diff}^m_{x_0} := \{j^m_{x_0}\tau : \tau \in \mathrm{Diff}_{x_0}\}$ as well as their subgroups $\mathrm{NDiff}^m_{x_0} := \{j^m_{x_0}\tau \in \mathrm{Diff}^m_{x_0} : j^1_{x_0}\tau = j^1_{x_0}\mathrm{Id}\}$. They are related by the following short exact sequence of groups:

$$1 \longrightarrow \mathrm{NDiff}^m_{x_0} \hookrightarrow \mathrm{Diff}^m_{x_0} \longrightarrow \mathrm{Gl} \longrightarrow 1\,, \tag{3.3}$$
$$j^m_{x_0}\tau \longmapsto \mathrm{d}_{x_0}\tau \tag{3.4}$$

where $\mathrm{Gl} := \mathrm{Diff}^1_{x_0} = \{\mathrm{d}_{x_0}\tau \colon \tau \in \mathrm{Diff}_{x_0}\}$.

Their inverse limits define groups

$$\mathrm{Diff}^\infty_{x_0} := \lim_{\leftarrow} \mathrm{Diff}^m_{x_0} \quad \text{and} \quad \mathrm{NDiff}^\infty_{x_0} := \lim_{\leftarrow} \mathrm{NDiff}^m_{x_0}\,,$$

with the corresponding short exact sequence of groups:

$$1 \longrightarrow \mathrm{NDiff}^\infty_{x_0} \longrightarrow \mathrm{Diff}^\infty_{x_0} \longrightarrow \mathrm{Gl} \longrightarrow 1\,. \tag{3.5}$$

Reduction Theorem (for linear connections). *The* $\mathrm{Diff}^{m+2}_{x_0}$*−equivariant smooth map*

$$\begin{aligned} \phi_m \colon J^m_{x_0}Conn &\longrightarrow N_0 \times \ldots \times N_m \\ j^m_{x_0}\nabla &\longmapsto (\mathbb{T}_{x_0}, \bar{\nabla}^1_{x_0}\mathbb{T}, , \ldots, \bar{\nabla}^m_{x_0}\mathbb{T}), \end{aligned}$$

is surjective, its fibres are the orbits of $\mathrm{NDiff}^{m+2}_{x_0}$ *and it admits smooth sections passing through any point of* $J^m_{x_0}Conn$*.*

As a consequence, ϕ_m *induces a* Gl*-equivariant isomorphism of ringed spaces:*

$$(J^m_{x_0}Conn\,)/\mathrm{NDiff}^{m+2}_{x_0} = N_0 \times \ldots \times N_m\,.$$

Proof: The proof can be checked at [18]. However, we have chosen to reproduce it here in greater detail, for the sake of completeness.

We begin by checking that the fibres of ϕ_m coincide with the orbits of $\mathrm{NDiff}^{m+2}_{x_0}$. Let $j^m_{x_0}\nabla' = j^{m+2}_{x_0}\tau \cdot j^m_{x_0}\nabla$, for $j^{m+2}_{x_0}\tau \in \mathrm{NDiff}^{m+2}_{x_0}$ and $j^m_{x_0}\nabla \in J^m_{x_0}\mathrm{Conn}$. As ϕ_m is $\mathrm{Diff}^{m+2}_{x_0}$−equivariant (thus also $\mathrm{NDiff}^{m+2}_{x_0}$−equivariant) and $j^{m+2}_{x_0}\tau$ acts as the identity in $N_0 \times \ldots \times N_m$,

$$\phi_m(j^m_{x_0}\nabla') = \phi_m(j^{m+2}_{x_0}\tau \cdot j^m_{x_0}\nabla) = j^{m+2}_{x_0}\tau \cdot \phi_m(j^m_{x_0}\nabla) = \phi_m(j^m_{x_0}\nabla)\,.$$

Let now $j^m_{x_0}\nabla, j^m_{x_0}\nabla' \in J^m_{x_0}\mathrm{Conn}$ such that $\phi_m(j^m_{x_0}\nabla) = \phi_m(j^m_{x_0}\nabla') = (T_0, \ldots, T_m)$. Let us fix a base of $T^*_{x_0}X$. This base induces two systems of normal coordinates at x_0, denoted by $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$, via the exponential maps associated to any representative of $j^m_{x_0}\nabla$ and any representative of $j^m_{x_0}\nabla'$, respectively. With both systems of coordinates, we can construct a diffeomorphism τ, defined by the equalities $\tau \cdot x_i = (\tau^{-1})^* x_i = x'_i$ for all $i \in \{1, \ldots, n\}$. As $\mathrm{d}_{x_0}x_i = \mathrm{d}_{x_0}x'_i$ for all $i \in \{1, \ldots, n\}$ by definition,

$$\tau^*(\mathrm{d}_{x_0}x'_i) = \mathrm{d}_{x_0}(\tau^* x'_i) = \mathrm{d}_{x_0}x_i = \mathrm{d}_{x_0}x'_i \,,$$

and thus $j^{m+2}_{x_0}\tau \in \mathrm{NDiff}^{m+2}_{x_0}$.

Let us check that $j^{m+2}_{x_0}\tau \cdot j^m_{x_0}\nabla = j^m_{x_0}\nabla'$. In the coordinates induced by $x_1, \ldots, x_n$ on $J^m_{x_0}\mathrm{Conn}$, the element $j^m_{x_0}\nabla$ is written as $j^m_{x_0}\nabla = (\Gamma^k_{ij}, \Gamma^k_{ij,a_1}, \ldots, \Gamma^k_{ij,a_1\ldots a_m})$, following the notation of Definition 3.1. Similarly, in the coordinates induced by $x'_1, \ldots, x'_n$ on $J^m_{x_0}\mathrm{Conn}$, the element $j^m_{x_0}\nabla'$ is written as $j^m_{x_0}\nabla' = ((\Gamma')^k_{ij}, (\Gamma')^k_{ij,a_1}, \ldots, (\Gamma')^k_{ij,a_1\ldots a_m})$ and the element $j^{m+2}_{x_0}\tau \cdot j^m_{x_0}\nabla = j^m_{x_0}(\tau\cdot\nabla)$ is written as $j^m_{x_0}(\tau\cdot\nabla) = ((\tau\cdot\Gamma)^k_{ij}, (\tau\cdot\Gamma)^k_{ij,a_1}, \ldots, (\tau\cdot\Gamma)^k_{ij,a_1\ldots a_m})$, where $\tau\cdot\Gamma$ are the Christoffel symbols of the connection $\tau\cdot\nabla$.

For all $r \in \{1, \ldots, m\}$, using that $j^{m+2}_{x_0}\tau \in \mathrm{NDiff}^{m+2}_{x_0}$ we obtain the following equalities:

$$\sum_{i,j,k,a_1,\ldots,a_r} (\Gamma')^k_{ij,a_1\ldots a_r} \cdot \left(\frac{\partial}{\partial x'_k}\right)_{x_0} \otimes \mathrm{d}_{x_0}x'_i \otimes \mathrm{d}_{x_0}x'_j \otimes \mathrm{d}_{x_0}x'_{a_1} \otimes \ldots \otimes \mathrm{d}_{x_0}x'_{a_r} = T_r$$

$$= \tau \cdot T_r = \sum_{i,j,k,a_1,\ldots,a_r} (\tau\cdot\Gamma)^k_{ij,a_1\ldots a_r} \cdot \left(\frac{\partial}{\partial x'_k}\right)_{x_0} \otimes \mathrm{d}_{x_0}x'_i \otimes \mathrm{d}_{x_0}x'_j \otimes \mathrm{d}_{x_0}x'_{a_1} \otimes \ldots \otimes \mathrm{d}_{x_0}x'_{a_r} \,,$$

and so $(\tau\cdot\Gamma)^k_{ij,a_1\ldots a_r} = (\Gamma')^k_{ij,a_1\ldots a_r}$ for all $r \in \{1, \ldots, m\}$. Therefore, the coordinates of $j^m_{x_0}\nabla'$ and $j^{m+2}_{x_0}\tau \cdot j^m_{x_0}\nabla$ coincide in the system induced by $x'_1, \ldots, x'_n$ on $J^m_{x_0}\mathrm{Conn}$, and thus are equal.

To finish the proof, let us fix any $j^m_{x_0}\nabla \in J^m_{x_0}\mathrm{Conn}$ and let us check that we can construct a section of ϕ_m passing through $j^m_{x_0}\nabla$. Let $x_1, \ldots, x_n$ be normal coordinates at x_0 for any representative of $j^m_{x_0}\nabla$. As before, they induce coordinates in $J^m_{x_0}\mathrm{Conn}$.

The section s is defined as follows: we assign to any $(T_0, \ldots, T_m) \in N_0 \times \ldots \times N_m$ the element of $J^m_{x_0}\mathrm{Conn}$ that is written, in the previously mentioned coordinates, as $(T_0, \ldots, T_m)$. This map is well defined thanks to the symmetries of the space $N_0 \times \ldots \times N_m$, is clearly smooth and is a section of ϕ_m, as the coordinates $x_1, \ldots, x_n$ are normal for any jet in the image of s, again due to the symmetries of $N_0 \times \ldots \times N_m$.

□

As the following diagrams commute for all m,

$$\begin{tikzcd}
J^{m+1}_{x_0}\mathrm{Conn}/\mathrm{NDiff}^{m+3}_{x_0} \arrow[r, "\phi_{m+1}", "\sim"'] \arrow[d] & N_1 \times \ldots \times N_{m+1} \arrow[d] \\
J^{m}_{x_0}\mathrm{Conn}/\mathrm{NDiff}^{m+2}_{x_0} \arrow[r, "\phi_{m}", "\sim"'] & N_1 \times \ldots \times N_{m},
\end{tikzcd}$$

we obtain the following result:

Corollary 3.3. *The* $\mathrm{Diff}^{\infty}_{x_0}$*−equivariant morphism of ringed spaces*

$$\begin{aligned} \phi_\infty \colon J^\infty_{x_0}Conn &\longrightarrow \prod_{i=0}^{\infty} N_i \\ j^\infty_{x_0}\nabla &\longmapsto (\mathbb{T}_{x_0}, \bar\nabla^1_{x_0}\mathbb{T}, \bar\nabla^2_{x_0}\mathbb{T}, \dots) \end{aligned}$$

induces a Gl*-equivariant isomorphism of ringed spaces:*

$$(J^\infty_{x_0}Conn\,)/\mathrm{NDiff}^\infty_{x_0} = \prod_{i=0}^{\infty} N_i\,.$$

Remark 3.4. Even though the corolary above is everything that we will need during this work, there exists a similar statement of the Reduction Theorem for ∞-jets of linear connections, adding that ϕ_∞ is surjective, its fibres are the orbits of $\mathrm{NDiff}^\infty_{x_0}$ and it admits smooth sections passing through any point of $J^\infty_{x_0}\mathrm{Conn}$. Its proof is routine and can be checked at [20], Appendix E.

Remark 3.5. The same theory can be developed for symmetric linear connections: let N_m^{sym} be the subspace of N_m formed by those tensors $T \in N_m$ such that

$$T^l_{ijk_1\dots k_m} = T^l_{jik_1\dots k_m}\,.$$

Then, it is clear that the m-th normal tensor of a symmetric linear connection belongs in N_m^{sym}. Repeating the process above leads to a modified version of the Reduction Theorem, obtaining a Gl-equivariant isomorphism of ringed spaces:

$$(J^\infty_{x_0}\mathrm{Conn}^{\,\mathrm{sym}})/\mathrm{NDiff}^\infty_{x_0} = \prod_{i=1}^{\infty} N_i^{\mathrm{sym}}\,,$$

where $\mathrm{Conn}^{\,\mathrm{sym}} \to X$ is the fibre bundle of symmetric linear connections on X and $N_0^{\mathrm{sym}} = 0$ due to its symmetries, and so it has been omitted from the product on the right-hand side of the equality above.

3.2 Normal extensions of metrics

Let X be a smooth manifold of dimension n. Let $M_{(s_+,s_-)} \to X$ be the natural bundle of pseudo-riemannian metrics of a fixed signature (s_+, s_-).

Let g be the germ of a pseudo-riemannian metric with fixed signature (s_+, s_-) at $x_0 \in X$ and let ∇ be the germ of its Levi-Civita connection (that is, the only symmetric linear connection compatible with g) at x_0. Let $\bar\nabla$ be the germ of the flat connection at x_0 corresponding, via the exponential map, to the flat connection of $T_{x_0}X$.

Definition 3.6. For any integer $m \geq 0$, the m-th normal tensor of g at x_0 is $\bar\nabla^m_{x_0} g$.

In a system of normal coordinates $(x_1, \ldots, x_n)$ for the Levi-Civita connection ∇ at x_0, the tensor $\bar{\nabla}^m_{x_0} g$ is written as follows:

$$\bar{\nabla}^m_{x_0} g = \sum_{i,j,k,a_1,\ldots,a_m} g_{ij,a_1\ldots a_m} \cdot \mathrm{d}_{x_0} x_i \otimes \mathrm{d}_{x_0} x_j \otimes \mathrm{d}_{x_0} x_{a_1} \otimes \ldots \otimes \mathrm{d}_{x_0} x_{a_m} \ ,$$

where $g_{ij,a_1\ldots a_m} := \frac{\partial^m g_{ij}}{\partial x_{a_1} \ldots \partial x_{a_m}}(x_0)$.

Definition 3.7. Let $m \geq 1$ be an integer. The space N_m of normal tensors of order m at x_0 is the vector subspace of $m+2$-covariant tensors T at x_0 satisfying the following symmetries:

1. they are symmetric in the first two and the last m covariant indices:

$$T_{ijk_1\ldots k_m} = T_{jik_1\ldots k_m} \ , \quad T_{ijk_1\ldots k_m} = T_{ijk_{\sigma(1)}\ldots k_{\sigma(m)}} \ , \quad \forall\, \sigma \in S_m \, ; \tag{3.6}$$

2. the cyclic sum of the last $m+1$ covariant indices is zero:

$$T_{ijk_1\ldots k_m} + T_{ik_m jk_1\ldots k_{m-1}} + \ldots + T_{ik_1\ldots k_m j} = 0. \tag{3.7}$$

For $m = 0$, the space N_0 is defined as the set of pseudo-riemannian metrics of the fixed signature (s_+, s_-) at x_0 (which is not a vector subspace of $S^2 T_{x_0} X$, but rather an open subset of it).

Observe that $N_1 = 0$, due to its symmetries.

For any $m \geq 0$, we get a map

$$\begin{aligned} \phi_m \colon J^m_{x_0} M_{(s_+,s_-)} &\longrightarrow N_0 \times N_2 \times \ldots \times N_m \\ j^m_{x_0} \nabla &\longmapsto (\bar{\nabla}^1_{x_0} \mathbb{T}, \ldots, \bar{\nabla}^m_{x_0} \mathbb{T}) \end{aligned}$$

that is well defined, as normal tensors $\bar{\nabla}^m_{x_0} g$ lie in N_m due to their expression in a system of normal coordinates at x_0. They are compatible: the diagram

$$\begin{tikzcd} J^{m+1}_{x_0} M_{(s_+,s_-)} \arrow[r, "\phi_{m+1}"] \arrow[d] & N_1 \times \ldots \times N_{m+1} \arrow[d] \\ J^m_{x_0} M_{(s_+,s_-)} \arrow[r, "\phi_m"] & N_1 \times \ldots \times N_m, \end{tikzcd}$$

is commutative for any m, and so we obtain a map between the inverse limits:

$$\begin{aligned} \phi_\infty \colon J^\infty_{x_0} M_{(s_+,s_-)} &\longrightarrow \prod_{i=1}^{\infty} N_i \\ j^\infty_{x_0} \nabla &\longmapsto (\bar{\nabla}^1_{x_0} \mathbb{T}, \bar{\nabla}^2_{x_0} \mathbb{T}, \ldots). \end{aligned}$$

We state the corresponding Reduction Theorem for riemannian metrics, which holds a strong resemblance to the Reduction Theorem for linear connections:

Reduction Theorem (for metrics). *The $\mathrm{Diff}^{m+1}_{x_0}-$equivariant morphism of ringed spaces*

$$\begin{aligned} \phi_m \colon J^m_{x_0} M_{(s_+,s_-)} &\longrightarrow N_0 \times N_2 \times \ldots \times N_m \\ j^m_{x_0} g &\longmapsto (g_{x_0}, \bar{\nabla}^2_{x_0} g, , \ldots, \bar{\nabla}^m_{x_0} g), \end{aligned}$$

is surjective, its fibres are the orbits of $\mathrm{NDiff}^{m+1}_{x_0}$ *and it admits smooth sections passing through any point of* $J^m_{x_0} M_{(s_+,s_-)}$.

As a consequence, ϕ_m *induces a* Gl*-equivariant isomorphism of ringed spaces:*

$$(J^m_{x_0} M_{(s_+,s_-)}) / \mathrm{NDiff}^{m+1}_{x_0} = N_0 \times N_2 \times \ldots \times N_m \, .$$

Proof: The proof of this result is similar to the Reduction Theorem for linear connections (see [33]). □

As these isomorphisms are compatible for all m, they define a Gl-equivariant isomorphism

$$(J^\infty_{x_0} M_{(s_+,s_-)}) / \mathrm{NDiff}^\infty_{x_0} = N_0 \times \prod_{i=2}^{\infty} N_i \, .$$

Remark 3.8. A similar (and simpler) argument to that of Proposition 2.21 gives that, by fixing a pseudo-riemannian metric $g_{x_0} \in N_0$ at x_0, the following bijection is produced:

$$\left\{ \begin{array}{c} \text{Gl-equivariant smooth maps} \\ N_0 \times \prod_{i=2}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{O}(s_+, s_-)\text{-equivariant smooth maps} \\ \prod_{i=2}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\} .$$

3.3 Jets of Fedosov structures

Let us briefly review the reductions that have been performed on the set of differential invariants associated to Fedosov structures. As it has been explained, the sheaf of Fedosov structures is not the sheaf of smooth sections of any fibre bundle, and so our main weapon, the Peetre-Slovak Theorem 2.16 cannot be employed.

This problem can be circumvented by fixing a symplectic form, obtaining the sheaf of symplectic connections compatible with the fixed symplectic form, which is the sheaf of smooth sections of a fibre bundle, and so the Peetre-Slovak Theorem is applied, leaving us with the space of ∞-jets of symplectic connections compatible with the fixed form. The details of this process will be given in Chapter 4.

Unfortunately, this resulting space is not easily reduced. If proceeding as usual, one would choose normal coordinates for a given symplectic connection. However, it cannot be assured that these coordinates are also Darboux coordinates for the fixed symplectic form, and so the compatibility condition of the symplectic connections messes the definition of the spaces of normal tensors.

It turns out that the best approach is to unfix the symplectic form, so that a compatibility condition that does not depend on the symplectic form can be given. This approach, however, raises its share of questions, namely: how should one define jets of Fedosov structures, considering they do not form a fibre bundle? There would be two possible definitions[1]

$$J^m_{x_0}\mathcal{F} := \{(j^m_{x_0}\omega, j^m_{x_0}\nabla) : (\omega,\nabla) \in \mathcal{F}_{x_0}\} = \{(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla) : (\omega,\nabla) \in \mathcal{F}_{x_0}\}\ ,$$

$$\mathrm{Fed}^m_{x_0} := \{(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla) \in J^{m+1}_{x_0}\Lambda^2 \times J^m_{x_0}\mathrm{Conn}^{\,\mathrm{sym}} : j^m_{x_0}(\nabla\omega) = 0\}\ ,$$

and their inverse limits

$$J^\infty_{x_0}\mathcal{F} := \lim_{\leftarrow} J^m_{x_0}\mathcal{F}\ ,$$

$$\mathrm{Fed}^\infty_{x_0} := \lim_{\leftarrow}\mathrm{Fed}^m_{x_0}\ ,$$

where $\mathcal{F}_{x_0}$ denotes the germs of Fedosov structures at x_0, $\Lambda^2 \to X$ is the open set of non-singular 2-forms over X and $\mathrm{Conn}^{\,\mathrm{sym}}$ is the bundle of symmetric linear connections over X.

Observe that both of these spaces live in $J^{m+1}_{x_0}\Lambda^2 \times J^m_{x_0}\mathrm{Conn}^{\,\mathrm{sym}}$. In fact, it is easy to see that $J^\infty_{x_0}\mathcal{F} \subseteq \mathrm{Fed}^\infty_{x_0}$: for any $(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla) \in J^m_{x_0}\mathcal{F}$, there exists $(\omega,\nabla) \in \mathcal{F}(U)$ for some open neighbourhood U of x_0 verifying the compatibility condition $\nabla\omega = 0$. Then, taking jets it must hold that $j^m_{x_0}(\nabla\omega) = 0$, and thus $(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla) \in \mathrm{Fed}^m_{x_0}$.

It turns out that, in fact, both definitions coincide due to a formal version of the Poincaré Lemma:

Lemma 3.9. *Let ω be a 2-form on $\mathbb{R}^{2n}$ such that $j^m_0(\mathrm{d}\omega) = 0$, where m is either a positive integer or ∞. Then, there exists a 1-form θ such that $j^m_0(\mathrm{d}\theta) = j^m_0\omega$.*

Proof: During one of the proofs of the Poincaré Lemma (see [39]), it is shown that there exists an operator $\Theta : \Omega^2 \to \Omega^1$ such that $\Theta 0 = 0$ and $\Theta(\mathrm{d}\omega) + \mathrm{d}(\Theta\omega) = \omega$, where Ω^p denotes the set of p-forms on $\mathbb{R}^{2n}$. It is defined as follows: if $\omega =$

[1] Observe that we are committing an abuse of notation with the first definition, as the notation $J^m_{x_0}$ usually indicates jets of fibre bundles. However, such a definition could be extended with ease to the setting of natural sheaves, as what is used to define a jet are the smooth sections of the fibre bundle and not the elements of the fibre bundle itself. Thus, given a natural bundle $E \to X$ and its sheaf of smooth sections $\mathcal{E}$, then

$$J^m_{x_0}\mathcal{E} := \{j^m_{x_0}s : s \in \mathcal{E}_{x_0}\} = J^m_{x_0}E\ .$$

$\sum_{i<j} \omega_{ij} \mathrm{d}x_i \wedge \mathrm{d}x_j$, then $\Theta\omega$ is defined, on any point $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$ as

$$(\Theta\omega)_x = \sum_{i<j} \left[\left(x_i \int_0^1 t\omega_{ij}(tx)\mathrm{d}_x t \right) \mathrm{d}_x x_j - \left(x_j \int_0^1 t\omega_{ij}(tx)\mathrm{d}t \right) \mathrm{d}_x x_i \right] .$$

By computing the partial derivatives of any order of $\Theta\omega$ and valuing at $x = 0$, it is easily checked that if $j_0^m \omega = 0$ then $j_0^m \Theta\omega = 0$. Therefore, if ω is a 2-form verifying $j_0^m(\mathrm{d}\omega) = 0$, then

$$j_0^m \omega = j_0^m(\Theta(\mathrm{d}\omega) + \mathrm{d}(\Theta\omega)) = j_0^m(\Theta(\mathrm{d}\omega)) + j_0^m(\mathrm{d}(\Theta\omega)) = j_0^m(\mathrm{d}(\Theta\omega)) .$$

□

Theorem 3.10. *$J_{x_0}^\infty \mathcal{F} = Fed_{x_0}^\infty$.*

Proof: Let us prove that $\mathrm{Fed}_{x_0}^\infty \subseteq J_{x_0}^\infty \mathcal{F}$ (recall that $J_{x_0}^\infty \mathcal{F} \subseteq \mathrm{Fed}_{x_0}^\infty$, due to the explanation above). Let $(j_{x_0}^\infty \omega, j_{x_0}^\infty \nabla) \in \mathrm{Fed}_{x_0}^\infty$, and let us consider a chart $(U; x_1, \ldots, x_{2n})$ centred at x_0, so that $x_0 = 0$ in the coordinates $x_1, \ldots, x_{2n}$. Let us consider any representative ω of $j_{x_0}^\infty \omega$ which, without loss of generality, can be defined in all points of U, reducing U if needed.

As ω verifies $j_{x_0}^\infty(\nabla\omega) = 0$, it holds that $j_{x_0}^\infty(\mathrm{d}\omega) = 0$, as ∇ is a symmetric linear connection and thus $h(\nabla\omega) = \mathrm{d}\omega$, where h denotes the skew-symmetrization operator. Therefore, the previous lemma can be applied to ω, and so there exists a 1-form θ in U such that $j_{x_0}^\infty(\mathrm{d}\theta) = j_{x_0}^\infty \omega$. The 2-form $\mathrm{d}\theta$ is a closed form that extends $j_{x_0}\omega$ to U.

As $j_{x_0}^\infty \nabla \in J_{x_0}^\infty \mathrm{Conn}^{\mathrm{d}\theta}$, it is now apparent that $j_{x_0}^\infty \nabla$ is an ∞-jet of a fibre bundle (the fibre bundle $\mathrm{Conn}^{\mathrm{d}\theta} \to U$ of symplectic connections compatible with $\mathrm{d}\theta$), and so it can be extended to a symplectic connection in U (reducing U again, if needed), that is, a symmetric linear connection ∇ in U such that $\nabla(\mathrm{d}\theta) = 0$, and so $(\mathrm{d}\theta, \nabla) \in \mathcal{F}_{x_0}$ and $(j_{x_0}^\infty \omega, j_{x_0}^\infty \nabla) \in J_{x_0}^\infty \mathcal{F}$. □

Another point of view on what this result represents is that 'formal Fedosov jets' (that is, elements of $\mathrm{Fed}_{x_0}^\infty$) can be 'realised' by germs of Fedosov structures, that is, for any jet $(j_{x_0}^\infty \omega, j_{x_0}^\infty \nabla) \in \mathrm{Fed}_{x_0}^\infty$ there exists $(\omega', \nabla') \in \mathcal{F}(U)$, where U is an open neighbourhood of x_0, such that $(j_{x_0}^\infty \omega', j_{x_0}^\infty \nabla') = (j_{x_0}^\infty \omega, j_{x_0}^\infty \nabla)$.

This property is verified automatically for jets of fibre bundles due to Borel's Lemma, and is unknown to us whether it holds for other kind of sheaves, such as natural sheaves. This topic will be discussed further on Chapter 6.

3.4 Normal extensions of symplectic connections

Let $x_0 \in X$, let (ω, ∇) be the germ of a Fedosov structure at x_0, and let $\bar{\nabla}$ be the germ of the flat connection at $x_0 \in X$ corresponding, via the exponential map, to the flat connection of $T_{x_0}X$. Let $\mathbb{T} := C_2^1(\omega \otimes \mathbb{T})$, where C_i^j denotes the tensor contraction of the i-th covariant index with the j-th contravariant index.

Definition 3.11. For any integer $m \geq 0$, the m-th normal tensor of ∇ at x_0 is $\bar{\nabla}^m_{x_0}\mathbb{T}$.

In a system of normal coordinates $(x_1, \ldots, x_n)$ around the point x_0 for ∇, the tensor $\bar{\nabla}^m_{x_0}\mathbb{T}$ is written as

$$\bar{\nabla}^m_{x_0}\mathbb{T} = \sum_{i,j,k,a_1,\ldots,a_m} \Gamma_{ijk,a_1\ldots a_m} \cdot \mathrm{d}_{x_0}x_i \otimes \mathrm{d}_{x_0}x_j \otimes \mathrm{d}_{x_0}x_k \otimes \mathrm{d}_{x_0}x_{a_1} \otimes \ldots \otimes \mathrm{d}_{x_0}x_{a_m} \ ,$$

where $\Gamma^k_{ijk,a_1\ldots a_m} := \frac{\partial^m \Gamma_{ijk}}{\partial x_{a_1}\ldots\partial x_{a_m}}(x_0)$ and $\Gamma_{ijk} = \sum_{l=1}^{2n} \omega_{il}\Gamma^l_{jk}$.

Remark 3.12. Notice that the collection of tensors $(\bar{\nabla}^1_{x_0}\mathbb{T}, \ldots, \bar{\nabla}^m_{x_0}\mathbb{T})$ and $(\bar{\nabla}^1_{x_0}(\nabla - \bar{\nabla}), \ldots, \bar{\nabla}^m_{x_0}(\nabla - \bar{\nabla}))$ mutually determine each other, as ω is non-singular. Following the notations above, the tensor $\bar{\nabla}^m_{x_0}(\nabla - \bar{\nabla})$ is written as usual:

$$\bar{\nabla}^m_{x_0}\mathbb{T} = \sum_{i,j,k,a_1,\ldots,a_m} \Gamma^k_{ij,a_1\ldots a_m} \cdot \left(\frac{\partial}{\partial x_k}\right)_{x_0} \otimes \mathrm{d}_{x_0}x_i \otimes \mathrm{d}_{x_0}x_j \otimes \mathrm{d}_{x_0}x_{a_1} \otimes \ldots \otimes \mathrm{d}_{x_0}x_{a_m} \ ,$$

where $\Gamma^k_{ij,a_1\ldots a_m} := \frac{\partial^m \Gamma^k_{ij}}{\partial x_{a_1}\ldots\partial x_{a_m}}(x_0)$.

Definition 3.13. The space N_m of normal tensors of order m at $x_0 \in X$ is the vector subspace of $(m+3)$-tensors whose elements T verify the following symmetries:

1. they are symmetric in the second and third indices, and in the last m:

$$T_{ikja_1\ldots a_m} = T_{ijka_1\ldots a_m}, \quad T_{ijka_{\sigma(1)}\ldots a_{\sigma(m)}} = T_{ijka_1\ldots a_m}, \quad \forall \sigma \in S_m \ ;$$

2. the symmetrization of the last $m+2$ covariant indices is zero:

$$\sum_{\sigma \in S_{m+2}} T_{i\sigma(j)\sigma(k)\sigma(a_1)\ldots\sigma(a_m)=0} \ ;$$

3. the following tensor is symmetric in k and a_1:

$$T_{ikja_1\ldots a_m} - T_{jkia_1\ldots a_m} \ .$$

Due to its symmetries, it is immediate that $N_0 = 0$.

Normal tensors belong in N_m, that is, $\bar{\nabla}^m_{x_0}\mathbb{T} \in N_m$, due to its expression in normal coordinates ([11]). The tensor $\bar{\nabla}^m_{x_0}\mathbb{T}$ depends only on the value of the m-jet $j^m_{x_0}\nabla$, and so we define the following maps, reminiscent of the one defined in Section 3.1:

$$
\begin{aligned}
\phi_m \colon J^m_{x_0}\mathcal{F} &\longrightarrow \Lambda_0 \times \prod_{i=1}^{m} N_i \\
(j^{r+1}_{x_0}\omega, j^r_{x_0}\nabla) &\longmapsto (\omega_{x_0}, \bar{\nabla}^1_{x_0}\mathbb{T}, \bar{\nabla}^2_{x_0}\mathbb{T}, \ldots, \bar{\nabla}^m_{x_0}\mathbb{T}\,),
\end{aligned}
$$

where Λ_0 denotes the open set of non-singular 2-forms at x_0. The maps ϕ_m are Diff_{x_0}-equivariant and also compatible, giving rise to a morphism of ringed spaces:

$$
\begin{aligned}
\phi_\infty \colon J^\infty_{x_0}\mathcal{F} &\longrightarrow \Lambda_0 \times \prod_{i=1}^{\infty} N_i \\
(j^\infty_{x_0}\omega, j^\infty_{x_0}\nabla) &\longmapsto (\omega_{x_0}, \bar{\nabla}^1_{x_0}\mathbb{T}, \bar{\nabla}^2_{x_0}\mathbb{T}, \ldots).
\end{aligned}
$$

Reduction Theorem (for Fedosov structures). *The* $\mathrm{Diff}^{m+2}_{x_0}$*−equivariant morphism of ringed spaces*

$$
\begin{aligned}
\phi_m \colon J^m_{x_0}\mathcal{F} &\longrightarrow \Lambda_0 \times \prod_{i=1}^{m} N_i \\
(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla) &\longmapsto (\omega_{x_0}, \bar{\nabla}^1_{x_0}\mathbb{T}, \bar{\nabla}^2_{x_0}\mathbb{T}, \ldots, \bar{\nabla}^m_{x_0}\mathbb{T}\,).
\end{aligned}
$$

is surjective, its fibres are the orbits of $\mathrm{NDiff}^{m+2}_{x_0}$ *and it admits smooth sections passing through any point of* $J^m_{x_0}\mathcal{F}$*.*

As a consequence, ϕ_m *induces a* Gl*-equivariant isomorphism of ringed spaces:*

$$
(J^m_{x_0}\mathcal{F}\,)/\mathrm{NDiff}^{m+2}_{x_0} = N_1 \times \ldots \times N_m\ .
$$

Proof: Let us first prove that the fibres of ϕ_m are the orbits of $\mathrm{NDiff}^{m+2}_{x_0}$. Any two points in the orbit of $\mathrm{NDiff}^{m+2}_{x_0}$ belong in the same fibre of ϕ_m due to the same arguments that were given in the Reduction Theorem for linear connections 3.1.

Let $(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla)$, $(j^{m+1}_{x_0}\omega', j^m_{x_0}\nabla') \in J^m_{x_0}\mathcal{F}$ be two points in the same fibre of ϕ_m, that is, $\phi_m((j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla)) = \phi_m((j^{m+1}_{x_0}\omega', j^m_{x_0}\nabla')) = (T_1, \ldots, T_r)$. As it was done in the Reduction Theorem for linear connections, we can construct a diffeomorphism τ such that $\tau \cdot x_i = x'_i$ and $\mathrm{d}_{x_0}x_i = \mathrm{d}_{x_0}x'_i$ for all $i \in \{1, \ldots, 2n\}$, where $x_1, \ldots, x_{2n}$ and $x'_1, \ldots, x'_{2n}$ are systems of normal coordinates at x_0 for $j^m_{x_0}\nabla$ and $j^m_{x_0}\nabla'$ respectively. It is deduced that $j^{m+2}_{x_0}\tau \in \mathrm{NDiff}^{m+2}_{x_0}$.

Following the proof of the Reduction Theorem for linear connections, let us write[2] $j^m_{x_0}\nabla = (0, \Gamma^k_{ij,a_1}, \ldots, \Gamma^k_{ij,a_1\ldots a_m})$ and $j^{m+1}_{x_0}\omega = (\omega_{ij}, \omega_{ij,k}, \ldots, \omega_{ij,ka_1\ldots a_m})$ in the coordinates induced by $x_1, \ldots, x_{2n}$ on $J^m_{x_0}\mathcal{F}$. Similarly, in the coordinates induced by

[2] Recall that we are considering symmetric connections only.

$x'_1, \ldots, x'_{2n}$ on $J^m_{x_0}\mathcal{F}$, we write $j^m_{x_0}\nabla' = (0, (\Gamma')^k_{ij,a_1}, \ldots, (\Gamma')^k_{ij,a_1\ldots a_m})$, $j^{m+1}_{x_0}\omega' = (\omega'_{ij}, \omega'_{ij,k}, \ldots, \omega'_{ij,ka_1\ldots a_m})$, $j^m_{x_0}(\tau \cdot \nabla) = (0, (\tau \cdot \Gamma)^k_{ij,a_1}, \ldots, (\tau \cdot \Gamma)^k_{ij,a_1\ldots a_m})$ and $j^{m+1}_{x_0}(\tau \cdot \omega) = ((\tau \cdot \omega)_{ij}, (\tau \cdot \omega)_{ij,k}, \ldots, (\tau \cdot \omega)_{ij,ka_1\ldots a_m})$.

Observe that, due to the same arguments as before, it is proven that $(\tau \cdot \Gamma)_{ijk} = \Gamma'_{ijk}, \ldots, (\tau \cdot \Gamma)_{ijk,a_1\ldots a_m} = \Gamma'_{ijk,a_1\ldots a_m}$. By Remark 3.12, it is now enough to check that $j^m_{x_0}(\tau \cdot \omega) = j^m_{x_0}(\omega')$:

$$\sum_{i<j}(\tau \cdot \omega)_{ij} \mathrm{d}_{x_0}x'_i \wedge \mathrm{d}_{x_0}x'_j = (\tau \cdot \omega)_{x_0} = \omega_{x_0} = \omega'_{x_0} = \sum_{i<j} \omega_{ij} \mathrm{d}_{x_0}x'_i \wedge \mathrm{d}_{x_0}x'_j \,,$$

$$(\tau \cdot \omega)_{ij,k} = (\tau \cdot \Gamma)_{ikj} - (\tau \cdot \Gamma)_{jki} = 0 = (\Gamma')_{ikj} - (\Gamma')_{jki} = \omega'_{ij,k} \,,$$

$$\vdots$$

$$(\tau \cdot \omega)_{ij,ka_1\ldots a_m} = (\tau \cdot \Gamma)_{ikj,a_1\ldots a_m} - (\tau \cdot \Gamma)_{jki,a_1\ldots a_m} = (\Gamma')_{ikj,a_1\ldots a_m} - (\Gamma')_{jki,a_1\ldots a_m} = \omega'_{ij,ka_1\ldots a_m} \,.$$

Lastly, let us prove the statement about the existence of smooth sections

$$s : \Lambda_0 \times N_1 \times \ldots \times N_m \longrightarrow J^m_{x_0}\mathcal{F}.$$

Let us fix a system of coordinates $x_1, \ldots, x_{2n}$ at x_0, and let $(B_{ij}, A_{ijka_1}, \ldots, A_{ijka_1\ldots a_m}) \in \Lambda_0 \times N_1 \times \ldots \times N_m$.

The jet $s((B_{ij}, A_{ijka_1}, \ldots, A_{ijka_1\ldots a_m})) = (j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla)$ is defined, in the coordinates induces by the fixed system in $J^m_{x_0}\mathcal{F}$, as follows:

$$\Gamma_{ijk} = 0, \Gamma_{ijk,a_1} = A_{ijka_1}, \ldots, \Gamma_{ijk,a_1\ldots a_m} = A_{ijka_1\ldots a_m} \,,$$

$$\omega_{ij} = B_{ij}, \omega_{ij,k} = 0, \omega_{ij,ka_1} = A_{ikja_1} - A_{jkia_1}, \ldots, \omega_{ij,ka_1\ldots a_m} = A_{ikja_1\ldots a_m} - A_{jkia_1\ldots a_m} \,,$$

and so the jet $j^m_{x_0}\nabla = (0, \Gamma^k_{ij,a_1}, \ldots, \Gamma^k_{ij,a_1\ldots a_m})$ is defined. The symmetries of the spaces N_i assure that $(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla) \in \mathrm{Fed}^m_{x_0} = J^m_{x_0}\mathcal{F}$ and that $x_1, \ldots, x_{2n}$ is a system of normal coordinates at x_0 for $j^m_{x_0}\nabla$.

□

Corollary 3.14. *The* $\mathrm{Diff}^\infty_{x_0}$*−equivariant morphism of ringed spaces*

$$\begin{aligned} \phi_\infty \colon J^\infty_{x_0}\mathcal{F} &\longrightarrow \Lambda_0 \times \prod_{i=1}^{\infty} N_i \\ (j^\infty_{x_0}\omega, j^\infty_{x_0}\nabla) &\longmapsto (\omega_{x_0}, \bar{\nabla}^1_{x_0}\mathbb{T}, \bar{\nabla}^2_{x_0}\mathbb{T}, \ldots), \end{aligned}$$

induces a Gl*-equivariant isomorphism of ringed spaces:*

$$(J^\infty_{x_0}\mathcal{F})/\mathrm{NDiff}^\infty_{x_0} = \Lambda_0 \times \prod_{i=1}^{\infty} N_i \,.$$

Corollary 3.15. *The choice of a non-singular 2-form η_{x_0} at x_0 produces a bijection:*

$$\left\{ \begin{array}{c} \mathrm{Gl}\text{-}\textit{equivariant smooth maps} \\ \Lambda_0 \times \prod\limits_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\} =\!=\!= \left\{ \begin{array}{c} \mathrm{Sp}(2n,\mathbb{R})\text{-}\textit{equivariant smooth maps} \\ \prod\limits_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\},$$

where $\mathrm{Sp}(2n,\mathbb{R}) := \{\mathrm{d}_{x_0}\tau \colon \tau \in \mathrm{Aut}(\eta)_{x_0}\}$.

Proof: The proof of this result is similar to Proposition 2.21. $\square$

Chapter 4

Differential invariants associated to geometric structures in presence of a linear connection

This chapter is dedicated to the exposition of the Main Theorems and their full proofs, linking together the different results obtained up until now, thus playing a central role in this memoir.

4.1 Linear connections

Main Theorem 4.1 ([19]). *Let X be a smooth manifold and let $\mathcal{C}$ denote the sheaf of linear connections on X.*

Let T be a natural sub-bundle of the bundle of (p,q)-tensors T_p^q and let $\mathcal{T}$ be its sheaf of smooth sections.

If we fix a point $x_0 \in X$, there exists an $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural tensors} \\ \mathcal{C} \longrightarrow \mathcal{T} \end{array} \right\} = \bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Gl}}(S^{d_0} N_0 \otimes \ldots \otimes S^{d_k} N_k, T_{x_0}) \ ,$$

where $d_0, \ldots, d_k$ run over the non-negative integer solutions of the equation

$$d_0 + \ldots + (k+1) d_k = p - q \ .$$

Proof: Theorem 2.16 yields the isomorphism:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{C} \longrightarrow \mathcal{T} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathrm{Conn} \longrightarrow T_{x_0} \end{array} \right\} .$$

Observe that the action of Diff_{x_0} over $J^\infty_{x_0}\mathrm{Conn}$ and T_{x_0} coincides with that of $\mathrm{Diff}^\infty_{x_0}$, so that, in the formula above, we may consider $\mathrm{Diff}^\infty_{x_0}$-equivariant maps instead.

In addition, notice that the following sequence of groups is exact:

$$1 \longrightarrow \mathrm{NDiff}^{\infty}_{x_0} \longrightarrow \mathrm{Diff}^{\infty}_{x_0} \longrightarrow \mathrm{Gl} \longrightarrow 1\,.$$

The subgroup $\mathrm{NDiff}^{\infty}_{x_0}$ acts by the identity over T_{x_0} so that Corollary 2.6, in conjunction with the exact sequence above, assures the existence of an isomorphism:

$$\left\{ \begin{array}{c} \mathrm{Diff}^{\infty}_{x_0}\text{-equivariant smooth maps} \\ J^{\infty}_{x_0}\mathrm{Conn} \longrightarrow T_{x_0} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ J^{\infty}_{x_0}\mathrm{Conn}\,/\mathrm{NDiff}^{\infty}_{x_0} \longrightarrow T_{x_0} \end{array} \right\}.$$

Now, the corolary of the Reduction Theorem 3.1 above allows us to replace this quotient ringed space with an infinite product of vector spaces via the isomorphism:

$$\left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ J^{\infty}_{x_0}\mathrm{Conn}\,/\mathrm{NDiff}^{\infty}_{x_0} \longrightarrow T_{x_0} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ t\colon \prod\limits_{i=0}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\}.$$

Finally, in the last step, we make use of the equivariance by homotheties $h_\lambda\colon T_{x_0}X \to T_{x_0}X$ of ratio $\lambda > 0$. As $h_{\lambda^{-1}} \in \mathrm{Gl}$, the equivariance of these maps t implies

$$t(\ldots,\lambda^{m+1}\Gamma^m_{x_0},\ldots) = t(h_{\lambda^{-1}}(\ldots,\Gamma^m_{x_0},\ldots)) = h_{\lambda^{-1}}\cdot t(\ldots,\Gamma^m_{x_0},\ldots) = \lambda^{r-s}t(\ldots,\Gamma^m_{x_0},\ldots)$$

for all $\lambda > 0$, $(\ldots,\Gamma^m_{x_0},\ldots) \in \prod\limits_{i=0}^{\infty} N_i$.

In view of this property of the smooth maps t, the Homogeneous Function Theorem (to be precise, Formula (A.4)), stated in the previous section, allows us to conclude with the isomorphism:

$$\left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ t\colon \prod\limits_{i=0}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\} = \bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Gl}}(S^{d_0}N_0 \otimes \ldots \otimes S^{d_k}N_k, T_{x_0})\,,$$

where $d_0,\ldots,d_k$ are non-negative integers running over the solutions of the equation

$$d_0 + \ldots + (k+1)d_k = p - q\,.$$

□

4.1.1 Linear connections and orientations

Let (X,Ω) be an oriented manifold of dimension n.

Recall the definitions given in Section 3.1 of the Lie groups $\mathrm{Diff}^m_{x_0}$ and $\mathrm{NDiff}^m_{x_0}$, as well as their inverse limits $\mathrm{Diff}^{\infty}_{x_0}$ and $\mathrm{NDiff}^{\infty}_{x_0}$. Let us define the corresponding Lie groups of this geometric structure: for any $m \geq 1$, let $\mathrm{Aut}(\Omega)^m_{x_0} := \{j^m_{x_0}\tau : \tau \in \mathrm{Aut}(\Omega)_{x_0}\}$, which is related to $\mathrm{NDiff}^m_{x_0}$ by the following short exact sequence of groups (observe that $\mathrm{NDiff}^m_{x_0} \subset \mathrm{Aut}(\Omega)^m_{x_0}$ for all m):

$$1 \longrightarrow \mathrm{NDiff}^m_{x_0} \hookrightarrow \mathrm{Aut}(\Omega)^m_{x_0} \longrightarrow \mathrm{Sl} \longrightarrow 1\,,$$
$$j^m_{x_0}\tau \longmapsto \mathrm{d}_{x_0}\tau$$

where $\mathrm{Sl} := \mathrm{Aut}(\Omega)^1_{x_0} = \{\mathrm{d}_{x_0}\tau \colon \tau \in \mathrm{Aut}(\Omega)_{x_0}\}$.

As before, we consider the inverse limit $\mathrm{Aut}(\Omega)^\infty_{x_0} := \varprojlim \mathrm{Aut}(\Omega)^m_{x_0}$ and the corresponding short exact sequence of groups:

$$1 \longrightarrow \mathrm{NDiff}^\infty_{x_0} \longrightarrow \mathrm{Aut}(\Omega)^\infty_{x_0} \longrightarrow \mathrm{Sl} \longrightarrow 1\,. \tag{4.1}$$

Main Theorem 4.2 ([20]). *Let X be a smooth manifold and let $\mathcal{C}$ and Or_X denote the sheaves of linear connections and orientations on X, respectively.*

Let T be a natural sub-bundle of the bundle of (p,q)-tensors T^q_p and let $\mathcal{T}$ be its sheaf of smooth sections.

If we fix a point $x_0 \in X$ and an orientation Ω at an open neighbourhood of x_0, there exists an $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{C} \times Or_X \longrightarrow \mathcal{T} \end{array} \right\} = \bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Sl}}(S^{d_0}N_0 \otimes \ldots \otimes S^{d_k}N_k, T_{x_0})\,,$$

where $d_0, \ldots, d_k$ run over the non-negative integer solutions of the equation

$$d_0 + \ldots + (k+1)d_k = p - q\,.$$

Proof: The proof of this result is similar to that of Theorem 4.1. In this case, we firstly use Theorem 2.22 to obtain the isomorphism:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{C} \times Or_X \longrightarrow \mathcal{T} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\Omega)_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathrm{Conn} \longrightarrow T_{x_0} \end{array} \right\}.$$

As the action of $\mathrm{Aut}(\Omega)_{x_0}$ over $J^\infty_{x_0}\mathrm{Conn}$ and T_{x_0} coincides with that of $\mathrm{Aut}(\Omega)^\infty_{x_0}$, we may replace these maps by $\mathrm{Aut}(\Omega)^\infty_{x_0}$-equivariant maps instead.

The exact sequence of groups 4.1, described above, allows the application of Corollary 2.6 and the Reduction Theorem 3.1 (the isomorphism obtained there is Gl-equivariant, hence Sl-equivariant), achieving the next two isomorphisms respectively:

$$\left\{ \begin{array}{c} \mathrm{Aut}(\Omega)^\infty_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathrm{Conn} \longrightarrow T_{x_0} \end{array} \right\} = \left\{ \begin{array}{c} \text{Sl-equivariant smooth maps} \\ J^\infty_{x_0}\mathrm{Conn}\,/\mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \end{array} \right\}.$$

$$\left\{ \begin{array}{c} \text{Sl-equivariant smooth maps} \\ J^\infty_{x_0}\text{Conn} \,/ \text{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \end{array} \right\} =\!=\!= \left\{ \begin{array}{c} \text{Sl-equivariant smooth maps} \\ t \colon \prod\limits_{i=0}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\} .$$

As homotheties $h_\lambda \colon T_{x_0}X \to T_{x_0}X$ of ratio $\lambda > 0$ also belong in Sl, the Homogeneous Function Theorem finishes the proof with the isomorphism:

$$\left\{ \begin{array}{c} \text{Sl-equivariant smooth maps} \\ t \colon \prod\limits_{i=0}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\} =\!=\!= \bigoplus_{d_i} \text{Hom}_{\text{Sl}}(S^{d_0}N_0 \otimes \ldots \otimes S^{d_k}N_k, T_{x_0}) ,$$

where $d_0, \ldots, d_k$ are non-negative integers running over the solutions of the equation

$$d_0 + \ldots + (k+1)d_k = r - s .$$

□

4.2 Riemannian metrics

Let X be a smooth manifold of dimension n. Let $M_{(s_+,s_-)} \to X$ be the natural bundle of pseudo-riemannian metrics of a fixed signature (s_+, s_-), and let $\mathcal{M}_{(s_+,s_-)}$ be its sheaf of smooth sections.

Definition 4.3. Let $\delta \in \mathbb{R}$. We say that a natural tensor $T : \mathcal{M}_{(s_+,s_-)} \to \mathcal{T}$ is homogeneous of weight δ if, for all non-zero $\lambda \in \mathbb{R}$, it holds that:

$$T(\lambda^2 g) = \lambda^\delta T(g) .$$

For example, the metric tensor $T(g) = g$, is a homogeneous natural tensor of weight 2, the $(3,1)$ curvature tensor R is a homogeneous natural tensor of weight 0 and the scalar curvature r is homogeneous of weight -2.

Notice that, if $T \neq 0$ and $\delta \in \mathbb{Z}$, the weight must be an even number: if T is an homogeneous natural tensor of odd weight δ, then the homogeneity condition for $\lambda = -1$ says:

$$T(g) = T((-1)^2 g) = (-1)^\delta T(g) = -T(g) ,$$

obtaining that $T = 0$.

Main Theorem 4.4 ([33]). *Let X be a smooth manifold of dimension n, and let $\mathcal{M}_{(s_+,s_-)}$ denote the sheaves of pseudo-riemannian metrics of fixed signature (s_+, s_-).*

Let $T \to X$ be a natural subbundle of the bundle of p-covariant tensors T_p and let $\mathcal{T}$ be its sheaf of smooth sections. Let $\delta \in \mathbb{Z}$.

If we fix a point $x_0 \in X$ and a pseudo-riemannian metric g_{x_0} of signature (s_+, s_-) at x_0, there exists a $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{M}_{(s_+,s_-)} \longrightarrow \mathcal{T} \\ \textit{homogeneous of weight } \delta \end{array} \right\} = \bigoplus_{d_2,\dots,d_r} \mathrm{Hom}_{\mathrm{O}(s_+,s_-)}(S^{d_2}N_2 \otimes \ldots \otimes S^{d_r}N_r, T_{x_0}) \,,$$

where $d_2, \dots, d_r$ run over the non-negative integer solutions of the equation

$$2d_2 + \ldots + rd_r = p - \delta \,,$$

and where $\mathrm{O}(s_+, s_-) := \{\mathrm{d}_{x_0}\tau \colon \tau \in \mathrm{Diff}_{x_0} : \tau_{*,x_0} g_{x_0} = g_{x_0}\}$.

Proof: Although the proof of this result can also be read at ([33]), let us reproduce the proof here, following the sketch of the proof of Theorem 4.1.

As pseudo-riemannian metrics of a fixed signature form a natural bundle, the Peetre-Slovak Theorem 2.16 can be applied, obtaining a bijection

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{M}_{(s_+,s_-)} \longrightarrow \mathcal{T} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0} M_{(s_+,s_-)} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\}$$

by fixing a point $x_0 \in X$.

As the action of Diff_{x_0} and $\mathrm{Diff}^\infty_{x_0}$ coincide over $J^\infty_{x_0} M_{(s_+,s_-)}$ and T_{x_0}, we may consider $\mathrm{Diff}^\infty_{x_0}$-equivariant maps instead.

The short exact sequence 3.3 and Corolary 2.6 produces a bijection

$$\left\{ \begin{array}{c} \mathrm{Diff}^\infty_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0} M_{(s_+,s_-)} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ J^\infty_{x_0} M_{(s_+,s_-)} / \mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\}$$

Now, the Reduction Theorem for metrics 3.2 and the remark after it gives:

$$\left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ J^\infty_{x_0} M_{(s_+,s_-)} / \mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{O}(s_+,s_-)\text{-equivariant smooth maps} \\ \prod_{i=2}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} .$$

Lastly, the homogeneity hypothesis lets us invoke the Homogeneous Function Theorem (Appendix A), obtaining the last bijection:

$$\left\{ \begin{array}{c} \mathrm{O}(s_+, s_-)\text{-equivariant smooth maps} \\ \prod_{i=2}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\}$$
$$\Big\|$$
$$\bigoplus_{d_2,\ldots,d_r} \mathrm{Hom}_{\mathrm{O}(s_+,s_-)}(S^{d_2} N_2 \otimes \ldots \otimes S^{d_r} N_r, T_{x_0}) \ ,$$

where $d_2, \ldots, d_r$ run over the non-negative integer solutions of the equation

$$2d_2 + \ldots + rd_r = p - \delta \ .$$

□

Remark 4.5. Without the homogeneity condition, natural tensors associated to metrics may fail to be polynomial. The following example was given by D.B.A. Epstein ([5]): consider the natural morphism of sheaves that assigns, to a metric g of Riemannian-Christoffel curvature tensor R, the function

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \frac{1}{1 + \langle \nabla^n R, \nabla^n R \rangle} \ ,$$

where ∇ denotes the Levi-Civita connection of g and $\langle \cdot , \cdot \rangle$ denotes the inner product induced by g over the corresponding space of tensors. It is a natural function associated to metrics, but its expression is not polynomial on the covariant derivatives of R, hence neither on the normal tensors associated to g.

4.2.1 Riemannian metrics and orientations

Let $M \to X$ be the bundle of riemannian metrics over X, and let $\mathcal{M}$ be its sheaf of smooth sections.

Definition 4.6. Let $\delta \in \mathbb{R}$. We say that a natural tensor $T : \mathcal{M} \times Or \to \mathcal{T}$ is homogeneous of weight δ if, for all non-zero $\lambda \in \mathbb{R}$, it holds that:

$$T(\lambda^2 g, \Omega) = \lambda^\delta T(g, \Omega) \ .$$

Main Theorem 4.7. *Let X be a smooth manifold of dimension n, let $\mathcal{M}$ denote the sheaves of pseudo-riemannian metrics and let Or be the sheaf of orientations over X.*

Let $T \to X$ be a natural subbundle of the bundle of p-covariant tensors T_p and let $\mathcal{T}$ be its sheaf of smooth sections. Let $\delta \in \mathbb{Z}$.

If we fix a point $x_0 \in X$, a pseudo-riemannian metric g_{x_0} of signature (s_+, s_-) at x_0 and an orientation Ω at an open neighbourhood of x_0, there exists a $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{M} \times \mathcal{O}r \longrightarrow \mathcal{T} \\ \textit{homogeneous of weight } \delta \end{array} \right\} = \bigoplus_{d_2,\ldots,d_r} \mathrm{Hom}_{\mathrm{SO}(s_+,s_-)}(S^{d_2}N_2 \otimes \ldots \otimes S^{d_r}N_r, T_{x_0}) \,,$$

where $d_2, \ldots, d_r$ *run over the non-negative integer solutions of the equation*

$$2d_2 + \ldots + rd_r = p - \delta \,,$$

and where $\mathrm{SO}(s_+,s_-) := \{\mathrm{d}_{x_0}\tau \colon \tau \in \mathrm{Aut}(\Omega)_{x_0} : \tau_{*,x_0} g_{x_0} = g_{x_0}\}$.

Proof: By Proposition 2.22, there exists a bijection

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{M} \times \mathcal{O}r \longrightarrow \mathcal{T} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\Omega)_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0} M \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} ,$$

The action of $\mathrm{Aut}(\Omega)_{x_0}$ on $J^\infty_{x_0}M$ and T_{x_0} coincides with the action of $\mathrm{Aut}(\Omega)^\infty_{x_0}$, and so we will consider $\mathrm{Aut}(\Omega)^\infty_{x_0}$-equivariant morphisms instead.

The short exact sequence 4.1 and Corolary 2.6 produces a bijection

$$\left\{ \begin{array}{c} \mathrm{Aut}(\Omega)^\infty_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0} M \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Sl}\text{-equivariant smooth maps} \\ J^\infty_{x_0} M/\mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\}$$

Per the Reduction Theorem for metrics 3.2, we obtain

$$\left\{ \begin{array}{c} \mathrm{Sl}\text{-equivariant smooth maps} \\ J^\infty_{x_0} M/\mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Sl}\text{-equivariant smooth maps} \\ N_0 \times \prod\limits_{i=2}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} .$$

As the group Sl acts transitively on N_0, there exists a bijection

$$\left\{ \begin{array}{c} \mathrm{Sl}\text{-equivariant smooth maps} \\ N_0 \times \prod\limits_{i=2}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{SO}(s_+,s_-)\text{-equivariant smooth maps} \\ \prod\limits_{i=2}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} .$$

Finally, the Homogeneous Function Theorem (Appendix A) gives the desired result.

□

4.3 Fedosov structures

Unlike the case of metrics, let us present the Main Theorem for Fedosov structures without and with homogeneity. The Main Theorem without homogeneity is of special interest in this case, as it can be compared in a more direct way to results by Gelfand-Retakh-Shubin [11]:

Main Theorem 4.8. *Let X be a smooth manifold of dimension $2n$, and let $\mathcal{F}$ denote the sheaves of Fedosov structures. Let $\mathcal{T}$ be the sheaf of smooth sections of a natural subbundle $T \to X$ of the bundle p-covariant tensors on X.*

Fixing a point $x_0 \in X$ and a non-singular 2-form η_{x_0} at x_0 produces a $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Sp}(2n,\mathbb{R})\textit{-equivariant smooth maps} \\ \prod_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\}.$$

where $\mathrm{Sp}(2n,\mathbb{R}) := \{\mathrm{d}_{x_0}\tau \colon \tau \in \mathrm{Aut}(\eta)_{x_0}\}$.

Proof: Let us fix a point $x_0 \in X$. Choose a chart $U \simeq \mathbb{R}^{2n}$ around x_0, so that Proposition 2.19 produces a bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \end{array} \right\} = \left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F}_{\mathbb{R}^{2n}} \longrightarrow \mathcal{T}_{\mathbb{R}^{2n}} \end{array} \right\},$$

where $\mathcal{F}_{\mathbb{R}^{2n}}$ and $\mathcal{T}_{\mathbb{R}^{2n}}$ denote the sheaves $\mathcal{F}$ and $\mathcal{T}$ restricted to U and passed through the diffeomorphism $U \simeq \mathbb{R}^{2n}$.

Fixing the canonical symplectic form η on $\mathbb{R}^{2n}$ lets us invoke Proposition 2.24, which gives the bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F}_{\mathbb{R}^{2n}} \longrightarrow \mathcal{T}_{\mathbb{R}^{2n}} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\eta)_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathrm{Conn}_\eta \longrightarrow T_{x_0} \end{array} \right\}.$$

Let us now unfix the symplectic form (recall from the example of Definition 2.20 that diffeomorphisms act transitively on symplectic forms due to the existence of Darboux coordinates):

$$\left\{ \begin{array}{c} \mathrm{Aut}(\eta)_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathrm{Conn}_\eta \longrightarrow T_{x_0} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F} \longrightarrow T_{x_0} \end{array} \right\}.$$

As the action of both Diff_{x_0} and $\mathrm{Diff}^\infty_{x_0}$ coincide over $J^\infty_{x_0}\mathcal{F}$ and T_{x_0}, we may consider $\mathrm{Diff}^\infty_{x_0}$-equivariant maps instead in the set above.

For the next step, recall that the following sequence of groups is exact:

$$1 \longrightarrow \mathrm{NDiff}^\infty_{x_0} \longrightarrow \mathrm{Diff}^\infty_{x_0} \longrightarrow \mathrm{Gl} \longrightarrow 1$$

As the subgroup $\mathrm{NDiff}^\infty_{x_0}$ acts by the identity over T_{x_0}, Corollary 2.6 in conjunction with the exact sequence above assures the existence of an isomorphism:

$$\left\{ \begin{array}{c} \mathrm{Diff}^\infty_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F} \longrightarrow T_{x_0} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F}/\mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \end{array} \right\} .$$

Now, the corolary of the Reduction Theorem for Fedosov structures 3.4 allows us to replace this quotient ringed space via the bijection:

$$\left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F}/\mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ \Lambda_0 \times \prod\limits_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\} .$$

The choice of a non-singular 2-form η_{x_0} on x_0 allows us to remove the space Λ_0, finishing the proof:

$$\left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ \Lambda_0 \times \prod\limits_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Sp}(2n,\mathbb{R})\text{-equivariant smooth maps} \\ \prod\limits_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \end{array} \right\} .$$

□

Definition 4.9. Let $\delta \in \mathbb{R}$. We say that a natural tensor $T : \mathcal{F} \to \mathcal{T}$ is homogeneous of weight δ if, for all non-zero $\lambda \in \mathbb{R}$, it holds that[1]:

$$T(\lambda^2\omega, \nabla) = \lambda^\delta T(\omega, \nabla) .$$

For example, the symplectic form seen as a natural tensor, $T(\omega, \nabla) = \omega$, is a homogeneous natural tensor of weight 2. The $(3,1)$ curvature tensor R is also a homogeneous natural tensor, of weight 0.

As it happened with homogeneity in the case of pseudo-riemannian metrics, notice that, if $T \neq 0$ and $\delta \in \mathbb{Z}$, the weight must be an even number: if T is an homogeneous natural tensor of odd weight δ, then the homogeneity condition for $\lambda = -1$ says:

$$T(\omega, \nabla) = T((-1)^2\omega, \nabla) = (-1)^\delta T(\omega, \nabla) = -T(\omega, \nabla) ,$$

obtaining that $T = 0$.

Main Theorem 4.10. *Let X be a smooth manifold of dimension $2n$, and let $\mathcal{F}$ denote the sheaves of Fedosov structures. Let $\mathcal{T}$ be the sheaf of smooth sections of a natural subbundle $T \to X$ of the bundle p-covariant tensors on X. Let $\delta \in \mathbb{Z}$.*

[1]Observe that if (ω, ∇) is a Fedosov structure, then $(\lambda\omega, \nabla)$ is also a Fedosov structure for any $\lambda \in \mathbb{R} \setminus \{0\}$.

Fixing a point $x_0 \in X$ and a non-singular 2-form η_{x_0} at x_0 produces a $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \\ \textit{homogeneous of weight } \delta \end{array} \right\} = \bigoplus_{d_1,\ldots,d_r} \mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}(S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r, T_{x_0}) \,,$$

where $d_1, \ldots, d_r$ run over the non-negative integer solutions of the equation

$$2d_1 + \ldots + (r+1)d_r = p - \delta \,,$$

and $\mathrm{Sp}(2n, \mathbb{R}) := \{\mathrm{d}_{x_0}\tau \colon \tau \in \mathrm{Aut}(\eta)_{x_0}\}$.

Proof: The proof of this result begins in the same way as the proof of Theorem 4.8. Let us fix a point $x_0 \in X$. Choose a chart $U \simeq \mathbb{R}^{2n}$ around x_0, so that Proposition 2.19 produces a bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F}_{\mathbb{R}^{2n}} \longrightarrow \mathcal{T}_{\mathbb{R}^{2n}} \\ \text{homogeneous of weight } \delta \end{array} \right\},$$

where $\mathcal{F}_{\mathbb{R}^{2n}}$ and $\mathcal{T}_{\mathbb{R}^{2n}}$ denote the sheaves $\mathcal{F}$ and $\mathcal{T}$ restricted to U and passed through the diffeomorphism $U \simeq \mathbb{R}^{2n}$.

Fixing the canonical symplectic form η on $\mathbb{R}^{2n}$ lets us invoke Proposition 2.24, which gives the bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F}_{\mathbb{R}^{2n}} \longrightarrow \mathcal{T}_{\mathbb{R}^{2n}} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\eta)_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathrm{Conn}_\eta \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\},$$

where an $\mathrm{Aut}(\eta)_{x_0}$-equivariant smooth map $T : J^\infty_{x_0}\mathrm{Conn}_\omega \to T_{x_0}$ being homogeneous of weight δ means that it verifies the following property:

$$T(h_\lambda \cdot (j^\infty_{x_0}\nabla)) = \lambda^{p-\delta}T(j^\infty_{x_0}\nabla) \,,$$

for any homothety[2] h_λ of ratio $\lambda \neq 0$.

Let us now unfix the symplectic form (recall from the example of Definition 2.20 that diffeomorphisms act transitively on symplectic forms due to the existence of Darboux coordinates):

$$\left\{ \begin{array}{c} \mathrm{Aut}(\eta)_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathrm{Conn}_\eta \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\}.$$

[2] We say that $\tau \in \mathrm{Diff}_{x_0}$ is a homothety of ratio $\lambda \neq 0$ if $\mathrm{d}_{x_0}\tau = \lambda \cdot \mathrm{Id}$.

As the action of both Diff_{x_0} and $\mathrm{Diff}^\infty_{x_0}$ coincide over $J^\infty_{x_0}\mathcal{F}$ and T_{x_0}, we may consider $\mathrm{Diff}^\infty_{x_0}$-equivariant maps instead in the set above.

For the next step, recall that the following sequence of groups is exact:

$$1 \longrightarrow \mathrm{NDiff}^\infty_{x_0} \longrightarrow \mathrm{Diff}^\infty_{x_0} \longrightarrow \mathrm{Gl} \longrightarrow 1$$

As the subgroup $\mathrm{NDiff}^\infty_{x_0}$ acts by the identity over T_{x_0}, Corollary 2.6 in conjunction with the exact sequence above assures the existence of an isomorphism:

$$\left\{ \begin{array}{c} \mathrm{Diff}^\infty_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F}/\mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} .$$

Now, the corolary of the Reduction Theorem for Fedosov structures 3.4 allows us to replace this quotient ringed space via the bijection:

$$\left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F}/\mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ \Lambda_0 \times \prod_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} .$$

The choice of a non-singular 2-form η_{x_0} on x_0 allows us to remove the space Λ_0, due to the bijection:

$$\left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ \Lambda_0 \times \prod_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Sp}(2n,\mathbb{R})\text{-equivariant smooth maps} \\ \prod_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} ,$$

where, following the previous bijections, a $\mathrm{Sp}(2n,\mathbb{R})$-equivariant smooth map $T : \prod_{i=1}^{\infty} N_i \longrightarrow T_{x_0}$ is said to be homogeneous of weight δ if, for any $\lambda \neq 0$, it holds that $T(\lambda^2 T_1, \lambda^3 T_2, \dots) = \lambda^{p-\delta} T(T_1, T_2, \dots)$.

Therefore, the homogeneity allows us to make the final reduction by applying the Homogeneous Function Theorem (Appendix A), producing the isomorphism:

$$\left\{ \begin{array}{c} \mathrm{Sp}(2n,\mathbb{R})\text{-equivariant smooth maps} \\ \prod_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \bigoplus_{d_1,\dots,d_r} \mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}(S^{d_1}N_1 \otimes \dots \otimes S^{d_r}N_r, T_{x_0}) ,$$

where $d_1, \ldots, d_r$ are non-negative integers running over the solutions of the equation

$$2d_1 + \ldots + (r+1)d_r = p - \delta\,.$$

$\square$

An immediate corolary of the Main Theorem 4.10 is that, if the left side of Equation 4.10 is negative, there are no natural tensors:

Corollary 4.11. *There are no homogeneous natural p-tensors associated to Fedosov structures of weight $\delta > p$.*

Remark 4.12. By considering the polarity isomorphism of the symplectic form, or through direct computation as it has been done above, it is easy to check that with the notations and hypotheses of the Main Theorem there exists a bijection

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T}_p^q \\ \text{homogeneous of weight } \delta \end{array} \right\} = \bigoplus_{d_1,\ldots,d_r} \mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}(S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r, T^q_{p,x_0})\,,$$

where $\mathcal{T}_p^q$ denotes the sheaf of smooth sections of a natural bundle of (p,q)-tensors $T_p^q \to X$ and $d_1, \ldots, d_r$ run over the non-negative integer solutions of the equation

$$2d_1 + \ldots + (r+1)d_r = p - q - \delta\,.$$

Remark 4.13. Observe that the process described in this work to compute natural tensors associated to Fedosov structures also gives the description of differential invariants associated to symplectic forms: if we denote by $\mathcal{S}$ the sheaf of symplectic forms over a smooth manifold X, then choosing a point $x_0 \in X$ produces a bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{S} \longrightarrow \mathcal{T} \end{array} \right\} = T_{x_0}\,.$$

In other words, there are essentially no differential invariants associated to symplectic forms, as expected.

Chapter 5

Applications

We dedicate this Chapter to the application of the Main Theorems proved in Chapter 4, obtaining various characterizations of the torsion and curvature operators, computing scalar differential invariants associated to Fedosov structures and describing dimensional identities of the curvature operator in Fedosov geometry.

5.1 Invariants of linear connections

Let X be a smooth manifold of dimension $n \geq 3$. Whenever it is necessary, x_0 will also be considered as a fixed point in X.

Definition 5.1. Let Λ^p be the sheaf of differential p-forms over X. Let $E \to X$ be a bundle of tensors and let $\mathcal{E}$ be its sheaf of smooth sections. An E-valued natural p-form (associated to linear connections) is a natural morphism of sheaves

$$\mathcal{C} \longrightarrow \Lambda^p \otimes \mathcal{E} \, .$$

The Main Theorem 4.1 assures that the space of E-valued natural forms associated to linear connections is a finite-dimensional real vector space.

Examples. Let $\mathcal{D}$ be the sheaf of smooth sections of the tangent bundle $TX \to X$. The torsion tensor of a linear connection can be understood as a vector-valued natural 2-form; that is to say, as a natural morphism of sheaves

$$\mathrm{Tor}\colon \mathcal{C} \longrightarrow \Lambda^2 \otimes \mathcal{D} \, ,$$

whose value on a linear connection ∇ defined on an open set $U \subset X$ is the following vector-valued 2-form on U:

$$\mathrm{Tor}_\nabla(D_1, D_2) := \nabla_{D_1} D_2 - \nabla_{D_2} D_1 - [D_1, D_2] \, .$$

In a similar manner, let $\mathcal{E}nd(\mathcal{D})$ be the sheaf of smooth sections of the fibre bundle $\mathrm{End}(TX) \to X$. The curvature tensor of a linear connection can be thought of as an endomorphism-valued natural 2-form; that is to say, as a natural morphism of sheaves

$$R\colon \mathcal{C} \longrightarrow \Lambda^2 \otimes \mathcal{E}\mathrm{nd}(\mathcal{D}) \,,$$

whose value on a linear connection ∇ defined on an open set $U \subset X$ is the following endomorphism-valued 2-form R_∇ on U:

$$R_\nabla(D_1, D_2)D_3 := \nabla_{D_1}\nabla_{D_2}D_3 - \nabla_{D_2}\nabla_{D_1}D_3 - \nabla_{[D_1,D_2]}D_3 \,.$$

Given a E-valued natural p-form ω, we may consider its exterior differential $\mathrm{d}\omega\colon \mathcal{C} \to \Lambda^{p+1} \otimes \mathcal{E}$ as a natural $(p+1)$-form defined, on each section ∇ of $\mathcal{C}$, with respect to the linear connection on E induced by ∇.

Moreover, as the exterior differential commutes with diffeomorphisms, it induces $\mathbb{R}$-linear maps

$$\begin{bmatrix} E\text{-valued natural} \\ p\text{-forms} \end{bmatrix} \xrightarrow{\ \mathrm{d}\ } \begin{bmatrix} E\text{-valued natural} \\ (p+1)\text{-forms} \end{bmatrix} .$$

Definition 5.2. A closed E-valued natural p-form (associated to linear connections) is an element in the kernel of the map above.

5.1.1 Closed vector-valued natural forms

In view of the decomposition of Lemma B.2, the torsion tensor also produces this vector-valued 2-form, naturally associated to linear connections:

$$H := c_1^1(\mathrm{Tor}) \wedge I \,.$$

Lemma 5.3. *Tor and H are a basis of the $\mathbb{R}$-vector space of vector-valued natural 2-forms.*

Proof: In virtue of the Main Theorem 4.1, the vector space under consideration comes from integer solutions $\{d_0, \ldots, d_k\}$ to the equation

$$d_0 + 2d_1 + \ldots + (k+1)d_k \ = \ 2 - 1 = 1 \,.$$

As there is only one solution, $d_0 = 1$, $d_1 = \ldots = d_k = 0$, we are led to describe all possible Gl-equivariant linear endomorphisms

$$N_0 = \Lambda^2 T^*_{x_0}X \otimes T_{x_0}X \ \longrightarrow \ \Lambda^2 T^*_{x_0}X \otimes T_{x_0}X \,.$$

A simple computation using the the First Fundamental Theorem of the general linear group Gl B.1 allows to prove that this vector space has two generators.

Then, the task is reduced to check that Tor and H are $\mathbb{R}$-linearly independent natural tensors.

To this end, it is enough to find, for any $n \geq 3$, a linear connection ∇ on a smooth manifold of dimension n for which the tensors Tor_∇ and H_∇ are not $\mathbb{R}$-proportional.

For example, let ∇ be the linear connection on $\mathbb{R}^n$ whose only non-zero Christoffel symbols in cartesian coordinates are $\Gamma^1_{12} = \frac{1}{2}x_1 + x_3$ and $\Gamma^1_{21} = -\frac{1}{2}x_1 - x_2$. Direct computation shows that

$$\mathrm{Tor}_\nabla \ = \ (x_1 + x_2 + x_3)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \partial_{x_1} \ ,$$

whereas

$$H_\nabla = -\,(x_1 + x_2 + x_3)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \partial_{x_1} + \sum_{i \geq 3}^{n} (x_1 + x_2 + x_3)\,\mathrm{d}x_2 \wedge \mathrm{d}x_i \otimes \partial_{x_i} \ ,$$

and so the proof is finished. □

Proposition 5.4. *There are no non-zero closed vector-valued natural 2-forms.*

In other words, the exterior differential is an injective $\mathbb{R}$*-linear map:*

$$\left[\begin{array}{c}\textit{Vector-valued natural} \\ \textit{2-forms}\end{array}\right] \xrightarrow{\ \mathrm{d}\ } \left[\begin{array}{c}\textit{Vector-valued natural} \\ \textit{3-forms}\end{array}\right] .$$

Proof: In view of Lemma 5.3, it is enough to prove that $\mathrm{d}H$ and $\mathrm{d}\mathrm{Tor}$ are $\mathbb{R}$-linearly independent vector-valued natural 3-forms.

If we choose the same connection ∇ on $\mathbb{R}^n$ ($n \geq 3$) considered in the proof of Lemma 5.3, then we obtain:

$$\mathrm{d}_\nabla H \ = \ \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes (-\partial_{x_1} + \partial_{x_3}) + \sum_{i>3}^{n} (\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_i \otimes \partial_{x_i} - \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_i \otimes \partial_{x_i}) \ ,$$

and

$$\mathrm{d}_\nabla \mathrm{Tor} = R_\nabla \wedge I \ = \ \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \partial_{x_1} \, .$$

Since $\mathrm{d}_\nabla \mathrm{Tor}$ and $\mathrm{d}_\nabla H$ are not $\mathbb{R}-$proportional, this example suffices to end the proof. □

Definition 5.5. A vector-valued 2-form α naturally associated to linear connections is said to satisfy the first Bianchi identity if the following equality of vector-valued natural 3-forms holds:

$$\mathrm{d}\alpha \ = \ R \wedge I \, .$$

Examples. The torsion tensor Tor of a linear connection ∇ verifies the first Bianchi identity:

$$\mathrm{dTor} \;=\; R \wedge I \,.$$

By Proposition 5.4 there exists, at most, one vector-valued 2-form that verifies the first Bianchi identity. Therefore:

Theorem 5.6. *The only vector-valued 2-form naturally associated to linear connections satisfying the first Bianchi identity is the torsion tensor.*

5.1.2 Closed endomorphism-valued natural forms

For the remainder of this section, let us consider symmetric linear connections.

Definition 5.7. An endomorphism-valued 2-form α naturally associated to symmetric linear connections is said to satisfy the first Bianchi identity if, for any symmetric linear connection ∇ and any vector fields D_1, D_2, D_3:

$$\alpha_\nabla(D_1, D_2)D_3 + \alpha_\nabla(D_2, D_3)D_1 + \alpha_\nabla(D_3, D_1)D_2 \;=\; 0 \,.$$

Examples. The curvature tensor R_∇ of a symmetric linear connection ∇ satisfies this identity.

Also, if Ric^s and Ric^h stand for the symmetric and skew-symmetric part of the Ricci tensor, respectively, then the following $(3,1)$-tensors also satisfy the first Bianchi identity:

$$C_1(D_1, D_2, D_3, \omega) := \;\mathrm{Ric}^s(D_1, D_3)\omega(D_2) - \mathrm{Ric}^s(D_2, D_3)\omega(D_1) \,,$$

and

$$C_2(D_1, D_2, D_3, \omega) := \;\mathrm{Ric}^h(D_1, D_3)\omega(D_2) - \mathrm{Ric}^h(D_2, D_3)\omega(D_1) + 2\mathrm{Ric}^h(D_1, D_2)\omega(D_3) \,.$$

Lemma 5.8. *C_1, C_2 and R are a basis of the $\mathbb{R}$-vector space of endomorphism-valued natural 2-forms (associated to symmetric linear connections) that satisfy the first Bianchi identity.*

Proof: The Main Theorem 4.1 reduces the problem to that of describing the following vector space[1]:

$$\bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Gl}} \left(S^{d_1} N_1^{\mathrm{sym}} \otimes \cdots \otimes S^{d_r} N_r^{\mathrm{sym}} \,,\, \mathcal{R} \right) \,,$$

where the summation is over all sequences $\{d_1, \ldots, d_r\}$ of non-negative integers satisfying:

$$2d_1 + \ldots + (r+1)\, d_r = 3 - 1 = 2 \,. \tag{5.1}$$

[1]Please recall Remark 3.5, since we are now considering symmetric linear connections.

There is only one solution, $d_1 = 1, d_2 = \ldots = d_r = 0$, and therefore our task consists in computing the vector space of Gl-equivariant linear maps

$$N_1^{\mathrm{sym}} \longrightarrow \mathcal{R} \,.$$

It is not difficult to check that the formula $T^l_{ijk} = \Gamma^l_{jki} - \Gamma^l_{ikj}$ establishes a Gl-equivariant linear isomorphism $N_1^{\mathrm{sym}} \simeq \mathcal{R}$. Thus, the problem is then to compute the equivariant endomorphisms of the Gl-module $\mathcal{R}$.

As this module decomposes into three non-isomorphic irreducible components (Lemma B.3), the vector space of equivariant endomorphisms has dimension 3. Moreover, due to the explicit description of these components, it follows that the elements that produce C_1, C_2 and R are a basis of this vector space.

Again, it is enough to find, for any $n \geq 3$, a symmetric linear connection ∇ on a smooth manifold of dimension n for which the tensors $(C_1)_\nabla$, $(C_2)_\nabla$ and R_∇ are linearly independent.

For example, let ∇ be the linear connection on $\mathbb{R}^n$ ($n \geq 3$) whose only non-zero Christoffel symbols in cartesian coordinates are $\Gamma^1_{11} = x_2x_3$ and $\Gamma^2_{23} = \Gamma^2_{32} = x_1$.

By using the notation $T_{ij} := \mathrm{d}x_i \otimes \partial_{x_j}$, straightforward computations give these linearly independent tensors:

$$\begin{aligned}
R_\nabla &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes (-x_3T_{11} + T_{32}) + \\
&\quad + \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes (-x_2T_{11} + T_{22}) + \\
&\quad + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes (-x_1^2T_{32}) \,, \\
(C_1)_\nabla &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \left(\frac{1}{2}x_3T_{11} - \frac{1}{2}x_3T_{22} - \frac{1}{2}(x_2+1)T_{32} \right) + \\
&\quad + \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes \left(\frac{1}{2}(x_2+1)T_{11} - \frac{1}{2}x_3T_{23} + x_1^2T_{31} - \frac{1}{2}(x_2+1)T_{33} \right) + \\
&\quad + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(\frac{1}{2}(x_2+1)T_{12} - \frac{1}{2}x_3T_{13} + x_1^2T_{32} \right) , \\
(C_2)_\nabla &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \left(\frac{3}{2}x_3T_{11} + \frac{3}{2}x_3T_{22} + \frac{1}{2}(x_2-1)T_{32} + x_3T_{33} \right) + \\
&\quad + \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes \left(\frac{3}{2}(x_2-1)T_{11} + (x_2-1)T_{22} + \frac{1}{2}x_3T_{23} + \frac{3}{2}(x_2-1)T_{33} \right) + \\
&\quad + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(\frac{1}{2}(x_2-1)T_{12} - \frac{1}{2}x_3T_{13} \right) .
\end{aligned}$$

□

Definition 5.9. An endomorphism-valued natural 2-form α is said to satisfy the second Bianchi identity if it is a closed endomorphism-valued natural 2-form, in the sense of Definition 5.2.

Theorem 5.10. *For any smooth $n-$manifold (with $n \geq 3$), the constant multiples of the curvature are the only endomorphism-valued natural 2-forms (associated to symmetric linear connections) that satisfy both the first and second Bianchi identities.*

Proof: Since $dR = 0$, and because of Lemma 5.8, it suffices to prove that dC_1 and dC_2 are $\mathbb{R}$-linearly independent natural tensors.

As we did before, it is enough to find a symmetric linear connection ∇ on a smooth manifold of dimension $n \geq 3$ whose tensors $d_\nabla C_1$ and $d_\nabla C_2$ are not $\mathbb{R}$-proportional.

For example, choosing the same connection ∇ on $\mathbb{R}^n$ considered in the proof of Lemma 5.8, we obtain the following non-proportional tensors, which finishes the proof:

$$
\begin{aligned}
d_\nabla C_1 = dx_1 \wedge dx_2 \wedge dx_3 \otimes & \left(-\frac{1}{2} x_2 x_3 (x_2+1) T_{12} + \frac{1}{2} x_2 x_3^2 T_{13} + \frac{1}{2}(x_1 x_3 - 1) T_{22} + \right. \\
& \left. +2 x_1 T_{32} - \frac{1}{2}(x_1 x_3 - 1) T_{33} \right) , \\
d_\nabla C_2 = dx_1 \wedge dx_2 \wedge dx_3 \otimes & \left(-\frac{1}{2} x_2 x_3 (x_2-1) T_{12} + \frac{1}{2} x_2 x_3^2 T_{13} - \frac{1}{2}(x_1 x_3 - 1) T_{22} + \right. \\
& \left. +\frac{1}{2}(x_1 x_3 - 1) T_{33} \right) .
\end{aligned}
$$

□

5.1.3 Ordinary natural forms

For completeness, let us reproduce the characterization of ordinary differential forms associated to symmetric linear connections, due to Kolář-Michor-Slovák ([25]), albeit with a slightly modified proof, adapted to our theory.

For the following definition, recall that the curvature tensor R associated to a linear connection ∇ is an endomorphism-valued natural 2-form:

Definition 5.11. A Chern[2] $2q$-form c_q over X is a natural q-form (associated to symmetric linear connections) generated by exterior products of (real-valued) natural forms

$$\mathrm{tr}\,(R \wedge \ldots \wedge R) \ ,$$

where the exterior product of endomorphism-valued 2-forms $\wedge$ is defined by the composition of endomorphisms.

In index notation, a Chern form can be expressed as the alternation of the indices j and k of the expression

$$R^{i_q}_{i_1 j_1 k_1} R^{i_1}_{i_2 j_2 k_2} \ldots R^{i_{q-1}}_{i_q j_q k_q} \ .$$

[2]These forms are closely related to the usual Chern forms, defined in the context of cohomology theory.

Theorem 5.12. *The Chern forms c_q are the only (real-valued) natural differential forms associated to symmetric linear connections.*

In particular, there are no non-zero natural forms with odd degrees associated to symmetric linear connections, and hence all natural forms associated to symmetric linear connections are closed.

Proof: For any $p \in \mathbb{N}$, let us compute all non-zero natural p-forms associated to symmetric linear connections. The Main Theorem 4.1 leads us to describe the space of Gl-equivariant linear maps

$$S^{d_1} N_1^{\mathrm{sym}} \otimes \ldots \otimes S^{d_r} N_r^{\mathrm{sym}} \longrightarrow \Lambda^p (T^*_{x_0} X) \,,$$

where $d_1, \ldots, d_r$ are non-negative integers running over the solutions of the equation

$$2d_1 + \ldots + (r+1) d_r = p$$

Due to Proposition B.1, it is equivalent to describe the space of Gl-invariant linear maps

$$S^{d_1} N_1^{\mathrm{sym}} \otimes \ldots \otimes S^{d_r} N_r^{\mathrm{sym}} \otimes \Lambda^p (T_{x_0} X) \longrightarrow \mathbb{R} \,.$$

The First Fundamental Theorem of Gl B.1 explicitly states the generators of such a space: the total contractions ϕ_σ.

Observe that any $T^l_{ijk} \in N_1^{\mathrm{sym}}$ possesses a symmetric pair of covariant indices (the pair (i,j)), and for $m > 1$ any $T^l_{ijk_1 \ldots k_m} \in N_m^{\mathrm{sym}}$ possesses at least two pairs of symmetric covariant indices (the pairs (i,j) and (k_1, k_2)).

However, contracting such a symmetric pair with contravariant indices of any $e^{a_1 \ldots a_p} \in \Lambda^p (T_{x_0} X)$ is null. Therefore, we must 'break' all symmetric pairs by contracting one index in each symmetric pair with a contravariant index that does not belong to $e^{a_1 \ldots a_p}$.

The remaining contravariant indices are the indices l of elements in the spaces of normal tensors. As the elements of N_m^{sym} have two symmetric pair of indices for all $m > 1$, it must hold that $d_2 = \ldots = d_r = 0$, in order to have enough contravariant indices to break the symmetric pairs, and thus $2d_1 = p$.

Summarizing the situation, the generators of the space of natural p-forms associated to symmetric linear connections are zero if p is odd, whereas if $p = 2q$ the generators can be expressed as total contractions ϕ_σ applied to elements

$$T^{l_1}_{i_1 j_1 k_1} \ldots T^{l_q}_{i_q j_q k_q} e^{a_1 \ldots a_{2q}} \in S^q N_1^{\mathrm{sym}} \otimes \Lambda^{2q} (T_{x_0} X) \,.$$

The contraction of the contravariant indices l with a covariant index in each symmetric pair (i,j) (which is necessary, as explained above) leaves $2q$ covariant indices, which must be contracted with the indices of the element in $\Lambda^{2q}(T_{x_0} X)$.

Finally, considering the Gl-equivariant linear isomorphism $N_1^{\text{sym}} \simeq \mathcal{R}$ stated in the proof of Lemma 5.8 and invoking Proposition B.1 again, we obtain that the generators of the space of natural $2q$-forms are the Chern forms c_q. $\square$

5.2 Invariants of linear connections and orientations

The definitions introduced in the last section are easily generalised to other settings; namely to differential invariants associated to linear connections and orientations. For example, let X be a smooth manifold of dimension $n \geq 3$, let Λ^p be the sheaf of differential p-forms over X and let $E \to X$ be a bundle of tensors and let $\mathcal{E}$ be its sheaf of smooth sections. Then, we say that an E-valued natural p-form (associated to linear connections and orientations) is a natural morphism of sheaves

$$\mathcal{C} \times \mathrm{Or}_X \longrightarrow \Lambda^p \otimes \mathcal{E} .$$

As it happened with the case of differential invariants associated to linear connections, the space of E-valued natural p-forms is a finite-dimensional real vector space, now due to the Main Theorem 4.2.

Therefore, we can study whether we can obtain similar theorems to Theorem 5.4 and Theorem 5.10 in this setting. The answer is affirmative, and we will prove them by repeating the same steps:

Lemma 5.13. *If* $\dim X \geq 3$, *then Tor and H are a basis of the* $\mathbb{R}$*-vector space of vector-valued natural 2-forms (associated to linear connections and orientations).*

Proof: Looking at the Main Theorem 4.2, we first compute the non-negative integer solutions of

$$d_0 + 2d_1 + \ldots + (k+1)d_k = 2 - 1 = 1 .$$

There is only one solution, namely $d_0 = 1$, $d_i = 0$, for $i > 0$, so the Main Theorem 4.2 assures that, after choosing a point $x_0 \in X$ and an orientation Ω at x_0, the vector space under consideration is isomorphic to the space of Sl-equivariant linear maps:

$$N_0 = \Lambda^2 T^*_{x_0} X \otimes T_{x_0} X \longrightarrow \Lambda^2 T^*_{x_0} X \otimes T_{x_0} X .$$

Thus, the problem is reduced to a question of invariants for the special linear group, and we can invoke the First Fundamental Theorem of Sl B.1 and Proposition B.1 to obtain generators for this vector space.

According to those results, if $\dim X > 3$, then the space of Sl-equivariant linear maps that we are considering coincides with the space of Gl-equivariant linear maps, which, in turn, are proved in 5.3 to be spanned by H and Tor.

If $\dim X = 3$, there could exist another generator: the map $\varphi\colon \Lambda^2 T^*_{x_0}X \otimes T_{x_0}X \longrightarrow \Lambda^2 T^*_{x_0}X \otimes T_{x_0}X$, which, in the coordinates x_1, x_2, x_3 around p for which $\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$, reads:

$$(\mathrm{d}x_i \wedge \mathrm{d}x_j) \otimes \partial_{x_k} \quad \longmapsto \quad \Omega(\partial_{x_k}, \_\_, \_\_) \cdot e(\mathrm{d}x_i, \mathrm{d}x_j, \_\_)\,,$$

where e is the dual 3-vector of Ω.

If Γ^k_{ij} denote the Christoffel symbols, then a trivial computation allows us to express

$$\begin{aligned}
\varphi \;=\; & \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \left(\Gamma^3_{23} \cdot \partial_{x_1} + \Gamma^3_{31} \cdot \partial_{x_2} + \Gamma^3_{12} \cdot \partial_{x_3}\right) \\
& + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(\Gamma^1_{23} \cdot \partial_{x_1} + \Gamma^1_{31} \cdot \partial_{x_2} + \Gamma^1_{12} \cdot \partial_{x_3}\right) \\
& + \mathrm{d}x_3 \wedge \mathrm{d}x_1 \otimes \left(\Gamma^2_{23} \cdot \partial_{x_1} + \Gamma^2_{31} \cdot \partial_{x_2} + \Gamma^2_{12} \cdot \partial_{x_3}\right)\,,
\end{aligned}$$

as well as the linear relation $\varphi = \mathrm{Tor} + H$. □

Proposition 5.14. *If* $\dim X \geq 3$, *then the exterior differential is an injective* $\mathbb{R}$*-linear map:*

$$\left[\begin{array}{c}\textit{Vector-valued natural}\\ \textit{2-forms}\\ \textit{associated to linear}\\ \textit{connections and orientations}\end{array}\right] \xrightarrow{\ \mathrm{d}\ } \left[\begin{array}{c}\textit{Vector-valued natural}\\ \textit{3-forms}\\ \textit{associated to linear}\\ \textit{connections and orientations}\end{array}\right].$$

Proof: It is a consequence of both Lemma 5.13 and the fact that $\mathrm{d}H$ and $\mathrm{d}\mathrm{Tor}$ are $\mathbb{R}$-linearly independent by Theorem 5.4. □

Once again, as an immediate corollary of Proposition 5.14 we obtain:

Theorem 5.15. *The torsion tensor is the only vector-valued natural 2-form (associated to linear connections and orientations)* ω *that satisfies the first Bianchi identity, i. e., such that* $\mathrm{d}\,\omega = R \wedge I$.

Lemma 5.16. *If* $\dim X > 3$, *then the tensors* C_1, C_2, *and* R *are a basis of the* $\mathbb{R}$*-vector space of endomorphism-valued natural 2-forms (associated to symmetric linear connections and orientations) that satisfy the first Bianchi identity.*

If $\dim X = 3$, *then that vector space has dimension four.*

Proof: Let $\mathcal{R}$ be the vector space of endomorphism-valued 2-forms at a point that satisfies the first Bianchi identity. The Main Theorem 4.2 describes the space of the natural 2-forms under consideration as the vector space:

$$\bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Sl}(T_{x_0}X)}\left(S^{d_1}N_1^{\mathrm{sym}} \otimes \ldots \otimes S^{d_k}N_k^{\mathrm{sym}}, \mathcal{R}\right),$$

where $d_1, \dots, d_k$ are non-negative integers verifying the equation:

$$2d_1 + \ldots + (k+1)d_k = 3 - 1 = 2\,.$$

The only solution to this equation is $d_1 = 1$, $d_2 = \ldots = d_k$, so that the vector space to analyse is the space of Sl-equivariant linear maps:

$$N_1^{\mathrm{sym}} \longrightarrow \mathcal{R}\,. \tag{5.2}$$

First of all, recall that Lemma 5.8 assures that the maps induced by the tensors R, C_1, and C_2 are a basis of the space of Gl-equivariant linear maps $N_1^{\mathrm{sym}} \longrightarrow \mathcal{R}$.

A systematic application of the First Fundamental Theorem of Sl B.1 now allows us to find generators for the space of Sl-equivariant maps.

If $\dim X > 5$, then the vector space of Sl-equivariant maps coincides with the space of Gl-equivariant maps and, hence, is generated by these three elements.

In case $\dim X = 4$, there is another possible generator: the map $N_1^{\mathrm{sym}} \to \mathcal{R}$ defined as

$$\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l} \longmapsto \Omega(\partial_{x_l}, \_\_, \_\_, \_\_) \cdot e(\mathrm{d}x_i, \mathrm{d}x_j, \mathrm{d}x_k, \_\_)\,.$$

However, as any tensor in N_1^{sym} is symmetric in the first two indices, it readily follows that this map is identically zero.

If $\dim X = 3$, let us first describe the Sl-equivariant endomorphisms $T_3^1 \to T_3^1$ that are not Gl-equivariant. Let x_1, x_2 and x_3 be coordinates centered at p such that $\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$, and let e be its dual 3-vector. Applying the First Fundamental Theorem of Sl B.1 gives a system of 16 such generators, which can all be expressed as the composition of a permutation of the factors of T_3^1 and one of these four maps:

(a) $\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l} \longmapsto e(\mathrm{d}x_i, \mathrm{d}x_j, \mathrm{d}x_k) \cdot \Omega \otimes \partial_{x_l}$,

(b) $\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l} \longmapsto e(\mathrm{d}x_i, \mathrm{d}x_j, \mathrm{d}x_k) \cdot \Omega(\partial_{x_l}, \_\_, \_\_) \otimes I$,

(c) $\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l} \longmapsto \delta^l_{\sigma(i)} \cdot \Omega \otimes e(\mathrm{d}x_{\sigma(j)}, \mathrm{d}x_{\sigma(k)}, \_\_)$, $\quad \sigma \in S_3$,

(d) $\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l} \longmapsto \Omega(\partial_{x_l}, \_\_, \_\_) \otimes \mathrm{d}x_{\sigma(i)} \otimes e(\mathrm{d}x_{\sigma(j)}, \mathrm{d}x_{\sigma(k)}, \_\_)$, $\quad \sigma \in S_3$.

By Proposition B.1, all Sl-equivariant maps from N_1^{sym} to $\mathcal{R}$ are restrictions of these maps. As the first two covariant indices of N_1^{sym} are symmetric, the following maps are identically zero: (a), (b), and "raising" indices i and j at (c) and (d) with the 3-vector e.

That leaves eight non-zero generators. However, this symmetry also makes raising the pairs of indices i, k or j, k indistinguishable, hence reducing the system to just four generators.

The last step is to check whether there exits maps linearly generated by them which take values at $\mathcal{R}$. Out of these four generators, the following two are skew-symmetric in the first two covariant indices:

$$\begin{aligned}
\varphi_1\left(\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l}\right) &= \Omega(\partial_{x_l}, __, __) \otimes \mathrm{d}x_j \otimes e(\mathrm{d}x_i, \mathrm{d}x_k, __) \\
\varphi_2\left(\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l}\right) &= \delta^l_j \cdot \Omega \otimes e(\mathrm{d}x_i, \mathrm{d}x_k, __) \ ,
\end{aligned}$$

and the skew-symmetrization of the remaining two is a linear combination of these.

Although neither of these tensors satisfy the first Bianchi identity, the linear combination $\varphi := 3\varphi_1 - \varphi_2$ does, giving rise to a Sl-equivariant map $\varphi : N_1^{\mathrm{sym}} \to \mathcal{R}$.

Finally, all that is left to prove is that φ is $\mathbb{R}$-linearly independent of R, C_1, and C_2. In order to do that, it is enough to find a symmetric linear connection and an orientation on a 3-manifold X such that the aforementioned tensors on X are $\mathbb{R}$-linearly independent.

The following example works: Let ∇ be the linear connection on $\mathbb{R}^3$ whose only non-zero Christoffel symbols in cartesian coordinates are

$$\Gamma^1_{11} = x_2x_3 \quad , \quad \Gamma^2_{23} = \Gamma^2_{32} = x_1x_2 \ .$$

Assume that $\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$, and denote $T_{ij} := \mathrm{d}x_i \otimes \partial_{x_j}$.

Direct computation gives the following linearly independent tensors, thus finishing the proof:

$$\begin{aligned}
R = {} & \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes (-x_3T_{11} + x_2T_{32}) + \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes (-x_2T_{11} + x_2T_{22}) \\
& + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(x_1T_{22} - x_1^2x_2^2T_{32}\right) ,
\end{aligned}$$

$$\begin{aligned}
C_1 = {} & \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \left(\frac{1}{2}x_3T_{11} - \frac{1}{2}x_3T_{22} - \frac{1}{2}x_1T_{31} - x_2T_{32}\right) + \\
& + \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes \left(x_2T_{11} - \frac{1}{2}x_1T_{21} - \frac{1}{2}x_3T_{23} + x_1^2x_2^2T_{31} - x_2T_{33}\right) + \\
& + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(x_2T_{12} - \frac{1}{2}x_3T_{13} - \frac{1}{2}x_1T_{22} + x_1^2x_2^2T_{32} + \frac{1}{2}x_1T_{33}\right) ,
\end{aligned}$$

$$\begin{aligned}
C_2 = {} & \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \left(\frac{3}{2}x_3T_{11} + \frac{3}{2}x_3T_{22} + \frac{1}{2}x_1T_{31} + x_3T_{33}\right) + \\
& + \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes \left(-\frac{1}{2}x_1T_{21}\frac{1}{2}x_3T_{23}\right) + \\
& + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(-x_1T_{11} - \frac{1}{2}x_3T_{13} - \frac{3}{2}x_1T_{22} - \frac{3}{2}x_1T_{33}\right) ,
\end{aligned}$$

$$\begin{aligned}\varphi = &\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes (x_1 T_{31} - x_3 T_{33}) + \\ &+ \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes (2x_1 T_{21} - 3x_2 T_{22} + x_3 T_{23} + 3x_2 T_{33}) + \\ &+ \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes (x_1 T_{11} - 3x_2 T_{12} + 2x_3 T_{13}) .\end{aligned}$$

□

Theorem 5.17. *For any smooth n-manifold (with $n \geq 3$), the constant multiples of the curvature are the only endomorphism-valued natural 2-forms (associated to symmetric linear connections and orientations) that satisfy both the first and second Bianchi identities.*

Proof: The curvature tensor R is always a closed natural 2-form, so, by the previous Lemma, it is enough to analyse the $\mathbb{R}$-linear span of the differentials of C_1, C_2, and, in dimension 3, of φ.

If $\dim X > 3$, then $\mathrm{d}C_1$ and $\mathrm{d}C_2$ are linearly independent by Theorem 5.10, and the statement follows.

If $\dim X = 3$, a direct computation, using the same example as in the previous Lemma, proves that $\mathrm{d}C_1$, $\mathrm{d}C_2$, and $\mathrm{d}\varphi$ are $\mathbb{R}$-linearly independent tensors:

$$\begin{aligned}\mathrm{d}_\nabla C_1 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes & \left(-\frac{1}{2} T_{11} - x_2^2 x_3 T_{12} + \frac{1}{2} x_2 x_3^2 T_{13} + \frac{1}{2}(x_1 x_2 x_3 - 2) T_{22} + \right. \\ & \left. -\frac{5}{2} x_1^2 x_2 T_{31} + 2x_1 x_2^2 T_{32} - \frac{1}{2}(x_1 x_2 x_3 - 3) T_{33} \right) , \\ \mathrm{d}_\nabla C_2 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes & \left(\frac{1}{2} T_{11} + \frac{1}{2} x_2 x_3^2 T_{13} - \frac{1}{2} x_1 x_2 x_3 T_{22} + \right. \\ & \left. -\frac{1}{2} x_1^2 x_2 T_{31} + \frac{1}{2}(x_1 x_2 x_3 - 1) T_{33} \right) , \\ \mathrm{d}_\nabla \varphi = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes & \Big(T_{11} + 3x_2^2 x_3 T_{12} - 2x_2 x_3^2 T_{13} - (x_1 x_2 x_3 - 3) T_{22} + \\ & + 2x_1^2 x_2 T_{31} - 6x_1 x_2^2 T_{32} + (x_1 x_2 x_3 - 4) T_{33} \Big) .\end{aligned}$$

□

5.3 Invariants of Fedosov structures

Although the expressions of differential invariants associated to pseudo-riemannian metrics (see Theorem 4.4) and those associated to Fedosov structures are similar, some differential invariants associated to Riemannian metrics do not have a match in the Fedosov setting. As examples, we will prove that there are no equivalent notion to the scalar curvature and the Laplacian of Riemannian geometry in Fedosov geometry.

5.3.1 Scalar differential invariants on Fedosov manifolds

Let X be a smooth manifold of dimension $2n$.

The following lemma is due to Gelfand-Retakh-Shubin ([11]):

Lemma 5.18. *There exists an* Sp*-equivariant linear isomorphism*

$$\begin{aligned} N_1 &\longrightarrow \mathcal{R} \\ T_{ijkl} &\longmapsto R_{ijkl} = T_{ijlk} - T_{ijkl}, \end{aligned}$$

where N_1 is the space of normal tensors of order 1 and $\mathcal{R} \subset S^2T^*_{x_0}X \otimes \Lambda^2 T^*_{x_0}X$ is the vector subspace of tensors R that satisfy the Bianchi identity:

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0\,.$$

Proposition 5.19. *There are no non-zero homogeneous natural functions associated to Fedosov structures of weight $\delta = -2$.*

If $\dim X \geq 4$*, then the space of homogeneous natural functions of weight $\delta = -4$ is a real vector space of dimension 3, generated by the natural functions:*

- $f_1 = R_{ijkl}R^{ijkl}$,
- $f_2 = R_{ijk}{}^{k}R^{ijl}{}_{l}$,
- $f_3 = \Gamma_{ijk}{}^{ijk}$.

Proof: Let us fix $x_0 \in X$ and a non-singular 2-form ω at x_0. Let us invoke the Main Theorem 4.10 for $p = 0$ and $\delta = -2$. The only non-negative integer solution of the equation

$$2d_1 + \ldots + (r+1)d_r = p - \delta = 2$$

is $d_1 = 1$.

Therefore, the problem is reduced to computing $\mathrm{Sp}(2n, \mathbb{R})$-equivariant maps $N_1 \to \mathbb{R}$. As the elements in N_1 are 4-covariant tensors symmetric in the second and third indices, by the First Fundamental theorem of Sp it is sufficient to check that the map

$$T_{ijka} \longrightarrow \omega^{ij}\omega^{ka}T_{ijka}$$

is zero:

$$\omega^{ij}\omega^{ka}T_{ijka} = \frac{1}{2}\omega^{ij}\omega^{ka}(T_{ijka} - T_{jika}) = 0\,,$$

as the elements in N_1 verify that

$$T_{ijka} - T_{jika} = T_{ijak} - T_{jiak}\,.$$

Repeating the arguments for $w = -4$, we obtain two solutions to the equation above: $d_1 = 2$ and $d_3 = 1$.

Let us begin with solution $d_1 = 2$: we need to compute total index contractions of the expression $T_{ijkl}T_{abcd}$. By Lemma 5.18 stated above, it is equivalent to compute the total index contractions of the expression $R_{ijkl}R_{abcd}$. As the contraction of the symmetric pair is zero, the possibilities are:

- $f_1 = R_{ijkl}R^{ijkl}$.
- $f_2 = R_{ijk}{}^{k}R^{ijl}{}_{l}$.
- $R_{ijkl}R^{ikjl}$, which by the Bianchi identity is equal to $f_1/2$.

For $d_3 = 1$, the last three indices of any tensor in N_3 are symmetric , so there is only one option: $f_3 = T_{ijk}{}^{ijk}$.

As for the linear independence of the three functions, by naturalness it is enough to check if they are independent at any given Fedosov manifold. For example, consider the Fedosov manifold $(\mathbb{R}^4, \eta, \nabla)$, where $\eta = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4$ and ∇ is the linear connection with the following Christoffel symbols (with the contravariant index lowered):

- $\Gamma_{ijk} = 1$, for any $\{i, j, k\}$ permutation of $\{1, 1, 2\}$.
- $\Gamma_{ijk} = x_1x_3x_4$, for any $\{i, j, k\}$ permutation of $\{2, 3, 4\}$.
- $\Gamma_{ijk} = 0$, for any other combination.

Computing the natural functions in this manifold gives:

- $f_1 = -4x_3^2x_4^2(-4x_1^2 + 4x_1 + 1)$.
- $f_2 = 2x_3^2x_4^2(4x_1^2 - 1)$.
- $f_3 = 6$,

which are clearly $\mathbb{R}$-linearly independent. $\square$

5.3.2 Differential operators in Fedosov geometry

More differences between Riemannian geometry and Fedosov geometry can be found in the context of differential operators. Let $\mathrm{Dif}^k_{\mathbb{R}}(\mathcal{C}^\infty_X, \mathcal{C}^\infty_X)$ be the sheaf of $\mathbb{R}$-linear differential operators of order less or equal than k, which is a natural sheaf.

Definition 5.20. Let $\mathcal{G}$ be a natural sheaf. A differential operator of order less or equal than k associated to sections of $\mathcal{G}$ is a natural morphism of sheaves $\mathcal{G} \to \mathrm{Dif}^k_{\mathbb{R}}(\mathcal{C}^\infty_X, \mathcal{C}^\infty_X)$.

Recall that the symbol of a differential operator is a natural morphism of sheaves

$$\sigma : \mathrm{Dif}^k_{\mathbb{R}}(\mathcal{C}^\infty_X, \mathcal{C}^\infty_X) \longrightarrow S^kTX\,,$$

and that a differential operator $P : \mathcal{G} \to \mathrm{Dif}^k_{\mathbb{R}}(\mathcal{C}^\infty_X, \mathcal{C}^\infty_X)$ is of order k if and only if its symbol is non-zero.

Example. The Laplacian $\Delta : \mathcal{M} \to \mathrm{Diff}^2_{\mathbb{R}}(\mathcal{C}^\infty_X, \mathcal{C}^\infty_X)$ is a differential operator of order 2 associated to metrics, defined as $\Delta(g)(f) := \mathrm{tr}_g \nabla^2 f$. It is homogeneous of weight -2.

It turns out that an operator equivalent to the Laplacian does not exist in Fedosov geometry:

Theorem 5.21. *There are no non-zero homogeneous differential operators of order* 2 *and weight* -2 *associated to Fedosov structures.*

Proof: The composition of a differential operator and the symbol results in a natural tensor

$$\mathcal{F} \longrightarrow S^2 TX \,,$$

and thus it is enough to check that there is no non-zero natural tensor with the form above to prove the statement.

Let us suppose that such a morphism exists. By the Main Theorem 4.10, it corresponds to a Sp-equivariant linear map

$$T : S^{d_1} N_1 \otimes \ldots \otimes S^{d_r} N_r \longrightarrow S^2 T_{x_0} X \,,$$

where $d_1, \ldots, d_r$ are non-negative integers running over the solutions of the equation

$$2d_1 + \ldots + (r+1)d_r = -2 - (-2) = 0 \,.$$

The only solution to the equation is $d_1 = \ldots = d_r = 0$. Thus, we are looking for Sp-equivariant morphisms

$$\mathbb{R} \longrightarrow S^2 T_{x_0} X \,,$$

or, equivalently,

$$S^2 T^*_{x_0} X \longrightarrow \mathbb{R} \,.$$

However, due to the First Fundamental Theorem of Sp, both symmetric indices must be contracted with the fixed symplectic form, which is null, leading to a contradiction. □

5.3.3 Dimensional curvature identities

As natural tensors can be computed locally (demonstrated in Chapter 3), the choice of the base smooth manifold X is inconsequential, with a small caveat: the dimension of the manifold does matter.

However, our Main Theorems prove that natural tensors can be described as equivariant maps by the action of classical groups (adding an homogeneity condition, if necessary). Taking a closer look at the First Fundamental Theorems of the classical groups, one sees that the description of the generators of these equivariant maps does not depend on the dimension of the vector space. We will elaborate more on this later.

Moreover, one can define 'dimensional reduction' maps[3]

$$\dots \xrightarrow{r_{n+1}} T_{p,\delta}[n+1] \xrightarrow{r_n} T_{p,\delta}[n] \xrightarrow{r_{n-1}} T_{p,\delta}[n-1] \xrightarrow{r_{n-2}} \dots ,$$

where $T_{p,\delta}[n]$ denotes the space of homogeneous natural p-covariant tensors of weight δ in dimension n:

$$T_p[n] := \left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{G} \to \mathcal{T}^p \\ \text{homogeneous of weight } \delta \end{array} \right\},$$

where $\mathcal{G}$ is either the sheaf of metrics $\mathcal{M}$ or the sheaf of Fedosov structures $\mathcal{F}$.

These maps are surjective, due to the First Fundamental Theorems, stated in Section B.1. However, they might not be injective: there exists natural tensors in higher dimension which, when reduced to a lower dimension, become zero. Such natural tensors are called 'dimensional identities', and the Second Fundamental Theorems will allow us to describe them.

We will start with a brief review of dimensional identities in the context of Riemannian geometry, and then we will compute dimensional identities in Fedosov geometry, following the sketch of the known Riemannian theory.

Review of dimensional curvature identities on Riemannian geometry

Let (X, g) be a Riemannian manifold of dimension n.

Let us fix an amount of indices $p \in \mathbb{N}$ and a weight $\delta \in \mathbb{N}$, and denote

$$T_{p,\delta}[n] := \left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{M} \to \mathcal{T}^p \\ \text{homogeneous of weight } \delta \end{array} \right\},$$

where $\mathcal{M}$ is the sheaf of riemannian metrics of fixed signature over X and $\mathcal{T}^p$ is the sheaf of p-covariant tensors over X.

Let us consider the $(n+1)$ Riemannian manifold $(X \times \mathbb{R}, g + \mathrm{d}x_{n+1} \otimes \mathrm{d}x_{n+1})$, and let i denote the embedding $i : X \hookrightarrow X \times \mathbb{R}$, $x \to (x, 0)$.

[3]An explicit description of these maps will be given later.

Definition 5.22. With the notations described above, the dimensional reduction of natural tensors associated to metrics to dimension n is defined as the map

$$\begin{aligned} r_n : T_{p,\delta}[n+1] &\longrightarrow T_{p,\delta}[n] \\ T &\longmapsto r_n(T)(g) := i^*(T(g + \mathrm{d}x_{n+1} \otimes \mathrm{d}x_{n+1})) \, . \end{aligned}$$

The dimension reduction maps r_n are always surjective, and they are isomorphisms for n sufficiently large ([30]).

Definition 5.23. The inverse limit $T_{p,\delta} = \varprojlim T_{p,\delta}[n]$ of the sequence of maps $r_n : T_{p,\delta}[n+1] \to T_{p,\delta}[n]$ for $n \in \mathbb{N}$ is called the space of universal tensors homogeneous of weight δ.

Definition 5.24. A universal tensor $T \in T_{p,\delta}$ is said to be a dimensional curvature identity in dimension n if it is an element of the space $K_{p,\delta}[n] := \mathrm{Ker}(T_{p,\delta} \to T_{p,\delta}[n])$.

As said above, for any amount of indices p and any weight δ there exists a sequence

$$\ldots \xrightarrow{r_{n+1}} T_{p,\delta}[n+1] \xrightarrow{r_n} T_{p,\delta}[n] \xrightarrow{r_{n-1}} T_{p,\delta}[n-1] \xrightarrow{r_{n-2}} \ldots \, ,$$

where the morphisms r_n are isomorphisms for $n >> 0$, and thus $K_{p,\delta}[n] = 0$ for $n >> 0$. The following theorems compute the first non-trivial kernels that appear in the sequence above by reducing the dimension n, maintaining p and δ fixed (their proof can be found in [15]):

Theorem 5.25 (Scalar identities)**.** *The space $K_{0,-2}[1]$ of scalar (i.e. $p = 0$) dimensional curvature identities of weight $\delta = -2$ in dimension 1 is generated by the scalar curvature.*

In general, for any even weight $\delta = -(n+1)$, the space $K_{0,-(n+1)}[n]$ of scalar dimensional curvature identities of weight $-(n+1)$ in dimension n is generated by the Pfaffian function of order $n+1$.

Theorem 5.26. *Let $p = 2$. The space $K_{2,0}[2]$ of dimensional curvature identities with 2 indices of weight $\delta = 0$ in dimension 2 is generated by the Einstein tensor:*

$$\mathrm{Ric} - \frac{r}{2} g \, ,$$

where Ric denotes the Ricci curvature tensor and r denotes the scalar curvature.

In general, for any even weight $\delta = 2 - n$, the space $K_{2,2-n}[n]$ of scalar dimensional curvature identities of weight $2 - n$ in dimension n is generated by the Lovelock tensor of order $\frac{n}{2}$.

Example. For $p = 2$ and $\delta = 0$, the entire sequence can be described, as all kernels can be computed in a simple way: by the Main Theorem 4.4 and the First Fundamental Theorem of the orthogonal group B.1, it is easy to prove that $T_{2,0}[n]$ is spanned by Ric and rg (see also ([33])). Then, Theorem 5.26 implies that $\dim T_{2,0}[n] = 2$ for all $n > 2$ and that $\dim T_{2,0}[2] = 1$, as in dimension 2 the generators are related by the identity $\mathrm{Ric} - \frac{r}{2}g$.

Moreover, it holds that $K_{2,0}[1]$ is spanned by Ric, as the curvature tensor is zero in dimension 1, and so $\dim T_{2,0}[1] = 0$.

Dimensional curvature identities on Fedosov geometry

Let $(\mathbb{R}^{2n}, \omega, \nabla)$ be a Fedosov manifold of dimension $2n$ [4]. As it was done in Riemannian geometry, let us fix an amount of indices $p \in \mathbb{N}$ and a weight $\delta \in \mathbb{N}$, and denote

$$T_{p,\delta}[2n] := \left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F} \to \mathcal{T}^p \\ \text{homogeneous of weight } \delta \end{array} \right\},$$

where $\mathcal{F}$ is the sheaf of Fedosov structures over $\mathbb{R}^{2n}$ and $\mathcal{T}^p$ is the sheaf of p-covariant tensors over $\mathbb{R}^{2n}$.

Let Γ_{ij}^k denote the Christoffel symbols of ∇. Let us consider the $2(n+1)$-dimensional Fedosov manifold $(\mathbb{R}^{2n} \times \mathbb{R}^2, \omega', \nabla')$, where $\omega' = \omega + \mathrm{d}x_{n+1} \wedge \mathrm{d}y_{n+1}$ and the connection ∇' is defined by the following Christoffel symbols:

$$\begin{aligned} (\Gamma')_{ij}^k &= \Gamma_{ij}^k, \quad 1 \leq i, j, k \leq 2n\,. \\ (\Gamma')_{ij}^k &= 0, \quad \text{in any other case}\,. \end{aligned}$$

In other words, $(\mathbb{R}^{2n} \times \mathbb{R}^2, \omega', \nabla')$ is defined as the product (as Fedosov manifolds) of $(\mathbb{R}^{2n}, \omega, \nabla)$ and $(\mathbb{R}^2, \eta, \bar{\nabla})$, where η is the canonical symplectic form of $\mathbb{R}^2$ and $\bar{\nabla}$ is the flat linear connection. Let i denote the embedding $i : \mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n} \times \mathbb{R}^2$, $x \mapsto (x, 0, 0)$.

Definition 5.27. With the notations described above, the dimensional reduction of natural tensors associated to Fedosov structures to dimension $2n$ is defined as the map

$$\begin{aligned} r_n : T_{p,\delta}[2(n+1)] &\longrightarrow T_{p,\delta}[2n] \\ T &\longmapsto r_n(T)(\omega, \nabla) := i^*(T(\omega', \nabla'))\,. \end{aligned}$$

This map is well defined: it is easy to check that $r_n(T)$ is natural, and as $\lambda^2\omega + \mathrm{d}x_{n+1} \wedge \mathrm{d}y_{n+1}$ and $\lambda^2\omega' = \lambda^2\omega + \lambda^2(\mathrm{d}x_{n+1} \wedge \mathrm{d}y_{n+1})$ coincide on $i(X)$, we obtain

$$\begin{aligned} r_n(T)(\lambda^2\omega, \nabla) =& i^*(T(\lambda^2\omega + \mathrm{d}x_{n+1} \wedge \mathrm{d}y_{n+1}, \nabla')) = \\ & i^*(T(\lambda^2\omega', \nabla')) = \\ & i^*(\lambda^\delta T(\omega', \nabla')) = \lambda^\delta r_n(T)(\omega, \nabla)\,, \end{aligned}$$

[4] Recall that natural tensors are computed locally, hence we can always assume that the base smooth manifold is $\mathbb{R}^m$ for some $m \in \mathbb{N}$.

so $r_n(T)$ is homogeneous of weight δ.

Proposition 5.28. *The maps r_n are surjective for all $n \in \mathbb{N}$.*

Proof: By the Main Theorem 4.10, given any fixed non-singular 2-form ω at $0 \in \mathbb{R}^{2n}$, any $T_{2n} \in T_{p,\delta}[2n]$ can be expressed as a Sp-equivariant linear map $t_{2n} : S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r \to \bigotimes^p T_0^*\mathbb{R}^{2n}$, where $d_1, \ldots, d_r$ are non-negative integers running over the solutions of Equation 4.10.

Due to Proposition B.1 and using the polarity isomorphism of ω, t_{2n} is the restriction to $S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r \bigotimes^p T_0^*\mathbb{R}^{2n}$ of a Sp-equivariant map $\bigotimes^N T_0^*\mathbb{R}^{2n} \to \mathbb{R}$, where

$$N = 4d_1 + \ldots + (3+r)d_r + p \, .$$

Applying the First Fundamental Theorem of Sp B.1 and restricting, $t_{2n} = \sum_{\sigma \in S_N} \lambda_\sigma \omega_\sigma$, where $\sigma \in S_N$ and $\lambda_\sigma \in \mathbb{R}$ for all $\sigma \in S_N$ [5].

Then, denoting by $\bar{N}_1, \ldots, \bar{N}_r$ the spaces of normal tensors in $\mathbb{R}^{2(n+1)}$ and defining the Sp-equivariant map $t_{2(n+1)} : S^{d_1}\bar{N}_1 \otimes \ldots \otimes S^{d_r}\bar{N}_r \bigotimes^p T_0^*\mathbb{R}^{2(n+1)}$ as $t_{2(n+1)} := \sum_{\sigma \in S_N} \lambda_\sigma \omega_\sigma$, it is easy to compute that $r_n(T_{2(n+1)}) = T_{2n}$, where $T_{2(n+1)} \in T_{p,\delta}[2(n+1)]$ is the natural tensor that corresponds to $t_{2(n+1)}$ by the Main Theorem 4.10 (fixing the non-singular 2-form $\omega + \mathrm{d}_0 x_{n+1} \wedge \mathrm{d}_0 y_{n+1}$ at $0 \in \mathbb{R}^{2(n+1)}$). $\square$

Definition 5.29. The inverse limit $T_{p,\delta} = \varprojlim T_{p,\delta}[2n]$ of the sequence of maps $r_n : T_{p,\delta}[2(n+1)] \to T_{p,\delta}[2n]$ for $n \in \mathbb{N}$ is called the space of universal tensors homogeneous of weight δ.

Examples. The symplectic form $T(\omega, \nabla) := \omega$, the $(4,0)$ curvature tensor R_{ijkl} and the natural functions obtained in Corollary 5.19 are universal tensors.

However, the tensor $T(\omega, \nabla) := (\dim X)\, \omega$ is not universal.

Definition 5.30. A universal tensor T homogeneous of weight δ is said to be a dimensional curvature identity in dimension $2n$ if it is an element of the space $K_{p,\delta}[2n] := \mathrm{Ker}(T_{p,\delta} \to T_{p,\delta}[2n])$.

Let ω be a fixed non-singular 2-form at a point $x_0 \in X$. As it was done in Riemannian geometry, let us compute the first non-trivial kernels (thus also the first dimensional identities) that appear by reducing the dimension in the sequence

$$\ldots \xrightarrow{r_{n+1}} T_{p,\delta}[2(n+1)] \xrightarrow{r_n} T_{p,\delta}[2n] \xrightarrow{r_{n-1}} T_{p,\delta}[2(n-1)] \xrightarrow{r_{n-2}} \ldots \, ,$$

for some amount of indices p and weight δ fixed.

[5] Recall that if ω is a non-singular 2-form on a vector space V, then the Sp-equivariant linear maps $\omega_\sigma : V \otimes \overset{N}{\ldots} \otimes V \to \mathbb{R}$ are defined as

$$\omega_\sigma((e_1, \ldots, e_N)) := \omega(e_{\sigma(1)}, e_{\sigma(2)}) \ldots \omega(e_{\sigma(N-1)}, e_{\sigma(N)}) \, .$$

The following results are corollaries of a more general statement, which will be proven in the next section.

Theorem 5.31. *For $\delta \geq -2$ it holds that $T_{0,\delta} = 0$, and so there are no scalar dimensional identities of the curvature of weight $\delta \geq -2$ in any dimension.*

There are no scalar dimension identities of the curvature of weight $\delta = -2k$ in dimension $2k-2$, for any $k \in \mathbb{Z}$ odd.

The space $K_{0,-4}[2]$ of scalar dimensional curvature identities for $\delta = -4$ in dimension 2 is generated by the natural function

$$R_{i_1}^{\ i_2 j_1 k_1} R_{i_2}^{\ i_1 j_2 k_2} (\omega \wedge \omega)_{j_1 k_1 j_2 k_2} \, .$$

Theorem 5.32. *For $\delta \geq 0$ it holds that $T_{2,\delta} = 0$, and so there are no dimensional identities of the curvature with $p = 2$ indices of weight $\delta \geq 0$ in any dimension.*

There are no dimension identities of the curvature with $p = 2$ indices of weight $\delta = -2k$ in dimension $2k+2$, for any $k \in \mathbb{Z}$ even.

The space $K_{2,-2}[4]$ of scalar dimensional curvature identities for $\delta = -2$ in dimension 4 is generated by one natural 2-tensor, which can be expressed as

$$T_{ab} = R_{i_1}^{\ i_2 j_1 k_1} R_{i_2}^{\ i_1 j_2 k_2} (\omega \wedge \omega \wedge \omega)_{j_1 k_1 j_2 k_2 ab} \, .$$

Expanding the expression of the tensor T above produces:

$$T_{ab} = 2K_i^j K_j^i \omega_{ab} - R^l_{\ ijk} R_l^{\ ijk} \omega_{ab} + 4K_i^j R^i_{\ jab} - 4R^j_{\ iak} R^i_{\ jb}{}^k \, ,$$

where $K_{ij} := R^k_{\ ikj}$. The theorem above says that T is a non-zero natural tensor in dimension $2n > 4$, and that $T = 0$ as a natural tensor in dimension $2n \leq 4$.

Remark 5.33. Unlike in the Riemannian case, the first non-trivial kernel might not have a nice description for certain amount of indices p and weight δ, specifically whenever $\frac{p-\delta}{2}$ is odd. The reason why this occurs will become apparent during the next section and involves the nature of the Chern forms associated to a symplectic connection.

Proof of the general statement

Let $X = \mathbb{R}^{2n}$. We will begin this section by observing that the notion of dimensional identity we have defined is closely related to the Second Fundamental Theorem of Sp: let $T = \{T_{2m}\}_{m \in \mathbb{N}}$ be a dimensional curvature identity in dimension $2n$, for some $n \in \mathbb{N}$ (and so $T_{2n} = 0$). By the Main Theorem 4.10, given any fixed non-singular

2-form ω, any T_{2m} can be expressed as a Sp-equivariant linear map

$$t_{2m} : S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r \to \bigotimes^p T^*_{x_0}X \, .$$

As $T_{2n} = 0$, it must also hold that $t_{2n} = 0$.

The key fact is that, due to Proposition B.1 and using the polarity isomorphism of ω, t_{2n} is the restriction to $S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r \bigotimes^p T^*_{x_0}X$ of a Sp-equivariant map $\bigotimes^N T^*_{x_0}X \to \mathbb{R}$, where

$$N = 4d_1 + \ldots + (3+r)d_r + p \, .$$

Applying the First Fundamental Theorem of Sp B.1, such a map is a linear combination

$$\sum_{\sigma \in S_N} \lambda_\sigma \omega_\sigma$$

of maps ω_σ (defined in the First Fundamental Theorem of Sp B.1), with $\sigma \in S_N$, that is null when restricted to $S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r$.

As the symmetries of the spaces of normal tensors N_i do not depend on the dimension of the base manifold, they cannot be the reason why t_{2n} is null, as if that were the case then $t_{2m} = 0$ for all $m \in \mathbb{N}$ (as $t_{2m} = \sum_{\sigma \in S_N} \lambda_\sigma \omega_\sigma$ for all $m \in \mathbb{N}$, see the proof of Proposition 5.28) and $T = 0$, leading to a contradiction.

Therefore, $\sum_{\sigma \in S_N} \lambda_\sigma \omega_\sigma = 0$ before restricting to the spaces of normal tensors N_i. Thus we can invoke the Second Fundamental Theorem of Sp B.1, which says that $\sum_{\sigma \in S_N} \lambda_\sigma \omega_\sigma$ (and so any t_{2m}) can be expressed as in Equation B.1:

$$\sum_{\sigma \in S_{|I|}} (\operatorname{sgn} \sigma)\, \omega_\sigma \, ,$$

where $I \subseteq \{1, \ldots, N\}$ is a set such that $|I| > 2n$.

Let us compute the maximum amount of indices in $S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r \bigotimes^p T^*_{x_0}X$ that can belong in the set I. Let $s \in \{1, \ldots, r\}$, let $T_{ijka_1 \ldots a_s} \in N_s$ be a normal tensor and suppose that there are three of the indices $i, j, k, a_1, \ldots, a_s$ in I, i.e. they are being alternated. As the indices j and k and the last s indices are symmetric, we may suppose without loss of generality that i, j and a_1 are the alternated indices. However, the symmetry

$$T_{ijka_1a_2 \ldots a_s} - T_{jika_1a_2 \ldots a_s} = T_{ija_1ka_2 \ldots a_s} - T_{jia_1ka_2 \ldots a_s}$$

assures that this alternation is zero.

As only a maximum of two indices in each N_s factor can belong in I, the maximum total amount of indices in $S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r \bigotimes^p T^*_{x_0}X$ that can belong in I is

$$m = 2(d_1 + \ldots + d_r) + p \, .$$

Lemma 5.34. *There are no dimensional identities of the curvature of p indices and weight δ in dimension $2n \geq 2p - \delta$.*

Proof: The Second Fundamental Theorem of Sp B.1 assures that there are no dimensional identities for $2n \geq m$. In our case,

$$m = 2(d_1 + \ldots + d_r) + p = 2p - \delta - (d_2 + \ldots + (r-1)d_r) \geq 2p - \delta\,,$$

finishing the proof. □

Lemma 5.35. *The dimensional identities of the curvature of p indices and weight δ in dimension $2n = 2p - \delta - 2$ are independent of derivatives of the curvature, that is, it corresponds to an* Sp*-equivariant map $S^{d_1}N_1 \to \bigotimes^p T^*_{x_0}X$.*

Proof: For a dimensional identity to exist, it must hold that $m > 2n$. As m is even, it holds that $m \geq 2n + 2$. Therefore,

$$\begin{aligned} 2(d_1 + \ldots + d_r) + p = m \geq 2n + 2 = 2p - \delta & \\ &= 2d_1 + \ldots + (r+1)d_r + p \\ &= 2(d_1 + \ldots + d_r) + p + (d_2 + \ldots + (r-1)d_r)\,, \end{aligned}$$

obtaining $d_2 = \ldots = d_r = 0$. □

Observe that if we denote by $k = d_1$ the amount of curvature operators involved in the dimensional identity, then $\delta = p - 2k$ and so we may rewrite $2n = 2p - \delta - 2 = 2k + p - 2$.

Recall that a non-singular 2-form ω induces a product of q-forms for any $q \in \mathbb{N}$, which we will denote by $\langle \cdot\,,\cdot \rangle$, due to the polarity isomorphism: the indices of one of the q-forms are raised with ω, obtaining a q-vector which is then contracted with the remaining q-form.

Moreover, given a q-form and a q'-form with $q' < q$, then we can define a $(q-q')$-form by raising the indices of the q'-form and contracting with the q-form.

Before we state the general description of the dimensional identities of the curvature of a Fedosov structure, notice that Chern forms (Definition 5.11) can be defined for symplectic connections.

However, in the same vein as in the case of Riemannian geometry and the Levi-Civita connection of a metric ([25]), Chern forms c_q with q odd become null:

Lemma 5.36. *The Chern forms c_q of a symplectic connection are null for all q odd.*

Proof: Observe that it is enough to check that $R^{a_q}_{a_1b_1c_1} R^{a_1}_{a_2b_2c_2} \ldots R^{a_{q-1}}_{a_qb_qc_q} = 0$, as all Chern forms c_q of degree $2q$ contain a factor $R^{a_{q'}}_{a_1b_1c_1} R^{a_1}_{a_2b_2c_2} \ldots R^{a_{q'-1}}_{a_{q'}b_{q'}c_{q'}}$, with $q' \leq q$ odd.

Due to the symmetries of the curvature tensor of a symplectic connection, it holds that

$$\begin{aligned} R^{a_q}_{a_1b_1c_1}R^{a_1}_{a_2b_2c_2}\dots R^{a_{q-1}}_{a_qb_qc_q} &= \omega^{a_qd_q}R_{d_qa_1b_1c_1}\omega^{a_1d_1}R_{d_1a_2b_2c_2}\dots\omega^{a_{q-1}d_{q-1}}R_{d_{q-1}a_qb_qc_q} \\ &= (-1)^q\omega^{d_qa_q}R_{d_qa_1b_1c_1}\omega^{d_1a_1}R_{d_1a_2b_2c_2}\dots\omega^{d_{q-1}a_{q-1}}R_{d_{q-1}a_qb_qc_q} \\ &= (-1)^q\omega^{d_1a_1}R_{d_qa_1b_1c_1}\omega^{d_2a_2}R_{d_1a_2b_2c_2}\dots\omega^{d_qa_q}R_{d_{q-1}a_qb_qc_q} \\ &= (-1)^qR^{d_1}_{d_qb_1c_1}R^{d_2}_{d_1b_2c_2}\dots R^{d_q}_{d_{q-1}b_qc_q}\,. \end{aligned}$$

As q is odd, $(-1)^q = -1$. Reordering the factors,

$$(-1)^qR^{d_1}_{d_qb_1c_1}R^{d_2}_{d_1b_2c_2}\dots R^{d_q}_{d_{q-1}b_qc_q} = -R^{d_q}_{d_{q-1}b_qc_q}R^{d_{q-1}}_{d_{q-2}b_{q-1}c_{q-1}}\dots R^{d_1}_{d_qb_1c_1}\,.$$

Recall that the indices b and c are being alternated, so that we can permute them in the following way:

$$-R^{d_q}_{d_{q-1}b_qc_q}R^{d_{q-1}}_{d_{q-2}b_{q-1}c_{q-1}}\dots R^{d_1}_{d_qb_1c_1} = -R^{d_q}_{d_{q-1}b_1c_1}R^{d_{q-1}}_{d_{q-2}b_2c_2}\dots R^{d_1}_{d_qb_qc_q}\,.$$

Renaming the indices d as a ($d_i \to a_{q-i}$ for all $i \in \{1,\dots,q-1\}$, $d_q \to a_q$), we are left with the original Chern form with opposite sign, and thus is null. □

In concordance with the Riemannian case, the Chern forms of a symplectic connection will be called Pontryagin forms, and will be denoted by $p_q := c_{2q}$.

Theorem 5.37. *Let $x_0 \in X$ and let ω be a non-singular 2-form at x_0. The space $K_{p,\delta}[2p-\delta-2]$ of dimensional identities of the curvature homogeneous of weight δ with p indices is spanned by the p-forms*

$$\left\langle \omega\wedge \overset{k+\frac{p}{2}}{\dots} \wedge\omega\,,\, p_{\bar{k}} \right\rangle\,,$$

where $\bar{k} = \frac{k}{2} = \frac{p-\delta}{4}$ and $p_{\bar{k}}$ is a Pontryagin form.

In particular, there are no dimensional identities of the curvature in dimension $2p-\delta-2$ when $\frac{p-\delta}{2}$ is odd.

Proof: By the previous lemma and the observation above, any dimensional identity T for $2n = 2k+p-2$ can be expressed as an Sp-invariant linear map of the form

$$T : S^kN_1 \otimes \bigotimes^p T^*_{x_0}X \longrightarrow \mathbb{R}$$

for a fixed $x_0 \in X$ and ω a non-singular 2-form at x_0.

Applying Lemma 5.18, we can replace N_1 by the vector space of curvature-like tensors $\mathcal{R}$.

As explained before, out of the $4k+p$ indices only a maximum of two indices per $\mathcal{R}$ factor and the p free indices can belong in I, summing up to $m = 2k+p = 2p-\delta$.

It is also the minimum, as $m > 2n = 2p - \delta - 2$ due to the Second Fundamental Theorem of Sp and m must be even due to the First Fundamental Theorem of Sp.

As the symmetric pair of any $\mathcal{R}$ factor cannot belong in I, by applying the Bianchi identity and reordaining indices we may suppose, without loss of generality, that the skew-symmetric pair of each $\mathcal{R}$ factor belongs in I, along with the free p indices. This fills the amount of indices needed in I.

The remaining indices (that is, the symmetric pairs of indices of the $\mathcal{R}$ factors) must be contracted with indices of different symmetric pairs, since contracting a symmetric pair of indices with the symplectic form would be null. Hence we obtain a Pontryagin form p_k, which is non-zero only if k is even, due to the lemma above.

All that is left is to express this map as an Sp-equivariant map

$$T : S^k\mathcal{R} \longrightarrow \bigotimes^{p} T^*_{x_0}X \,,$$

by invoking Proposition B.1 and applying the polarity isomorphism given by the non-singular 2-form ω. This produces a p-form proportional to the one in the statement. □

Remark 5.38. Recall the expanded expression of the tensor T of Theorem 5.32:

$$T_{ab} = 2K_i^j K_j^i \omega_{ab} - R^l{}_{ijk} R_l{}^{ijk} \omega_{ab} + 4K_i^j R^i{}_{jab} - 4R^j{}_{iak} R^i{}_{jb}{}^k \,.$$

Utilizing the identity $R^k{}_{imj,}{}^m = K^k_{i,j}$, it is easy to prove that $\operatorname{div} T = 0$[6]. This produces a similarity with the Riemannian setting, as the divergence of the first dimensional identities in Riemannian geometry is also null.

These arguments lead us to believe that the dimensional identities computed above all have null divergence:

Conjecture. $\operatorname{div} \left\langle \omega \wedge \overset{k+\frac{p}{2}}{\ldots} \wedge \omega \,,\, p_{\bar{k}} \right\rangle = 0\,.$

The condition of null divergence has great physical meaning, and it is usually imposed on field equations in general relativity (see [33]).

[6]The divergence of a p-covariant natural tensor T is defined as the $(p-1)$-covariant natural tensor $\operatorname{div} T = \operatorname{tr}_\omega(\nabla T)$, where tr_ω denotes the contraction of the first two covariant indices with ω.

Chapter 6

Discussion on open problems

Over the last year, there has been a recurring question on my head: can the Peetre-Slovak theorem 2.16 be extended to the setting of natural sheaves? That is, whether the following conjecture is true:

Conjecture. *Let X be a smooth manifold. Let $\mathcal{F}$ be a natural sheaf over X, and let $\mathcal{F}'$ be the sheaf of smooth sections of a natural bundle $F' \to X$.*

The choice of a point $x_0 \in X$ allows the definition of a bijection:

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \phi \colon \mathcal{F} \longrightarrow \mathcal{F}' \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_{x_0}\textit{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F} \longrightarrow F'_{x_0} \end{array} \right\},$$

where $J^\infty_{x_0}\mathcal{F}$ denotes the ringed space of ∞-jets of sections of $\mathcal{F}$ and Diff_{x_0} stands for the group of germs of diffeomorphisms τ between open sets of X such that $\tau(x_0) = x_0$.

This question, which can seem innocuous at first, proposes a fundamental change in the philosophy of the theorem, as the focus has now been shifted from fibre bundles to sheaves. Such a result would allow the study of differential invariant associated to many relevant geometric structures, such as Kähler and Einstein geometries, and more generally any geometry defined by natural PDEs. The case of Fedosov geometry, studied during this memoir, would be another example.

There exists, however, two major roadblocks in this line of thought: one in the proof of the conjecture itself, and one in the application of this result to the computations of differential invariants. Both have been solved (near miraculously, one might add) in the instance of Fedosov geometry, as the previous chapters show.

Let us give a brief overview of the proof of the Peetre-Slovak theorem: with the same notations as in the statement of the theorem, given a smooth morphism $P : J^\infty_{x_0}F \to F'_{x_0}$, a (regular) morphism of sheaves $\phi_P : \mathcal{F} \to \mathcal{F}'$ is defined as $\phi_P(s)(x) := P(j^\infty_x s)$. Conversely, given a regular morphism of sheaves ϕ, the corresponding smooth morphism is defined as $P_\phi(j^\infty_x s) := \phi(s)(x)$, where s is a representative of $j^\infty_x s$ defined on a neighbourhood of x.

Thus, a big part of the proof is to check that P_ϕ does not depend on the choice of the representative s. This is essentially done by 'stitching' together any two sections with the same ∞-jet at the chosen point. Such procedure is done by applying Whitney's Extension Theorem ([46, 36]):

Whitney's Extension Theorem). *Let $K \subset \mathbb{R}^n$ be a compact set, and let $\{T_x\}_{x\in K}$ be a family of Taylor expansions on K.*

There exists $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that, for all $x \in K$, the Taylor expansion of f at x is T_x if and only if the Taylor condition is verified[1].

This result works in the setting of fibre bundles because any smooth function defines a smooth section of a fibre bundle. However, the same does not hold as easily when defining sections of natural sheaves: although any section of a natural sheaf is a smooth section of a natural bundle, such section will need to verify a certain (natural) condition. Extending the morphism of sheaves, a stronger version of Whitney's Extension Theorem or a different approach to the proof would be needed to continue.

Let us give an example, to help visualise the problem: given two sections of the Fedosov sheaf with the same jet at a point, Whitney's Extension Theorem can produce a symmetric linear connection ∇ and a non-singular 2-form ω 'stitching' both sections, but there is no guarantee that the condition $\nabla\omega$ holds. As it was shown during Chapter 2, the Fedosov case can be resolved by fixing a symplectic form and reducing to a sheaf of symplectic connections, which form a fibre bundle, and so the usual Peetre-Slovak Theorem can be applied.

Even though such a result would have intrinsic interest, as it was said before there exists another obstacle in the computation of differential invariants associated to geometries defined by natural PDEs, and that obstacle is proving that the natural restrictions imposed by the geometry at the ∞-jet level can be specified by algebraic relations of ∞-jets of natural bundles, so that the dependence of germs is fully severed.

Going back to the example differential invariants in Fedosov geometry, during Chapter 3 we proved Theorem 3.10, which allowed us to define $J^\infty_{x_0}\mathcal{F}$ as the space of ∞-jets of symmetric linear connections and non-singular 2-forms $(j^\infty_{x_0}\nabla, j^\infty_{x_0}\omega)$ such that $j^\infty_{x_0}(\nabla\omega) = 0$. This step is crucial to the latter identification of a quotient of $J^\infty_{x_0}\mathcal{F}$ with an infinite product of real vector spaces: with the data given by the normal vector spaces, jets of linear connections verifying compatibility conditions can be easily defined by choosing coordinates, but nothing can be said at germ level. As such, defining a Fedosov section that realises the ∞-jet defined by normal tensors is a difficult task – one that is solved by Theorem 3.10.

[1]The Taylor condition refers to the necessary condition imposed by Taylor's Theorem of approximation of smooth functions by analytic functions. It is a technical condition that will not have any relevance on the present discussion.

In fact, Theorem 3.10 has proven to be one of the hardest hurdles I have faced in the making of this memoir. I would like to take the opportunity to thank Prof. Juan B. Sancho de Salas, from the University of Extremadura, for his magnificent inputs on this matter. As shown, realizing 'formally closed' jets of non-singular 2-forms with symplectic forms on a neighbourhood is key in solving the Fedosov case – although not due to what we were expecting. We had hoped that solving the problem for symplectic forms (which constitute one of the simplest geometric structures defined by a natural PDE) would illuminate the way in which the general case should be solved.

However, the argument we found relies on the Poincaré Lemma, a very specific property of closed forms, which unfortunately does not help in the resolution of the general case. On the other hand, the equation $\nabla\omega = 0$ is similar enough to $d\omega = 0$ that the Fedosov case is solved as a corolary.

Summing up, the study of differential invariants associated to geometries defined by natural PDEs with our approach faces two major obstacles. Their resolution should streamline the computation of invariants for a wide variety of geometries, as the steps that follow (simplifying the space of ∞-jets, for example) should often be solved in a straightforward way. We hope to make advancements on both problems in future works.

Appendix A

Polynomial character of homogeneous tensors

Our process of reduction of the spaces of natural tensors result in smooth maps coming from infinite products of normal tensor spaces. Whether these smooth maps depend only on a finite amount of variables is a priori uncertain. To that end, we state below the Homogeneous Function Theorem, which assures such finite dependence whenever we have homogeneity (in fact, it gives more: homogeneous functions turn out to be polynomial).

Homogeneous Function Theorem. *Let $\{E_i\}_{i\in\mathbb{N}}$ be finite-dimensional vector spaces.*

Let $f : \prod_{i=1}^{\infty} E_i \to \mathbb{R}$ be a smooth function such that there exist positive real numbers $a_i > 0$ and $w \in \mathbb{R}$ satisfying:

$$f(\lambda^{a_1}e_1,\ldots,\lambda^{a_i}e_i,\ldots) = \lambda^{w} f(e_1,\ldots,e_i,\ldots) \tag{A.1}$$

for any positive real number $\lambda > 0$ and any $(e_1,\ldots,e_i,\ldots) \in \prod_{i=1}^{\infty} E_i$.

Then, f depends on a finite number of variables $e_1,\ldots,e_k$, and it is a sum of monomials of degree d_i in e_i satisfying the relation

$$a_1d_1 + \cdots + a_kd_k = w\,. \tag{A.2}$$

If there are no natural numbers $d_1,\ldots,d_r \in \mathbb{N}\cup\{0\}$ satisfying this equation, then f is the zero map.

Remark A.1. Observe that, if $w < 0$, then necessarily $f = 0$: if not, let us fix $e \in \prod_{i=1}^{\infty}$ such that $f(e) \neq 0$ and let us take the limit from the right when $\lambda \to 0$ in A.1 for the fixed e. Then, the left-hand side would be equal to $f(0,0,\ldots)$, whereas the limit of the right-hand side would be improper, leading to a contradiction.

Proof: Let us recall that we define $\prod_{i=1}^{\infty} E_i$ as the inverse limit of the spaces $\prod_{i=1}^{k} E_i$. Therefore, as f is smooth, by definition there exists $k \in \mathbb{N}$, a neighbourhood $U_k \subseteq$

$\prod_{i=1}^{k} E_i$ of the origin $(0,\overset{k}{\ldots},0)$ and a smooth map $f_k : U_k \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{tikzcd} \pi_k^{-1}(U_k) \arrow[rr, "f"] \arrow[dr, "\pi_k"'] & & \mathbb{R} \\ & U_k \arrow[ur, "f_k"'] & \end{tikzcd} .$$

As $a_1, \ldots, a_k$ are positive, there exists a neighbourhood $0 \in V \subset \mathbb{R}$ such that, for any $(e_1, \ldots, e_k) \in U_k$ and any $\lambda \in V$, the vector $(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k)$ lies in U_k. Therefore, for any positive $\lambda \in V$ and $(e_1, \ldots, e_k) \in U_k$ the function f_k also satisfies the homogeneity condition:

$$f_k(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) = \lambda^w f_k(e_1, \ldots, e_k) . \tag{A.3}$$

If we take partial derivatives on both sides of this equation, we obtain that the partial derivatives of f_k also verify the homogeneity condition:

$$\frac{\partial f_k}{\partial x_i}(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) = \lambda^{w-a_i} \frac{\partial f_k}{\partial x_i}(e_1, \ldots, e_k) , \quad i \in \{1, \ldots, k\} .$$

Repeating this process enough times, we end up with a partial derivative of f_k that verifies the homogene ity condition with negative weight, and thus it is zero by the remark above. This implies that f_k is a polynomial, and so the homogeneity condition (A.3) is satisfied for any positive $\lambda \in V$ if and only if its monomials satisfy (A.2).

Finally, given any $e = (e_1, \ldots, e_n, \ldots) \in \prod_{i=1}^{\infty} E_i$, we take $\lambda \in \mathbb{R}^+$ such that the vector $(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k, \ldots)$ lies in $\pi_k^{-1}(U_k)$. Then:

$$f(e) = \lambda^{-w} f(\lambda^{a_1} e_1, \ldots, \lambda^{a_n} e_n, \ldots) = \lambda^{-w} f_k(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) = f_k(e_1, \ldots, e_k)$$

and f only depends on the first k variables. □

A simple corolary of the Homogeneous Function Theorem is that, for any finite dimensional vector space W, there exists an $\mathbb{R}$-linear isomorphism:

$$\begin{array}{c} \left[\text{Smooth maps } f \colon \prod_{i=1}^{\infty} E_i \to W \text{ satisfying } (A.1) \right] \\ \Big\| \\ \bigoplus_{d_1, \ldots, d_k} \mathrm{Hom}_{\mathbb{R}}(S^{d_1} E_1 \otimes \ldots \otimes S^{d_k} E_k, W) \end{array} \tag{A.4}$$

where $d_1, \ldots, d_k$ run over the non-negative integer solutions of (A.2).

Appendix B

Invariant theory of classical groups

Let us go over some results of representation theory and invariant theory that are used during this memoir.

Although we are considering both compact and non-compact real Lie groups, the usual invariant theory of classical groups in the setting of algebraic geometry (see [17] for a full exposition of the results that follow, in the context of algebraic geometry) is valid in our context, see ([22]).

Let us being by exposing the following proposition, which is frequently utilised in the computations:

Proposition B.1 ([22]). *Let E and F be (algebraic) linear representations of G, where $G = \mathrm{Gl}(V)$, $\mathrm{Sl}(V)$ or $\mathrm{Sp}(2n, \mathbb{R})$.*

- *There exists a linear isomorphism $\mathrm{Hom}_G(E, F) = \mathrm{Hom}_G(E \otimes F^*, \mathbb{R})$.*
- *If $W \subseteq E$ is a sub-representation, then any equivariant linear map $W \to F$ is the restriction of an equivariant linear map $E \to F$.*

B.1 Fundamental Theorems

Let us state the First Fundamental Theorems for the classical groups, which describe the generators of the maps that are invariant by the group action:

First Fundamental Theorem of Gl ([17]). *Let V be an $\mathbb{R}$-vector space of finite dimension n, and let $\mathrm{Gl}(V)$ be the real Lie group of its $\mathbb{R}$-linear automorphisms.*

The vector space $\mathrm{Hom}_{\mathrm{Gl}(V)}\left(V^* \otimes \overset{p}{\ldots} \otimes V^* \otimes V \otimes \overset{q}{\ldots} \otimes V, \mathbb{R}\right)$ of invariant linear forms on $V^* \otimes \ldots \otimes V$ by the action of $\mathrm{Gl}(V)$ is zero if $p \neq q$, whereas, if $p = q$, it is spanned by the following "total contractions":

$$\phi_\sigma(\omega_1 \otimes \ldots \otimes e_p) := \omega_1(e_{\sigma(1)}) \cdot \ldots \cdot \omega_p(e_{\sigma(p)}), \quad \sigma \in S_p .$$

First Fundamental Theorem of Sl **([17]).** *Let V be an oriented $\mathbb{R}$-vector space of finite dimension n, and let $\mathrm{Sl}(V)$ be the real Lie group of its orientation-preserving $\mathbb{R}$-linear automorphisms.*

Let $\Omega \in \Lambda^n V^*$ be a representative of the orientation, and let e be its dual n-vector; that is to say, the only element in $\Lambda^n V$ such that $\Omega(e) = 1$.

The real vector space $\mathrm{Hom}_{\mathrm{Sl}(V)}\left(V^* \otimes \overset{p}{\ldots} \otimes V^* \otimes V \otimes \overset{q}{\ldots} \otimes V\,,\, \mathbb{R}\right)$ of invariant linear forms on $V^* \otimes \ldots \otimes V$ by the action of $\mathrm{Sl}(V)$ is zero if $p \neq q + kn$ for any $k \in \mathbb{Z}$, whereas if $p = q + kn$ for some $k \in \mathbb{Z}$ it is generated by the composition of some copies of the operations

$$\otimes\, \Omega : V^* \otimes \overset{p}{\ldots} \otimes V^* \otimes V \otimes \overset{q}{\ldots} \otimes V \longrightarrow V^* \otimes \overset{p+n}{\ldots} \otimes V^* \otimes V \otimes \overset{q}{\ldots} \otimes V\,,$$

$$\otimes\, e : V^* \otimes \overset{p}{\ldots} \otimes V^* \otimes V \otimes \overset{q}{\ldots} \otimes V \longrightarrow V^* \otimes \overset{p}{\ldots} \otimes V^* \otimes V \otimes \overset{q+n}{\ldots} \otimes V\,,$$

and a total contraction ϕ_σ.

In particular, for $p, q < n$, the vector space of $\mathrm{Sl}(V)$-invariant linear maps coincides with the vector space of $\mathrm{Gl}(V)$-invariant linear maps.

First Fundamental Theorem of O **([17]).** *Let V be a real vector space of finite dimension n, let g be a non-degenerate symmetric bilinear form on V and let $\mathrm{O}(n, \mathbb{R})$ be the real Lie group of $\mathbb{R}$-linear automorphisms $V \to V$ that preserve g.*

The real vector space $\mathrm{Hom}_{\mathrm{O}(n,\mathbb{R})}\left(V \otimes \overset{p}{\ldots} \otimes V\,,\, \mathbb{R}\right)$ of invariant linear forms on $V \otimes \ldots \otimes V$ is null if p is odd, whereas if p is even it is spanned by

$$g_\sigma((e_1, \ldots, e_p)) := g(e_{\sigma(1)}, e_{\sigma(2)}) \ldots g(e_{\sigma(p-1)}, e_{\sigma(p)})\,,$$

where $\sigma \in S_p$.

First Fundamental Theorem of Sp **([17]).** *Let V be a real vector space of finite dimension $2n$, let ω be a non-degenerate skew-symmetric bilinear form on V and let $\mathrm{Sp}(2n, \mathbb{R})$ be the real Lie group of $\mathbb{R}$-linear automorphisms $V \to V$ that preserve ω.*

The real vector space $\mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}\left(V \otimes \overset{p}{\ldots} \otimes V\,,\, \mathbb{R}\right)$ of invariant linear forms on $V \otimes \ldots \otimes V$ is null if p is odd, whereas if p is even it is spanned by

$$\omega_\sigma((e_1, \ldots, e_p)) := \omega(e_{\sigma(1)}, e_{\sigma(2)}) \ldots \omega(e_{\sigma(p-1)}, e_{\sigma(p)})\,,$$

where $\sigma \in S_p$.

There may be linear relations between these generators, which are explicitly stated by the Second Fundamental Theorems. Let us enunciate the Second Fundamental Theorem of the symplectic group (although it is also proven in [17], the reader may consult [28] for an exposition closer to ours):

Second Fundamental Theorem of Sp **([17, 28]).** *The only linear relations between the generators of* $\mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}\left(V\otimes \overset{p}{\ldots} \otimes V\,,\, \mathbb{R}\right)$ *described above are the dimensional identities: if $p > 2n$, then for any $I \subseteq \{1,\ldots,p\}$ such that $|I| > 2n$ one has:*

$$\sum_{\sigma\in S_{|I|}} (\operatorname{sgn}\sigma)\, \omega_\sigma = 0 \tag{B.1}$$

where $\sigma \in S_{|I|}$ is seen as an element of S_p by leaving the indices $\{1,\ldots,p\}\setminus I$ intact.

B.2 Irreducible Gl-components

Let $\mathrm{Gl} = \mathrm{Gl}(T_{x_0}X)$ be the real Lie group of linear automorphisms of the tangent space $T_{x_0}X$.

Observe the following Gl-equivariant linear maps:

$$c_1^1\colon \Lambda^2 T^*_{x_0}X\otimes T_{x_0}X \longrightarrow T^*_{x_0}X\,, \qquad \wedge I\colon T^*_{x_0}X \longrightarrow \Lambda^2 T^*_{x_0}X\otimes T_{x_0}X\,,$$

where c_1^1 stands for the contraction of the first covariant and the first contravariant indices, and I denotes the vector-valued 1-form defined by the identity endomorphism of $T_{x_0}X$.

Lemma B.2. *The decomposition into irreducible* Gl*-submodules of $\Lambda^2 T^*_{x_0}X\otimes T_{x_0}X$ is*

$$\Lambda^2 T^*_{x_0}X\otimes T_{x_0}X \;\simeq\; \mathrm{Ker}\, c_1^1 \oplus \mathrm{Im}\,(\wedge I)\ .$$

Proof: An easy computation, using Theorem B.1, proves that the dimension of the vector space of linear, Gl-equivariant endomorphisms of $\Lambda^2 T^*_{x_0}X\otimes T_{x_0}X$ is less than or equal to two and, by Schur's Lemma ([17]), so is the number of irreducible submodules.

As c_1^1 is surjective and $\wedge I$ is injective, $\mathrm{Ker}\, c_1^1$ and $\mathrm{Im}\,(\wedge I)$ have different dimension (recall $n \geq 3$), and thus they are non isomorphic. Therefore they are the only irreducible submodules. □

On the other hand, let $\mathcal{R} \subset \Lambda^2 T^*_{x_0}X\otimes T^*_{x_0}X\otimes T_{x_0}X$ be the vector subspace of tensors T satisfying

$$T^l_{ijk} + T^l_{jki} + T^l_{kij} \;=\; 0\,.$$

The contraction c_1^1 (usually called "Ricci contraction" in this context) composed with the natural projections define Gl-equivariant maps:

$$\mathcal{R} \longrightarrow S^2 T^*_{x_0}X \quad , \quad \mathcal{R} \longrightarrow \Lambda^2 T^*_{x_0}X\,,$$

which have Gl-equivariant sections

$$S^2 T^*_{x_0} X \xrightarrow{I \otimes_1} \mathcal{R} \quad , \quad \Lambda^2 T^*_{x_0} X \xrightarrow{I \otimes_2} \mathcal{R}$$

defined in this way ([42], Sect. 4):

$$\begin{aligned}
(I \otimes_1 S)(D_1, D_2, D_3, \omega) &:= S(D_1, D_3)\omega(D_2) - S(D_2, D_3)\omega(D_1) \,, \\
(I \otimes_2 H)(D_1, D_2, D_3, \omega) &:= H(D_1, D_3)\omega(D_2) - H(D_2, D_3)\omega(D_1) + 2H(D_1, D_2)\omega(D_3) \,.
\end{aligned}$$

Lemma B.3. *[42, Thm. 4.1] The decomposition of $\mathcal{R}$ into irreducible* Gl*-submodules is as follows:*

$$\mathcal{R} \;=\; \mathrm{Im}\,(I \otimes_1) \,\oplus\, \mathrm{Im}\,(I \otimes_2) \,\oplus\, \mathrm{Ker}\left(c_1^1\right) \,.$$

Appendix C

Introduction to Fedosov manifolds

Let X be a smooth manifold of dimension n.

Definition C.1. A Fedosov structure on X is a pair (ω, ∇), where ω is a non-singular closed 2-form on X (called symplectic form) and ∇ is a symmetric linear connection on X that preserves ω (called symplectic connection), that is, such that the equation $\nabla\omega = 0$ is verified.

A smooth manifold equipped with a Fedosov structure will be called a Fedosov manifold.

Remark C.2. Recall that, for a non-singular 2-form to exist, the dimension of the manifold n must be even.

As it is done in Riemannian geometry, an operator analogous to the Riemann - Christoffel curvature tensor can be defined in any Fedosov manifold. Let (X, ω, ∇) be a Fedosov manifold, and let R be the curvature $(3, 1)$-tensor, defined by the linear connection ∇ as per usual. Then, a $(4, 0)$-tensor can be created by lowering the contravariant index with the symplectic form ω [1]:

$$R_{ijkl} := \omega_{im} R^m_{jkl}$$

As in Riemannian geometry, it is skew-symmetric in the last two indices and verifies the Bianchi identity in the last three:

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \,. \tag{C.1}$$

However, the first two indices are now symmetric, in contrast with the Riemannian case.

The first covariant derivative of the curvature $R_{ijkl,m}$ verifies the expected symmetries: the symmetries verified by the curvature tensor in the first four indices and the

[1] Recall that a fixed non-singular 2-form induces an isomorphism (known as the polarity or the musical isomorphism) which allows the lowering and raising of tensor indices.

second Bianchi identity:

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0\,. \tag{C.2}$$

In [11], another symmetry is considered as necessary, called the integrability condition:

$$R_{imkj,l} + R_{ijml,k} + R_{iljk,m} + R_{iklm,j} = 0\,. \tag{C.3}$$

Proposition C.3. *The symmetry C.3 can be deduced by the rest of the symmetries of the first covariant derivative of the curvature.*

Proof: Let us apply C.2 to the last term in the left-hand side of C.3:

$$R_{imkj,l} + R_{ijml,k} + R_{iljk,m} - R_{ikjl,m} - R_{ikmj,l}\,.$$

As $R_{ikjl,m} = -R_{iklj,m}$, utilise C.1 to the third and fourth terms gives:

$$R_{imkj,l} + R_{ijml,k} - R_{ijkl,m} - R_{ikmj,l}\,.$$

Reiterating the previous step, consider that $R_{ijkl,m} = -R_{ijlk,m}$ and use C.2 in the second and third terms:

$$R_{imkj,l} - R_{ijkm,l} - R_{ikmj,l}\,.$$

Now, a final application of C.1 finishes the proof:

$$R_{imkj,l} - R_{ijkm,l} - R_{ikmj,l} = R_{imkj,l} + R_{ijmk,l} + R_{ikjm,l} = 0\,.$$

□

Appendix D

On the uniqueness of the torsion and curvature operators

ON THE UNIQUENESS OF THE TORSION AND CURVATURE OPERATORS

ADRIÁN GORDILLO-MERINO, RAÚL MARTÍNEZ-BOHÓRQUEZ, AND JOSÉ NAVARRO

Abstract. We use the theory of natural operations to characterise the torsion and curvature operators as the only natural operators associated to linear connections that satisfy the Bianchi identities.

1. Introduction

The theory of natural operations in differential geometry has a long history, and its modern development culminated by the end of the last century with the monograph by Kolář-Michor-Slovák ([9]), which has become the standard reference in the subject since then.

Paradigmatic results in this theory produced explicit descriptions of all natural operations of a certaind kind; that way, there appeared characterisations for many various differential operations, such as those for the exterior differential ([14]), the Lie bracket ([10]) or the celebrated uniqueness result of characteristic classes in Riemannian geometry ([4], [1]). Later on, the powerful techniques set forth in [9] enhanced all those results to a more satisfactory level of generality, that only requires mild local hypotheses.

In recent years, there is a renewed interest in the theory, as there have appeared different applications ([2], [7], [8], [12]) and some other geometrical aspects of the theory have been developed ([3], [13]). Nevertheless, lots of these references include rewritings of the foremost results of the theory ([8], [3], [13]), since the functorial language and the generality of the book by Kolář-Michor-Slovák make it difficult for the non-specialist to capture the precise meaning of some results.

With the aim of providing statements accessible to a wider audience, in the first part of this paper we briefly outline how to describe the space of natural tensors associated to linear connections in terms of certain linear representations of the general linear group. Our Theorem 2.3 essentially reformulates the main result of [15], although we use the language of sheaves, ringed spaces and a more elementary—yet equivalent (*cf.* [11])—notion of natural bundle.

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The second part of this note makes use of this machinery to give a characterisation (for dimension ≥ 3) of the torsion tensor of linear connections, as well as another of the curvature tensor of symmetric linear connections, much in the spirit of the classical results mentioned above.

To be more precise, we first prove that there are no non-zero closed vector-valued natural 2-forms associated to linear connections (Theorem 3.6), and it easily follows that the torsion tensor is the only vector-valued natural 2-form which satisfies the first Bianchi identity (Corollary 3.8). Then, we turn our attention to the space of endomorphism-valued natural 2-forms associated to symmetric linear connections, and describe the space of those satisfying the first Bianchi identity (Lemma 3.11). Finally, we prove that the constant multiples of the curvature tensor are the only endomorphism-valued natural 2-forms satisfying both the first and second Bianchi identities (Theorem 3.13).

2. Local invariants of linear connections

Let X be a smooth manifold of constant dimension n (the particular choice of the manifold plays no role in our discussion, so the reader may just think of $\mathbb{R}^n$).

Let $\mathrm{Conn} \to X$ and $T^q_p \to X$ denote the bundles of linear connections and (p,q)-tensors on X, and let $\mathcal{C}$ and $\mathcal{T}^q_p$ denote their sheaves of smooth sections, respectively.

That is to say, both $\mathcal{C}$ and $\mathcal{T}^q_p$ are contravariant functors, defined over the category of open sets of X and inclusions between them, that assign, to each open set $U \subseteq X$, the spaces $\mathcal{C}(U)$ and $\mathcal{T}^q_p(U)$ of linear connections and (p,q)-tensors on U and, to each inclusion between open sets $V \hookrightarrow U$, the corresponding restriction maps.

The notion of local (tensor) invariant of linear connections is expressed as follows:

Definition 2.1. A **natural** (p,q)**-tensor** associated to linear connections is a regular morphism of sheaves

$$T\colon \mathcal{C} \longrightarrow \mathcal{T}^q_p \ ,$$

satisfying this condition of naturalness:

$$T(\tau^*\nabla) \ = \ \tau^* T(\nabla) \ ,$$

for any diffeomorphism $\tau\colon U \to V$ between open sets of X, and for any linear connection $\nabla\colon U \to \mathrm{Conn}$.

The notion of *regular* morphism of sheaves consists in a smoothness assumption; the precise definition is as follows: if Λ is a smooth manifold, let us write $X_\Lambda := X \times \Lambda$, so that any open set $U \subset X_\Lambda$ can be thought of as a family of open sets $U_\lambda \subset X$, where $U_\lambda = U \cap (X \times \{\lambda\})$.

A family of linear connections $\{\, \nabla_\lambda \colon U_\lambda \to \mathrm{Conn}\,\}_{\lambda\in\Lambda}$ is said *smooth* (with respect to the parameters $\lambda \in \Lambda$) whenever the following two conditions are satisfied:

- $U = \amalg_{\lambda\in\Lambda} U_\lambda$ is an open set in X_Λ.

- The map $\nabla : U \to \mathrm{Conn}$, defined by $\nabla(\lambda, x) := \nabla_\lambda(x)$, is smooth.

A morphism of sheaves $T\colon \mathcal{C} \to \mathcal{T}_p^q$ is said to be **regular** if, for any smooth manifold Λ and any smooth family of linear connections $\{\nabla_\lambda\colon U_\lambda \to \mathrm{Conn}\}_{\lambda\in\Lambda}$, the family $\{T(\nabla_\lambda)\colon U_\lambda \to T_p^q\}_{\lambda\in\Lambda}$ is also smooth.

Observe that this definition of natural tensor is equivalent ([11]) to the more standard one in terms of bundle functors ([9]).

2.1. Description of the space of natural tensors.

Definition 2.2. Let $m \geq 0$ be a fixed integer and let $x_0 \in X$. The space of **normal tensors** of order m at x_0, which we will denote by N_m, is the vector space of $(m+2,1)-$tensors T at x_0 having the following symmetries:

- they are symmetric in the last m covariant indices:

$$T^l_{ijk_1\ldots k_m} = T^l_{ijk_{\sigma(1)}\ldots k_{\sigma(m)}} \quad , \quad \forall\, \sigma \in S_m\,; \tag{2.1}$$

- the symmetrization over the $m+2$ covariant indices is zero:

$$\sum_{\sigma\in S_{m+2}} T^l_{\sigma(i)\sigma(j)\sigma(k_1)\ldots\sigma(k_m)} = 0\,. \tag{2.2}$$

In the statement of next theorem, which we will prove later, $\mathrm{Hom}_{\mathrm{Gl}(T_{x_0}X)}$ stands for the vector space of $\mathrm{Gl}(T_{x_0}X)$-equivariant linear maps. One of the main highlights of the theory of natural operations is that it unveils the relation between invariant operations on smooth manifolds and the invariant theory of certain real Lie groups; a theory which is, in most cases, algebraic (see [6]) and permits the understanding of these objects via classical invariant theory.

Theorem 2.3. *Fix a point $x_0 \in X$. There exists an $\mathbb{R}$-linear isomorphism:*

$$\left[\begin{array}{c} \textit{Natural tensors } T\colon \mathcal{C} \longrightarrow \mathcal{T}_p^q \\ \textit{associated to linear connections} \end{array}\right]$$

$$\Big\|$$

$$\bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Gl}(T_{x_0}X)}\left(S^{d_0}N_0 \otimes S^{d_1}N_1 \otimes \cdots \otimes S^{d_r}N_r \ , \ \otimes^p T^*_{x_0}X \otimes \otimes^q T_{x_0}X \ \right)$$

where the summation is over all sequences $\{d_0, d_1, \ldots, d_r\}$ of non-negative integers satisfying:

$$d_0 + 2d_1 + \ldots + (r+1)\, d_r = p - q\ . \tag{2.3}$$

If this equation has no solutions, the above vector space reduces to zero.

For the computations that we will perform in this paper, we will only use a couple of basic facts regarding the invariant theory of the general linear group, that we collect in the result below ([6]).

Theorem 2.4. *Let V be an $\mathbb{R}$-vector space of finite dimension n, and let Gl be the Lie group of its $\mathbb{R}$-linear automorphisms.*

The following two facts hold:

(1) The vector space $\mathrm{Hom}_{\mathrm{Gl}}(V^*\otimes \overset{p}{\ldots} \otimes V^* \otimes V \otimes \overset{q}{\ldots} \otimes V, \mathbb{R})$ of invariant linear forms on $V^* \otimes \ldots \otimes V$ is zero if $p \neq q$, whereas, if $p = q$, it is spanned by the following "total contractions":

$$\phi_\sigma(\omega_1 \otimes \ldots \otimes e_p) := \omega_1(e_{\sigma(1)}) \cdot \ldots \cdot \omega_p(e_{\sigma(p)}) , \quad \sigma \in S_p .$$

(2) If E and F are (algebraic) linear representations of Gl, and $E' \subset E$ is a sub-representation, then any equivariant linear map $E' \to F$ is the restriction of an equivariant linear map $E \to F$.

2.2. **Proof of Theorem 2.3.** This proof follows several steps that we proceed to explain.

2.2.1. *A Peetre-like theorem.* If $E \to X$ is a bundle over X, let us denote by $J^r E \to X$ the bundle of r-jets of sections of $E \to X$.

Definition 2.5. The **bundle of ∞-jets** is the inverse limit, in the category of ringed spaces[1], of the r-jet bundles:

$$J^\infty E := \varprojlim J^r E .$$

(That is to say, $J^\infty E$ is endowed with the initial topology of the canonical projections $\pi_r \colon J^\infty E \to J^r E$, and its sheaf of "smooth" functions is $\mathcal{O}_{J^\infty} = \lim\limits_{\to} \pi_r^* \mathcal{C}^\infty_{J^r}$).

In relation to this bundle of $\infty-$jets, we will only use the following fundamental property: if Y is a smooth manifold, a continuous map $\varphi \colon J^\infty E \to Y$ is said to be a **morphism of ringed spaces** if, for any ∞-jet $j^\infty_x s \in J^\infty E$ there exist a natural number r and a smooth map $\varphi_r \colon J^r E \supseteq U \to Y$, defined on an open neighbourhood U of $j^r_x s$, such that $\varphi = \varphi_r \circ \pi_r$ in the neighbourhood $\pi_r^{-1}(U)$ of $j^\infty_x s$.

Definition 2.6. Let $E \to X$ and $F \to X$ be two bundles. A **differential operator** $\tilde{P} \colon E \rightsquigarrow F$ is a morphism of ringed spaces over X, $\tilde{P} \colon J^\infty E \to F$. A differential operator $\tilde{P}$ is said to be of **order** $\leq r$ if there exists a morphism of bundles $\tilde{P}_r \colon J^r E \to F$ such that $\tilde{P} = \tilde{P}_r \circ \pi_r$.

Let $\mathcal{E}$ and $\mathcal{F}$ be the sheaves of sections of two bundles $E \to X$ and $F \to X$. A differential operator $\tilde{P} \colon E \rightsquigarrow F$ can be understood as a morphism of sheaves:

$$P \colon \mathcal{E} \longrightarrow \mathcal{F} \quad , \quad P(s)(x) := \tilde{P}(j^\infty_x s) ,$$

[1]Let us succintly recall that a ringed space is nothing but a topological space X, endowed with a certain subsheaf $\mathcal{O}_X$ of the sheaf $\mathcal{C}(X)$ of continuous real-valued functions on X. For any open set U in X, the functions belonging to $\mathcal{O}_X(U)$ are pointed at to be the *smooth* functions on U in the ringed space X.

and those morphisms of sheaves obtained with this procedure are precisely the regular morphisms of sheaves, as the following nonlinear Peetre-like theorem assures:

Theorem 2.7 ([9], Sect. 19.7). *The previous assignment $\tilde{P} \to P$ establishes a bijection*

$$\left[\begin{array}{c} \textit{Morphisms of ringed spaces} \\ J^\infty E \longrightarrow F \end{array} \right] = \mathrm{Hom}_{\mathrm{reg}}(\mathcal{E}, \mathcal{F}) \,,$$

where $\mathrm{Hom}_{\mathrm{reg}}(\mathcal{E}, \mathcal{F})$ stands for the set of regular morphisms of sheaves.

2.2.2. *Natural bundles.* For any smooth manifold X, let us denote by $\mathrm{Diff}\,(X)$ the set of diffeomorphisms $\tau \colon U \to V$ between open sets in X.

If $\pi \colon F \to X$ is a bundle, then a *lift to F* of a diffeomorphism $\tau \colon U \to V$ between open sets in X is any diffeomorphism $\tau_* \colon F_U := \pi^{-1}(U) \to F_V := \pi^{-1}(V)$ making the following square commutative:

$$\begin{array}{ccc} F_U & \xrightarrow{\tau_*} & F_V \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\tau} & V. \end{array}$$

A *natural bundle* over X is a pair formed by a bundle $F \to X$, together with a lifting of diffeomorphisms:

$$\begin{array}{ccc} \mathrm{Diff}\,(X) & \longrightarrow & \mathrm{Diff}\,(F) \\ \tau & \longmapsto & \tau_* \,, \end{array}$$

satisfying certain functorial ($\mathrm{Id}_* = \mathrm{Id}$ and $(\tau \circ \tau')_* = (\tau)_* \circ (\tau')_*$) and locality conditions (for any diffeomorphism $\tau \colon U \to V$ between open sets in X, and for any open set $U' \subset U$, $(\tau_{|U'})_* = (\tau_*)_{|F_{U'}}$).

If $F \to X$ is a natural bundle, then so are the k-jet prolongations $J^kF \to X$, for any k; hence, there is also a well-defined action of $\mathrm{Diff}\,(X)$ on the ringed space $J^\infty F$.

Moreover, as it is usual in the theory of natural constructions, the $\mathrm{Diff}\,(X)$-equivariance allows to reduce the question to a point. To this end, let us fix a point $x_0 \in X$ and consider the group:

$$\mathrm{Diff}_{x_0} := \{\text{Germs of diffeomorphisms } \tau : U \to V \text{ such that } \tau(x_0) = x_0\} \,.$$

Summing up, Theorem 2.7 can be rephrased in the realm of natural bundles as follows:

Corollary 2.8. *Let $F \to X, \bar{F} \to X$ be natural bundles, let $\mathcal{F}, \bar{\mathcal{F}}$ be their sheaves of smooth sections and fix a point $x_0 \in X$.*

The assignment of Theorem 2.7, together with restriction to the point x_0, establishes bijections:

$$\left[\begin{array}{c}\text{\textit{Regular and natural morphisms of sheaves}}\\ \mathcal{F}\longrightarrow\bar{\mathcal{F}}\end{array}\right]$$
$$\Big\|$$
$$\left[\begin{array}{c}\mathrm{Diff}(X)\text{\textit{-equivariant morphisms of ringed spaces}}\\ J^\infty F\longrightarrow \bar{F}\end{array}\right]$$
$$\Big\|$$
$$\left[\begin{array}{c}\mathrm{Diff}_{x_0}\text{\textit{-equivariant morphisms of ringed spaces}}\\ J^\infty_{x_0} F\longrightarrow \bar{F}_{x_0}\end{array}\right] .$$

2.2.3. *Normal developments.* In the particular case of linear connections, the structure of $J^\infty_{x_0}\mathrm{Conn}$ is closely related to the curvature tensor and its covariant derivatives at that point (see [9]). Nevertheless, we will use normal developments of linear connections, since that suffices for our purposes and simplifies calculations.

If ∇ is a germ of linear connection at $x_0\in X$, let $\overline{\nabla}$ be the germ of linear connection around x_0 that corresponds, via the exponential map, to the canonical flat connection of $T_{x_0}X$.

For each $m\geq 0$, the $m-$th **normal tensor** of the connection ∇ at the point x_0 is:

$$\Gamma^m_{x_0} := \overline{\nabla}^m_{x_0}\mathbb{T} ,$$

where $\mathbb{T}$ is the difference tensor between ∇ and $\overline{\nabla}$:

$$\mathbb{T}(D_1,D_2,\omega) := \omega\left(\nabla_{D_1}D_2-\overline{\nabla}_{D_1}D_2\right) .$$

If $(x_1,\dots,x_n)$ is a normal chart for ∇ around x_0, then:

$$\Gamma^m_{x_0} = \sum_{i,j,k,a_1,\dots a_m}\Gamma^k_{ij;a_1\dots a_m}(x_0)\ \mathrm{d}_{x_0}x_i\otimes\mathrm{d}_{x_0}x_j\otimes\mathrm{d}_{x_0}x_{a_1}\otimes\ldots\otimes\mathrm{d}_{x_0}x_{a_m}\otimes\left(\frac{\partial}{\partial x_k}\right)_{x_0} ,$$

where

$$\Gamma^k_{ij;a_1\dots a_m} := \frac{\partial\Gamma^k_{ij}}{\partial x_{a_1}\dots\partial x_{a_m}} .$$

It is easy to check that the tensor $\Gamma^m_{x_0}$ belongs to the space of normal tensors N_m defined in page 3, for each $m\geq 0$. The nature of the map assigning to each connection its sequence of normal tensors is given by the following result ([5]):

Theorem 2.9. *The following* Diff_{x_0}*-equivariant morphism of ringed spaces is surjective:*

$$J^\infty_{x_0}\mathrm{Conn}\xrightarrow{\pi}\prod_{i=0}^\infty N_i \quad , \quad j^\infty_{x_0}\nabla\longmapsto(\Gamma^0_{x_0},\Gamma^1_{x_0},\dots) ,$$

and its fibers are the orbits of the group

$$\mathrm{NDiff}_{x_0} := \{\tau\in\mathrm{Diff}_{x_0}\colon \tau_{*,x_0}=\mathrm{Id}\in\mathrm{Gl}(T_{x_0}X)\} .$$

Moreover, there exist smooth sections passing through any point of $J^\infty_{x_0}\mathrm{Conn}$ *and, thus, it induces a* $\mathrm{Gl}(T_{x_0}X)$*-equivariant isomorphism of ringed spaces:*

$$(J^\infty_{x_0}\mathrm{Conn})/\mathrm{NDiff}_{x_0} = \prod_{i=0}^{\infty} N_i \ .$$

Remark 2.10. If ∇ is a symmetric linear connection, then it is evident that the m-th normal tensor $\Gamma^m_{x_0}$ of ∇ at x_0 belongs to the vector subspace N^{sym}_m of N_m, formed by those tensors T in N_m which verify the following additional symmetry:

$$T^l_{ijk_1\ldots k_m} = T^l_{jik_1\ldots k_m} \, .$$

This new symmetry readily leads to $\Gamma^0_{x_0} = 0$. That is why, when working with symmetric connections, the statement of Thm. 2.9 changes slightly, as the map assigning to each symmetric linear connection its sequence of normal tensors at x_0 takes values in the product $\prod_{i=1}^\infty N_i^{\mathrm{sym}}$.

2.2.4. *Algebraic character.* In the last step of the proof, the algebraic character of the invariants under consideration will be a consequence of the following elementary fact, that is a reformulation of the analogous fact for a finite collection of vector spaces ([9], Sect. 24.1):

Proposition 2.11. *Let* $\{E_i\}_{i\in\mathbb{N}_0}$ *be a sequence of real vector spaces of finite dimension. Let* $f : \prod_{i=0}^\infty E_i \to \mathbb{R}$ *be a morphism of ringed spaces such that there exist* $a_i > 0$ *and* $\omega \in \mathbb{R}$*, so that:*

$$f(\lambda^{a_0} e_0, \ldots, \lambda^{a_m} e_m, \ldots) = \lambda^\omega f(e_0, \ldots, e_m, \ldots)$$

for all $\lambda > 0$ *and* $(e_0, \ldots, e_m, \ldots) \in \prod_{i=0}^\infty E_i$.

Then, either f *is the null map or* f *depends on a finite number of variables* $e_0, \ldots, e_k$*, and it can be written as a sum of monomials of degree* $d_i \in \mathbb{N}_0$ *at* e_i*, verifying the formula*

$$a_0 d_0 + \ldots + a_k d_k = \omega \, .$$

2.2.5. *End of the proof.* We now dispose all the previous elements together in order to accomplish the proof.

Proof of Theorem 2.3: Corollary 2.8 produces the following isomorphism of vector spaces:

$$\left[\begin{array}{c} \text{Natural tensors } T\colon \mathcal{C} \longrightarrow \mathcal{T}^q_p \\ \text{associated to linear connections} \end{array}\right]$$
$$\Big\|$$
$$\left[\begin{array}{c} \mathrm{Diff}_{x_0}\text{-equivariant morphisms of ringed spaces} \\ \mathsf{t}_{x_0}\colon J^\infty_{x_0}\mathrm{Conn} \ \longrightarrow \ \otimes^p T_{x_0}X \otimes \otimes^q T^*_{x_0}X \end{array}\right].$$

As the subgroup $\mathrm{NDiff}_{x_0} \subset \mathrm{Diff}_{x_0}$ acts by the identity on $\otimes^p T_{x_0}X \otimes \otimes^q T^*_{x_0}X$, any such morphism t_{x_0} factors through the quotient ringed space $J^\infty_{x_0}\mathrm{Conn}\,/\mathrm{NDiff}_{x_0}$, and therefore the above vector space is isomorphic, in virtue of Theorem 2.9, to the following vector space:

$$\begin{bmatrix} \mathrm{Gl}(T_{x_0}X)\text{-equivariant morphisms of ringed spaces} \\ \mathsf{t}_{x_0}\colon \prod_{i=0}^{\infty} N_i \longrightarrow \otimes^p T_{x_0}X \otimes \otimes^q T^*_{x_0}X \end{bmatrix} .$$

Now, the equivariance with respect to the homothety of ratio λ^{-1} implies that any such a morphism t_{x_0} satisfies the following homogeneity condition:

$$\mathsf{t}_{x_0}(\ldots, \lambda^{m+1}\Gamma^m_{x_0}, \ldots) \ = \ \lambda^{p-q}\, \mathsf{t}_{x_0}(\ldots, \Gamma^m_{x_0}, \ldots) \ ,$$

for any sequence $(\ldots, \Gamma^m_{x_0}, \ldots) \in \prod_{i=0}^{\infty} N_i$.

As this condition should be satisfied for all $\lambda > 0$, we can invoke Proposition 2.11, and conclude that the above vector space is isomorphic to

$$\bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Gl}(T_{x_0}X)} \left(S^{d_0}N_0 \otimes S^{d_1}N_1 \otimes \cdots \otimes S^{d_r}N_r \ , \ \otimes^p T^*_{x_0}X \otimes \otimes^q T_{x_0}X \ \right) ,$$

where the summation is over all sequences $\{d_0, d_1, \ldots, d_r\}$ of non-negative integers satisfying equation (2.3) in the statement of Theorem 2.3.

□

3. Main results

In this section, we keep writing X to denote an $n-$dimensional smooth manifold, and we now assume, for the rest of this note, $n \geq 3$. Whenever it is necessary, x_0 will also be considered as a fixed point in X.

Let $E \to X$ be a bundle of tensors (in what follows, we will study the cases $E = TX$ or $E = \mathrm{End}(TX)$) and let $\mathcal{E}$ be its sheaf of smooth sections.

According to Definition 2.1, an E-valued natural form (associated to linear connections) is a regular and natural morphism of sheaves

$$\mathcal{C} \ \longrightarrow \ \Omega^p \otimes \mathcal{E} \ ,$$

and Theorem 2.3 assures that the space of E-valued natural forms associated to linear connections is a finite-dimensional real vector space.

Moreover, as the exterior differential commutes with diffeomorphisms, it induces $\mathbb{R}$-linear maps

$$\begin{bmatrix} E\text{-valued natural} \\ p\text{-forms} \end{bmatrix} \xrightarrow{\ \mathrm{d}\ } \begin{bmatrix} E\text{-valued natural} \\ (p+1)\text{-forms} \end{bmatrix} ,$$

where it should be understood that, if ω is an E-valued natural p-form, the differential $\mathrm{d}\omega\colon \mathcal{C} \to \Omega^{p+1} \otimes \mathcal{E}$ is defined, on each section ∇, with respect to the linear connection on E induced by ∇.

Definition 3.1. A **closed E-valued natural p-form** (associated to linear connections) is an element in the kernel of the map above.

Example 3.2. The torsion tensor of a linear connection can be understood as a vector-valued natural 2-form in the sense of Definition 2.1; that is to say, as a morphism of sheaves (that can easily be checked to be regular and natural),

$$\mathrm{Tor}\colon \mathcal{C} \longrightarrow \Omega^2 \otimes \mathcal{D} \ ,$$

whose value on a linear connection ∇ defined on an open set $U \subset X$ is the following vector-valued 2-form on U:

$$\mathrm{Tor}_\nabla(D_1, D_2) := \nabla_{D_1} D_2 - \nabla_{D_2} D_1 - [D_1, D_2] \ .$$

In a similar manner, the curvature tensor of a linear connection can be thought of as an endomorphism-valued natural 2-form according to Definition 2.1; that is to say, as a (regular and natural) morphism of sheaves

$$R\colon \mathcal{C} \longrightarrow \Omega^2 \otimes \mathcal{E}\mathrm{nd}(\mathcal{D}) \ ,$$

whose value on a linear connection ∇ defined on an open set $U \subset X$ is the following endomorphism-valued 2-form R_∇ on U:

$$R_\nabla(D_1, D_2)D_3 := \nabla_{D_1}\nabla_{D_2} D_3 - \nabla_{D_2}\nabla_{D_1} D_3 - \nabla_{[D_1,D_2]} D_3 \ .$$

3.1. **Irreducible Gl-components.** Let $\mathrm{Gl} = \mathrm{Gl}(T_{x_0}X)$ the real Lie group of linear automorphisms of the tangent space $T_{x_0}X$.

Observe the following Gl-equivariant linear maps:

$$c_1^1\colon \Lambda^2 T^*_{x_0}X \otimes T_{x_0}X \longrightarrow T^*_{x_0}X \ , \qquad \wedge I\colon T^*_{x_0}X \longrightarrow \Lambda^2 T^*_{x_0}X \otimes T_{x_0}X \ ,$$

where c_1^1 stands for the contraction of the first covariant and the first contravariant indices, and I denotes the vector-valued 1-form defined by the identity endomorphism of $T_{x_0}X$.

Lemma 3.3. *The decomposition into irreducible* Gl*-submodules of* $\Lambda^2 T^*_{x_0}X \otimes T_{x_0}X$ *is*

$$\Lambda^2 T^*_{x_0}X \otimes T_{x_0}X \ \simeq \ \mathrm{Ker}\, c_1^1 \oplus \mathrm{Im}\,(\wedge I) \ .$$

Proof: An easy computation, using Theorem 2.4, proves that the dimension of the vector space of linear, Gl-equivariant endomorphisms of $\Lambda^2 T^*_{x_0}X \otimes T_{x_0}X$ is less than or equal to two, and, by Schur's Lemma, so is the number of irreducible submodules.

As the submodules $\mathrm{Ker}\, c_1^1$ and $\mathrm{Im}\,(\wedge I)$ have different dimension (recall $n \geq 3$), they are non isomorphic, and hence they are the only irreducible submodules. □

On the other hand, let $\mathcal{R} \subset \Lambda^2 T^*_{x_0}X \otimes T^*_{x_0}X \otimes T_{x_0}X$ be the vector subspace of tensors T satisfying

$$T^l_{ijk} + T^l_{jki} + T^l_{kij} \ = \ 0 \ .$$

The *Ricci contraction* c_1^1 composed with the natural projections define Gl-equivariant maps:

$$\mathcal{R} \longrightarrow S^2 T^*_{x_0}X \quad , \quad \mathcal{R} \longrightarrow \Lambda^2 T^*_{x_0}X \ ,$$

which have Gl-equivariant sections

$$S^2T^*_{x_0}X \xrightarrow{I\otimes_1} \mathcal{R} \quad , \quad \Lambda^2T^*_{x_0}X \xrightarrow{I\otimes_2} \mathcal{R}$$

defined in this way ([16], Sect. 4):

$$(I\otimes_1 S)(D_1,D_2,D_3,\omega) := S(D_1,D_3)\omega(D_2) - S(D_2,D_3)\omega(D_1) \ ,$$
$$(I\otimes_2 H)(D_1,D_2,D_3,\omega) := H(D_1,D_3)\omega(D_2) - H(D_2,D_3)\omega(D_1) + 2H(D_1,D_2)\omega(D_3) \ .$$

Lemma 3.4. [16, Thm. 4.1] *The decomposition of $\mathcal{R}$ into irreducible* Gl*-submodules is as follows:*

$$\mathcal{R} \;=\; \mathrm{Im}\,(I\otimes_1) \,\oplus\, \mathrm{Im}\,(I\otimes_2) \,\oplus\, \mathrm{Ker}\,\left(c_1^1\right) \ .$$

3.2. **Closed vector-valued natural forms.** In view of the decomposition of Lemma 3.3, the torsion tensor also produces this vector-valued 2-form, naturally associated to linear connections:

$$H := c_1^1(\mathrm{Tor}) \wedge I \ .$$

Lemma 3.5. *Tor and H are a basis of the $\mathbb{R}$-vector space of vector-valued natural 2-forms.*

Proof: In virtue of Theorem 2.3, the vector space under consideration is filtered with integer solutions $\{d_0,\dots,d_k\}$ to the equation

$$d_0 + 2d_1 + \ldots + (k+1)d_k \;=\; 2-1 = 1 \ .$$

As there is only one solution, $d_0 = 1$, $d_1 = \ldots = d_k = 0$, we are led to describe all possible Gl-equivariant linear maps

$$N_0 = \Lambda^2T^*_{x_0}X \otimes T_{x_0}X \;\longrightarrow\; \Lambda^2T^*_{x_0}X \otimes T_{x_0}X \ .$$

As we have already mentioned, a simple computation using the invariant theory of the general linear group allows to prove that this vector space has two generators.

Then, the task is reduced to check that Tor and H are $\mathbb{R}$-linearly independent natural tensors.

To this end, it is enough to find, for any $n \geq 3$, a linear connection ∇ on a smooth manifold of dimension n for which the tensors Tor_∇ and H_∇ are not $\mathbb{R}$-proportional.

For example, let ∇ be the linear connection on $\mathbb{R}^n$ whose only non-zero Christoffel symbols in cartesian coordinates are $\Gamma^1_{12} = \frac{1}{2}x_1 + x_3$ and $\Gamma^1_{21} = -\frac{1}{2}x_1 - x_2$. Direct computation shows that

$$\mathrm{Tor}_\nabla \;=\; (x_1+x_2+x_3)\,\mathrm{d}x_1\wedge\mathrm{d}x_2\otimes\partial_{x_1} \ ,$$

whereas

$$H_\nabla = -\,(x_1+x_2+x_3)\,\mathrm{d}x_1\wedge\mathrm{d}x_2\otimes\partial_{x_1} + \sum_{i\geq 3}^{n}(x_1+x_2+x_3)\,\mathrm{d}x_2\wedge\mathrm{d}x_i\otimes\partial_{x_i} \ ,$$

and so the proof is finished. □

Theorem 3.6. *There are no non-zero closed vector-valued natural 2-forms.*

In other words, the exterior differential is an injective $\mathbb{R}$-linear map:

$$\left[\begin{array}{c} \textit{Vector-valued natural} \\ \textit{2-forms} \end{array}\right] \xrightarrow{\mathrm{d}} \left[\begin{array}{c} \textit{Vector-valued natural} \\ \textit{3-forms} \end{array}\right] .$$

Proof: In view of Lemma 3.5, it is enough to prove that $\mathrm{d}H$ and dTor are $\mathbb{R}$-linearly independent vector-valued natural 3-forms.

If we choose the same connection ∇ on $\mathbb{R}^n$ $(n \geq 3)$ considered in the proof of Lemma 3.5, then we obtain:

$$\mathrm{d}_\nabla H \;=\; \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes (-\partial_{x_1} + \partial_{x_3}) + \sum_{i>3}^{n} (\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_i \otimes \partial_{x_i} - \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_i \otimes \partial_{x_i}) \ ,$$

and

$$\mathrm{d}_\nabla \mathrm{Tor} = R_\nabla \wedge I \;=\; \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \partial_{x_1} \ .$$

Since $\mathrm{d}_\nabla \mathrm{Tor}$ and $\mathrm{d}_\nabla H$ are not $\mathbb{R}-$proportional, this example suffices to end the proof.

□

Definition 3.7. A vector-valued 2-form α naturally associated to linear connections is said to satisfy the **first Bianchi identity** if the following equality of vector-valued natural 3-forms holds:

$$\mathrm{d}\alpha \;=\; R \wedge I \ .$$

Corollary 3.8. *The only vector-valued 2-form naturally associated to linear connections satisfying the first Bianchi identity is the torsion tensor.*

3.3. **Closed endomorphism-valued natural forms.**

Definition 3.9. An endomorphism-valued 2-form α naturally associated to symmetric linear connections is said to satisfy the **first Bianchi identity** if, for any symmetric linear connection ∇ and any vector fields D_1, D_2, D_3:

$$\alpha_\nabla(D_1, D_2)D_3 + \alpha_\nabla(D_2, D_3)D_1 + \alpha_\nabla(D_3, D_1)D_2 \;=\; 0 \ .$$

Example 3.10. The curvature tensor R_∇ of a symmetric linear connection ∇ satisfies this identity.

Also, if $Ricc^s$ and $Ricc^h$ stand for the symmetric and skew-symmetric part of the Ricci tensor, respectively, then the following $(3,1)$-tensors also satisfy the first Bianchi identity:

$$C_1(D_1, D_2, D_3, \omega) := \; Ric^s(D_1, D_3)\omega(D_2) - Ric^s(D_2, D_3)\omega(D_1) \ ,$$

and

$$C_2(D_1, D_2, D_3, \omega) := \; Ric^h(D_1, D_3)\omega(D_2) - Ric^h(D_2, D_3)\omega(D_1) + 2Ric^h(D_1, D_2)\omega(D_3) \ .$$

Lemma 3.11. *C_1, C_2 and R are a basis of the $\mathbb{R}$-vector space of endomorphism-valued natural 2-forms (associated to symmetric linear connections) that satisfy the first Bianchi identity.*

Proof: Theorem 2.3 reduces the problem to that of describing the following vector space[2]:

$$\bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Gl}}\left(S^{d_1} N_1^{\mathrm{sym}} \otimes \cdots \otimes S^{d_r} N_r^{\mathrm{sym}}\ ,\ \mathcal{R}\ \right)\ ,$$

where the summation is over all sequences $\{d_1, \ldots, d_r\}$ of non-negative integers satisfying:

$$2d_1 + \ldots + (r+1)\, d_r = 3 - 1 = 2\ . \tag{3.1}$$

There is only one solution, $d_1 = 1$, $d_2 = \ldots = d_r = 0$, and therefore our task consists in computing the vector space of Gl-equivariant linear maps

$$N_1^{\mathrm{sym}} \longrightarrow \mathcal{R}\ .$$

It is not difficult to check that the formula $T^l_{ijk} = \Gamma^l_{jki} - \Gamma^l_{ikj}$ establishes a Gl-equivariant linear isomorphism $N_1^{\mathrm{sym}} \simeq \mathcal{R}$. Thus, the problem is then to compute the equivariant endomorphisms of the Gl-module $\mathcal{R}$.

As this module decomposes into three non-isomorphic irreducible components (Lemma 3.4), the vector space of equivariant endomorphisms has dimension 3. Moreover, due to the explicit description of these components, it follows that the elements that produce C_1, C_2 and R are a basis of this vector space.

Again, it is enough to find, for any $n \geq 3$, a symmetric linear connection ∇ on a smooth manifold of dimension n for which the tensors $(C_1)_\nabla$, $(C_2)_\nabla$ and R_∇ are linearly independent.

For example, let ∇ be the linear connection on $\mathbb{R}^n$ ($n \geq 3$) whose only non-zero Christoffel symbols in cartesian coordinates are $\Gamma^1_{11} = x_2 x_3$ and $\Gamma^2_{23} = \Gamma^2_{32} = x_1$.

[2]Please recall Remark 2.10, since we are now considering symmetric linear connections.

By using the notation $T_{ij} := \mathrm{d}x_i \otimes \partial_{x_j}$, straightforward computations give these linearly independent tensors:

$$\begin{aligned}
R_\nabla &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes (-x_3 T_{11} + T_{32}) + \\
&+ \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes (-x_2 T_{11} + T_{22}) + \\
&+ \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(-x_1^2 T_{32}\right), \\
(C_1)_\nabla &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \left(\frac{1}{2}x_3 T_{11} - \frac{1}{2}x_3 T_{22} - \frac{1}{2}(x_2+1)T_{32}\right) + \\
&+ \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes \left(\frac{1}{2}(x_2+1)T_{11} - \frac{1}{2}x_3 T_{23} + x_1^2 T_{31} - \frac{1}{2}(x_2+1)T_{33}\right) + \\
&+ \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(\frac{1}{2}(x_2+1)T_{12} - \frac{1}{2}x_3 T_{13} + x_1^2 T_{32}\right), \\
(C_2)_\nabla &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \left(\frac{3}{2}x_3 T_{11} + \frac{3}{2}x_3 T_{22} + \frac{1}{2}(x_2-1)T_{32} + x_3 T_{33}\right) + \\
&+ \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes \left(\frac{3}{2}(x_2-1)T_{11} + (x_2-1)T_{22} + \frac{1}{2}x_3 T_{23} + \frac{3}{2}(x_2-1)T_{33}\right) + \\
&+ \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(\frac{1}{2}(x_2-1)T_{12} - \frac{1}{2}x_3 T_{13}\right).
\end{aligned}$$

□

Definition 3.12. An endomorphism-valued natural 2-form α is said to satisfy the **second Bianchi identity** if it is a closed endomorphism-valued natural 2-form, in the sense of Definition 3.1.

Theorem 3.13. *For any smooth $n-$manifold (with $n \geq 3$), the constant multiples of the curvature are the only endomorphism-valued natural 2-forms (associated to symmetric linear connections) that satisfy both the first and second Bianchi identities.*

Proof: Since $\mathrm{d}R = 0$, and because of Lemma 3.11, it suffices to prove that $\mathrm{d}C_1$ and $\mathrm{d}C_2$ are $\mathbb{R}$-linearly independent natural tensors.

As we did before, it is enough to find a symmetric linear connection ∇ on a smooth manifold of dimension $n \geq 3$ whose tensors $\mathrm{d}_\nabla C_1$ and $\mathrm{d}_\nabla C_2$ are not $\mathbb{R}$-proportional.

For example, choosing the same connection ∇ on $\mathbb{R}^n$ considered in the proof of Lemma 3.11, we obtain the following non-proportional tensors, which finishes the proof:

$$\begin{aligned}
\mathrm{d}_\nabla C_1 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \Big(& -\frac{1}{2}x_2x_3(x_2+1)T_{12} + \frac{1}{2}x_2x_3^2T_{13} + \frac{1}{2}(x_1x_3-1)T_{22} + \\
& +2x_1T_{32} - \frac{1}{2}(x_1x_3-1)T_{33}\Big), \\
\mathrm{d}_\nabla C_2 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \Big(& -\frac{1}{2}x_2x_3(x_2-1)T_{12} + \frac{1}{2}x_2x_3^2T_{13} - \frac{1}{2}(x_1x_3-1)T_{22} + \\
& +\frac{1}{2}(x_1x_3-1)T_{33}\Big).
\end{aligned}$$

□

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Appendix E

On invariant operations on a manifold with a linear connection and an orientation

ON INVARIANT OPERATIONS ON A MANIFOLD WITH A LINEAR CONNECTION AND AN ORIENTATION

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Abstract. We prove a theorem that describes all possible tensor-valued natural operations in presence of a linear connection and an orientation in terms of certain linear representations of the special linear group.

As an application of this result, we prove a characterisation of the torsion and curvature operators as the only natural operators that satisfy the Bianchi identities.

Contents

Since the very early days of differential geometry, the idea of natural operation played a mayor role in the development the theory. As an example, let us point out the applications of this notion of naturalness in the inception of general relativity (*cf.* [14]). In the course of the years, there also appeared some striking mathematical results, such as Gilkey's characterization of Pontryagin forms on Riemannian manifolds ([1], [4]) or his proof of the uniqueness of the Chern-Gauss-Bonnet formula ([5]). By the end of the last century, the modern development of this theory was summarized in the monograph by Kolář-Michor-Slovák ([13]).

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That book contained all the main results and techniques that were known so far, and thus became the standard reference in the subject since then.

On the other hand, the notion of covariance or naturalness is in some sense ubiquitous in physics and mathematics. For that reason, it is not surprising the renewed interest in this theory of natural operations that has raised in recent years, with the appearance of new results and applications to contact geometry ([2]), homotopy theory ([3], [19]), Riemannian and Kähler geometry ([6], [7], [15], [21]), general relativity ([17]) or quantum field theory ([11], [12]).

In this paper, we focus our attention on the vector space of tensor-valued natural operations that can be performed in presence of a linear connection and an orientation. Our main result, Theorem 3.3, establishes that such a vector space is isomorphic to the space of invariant maps between certain linear representations of the special linear group. Thus, the description of these spaces can, in certain cases, be completely achieved using classical invariant theory. As an example of this philosophy, in the final section we characterize the torsion and the curvature as the only natural tensors satisfying the Bianchi identities (Corollary 4.7 and Theorem 4.11).

These results generalize analogous statements recently proven in [10], where we studied natural tensors associated to a linear connection. This was also the situation considered in a landmark paper by Slovák ([20]), whose results were included —and expanded— in [13]. Nevertheless, the non-specialist may find difficult to understand the precise meaning of some statements of this book, due to the functorial language and the generality of its setting.

For this reason, we outlined in [10] the foundations of an alternative approach, that we hope will be accessible to a wider audience. The present paper lays out complete proofs of the main results of this approach, whose novelties are a systematic use of the language of sheaves, ringed spaces and a more elementary—yet equivalent (*cf.* [16])—notion of natural bundle. In our opinion, the heart of the matter in this theory is the existence of an analogue of a Galois theorem (*cf.* [16, Thm. 1.6]), that allows the use of group theory to infer theorems in many areas of differential geometry, in many of which (such as Fedosov, contact or Finsler geometry) this idea is still to be exploited.

1. The category of ringed spaces

In this section we firstly introduce the category of ringed spaces, that is a framework adequate for our purposes: it will allow us to treat certain "infinite dimensional" spaces — such as the ∞-jet space, or a countable product of vector spaces— and quotients of smooth manifolds by the actions of groups on equal footing as usual smooth, finite dimensional manifolds.

Secondly, we state Theorem 1.9, that is an important characterization of differential operators as those morphisms of sheaves that transform smooth families of sections into smooth families of sections.

Definition 1.1. A **ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a subalgebra of the sheaf of real-valued continuous functions on X.

A **morphism of ringed spaces** $\varphi\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $\varphi\colon X \to Y$ such that composition with φ induces a morphism of sheaves $\varphi^*\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$, that is, for any open set $V \subset Y$ and any function $f \in \mathcal{O}_Y(V)$, the composition $f \circ \varphi$ lies in $\mathcal{O}_X(\varphi^{-1}V)$.

Any smooth manifold X is a ringed space, where $\mathcal{O}_X = \mathcal{C}^\infty_X$ is the sheaf of smooth real-valued functions. If X and Y are smooth manifolds, a morphism of ringed spaces $X \to Y$ is just a smooth map.

By analogy with this example, on any ringed space $(X, \mathcal{O}_X)$ the sheaf $\mathcal{O}_X$ will be called the sheaf of *smooth* functions, and morphisms of ringed spaces $X \to Y$ will be often referred to as *smooth morphisms*.

Limits of ringed spaces. This category possesses limits; nevertheless, in what follows it will only appear this particular case:

Definition 1.2. The **inverse limit** of a sequence of smooth manifolds and smooth maps between them

$$\ldots \to X_{k+1} \xrightarrow{\varphi_{k+1}} X_k \xrightarrow{\varphi_k} X_{k-1} \to \ldots$$

is the ringed space $(X_\infty, \mathcal{O}_\infty)$ defined as follows:

- the underlying topological space is the inverse limit of the topological spaces X_k; i.e., the set:
$$X_\infty := \lim_{\leftarrow} X_k$$
endowed with the minimum topology for which the canonical projections $\pi_k\colon X_\infty \to X_k$ are continuous.
- its sheaf of smooth functions is the direct limit $\mathcal{O}_\infty := \lim\limits_{\rightarrow} \pi_k^* \mathcal{O}_{X_k}$.

That is to say, for any open set $U \subseteq X_\infty$, a continuous map $f\colon U \to \mathbb{R}$ lies in $\mathcal{O}_\infty(U)$ if and only if for any point $x \in U$, there exist $k \in \mathbb{N}$, an open neighbourhood $\pi_k(x) \in V_k \subseteq X_k$ and a smooth map $f_k\colon V_k \to \mathbb{R}$ such that the following triangle commutes:

$$\begin{tikzcd} \pi_k^{-1}(V_k) \arrow[rr, "f"] \arrow[dr, "\pi_k"'] & & \mathbb{R} \\ & V_k \arrow[ur, "f_k"'] & \end{tikzcd} .$$

Later we will need the following two properties regarding the smooth structure of this inverse limit:

Universal property of the inverse limit: *For any smooth manifold Y, the projections $\pi_k\colon X_\infty \to X_k$ induce a bijection, that is functorial on Y,*

$$\mathcal{C}^\infty(Y, X_\infty) \;=\; \lim_{\leftarrow} \;\mathcal{C}^\infty(Y, X_k)\;, \qquad \varphi \;\mapsto\; (\pi_k \circ \varphi)$$

where $\mathcal{C}^\infty(\_,\_)$ denotes the set of morphisms of ringed spaces.

Proof: The projections π_k are smooth maps, so one inclusion is trivial. As for the other, let $\varphi\colon Y \to X_\infty$ be a continuous map such that $\pi_k \circ \varphi$ is smooth, for any $k \in \mathbb{N}$.

Let $f \in \mathcal{O}_\infty(U)$ be a smooth function and let $y \in \varphi^{-1}(U)$. On a neighbourhood V of $\varphi(y)$, there exists an smooth map $f_k\colon X_k \to \mathbb{R}$ such that $f = f_k \circ \pi_k$, and therefore:

$$f \circ \varphi \;=\; (f_k \circ \pi_k) \circ \varphi \;=\; f_k \circ (\pi_k \circ \varphi)$$

that is smooth because $\pi_k \circ \varphi$ is a smooth map.

□

Proposition 1.3. *Let Z be a smooth manifold. A continuous map $\varphi\colon X_\infty \to Z$ is smooth if and only if it locally factors through a smooth map defined on some X_k.*

Proof: Let $\varphi\colon X_\infty \to Z$ be a smooth map; let $x \in X_\infty$ be a point and let $(U, z_1, \dots, z_n)$ be a coordinate chart around $\varphi(x)$ in Z. Each of the functions $z_1 \circ \varphi, \dots, z_n \circ \varphi \in \mathcal{O}_\infty(\varphi^{-1}U)$ locally factors through some X_j; as they are a finite number, there exists $k \in \mathbb{N}$ and an open neighbourhood V of x such that all of them, when restricted to V, factor through X_k. Hence, $\varphi_{|V} = (\varphi_k \circ \pi_k)_{|V}$, where $\varphi_k = (z_1 \circ \varphi, \dots, z_n \circ \varphi)$.

The converse is obvious because the composition of morphisms of ringed spaces is a morphism of ringed spaces.

□

As examples, the space $J^\infty F$ of ∞-jets of sections of a fibre bundle F is defined as the inverse limit of the sequence of k-jets fibre bundles:

$$\dots \to J^k F \to J^{k-1} F \to \dots \to F \to X\;.$$

Also, if $N_0, N_1, N_2, \dots$ is a countable family of finite dimensional $\mathbb{R}$-vector spaces, the vector space $\prod_{i=1}^\infty N_i$ is, by definition, the inverse limit of the projections:

$$\dots \to \prod_{i=1}^{k+1} N_i \to \prod_{i=1}^{k} N_i \to \dots \to N_2 \times N_1 \to N_1\;.$$

Quotients by the action of groups. Let G be a group acting on a ringed space X. Let us denote by X/G the quotient topological space and by $\pi : X \to X/G$ the quotient map.

Definition 1.4. The **quotient ringed space** $(X/G, \mathcal{O}_{X/G})$ is the ringed space whose underlying topological space is the quotient topological space X/G, and whose sheaf of smooth functions is defined, on any open set $U \subseteq X/G$ as:

$$\mathcal{O}_{X/G}(U) := \{f \in \mathcal{C}(U, \mathbb{R}) : f \circ \pi \in \mathcal{O}_X(\pi^{-1}(U))\} \;=\; \mathcal{O}_X(\pi^{-1}(U))^G\,,$$

where $\mathcal{O}_X(\pi^{-1}(U))^G$ stands for the set of maps $f \in \mathcal{O}_X^\infty(\pi^{-1}(U))$, such that $f(g\cdot p) = f(p)$, for any $g \in G$ and $p \in \pi^{-1}(U)$.

It is then routine to check that the quotient map $\pi : X \to X/G$ is a morphism of ringed spaces that satisfies the following property:

Universal property of the quotient: *For any ringed space Y, the quotient map $\pi\colon X \to X/G$ induces a functorial bijection:*

$$\left\{ \begin{array}{c} \text{Morphisms of ringed spaces } X \to Y \\ \\ \text{constant along the orbits of } G \end{array} \right\} \;=\!=\; \left\{ \begin{array}{c} \text{Morphisms of ringed spaces} \\ \\ X/G \longrightarrow Y \end{array} \right\} .$$

Corollary 1.5 (Orbit reduction)**.** *Let G be a group acting on a ringed space X, and let $f : X \to Y$ be a surjective morphism of ringed spaces that, locally on Y, admits smooth sections passing through any point of X.*

If the orbits of G coincide with the fibres of f, then the corresponding map $\bar{f} : X/G \to Y$ is an isomorphism of ringed spaces.

Proof: The hypothesis on the fibres assures that the induced morphism $\bar{f} : X/G \to Y$ is bijective. The inverse map $\bar{f}^{-1}$ is also a morphism of ringed spaces because locally coincides with the projection into the quotient of any smooth section of f. □

There is also the following corollary, whose proof is routine:

Corollary 1.6. *Let G be a group acting on two ringed spaces X and Y, and let $H \subseteq G$ be a subgroup that acts trivially on Y.*

Then, the universal property of the quotient restricts to a bijection:

$$\left\{ \begin{array}{c} G\textit{-equivariant morphisms} \\ \\ \textit{of ringed spaces } X \to Y \end{array} \right\} \;=\!=\; \left\{ \begin{array}{c} G/H\textit{-equivariant morphisms} \\ \\ \textit{of ringed spaces } X/H \longrightarrow Y \end{array} \right\} .$$

1.1. **Differential operators.** Let $F \to X$ and $F' \to X$ be fibre bundles over a smooth manifold X.

Definition 1.7. A **differential operator** is a morphism of ringed spaces $P\colon J^\infty F \to F'$ such that the following triangle commutes:

$$\begin{tikzcd} J^\infty F \arrow[rr, "P"] \arrow[dr, "j^\infty\pi"'] & & F' \arrow[dl, "\pi'"] \\ & X. & \end{tikzcd}$$

Let us denote by $\mathcal{F}$ and $\mathcal{F}'$ the sheaves of smooth sections of F and F', respectively.

Definition 1.8. A **family of sections** $\{s_t : U_t \to F\}_{t\in T}$ is **smooth** if T is a smooth manifold and the following conditions are satisfied:

(1) $U = \amalg_{t\in T} U_t$ is an open set of $X \times T$.
(2) The map $s : U \to F$, defined as $s(t,x) := s_t(x)$, is smooth.

A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{F}'$ is **regular** if, for any smooth family of sections $\{s_t : U_t \to F\}_{t\in T}$, the family $\{\phi(s_t) : U_t \to F'\}_{t\in T}$ is also smooth.

Any differential operator $P : J^\infty F \to F'$ defines a morphism of sheaves

$$\phi_P : \mathcal{F} \longrightarrow \mathcal{F}' , \qquad \phi_P(s)(x) := P(j^\infty_x s) ,$$

and the chain rule proves that it is a regular morphism of sheaves.

The following statement is a particular case of a deep result due to J. Slovák (see [13, Sect. 19.7] , or [18] for a proof of the specific statement below):

Theorem 1.9 (Peetre-Slovák)**.** *If $F \to X$ and $F' \to X$ are fibre bundles over a smooth manifold X, then the assignment $P \to \phi_P$ explained above establishes a bijection:*

$$\left\{ \begin{array}{c} \textit{Differential operators} \\ \\ J^\infty F \longrightarrow F' \end{array} \right\} \;=\!=\!=\; \left\{ \begin{array}{c} \textit{Regular morphisms of sheaves} \\ \\ \mathcal{F} \longrightarrow \mathcal{F}' \end{array} \right\} .$$

2. Natural operations in presence of an orientation

The purpose of this section is twofold: on the one hand, we present the notion of natural operation (Definition 2.2); our definition strongly differs from the standard one (*cf.* [13]), although it is equivalent to it ([16]). On the other hand, we prove a general result —Theorem 2.4— that relates these natural operations with certain smooth equivariant morphisms.

2.1. **Natural bundles.** Let $\mathrm{Diff}\,(X)$ denote the set of diffeomorphisms $\tau : U \to V$ between open sets of a smooth manifold X.

If $\pi\colon F \to X$ is a bundle over X, a *lifting of diffeomorphisms* is a map:

$$\begin{array}{rcl} \mathrm{Diff}\,(X) & \longrightarrow & \mathrm{Diff}\,(F) \\ \tau & \longmapsto & \tau_* \end{array}$$

such that if $\tau\colon U \to V$ is a diffeomorphism between open sets in X, then $\tau_*\colon F_U \to F_V$ is a diffeomorphism covering τ; that is to say, making the following square commutative

$$\begin{tikzcd} F_U \arrow[r, "\tau_*", "\sim"'] \arrow[d, "\pi"'] & F_V \arrow[d, "\pi"] \\ U \arrow[r, "\tau", "\sim"'] & V \end{tikzcd} ,$$

where $F_U := \pi^{-1}(U)$ and $F_V := \pi^{-1}(V)$.

Definition 2.1. A **natural bundle** over a smooth manifold X is a bundle $F \to X$ together with a lifting of diffeomorphisms satisfying the following properties:

(1) *Functorial character*: $\mathrm{Id}_* = \mathrm{Id}$ and $(\tau \circ \tau')_* = (\tau)_* \circ (\tau')_*$.
(2) *Local character*: for any diffeomorphism $\tau : U \to V$ and any open subset $U' \subset U$,
$$(\tau_{|U'})_* = (\tau_*)_{|F_{U'}} \ .$$
(3) *Regularity*: if $\{\tau_t : U_t \to V_t\}_{t\in T}$ is a smooth family of diffeomorphisms between open sets on X, then the family $\{(\tau_t)_* : F_{U_t} \to F_{V_t}\}_{t\in T}$ is also smooth.

A subbundle E of a natural bundle F is said to be a **natural** if it is a natural bundle and its lifting of diffeomorphisms is the restriction of the lifting of diffeomorphisms of F.

A **morphism of natural bundles** is a morphism of bundles $\varphi : F \to F'$ that commutes with the lifting of diffeomorphisms; that is, such that for any diffeomorphism $\tau : U \to V$, the following square commutes

$$\begin{tikzcd} F_U \arrow[r, "\varphi"] \arrow[d, "\tau_*"'] & F'_U \arrow[d, "\tau_*"] \\ F_V \arrow[r, "\varphi"] & F'_V . \end{tikzcd}$$

The tangent and cotangent bundles, or, more generally, the bundles of (r,s)-tensors T^s_r are examples of natural bundles. The subbundle of k-forms $\Omega^k \subset T^0_k$ is a natural subbundle of the bundle of k-covariant tensors T^0_k.

If $F \to X$ is a natural bundle, its k-jet prolongation $J^k F$ is also a natural bundle, for all $k \in \mathbb{N}$. Thus, if $\tau \colon U \to V$ is a diffeomorphism, its liftings to these jet spaces $J^k F$ allow to define a lifting to the ∞-jet space; in other words, a morphism of ringed spaces
$$\tau_* \colon J^\infty F_U \longrightarrow J^\infty F_V$$
covering the diffeomorphism τ.

Let $\pi : F \to X$ and $\pi' : F' \to X$ be natural bundles over X, and let $\mathcal{F}$ and $\mathcal{F}'$ be their sheaves of smooth sections, respectively.

Definition 2.2. A differential operator $P \colon J^\infty F \longrightarrow F'$ is **natural** if it is a morphism of ringed spaces that commutes with the lifting of diffeomorphisms.

A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{F}'$ is **natural** if it is a regular morphism of sheaves that commutes with the action of diffeomorphisms on sections; that is to say, if for any diffeomorphism $\tau : U \to V$ between open sets of X, the following square commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi"] \arrow[d, "\tau_*"'] & \mathcal{F}'(U) \arrow[d, "\tau_*"] \\ \mathcal{F}(V) \arrow[r, "\phi"] & \mathcal{F}'(V) . \end{tikzcd}$$

where $\tau_* \colon \mathcal{F}(U) \to \mathcal{F}(V)$ is defined as $\tau_*(s) := \tau_* \circ s \circ \tau^{-1}$, for any $s \in \mathcal{F}(U)$.

Theorem 2.3. *The choice of a point $p \in X$ allows to define a bijection*

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \\ \phi\colon \mathcal{F} \longrightarrow \mathcal{F}' \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_p\textit{-equivariant smooth maps} \\ \\ J^\infty_p F \longrightarrow F'_p \end{array} \right\} ,$$

where Diff_p *stands for the group of germs of diffeomorphisms τ between open sets of X such that $\tau(p) = p$.*

Proof: In this context, where both F and F' are natural bundles, the bijection of Theorem 1.9 specializes to a bijection

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \\ \phi\colon \mathcal{F} \longrightarrow \mathcal{F}' \end{array} \right\} = \left\{ \begin{array}{c} \text{Natural differential operators} \\ \\ P\colon J^\infty F \longrightarrow F' \end{array} \right\}.$$

Then, a standard argument—using that the pseudogroup $\mathrm{Diff}\,(X)$ acts transitively on X—allows to prove that restriction to the fibre of the point p establishes a bijection:

$$\left\{ \begin{array}{c} \text{Natural differential operators} \\ \\ P\colon J^\infty F \longrightarrow F' \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_p\text{-equivariant smooth maps} \\ \\ J^\infty_p F \longrightarrow F'_p \end{array} \right\} .$$

To be precise, if $f_p\colon J^\infty_p F \to F'_p$ is a Diff_p-equivariant map, the corresponding differential operator $P\colon J^\infty F \to F'$ is defined, over the fibre of any other point $q \in X$, as the composition $\tau_*^{-1} \circ f_p \circ \tau_*$, where $\tau\colon U_q \to V_p$ is any diffeomorphism such that $\tau(q) = p$. The choice of a different τ' produces the same P, due to the Diff_p-equivariance of f_p, whereas the smoothness of P is a consequence of the smoothness assumptions on the liftings on F and F'. □

2.2. **Natural operations in presence of an orientation.** Let us now explain how to generalize Theorem 2.3 to the case of natural operations that depend on an orientation.

First of all, observe that the orientation bundle $\mathrm{Or}_X \to X$ is a natural bundle: the lifting of a diffeomorphism τ at a point p is the identity, in case that $\det \tau_{*,p}$ is positive, and the other map otherwise.

On the other hand, let us also observe that the direct product $F \times F'$ of natural bundles is also a natural bundle, with the obvious lifting of diffeomorphisms.

Theorem 2.4. *Let F and F' be natural bundles over X, and let $\mathcal{F}$ and $\mathcal{F}'$ be their sheaves of smooth sections, respectively.*

The choice of a point $p \in X$ and an orientation or_p at p produces a bijection:

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \\ \mathcal{F} \times Or_X \longrightarrow \mathcal{F}' \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{SDiff}_p\textit{-equivariant smooth maps} \\ \\ J^\infty_p F \to F'_p \end{array} \right\} ,$$

where Or_X denotes the sheaf of orientations on X, and SDiff_p *stands for the group of germs at p of diffeomorphisms τ such that $\tau(p) = p$ and* $\det \tau_{*,p} > 0$.

Proof: Due to Theorem 2.3, the choice of a point p allows to define a bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \\ \mathcal{F} \times Or_X \longrightarrow \mathcal{F}' \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_p\text{-equivariant smooth maps} \\ \\ J^\infty_p F \times J^\infty_p \mathrm{Or}_X \longrightarrow F'_p \end{array} \right\} .$$

As the action of the group Diff_p on the ringed space $J^\infty_p \mathrm{Or}_X$ is transitive, a general statement about ringed spaces —Proposition 2.5 below— permits to conclude. □

Proposition 2.5. *Let G be a group acting on three ringed spaces X, Y and Z.*

If the action on Y is transitive, then the choice of a point $\delta \in Y$ allows to define a bijection:

$$\left\{ \begin{array}{c} G\text{-equivariant smooth maps} \\ \\ f\colon X \times Y \longrightarrow Z \end{array} \right\} = \left\{ \begin{array}{c} I_\delta\text{-equivariant smooth maps} \\ \\ \bar{f}\colon X \longrightarrow Z \end{array} \right\} ,$$

where $I_\delta \subseteq G$ denotes the isotropy group of δ.

Proof: For any smooth map $f\colon X \times Y \longrightarrow Z$, the restriction to the subspace $X \times \{\delta\}$ defines a smooth I_δ-equivariant map $\bar{f}\colon X \times \{\delta\} = X \longrightarrow Z$.

Conversely, any smooth I_δ-equivariant map $\bar{f}\colon X \longrightarrow Z$, can be extended to a smooth G-equivariant map as follows:

$$f\colon X \times Y \longrightarrow Z \quad , \quad f(x,y) := g \cdot (\bar{f}(g^{-1} \cdot y)) ,$$

where $g \in G$ is any element such that $x = g \cdot \delta$.

Finally, it is not difficult to check that this extension is well-defined, as well as that both assignments are mutually inverse. □

3. Invariants of linear connections and an orientation

This section is devoted to prove Theorem 3.3, which is a description of the space of natural tensors associated to a linear connection and an orientation.

Let ∇ be the germ of a linear connection at a point $p \in X$, and let $\bar{\nabla}$ be the germ of the flat connection at $p \in X$ corresponding, via the exponential map, to the flat connection of T_pX.

Let $\mathbb{T} := \nabla - \bar{\nabla}$ be the $(2,1)$-tensor:

$$\mathbb{T}(\omega, D_1, D_2) := \omega(D_1^\nabla D_2 - D_1^{\bar{\nabla}} D_2) .$$

Definition 3.1. For any integer $m \geq 0$, the **m-th normal tensor of ∇ at p** is $\bar{\nabla}_p^m \mathbb{T}$.

In a system of normal coordinates $(x_1, \dots, x_n)$ for ∇ at p:

$$\bar{\nabla}_p^m \mathbb{T} = \sum_{i,j,k,a_1,\dots,a_m} \frac{\partial^m \Gamma_{ij}^k}{\partial x_{a_1} \dots \partial x_{a_m}}(p) \left(\frac{\partial}{\partial x_k} \right)_p \otimes \mathrm{d}_p x_i \otimes \mathrm{d}_p x_j \otimes \mathrm{d}_p x_{a_1} \otimes \dots \otimes \mathrm{d}_p x_{a_m} \ .$$

Definition 3.2. The **space N_m of normal tensors of order m at p** is the vector subspace of $(m+2,1)$-tensors T at p satisfying the following symmetries:

(1) they are symmetric in the last m covariant indices:

$$T^l_{ijk_1\dots k_m} = T^l_{ijk_{\sigma(1)}\dots k_{\sigma(m)}} \quad , \quad \forall\, \sigma \in S_m \, ; \tag{3.1}$$

(2) the symmetrization of the $m+2$ covariant indices is zero:

$$\sum_{\sigma \in S_{m+2}} T^l_{\sigma(i)\sigma(j)\sigma(k_1)\dots\sigma(k_m)} = 0. \tag{3.2}$$

Normal tensors $\bar{\nabla}_p^m \mathbb{T}$ lie in N_m ([8, Prop. 3.4]). Thus, it makes sense to consider the following maps, for any $m \geq 0$:

$$\begin{aligned} \phi_m \colon J_p^m \mathrm{Conn} \ &\longrightarrow \ N_0 \times \dots \times N_m \\ j_p^m \nabla \ &\longmapsto \ (\mathbb{T}_p, \dots, \bar{\nabla}_p^m \mathbb{T}) \ , \end{aligned}$$

where $\mathrm{Conn} \to X$ denotes the bundle of linear connections on X (not necessarily symmetric).

These maps ϕ_m are compatible, in the sense that the following diagrams commute:

$$\begin{tikzcd} J_p^{m+1}\mathrm{Conn} \arrow[r, "\phi_{m+1}"] \arrow[d] & N_0 \times \dots \times N_{m+1} \arrow[d] \\ J_p^m \mathrm{Conn} \arrow[r, "\phi_m"] & N_0 \times \dots \times N_m, \end{tikzcd}$$

and hence they define a morphism of ringed spaces between the corresponding inverse limits:

$$\begin{aligned} \phi_\infty \colon J_p^\infty \mathrm{Conn} \ &\longrightarrow \ \prod_{i=0}^{\infty} N_i \\ j_p^\infty \nabla \ &\longmapsto \ (\mathbb{T}_p, \bar{\nabla}_p^1 \mathbb{T}, \dots). \end{aligned}$$

For any $m \geq 1$, let us consider the Lie groups $\mathrm{Diff}_p^m := \{ j_p^m \tau : \tau \in \mathrm{Diff}_p \}$ as well as their subgroups $\mathrm{NDiff}_p^m := \{ j_p^m \tau \in \mathrm{Diff}_p^m : j_p^1 \tau = j_p^1 \mathrm{Id} \}$.

Their inverse limits define groups

$$\mathrm{Diff}_p^\infty := \lim_{\leftarrow} \mathrm{Diff}_p^m \qquad \text{and} \qquad \mathrm{NDiff}_p^\infty := \lim_{\leftarrow} \mathrm{NDiff}_p^m \ ,$$

that can be related via a short exact sequence of groups:

$$1 \longrightarrow \mathrm{NDiff}_p^\infty \longrightarrow \mathrm{Diff}_p^\infty \longrightarrow \mathrm{Gl} \longrightarrow 1\,, \tag{3.3}$$

where $\mathrm{Gl} := \mathrm{Diff}_p^1 = \{\mathrm{d}_p\tau \colon \tau \in \mathrm{Diff}_p\}$.

Reduction Theorem. *The* Diff_p^∞*−equivariant morphism of ringed spaces*

$$\begin{aligned} \phi_\infty \colon J_p^\infty Conn &\longrightarrow \prod_{i=0}^{\infty} N_i \\ j_p^\infty \nabla &\longmapsto (\mathbb{T}_p, \bar{\nabla}_p^1 \mathbb{T}, \dots), \end{aligned}$$

is surjective, its fibres are the orbits of NDiff_p^∞ *and it admits smooth sections passing through any point of* $J_p^\infty Conn$*.*

As a consequence, ϕ_∞ *induces a* Gl*-equivariant isomorphism of ringed spaces:*

$$(J_p^\infty Conn)/\mathrm{NDiff}_p^\infty = \prod_{i=0}^{\infty} N_i\,.$$

Proof: By [8, Theorem 3.6], the Diff_p^{m+2} -equivariant maps

$$\begin{aligned} \phi_m \colon J_p^m \mathrm{Conn} &\longrightarrow N_0 \times \ldots \times N_m \\ j_p^m \nabla &\longmapsto (\bar{\nabla}_p^0 \mathbb{T}, \dots, \bar{\nabla}_p^m \mathbb{T}) \end{aligned}$$

are surjective, regular projections whose fibres are the orbits of NDiff_p^{m+2} , for any $m \geq 0$.

Let us explain how these facts imply the statement above that deals with formal developments of connections. Firstly, as ϕ_m is Diff_p^{m+2}-equivariant and surjective for all m, it follows that ϕ_∞ is Diff_p^∞-equivariant and surjective.

Next, let us check that the fibres of ϕ_∞ are the orbits of NDiff_p^∞. On the one hand, if $j_p^\infty\nabla = \tau_\infty \cdot j_p^\infty \nabla'$ for some $\tau_\infty \in \mathrm{NDiff}_p^\infty$, the condition of ϕ_∞ being Diff_p^∞-equivariant implies

$$\phi_\infty(j_p^\infty\nabla) = \phi_\infty(\tau_\infty \cdot j_p^\infty\nabla') = \tau_\infty \cdot \phi_\infty(j_p^\infty\nabla') = \phi_\infty(j_p^\infty\nabla')\,.$$

Conversely, if $\phi_\infty(j_p^\infty\nabla) = \phi_\infty(j_p^\infty\nabla')$, then $\phi_m(j_p^m\nabla) = \phi_m(j_p^m\nabla')$ for all m. Therefore, there exists $\tau_m \in \mathrm{NDiff}_p^{m+2}$ such that $j_p^m\nabla' = \tau_m \cdot j_p^m\nabla$. The sequence $(\tau_m)_{m\in\mathbb{N}}$ defines an element $\tau_\infty \in \mathrm{NDiff}_p^\infty$ that verifies

$$j_p^\infty\nabla = \tau_\infty \cdot j_p^\infty\nabla'\,,$$

so that both formal developments are in the same orbit of NDiff_p^∞.

As for the existence of smooth sections, let us choose a local coordinate system centred at p. For any given formal development $j_p^\infty\nabla$, the proof of [8, Theorem 3.6] shows how these coordinates define a global section σ_m of ϕ_m that passes through $j_p^m\nabla$. These sections are easily checked to be compatible with the projections $J_p^{m+1}\mathrm{Conn} \to J_p^m\mathrm{Conn}$ and $\prod_{i=0}^{m+1} N_i \to \prod_{i=0}^{m} N_i$ for all m, so that they in turn define a morphism of ringed spaces that is a section of ϕ_∞ and passes through $j_p^\infty\nabla$.

Finally, the last assertion of the statement is a consequence of Corollary 1.5. □

Theorem 3.3. *Let X be a smooth manifold and let $\mathcal{C}$ and Or_X denote the sheaves of connections and orientations on X, respectively.*

Let F be a natural subbundle of the bundle of (r,s)-tensors T^s_r and let $\mathcal{F}$ be its sheaf of smooth sections.

If we fix a point $p \in X$ and an orientation or_p at p, there exists an $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \\ \mathcal{C} \times Or_X \longrightarrow \mathcal{T} \end{array} \right\} = \bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Sl}}(S^{d_0}N_0 \otimes \ldots \otimes S^{d_k}N_k, T_p) \ ,$$

where $d_0, \ldots, d_k$ run over the non-negative integer solutions of the equation

$$d_0 + \ldots + (k+1)d_k = r - s \ ,$$

and where $\mathrm{Gl} := \{\mathrm{d}_p\tau \colon \tau \in \mathrm{Diff}_p\}$ and $\mathrm{Sl} := \{\mathrm{d}_p\tau \colon \tau \in \mathrm{SDiff}_p\}$.

Proof: Theorem 2.4 yields the isomorphism:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \\ \mathcal{C} \times Or_X \longrightarrow \mathcal{T} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{SDiff}_p\text{-equivariant smooth maps} \\ \\ J^\infty_p\mathrm{Conn} \longrightarrow T_p \end{array} \right\} .$$

Observe that the action of SDiff_p over $J^\infty_p\mathrm{Conn}$ and F_p coincides with that of SDiff^∞_p, so that, in the formula above, we may consider SDiff^∞_p-equivariant maps instead.

Also, notice that the following sequence of groups is exact:

$$1 \longrightarrow \mathrm{NDiff}^\infty_p \longrightarrow \mathrm{SDiff}^\infty_p \longrightarrow \mathrm{Sl} \longrightarrow 1$$

The subgroup NDiff^∞_p acts by the identity over T_p, so that Corollary 1.6, in conjunction with the exact sequence above, assures the existence of an isomorphism:

$$\left\{ \begin{array}{c} \mathrm{SDiff}^\infty_p\text{-equivariant smooth maps} \\ \\ J^\infty_p\mathrm{Conn} \longrightarrow T_p \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Sl}\text{-equivariant smooth maps} \\ \\ J^\infty_p\mathrm{Conn}\,/\mathrm{NDiff}^\infty_p \longrightarrow T_p \end{array} \right\} .$$

Now, the Reduction Theorem above allows us to replace this quotient ringed space by an infinite product of vector spaces, via the isomorphism:

$$\left\{ \begin{array}{c} \mathrm{Sl}\text{-equivariant smooth maps} \\ \\ J^\infty_p\mathrm{Conn}\,/\mathrm{NDiff}^\infty_p \longrightarrow T_p \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Sl}\text{-equivariant smooth maps} \\ \\ t \colon \prod_{i=0}^{\infty} N_i \longrightarrow T_p \end{array} \right\} .$$

Finally, in the last step we make use of the equivariance by homotheties $h_\lambda \colon T_pX \to T_pX$ of ratio $\lambda > 0$. As $h_{\lambda^{-1}} \in \mathrm{Sl}$, the equivariance of these maps t implies

$$t(\ldots, \lambda^{m+1}\Gamma^m_p, \ldots) = t(h_{\lambda^{-1}}(\ldots, \Gamma^m_p, \ldots)) = h_{\lambda^{-1}} \cdot t(\ldots, \Gamma^m_p, \ldots) = \lambda^{r-s} t(\ldots, \Gamma^m_p, \ldots)$$

for all $\lambda > 0$, $(\ldots, \Gamma^m_p, \ldots) \in \prod_{i=0}^\infty N_i$.

In view of this property of the smooth maps t, the Homogeneous Function Theorem stated below (to be precise, formula (3.7)) allows to conclude the isomorphism:

$$\left\{ \begin{array}{c} \text{Sl-equivariant smooth maps} \\ t \colon \prod_{i=0}^{\infty} N_i \longrightarrow T_p \end{array} \right\} = \bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Sl}}(S^{d_0} N_0 \otimes \ldots \otimes S^{d_k} N_k, T_p) \ ,$$

where $d_0, \ldots, d_k$ are non-negative integers running over the solutions of the equation

$$d_0 + \ldots + (k+1) d_k = r - s \ .$$

□

Homogeneous Function Theorem. *Let $\{E_i\}_{i\in\mathbb{N}}$ be finite dimensional vector spaces.*

Let $f : \prod_{i=1}^\infty E_i \to \mathbb{R}$ be a smooth function such that there exist positive real numbers $a_i > 0$, and $w \in \mathbb{R}$ satisfying:

$$f(\lambda^{a_1} e_1, \ldots, \lambda^{a_i} e_i, \ldots) = \lambda^w f(e_1, \ldots, e_i, \ldots) \tag{3.4}$$

for any positive real number $\lambda > 0$ and any $(e_1, \ldots, e_i, \ldots) \in \prod_{i=1}^\infty E_i$.

Then, f depends on a finite number of variables $e_1, \ldots, e_k$ and it is a sum of monomials of degree d_i in e_i satisfying the relation

$$a_1 d_1 + \cdots + a_k d_k = w \ . \tag{3.5}$$

If there are no natural numbers $d_1, \ldots, d_r \in \mathbb{N} \cup \{0\}$ satisfying this equation, then f is the zero map.

Proof: Firstly, if f is not the zero map, then observe $w \geq 0$ because, otherwise, (3.4) is contradictory when $\lambda \to 0$.

As f is smooth, there exists a neighbourhood $U = \{|e_1| < \epsilon_1, \ldots, |e_k| < \epsilon_k\} \subset \prod_{i=1}^\infty E_i$ of the origin and a smooth map $\overline{f} : \pi_k(U) \to \mathbb{R}$ such that $f_{|U} = (\overline{f} \circ \pi_k)_{|U}$.

As the $a_1, \ldots, a_k$ are positive, there exist a neighbourhood of zero, $\overline{V}_0 \subset \mathbb{R}$, and a neighbourhood of the origin $V \subset \pi_k(U)$, such that for any $(e_1, \ldots, e_k) \in V$, and any $\lambda \in \overline{V}_0$ positive, the vector $(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k)$ lies in V.

On that neighbourhood V, the function $\overline{f}$ satisfies the homogeneity condition:

$$\overline{f}(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) = \lambda^w \overline{f}(e_1, \ldots, e_k) \tag{3.6}$$

for any positive real number $\lambda \in \overline{V}_0$.

Differentiating this equation, we obtain analogous conditions for the partial derivatives of $\overline{f}$; v.gr.:

$$\frac{\partial \overline{f}}{\partial x_1}(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) = \lambda^{w-a_1} \frac{\partial \overline{f}}{\partial x_1}(e_1, \ldots, e_k) \ .$$

If the order of derivation is big enough, the corresponding partial derivative is homogeneous of negative weight, and hence zero. This implies that $\overline{f}$ is a polynomial; the homogeneity condition (3.6) is then satisfied for any positive $\lambda \in \overline{V}_0$ if and only if its monomials satisfy (3.5).

Finally, given any $e = (e_1, \ldots, e_n, \ldots) \in \prod_{i=1}^{\infty} E_i$, take $\lambda \in \mathbb{R}^+$ such that the vector $(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k, \ldots)$ lies in U. Then:

$$f(e) = \lambda^{-w} f(\lambda^{a_1} e_1, \ldots, \lambda^{a_n} e_n, \ldots) = \lambda^{-w} \overline{f}(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) = \overline{f}(e_1, \ldots, e_k)$$

and f only depends on the first k variables.

□

This statement readily generalizes to say that, for any finite dimensional vector space W, there exists an $\mathbb{R}$-linear isomorphism:

$$\left[\text{Smooth maps } f \colon \prod_{i=1}^{\infty} E_i \to W \text{ satisfying (3.4)} \right] \tag{3.7}$$
$$\Big\|$$
$$\bigoplus_{d_1, \ldots, d_k} \mathrm{Hom}_{\mathbb{R}}(S^{d_1} E_1 \otimes \ldots \otimes S^{d_k} E_k, \, W)$$

where $d_1, \ldots, d_k$ run over the non-negative integer solutions of (3.5).

4. An application

Finally, as an application of Theorem 3.3, we compute in this section some spaces of vector-valued and endomorphism-valued natural forms associated to linear connections and orientations, thus obtaining characterizations of the torsion and curvature operators (Corollary 4.7 and Theorem 4.11).

4.1. **Invariant theory of the special linear group.** Let V be an oriented $\mathbb{R}$-vector space of finite dimension n, and let $\mathrm{Sl}(V)$ be the real Lie group of its orientation-preserving $\mathbb{R}$-linear automorphisms.

Our aim is to describe the vector space of $\mathrm{Sl}(V)$-invariant linear maps

$$V^* \otimes \overset{p}{\ldots} \otimes V^* \otimes V \otimes \overset{p}{\ldots} \otimes V \longrightarrow \mathbb{R} \ .$$

For any permutation $\sigma \in S_p$, there exists the so called total contraction maps, which are defined as follows

$$C_\sigma(\omega_1 \otimes \ldots \otimes \omega_p \otimes e_1 \otimes \ldots \otimes e_p) := \ \omega_1(e_{\sigma(1)}) \ldots \omega_p(e_{\sigma(p)}) \ .$$

Moreover, let $\Omega \in \Lambda^n V^*$ be a representative of the orientation, and let $\mathfrak{e}$ be the dual n-vector; that is to say, the only element in $\Lambda^n V$ such that $\Omega(\mathfrak{e}) = 1$. For any permutation $\sigma \in S_{p+kn}$, the following linear maps are also $\mathrm{Sl}(V)$-invariant:

$$(\omega_1, \ldots, \omega_p, e_1, \ldots, e_p) \longmapsto C_\sigma(\Omega \otimes \overset{k}{\ldots} \otimes \Omega \otimes \omega_1 \otimes \ldots \otimes \omega_p \otimes \mathfrak{e} \otimes \overset{k}{\ldots} \otimes \mathfrak{e} \otimes e_1 \otimes \ldots \otimes e_p) \ .$$

Classical invariant theory proves that these maps suffice to generate the vector space under consideration:

Theorem 4.1. *The real vector space* $\mathrm{Hom}_{\mathrm{Sl}(V)}(V^* \otimes \overset{p}{\ldots} \otimes V^* \otimes V \otimes \overset{p}{\ldots} \otimes V\ , \mathbb{R})$ *of invariant linear forms on* $V^* \otimes \ldots \otimes V$ *is spanned by*

$$(\omega_1, \ldots, \omega_p, e_1, \ldots, e_p) \longmapsto C_\sigma(\Omega \otimes \overset{k}{\ldots} \otimes \Omega \otimes \omega_1 \otimes \ldots \otimes \omega_p \otimes \mathfrak{e} \otimes \overset{k}{\ldots} \otimes \mathfrak{e} \otimes e_1 \otimes \ldots \otimes e_p) \ ,$$

where k is a non-negative integer such that $0 \leq k \leq p/n$.

In particular, for $p < n$, the vector space of $\mathrm{Sl}(V)$*-invariant linear maps coincides with the vector space of* $\mathrm{Gl}(V)$*-invariant linear maps.*

In the applications, we will also require the following facts:

Proposition 4.2. *Let E and F be (algebraic) linear representations of* $\mathrm{Sl}(V)$*.*

- *There exists a linear isomorphism* $\mathrm{Hom}_{\mathrm{Sl}(V)}(E, F) = \mathrm{Hom}_{\mathrm{Sl}(V)}(E \otimes F^*, \mathbb{R})$.
- *If $W \subseteq E$ is a sub-representation, then any equivariant linear map $W \to F$ is the restriction of an equivariant linear map $E \to F$.*

4.2. **Uniqueness of the torsion and curvature operators.**

Definition 4.3. Let $E \to X$ be a natural vector bundle. An **E-valued natural k-form** (associated to linear connections and orientations) is a regular and natural morphism of sheaves

$$\omega \,:\, \mathcal{C} \times Or_X \longrightarrow \Omega^k \otimes \mathcal{E} \ ,$$

where Ω^k denotes the sheaf of differential k-forms on X and $\mathcal{E}$ stands for the sheaf of smooth sections of E.

Theorem 3.3 implies, in particular, that the space of E-valued natural forms associated to linear connections and orientations is a finite-dimensional real vector space. Moreover, as the exterior differential commutes with diffeomorphisms, it induces $\mathbb{R}$-linear maps

$$\begin{bmatrix} E\text{-valued natural} \\ k\text{-forms} \end{bmatrix} \xrightarrow{\ \mathrm{d}\ } \begin{bmatrix} E\text{-valued natural} \\ (k+1)\text{-forms} \end{bmatrix},$$

where it should be understood that, if ω is an E-valued natural k-form, the differential $\mathrm{d}\omega \colon \mathcal{C} \times Or_X \to \Omega^{k+1} \otimes \mathcal{E}$ is defined, on each section (∇, or), with respect to the linear connection on E induced by ∇.

Definition 4.4. A **closed E-valued natural k-form** (associated to linear connections and orientations) is an element in the kernel of the map above.

Vector-valued natural forms. The torsion tensor of a linear connection can be understood as a vector-valued natural 2-form; that is to say, as a regular and natural morphism of sheaves

$$\mathrm{Tor}\colon \mathcal{C} \times Or_X \longrightarrow \Omega^2 \otimes \mathcal{D} \ ,$$

where $\mathcal{D}$ stands for the sheaf of vector fields on X.

To be precise, the value of that tensor on a linear connection ∇ and an orientation *or* on an open set $U \subseteq X$ is

$$\mathrm{Tor}_\nabla(D_1, D_2) := \nabla_{D_1} D_2 - \nabla_{D_2} D_1 - [D_1, D_2] \ ,$$

so that, in particular, it is independent of the orientation.

On the other hand, if $I\colon \mathcal{D} \to \mathcal{D}$ denotes the identity map, and c_1^1 stands for the trace of the first covariant and contravariant indices, the tensor $H := c_1^1(\mathrm{Tor}) \wedge I$ defines another vector-valued natural 2-form:

$$H\colon \mathcal{C} \times Or_X \longrightarrow \Omega^2 \otimes \mathcal{D} \ .$$

Lemma 4.5. *If* $\dim X \geq 3$*, then Tor and* H *are a basis of the* $\mathbb{R}$*-vector space of vector-valued natural 2-forms.*

Proof: Looking at Theorem 3.3, we first compute the non-negative integers solutions of

$$d_0 + 2d_1 + \ldots + (k+1)d_k = 2 - 1 = 1 \ .$$

There is only one solution, namely $d_0 = 1$, $d_i = 0$, for $i > 0$, so Theorem 3.3 assures that the vector space under consideration is isomorphic to the space of Sl-equivariant linear maps:

$$N_0 = \Lambda^2 T_p^* X \otimes T_p X \ \longrightarrow \ \Lambda^2 T_p^* X \otimes T_p X \ .$$

Thus, the problem is reduced to a question of invariants for the special linear group, and we can invoke Theorem 4.1 and Proposition 4.2 to obtain generators for this vector space.

According to those results, if $\dim X > 3$ then the space of Sl-equivariant linear maps that we are considering coincides with the space of Gl-equivariant linear maps, which in turn is proved in [10, Lemma 3.5] to be spanned by H and Tor.

If $\dim X = 3$, there may exist another generator; namely the map $\varphi\colon \Lambda^2 T_p^* X \otimes T_p X \longrightarrow \Lambda^2 T_p^* X \otimes T_p X$ that in coordinates around p reads:

$$(\mathrm{d}x_i \wedge \mathrm{d}x_j) \otimes \partial_{x_k} \quad \longmapsto \quad \Omega(\partial_{x_k}, \_\_, \_\_) \cdot \mathfrak{e}(\mathrm{d}x_i, \mathrm{d}x_j, \_\_) \ ,$$

where $\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$ and $\mathfrak{e}$ is its dual 3-vector.

If Γ_{ij}^k denote the Christoffel symbols, then a trivial computation allows to express

$$\begin{aligned}\varphi \;=\;& \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \Big(\Gamma_{23}^3 \cdot \partial_{x_1} + \Gamma_{31}^3 \cdot \partial_{x_2} + \Gamma_{12}^3 \cdot \partial_{x_3}\Big) \\ +\;& \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \Big(\Gamma_{23}^1 \cdot \partial_{x_1} + \Gamma_{31}^1 \cdot \partial_{x_2} + \Gamma_{12}^1 \cdot \partial_{x_3}\Big) \\ +\;& \mathrm{d}x_3 \wedge \mathrm{d}x_1 \otimes \Big(\Gamma_{23}^2 \cdot \partial_{x_1} + \Gamma_{31}^2 \cdot \partial_{x_2} + \Gamma_{12}^2 \cdot \partial_{x_3}\Big) \ ,\end{aligned}$$

as well as the linear relation $\varphi = \mathrm{Tor} + H$. □

Theorem 4.6. *If* $\dim X \geq 3$*, then the exterior differential is an injective* $\mathbb{R}$*-linear map*

$$\begin{bmatrix} \textit{Vector-valued natural} \\ \textit{2-forms} \end{bmatrix} \xrightarrow{\ \mathrm{d}\ } \begin{bmatrix} \textit{Vector-valued natural} \\ \textit{3-forms} \end{bmatrix} .$$

Proof: It is a consequence of both Lemma 4.5 and the fact that $\mathrm{d}H$ and $\mathrm{d}\mathrm{Tor}$ are $\mathbb{R}$-linearly independent ([10, Theorem 3.6]). □

The so-called *first Bianchi identity* for the torsion tensor describes its differential in terms of the curvature, R, and the identity map, I: it is the following equality of vector-valued natural 3-forms

$$\mathrm{d}\,\mathrm{Tor} \;=\; R \wedge I \ .$$

Therefore, an immediate corollary of Theorem 4.6 is:

Corollary 4.7. *The torsion tensor is characterized as the only vector-valued natural 2-form* ω *that satisfies the first Bianchi identity; i. e., such that* $\mathrm{d}\,\omega = R \wedge I$.

Endomorphism-valued natural forms. In this section, we restrict our attention to symmetric linear connections.

As in the case of the torsion tensor, the curvature tensor can also be thought of as an endomorphism-valued natural 2-form; that is to say, as a (regular and natural) morphism of sheaves

$$R \colon \mathcal{C}^s \times Or_X \longrightarrow \Omega^2 \otimes \mathcal{E}\mathrm{nd}(\mathcal{D}) \ ,$$

whose value on a symmetric linear connection ∇ and an orientation *or* defined on an open set $U \subset X$ is the following endomorphism-valued 2-form R_∇ on U:

$$R_\nabla(D_1, D_2)D_3 := \nabla_{D_1}\nabla_{D_2}D_3 - \nabla_{D_2}\nabla_{D_1}D_3 - \nabla_{[D_1,D_2]}D_3 \ .$$

Definition 4.8. An endomorphism-valued natural 2-form ω satisfies the **first Bianchi identity** if, for any symmetric linear connection ∇, any orientation *or* and any vector fields D_1, D_2, D_3:

$$\omega_{(\nabla,or)}(D_1, D_2)D_3 + \omega_{(\nabla,or)}(D_2, D_3)D_1 + \omega_{(\nabla,or)}(D_3, D_1)D_2 \;=\; 0 \ .$$

The curvature tensor satisfies the first Bianchi identity. Moreover, if Ric^s and Ric^h denote the symmetric and skew-symmetric parts of the Ricci tensor Ric, then the following tensors also satisfy the first Bianchi identity:

$$\begin{aligned} C_1(D_1, D_2, D_3, \omega) &:= Ric^s(D_1, D_3)\omega(D_2) - Ric^s(D_2, D_3)\omega(D_1) \\ C_2(D_1, D_2, D_3, \omega) &:= Ric^h(D_1, D_3)\omega(D_2) - Ric^h(D_2, D_3)\omega(D_1) + 2Ric^h(D_1, D_2)\omega(D_3) \ . \end{aligned}$$

Lemma 4.9. *If* $\dim X > 3$*, then the tensors* C_1, C_2 *and* R *are a basis of the* $\mathbb{R}$*-vector space of endomorphism-valued natural 2-forms that satisfy the first Bianchi identity.*

If $\dim X = 3$*, then that vector space has dimension four.*

Proof: Let $\mathcal{R}$ be the vector space of endomorphism-valued 2-forms at a point that satisfy the first Bianchi identity. Theorem 3.3 describes the space of natural 2-forms under consideration as the vector space

$$\bigoplus_{d_i} \mathrm{Hom}_{\mathrm{Sl}(T_pX)}(S^{d_1}N_1^s \otimes \ldots \otimes S^{d_k}N_k^s, \mathcal{R}) \ ,$$

where $d_1, \ldots, d_k$ are non-negative integers verifying the equation

$$2d_1 + \ldots + (k+1)d_k = 3 - 1 = 2 \ .$$

The only solution to this equation is $d_1 = 1$, $d_2 = \ldots = d_k$, so that the vector space to analyse is the space of Sl-equivariant linear maps

$$N_1^s \longrightarrow \mathcal{R} \ . \tag{4.1}$$

First of all, recall that the maps induced by the tensors R, C_1 and C_2 are a basis of the space of Gl-equivariant linear maps $N_1^s \longrightarrow \mathcal{R}$, see [10, Lemma 3.11].

A systematic application of Theorem 4.1 now allows to find generators for the space of Sl-equivariant maps.

If $\dim X > 5$, then the vector space of Sl-equivariant maps coincides with the space of Gl-equivariant maps, and hence is generated by these three elements.

In case $\dim X = 4$, there is another possible generator: the map $N_1^s \to \mathcal{R}$ defined as

$$\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l} \longmapsto \Omega(\partial_{x_l}, \_\_, \_\_, \_\_) \cdot \mathfrak{e}(\mathrm{d}x_i, \mathrm{d}x_j, \mathrm{d}x_k, \_\_) \ .$$

However, as any tensor in N_1^s is symmetric in the first two indices, it readily follows that this map is identically zero.

If $\dim X = 3$, let us first describe the Sl-equivariant endomorphisms $T_3^1 \to T_3^1$.

To this end, let x_1, x_2 and x_3 be coordinates centred at p such that $\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$ is positively oriented, and let $\mathfrak{e}$ be its dual 3-vector.

Using Ω and $\mathfrak{e}$ we can construct 16 generators, and they can all be expressed as a permutation of the factors of T_3^1 followed by one of these 4 maps:

(a) $\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l} \longmapsto \mathfrak{e}(\mathrm{d}x_i, \mathrm{d}x_j, \mathrm{d}x_k) \cdot \Omega \otimes \partial_{x_l}$,

(b) $\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l} \longmapsto \mathfrak{e}(\mathrm{d}x_i, \mathrm{d}x_j, \mathrm{d}x_k) \cdot \Omega(\partial_{x_l}, \_\_, \_\_) \otimes I$,

(c) $\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l} \longmapsto \delta^l_{\sigma(i)} \cdot \Omega \otimes \mathfrak{e}(\mathrm{d}x_{\sigma(j)}, \mathrm{d}x_{\sigma(k)}, \_\_)$, $\quad \sigma \in S_3$,
(d) $\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l} \longmapsto \Omega(\partial_{x_l}, \_\_, \_\_) \otimes \mathrm{d}x_{\sigma(i)} \otimes \mathfrak{e}(\mathrm{d}x_{\sigma(j)}, \mathrm{d}x_{\sigma(k)}, \_\_)$, $\quad \sigma \in S_3$.

As the first two covariant indices of N^s_1 are symmetric, the following maps are identically zero: (a), (b) and raising the first two indices at (c) and (d).

That leaves 8 non-zero generators. But this symmetry also makes raising indices 1, 3 and 2, 3 indistinguishable, hence reducing to just 4 generators.

The last step is to check which of these maps take values in $\mathcal{R}$. Out of these four generators, only the following two produce tensors that are skew-symmetric in the first two covariant indices:

$$\varphi_1 \left(\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l}\right) = \Omega(\partial_{x_l}, \_\_, \_\_) \otimes \mathrm{d}x_j \otimes \mathfrak{e}(\mathrm{d}x_i, \mathrm{d}x_k, \_\_) \tag{4.2}$$

$$\varphi_2 \left(\mathrm{d}x_i \otimes \mathrm{d}x_j \otimes \mathrm{d}x_k \otimes \partial_{x_l}\right) = \delta^l_j \cdot \Omega \otimes \mathfrak{e}(\mathrm{d}x_i, \mathrm{d}x_k, \_\_) \ , \tag{4.3}$$

and the skew-symmetrization of the remaining two is a linear combination of these.

None of these two tensors satisfy the first Bianchi identity, but the linear combination $\varphi := 3\varphi_1 - \varphi_2$ does.

Finally, all that is left to prove is that φ is $\mathbb{R}$-linearly independent of R, C_1 and C_2. In order to do that, it is enough to find a symmetric linear connection and an orientation on a 3-manifold X such that the aforementioned tensors on X are $\mathbb{R}$-linearly independent.

The following example works: let ∇ be the linear connection on $\mathbb{R}^3$ whose only non-zero Christoffel symbols in cartesian coordinates are

$$\Gamma^1_{11} = x_2 x_3 \quad , \quad \Gamma^2_{23} = \Gamma^2_{32} = x_1 x_2 \ .$$

Assume $\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$ is positively oriented, and denote $T_{ij} := \mathrm{d}x_i \otimes \partial_{x_j}$.

Direct computation gives the following linearly independent tensors, thus finishing the proof:

$$\begin{aligned} R = {} & \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes (-x_3 T_{11} + x_2 T_{32}) + \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes (-x_2 T_{11} + x_2 T_{22}) \\ & + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(x_1 T_{22} - x_1^2 x_2^2 T_{32}\right) , \end{aligned}$$

$$\begin{aligned} C_1 = {} & \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes \left(\frac{1}{2} x_3 T_{11} - \frac{1}{2} x_3 T_{22} - \frac{1}{2} x_1 T_{31} - x_2 T_{32}\right) + \\ & + \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes \left(x_2 T_{11} - \frac{1}{2} x_1 T_{21} - \frac{1}{2} x_3 T_{23} + x_1^2 x_2^2 T_{31} - x_2 T_{33}\right) + \\ & + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes \left(x_2 T_{12} - \frac{1}{2} x_3 T_{13} - \frac{1}{2} x_1 T_{22} + x_1^2 x_2^2 T_{32} + \frac{1}{2} x_1 T_{33}\right) , \end{aligned}$$

$$
\begin{aligned}
C_2 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes & \left(\frac{3}{2}x_3T_{11} + \frac{3}{2}x_3T_{22} + \frac{1}{2}x_1T_{31} + x_3T_{33}\right) + \\
+ \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes & \left(-\frac{1}{2}x_1T_{21}\frac{1}{2}x_3T_{23}\right) + \\
+ \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes & \left(-x_1T_{11} - \frac{1}{2}x_3T_{13} - \frac{3}{2}x_1T_{22} - \frac{3}{2}x_1T_{33}\right),
\end{aligned}
$$

$$
\begin{aligned}
\varphi = & \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes (x_1T_{31} - x_3T_{33}) + \\
& + \mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes (2x_1T_{21} - 3x_2T_{22} + x_3T_{23} + 3x_2T_{33}) + \\
& + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes (x_1T_{11} - 3x_2T_{12} + 2x_3T_{13}).
\end{aligned}
$$

□

Definition 4.10. An endomorphism-valued natural 2-form ω is said to satisfy the **second Bianchi identity** if it is closed, in the sense of Definition 4.4.

Theorem 4.11. *The constant multiples of the curvature are the only endomorphism-valued natural 2-forms that satisfy both the first and second Bianchi identities.*

Proof: The curvature tensor R is always a closed natural 2-form, so, by the previous Lemma, it is enough to analyse the $\mathbb{R}$-linear span of the differentials of C_1, C_2 and, in dimension 3, also of φ.

If $\dim X > 3$, then $\mathrm{d}C_1$ and $\mathrm{d}C_2$ are linearly independent, by [10, Thm. 3.13], and the statement follows.

If $\dim X = 3$, a direct computation, using the same example as in the previous Lemma, proves that $\mathrm{d}C_1$, $\mathrm{d}C_2$ and $\mathrm{d}\varphi$ are $\mathbb{R}$-linearly independent tensors:

$$
\begin{aligned}
\mathrm{d}_\nabla C_1 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes & \left(-\frac{1}{2}T_{11} - x_2^2x_3T_{12} + \frac{1}{2}x_2x_3^2T_{13} + \frac{1}{2}(x_1x_2x_3 - 2)T_{22} + \right. \\
& \left. -\frac{5}{2}x_1^2x_2T_{31} + 2x_1x_2^2T_{32} - \frac{1}{2}(x_1x_2x_3 - 3)T_{33}\right), \\
\mathrm{d}_\nabla C_2 = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes & \left(\frac{1}{2}T_{11} + \frac{1}{2}x_2x_3^2T_{13} - \frac{1}{2}x_1x_2x_3T_{22} + \right. \\
& \left. -\frac{1}{2}x_1^2x_2T_{31} + \frac{1}{2}(x_1x_2x_3 - 1)T_{33}\right), \\
\mathrm{d}_\nabla \varphi = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \otimes & \Big(T_{11} + 3x_2^2x_3T_{12} - 2x_2x_3^2T_{13} - (x_1x_2x_3 - 3)T_{22} + \\
& + 2x_1^2x_2T_{31} - 6x_1x_2^2T_{32} + (x_1x_2x_3 - 4)T_{33}\Big).
\end{aligned}
$$

□

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Appendix F

On invariant operations of Fedosov structures

ON INVARIANT OPERATIONS OF FEDOSOV STRUCTURES

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Abstract. In this paper we study invariant local operations that can performed on a Fedosov manifold, with a particular emphasis on tensor-valued operations (also known as natural tensors). Our main result describes the spaces of homogeneous natural tensors as certain finite dimensional linear representations of the symplectic group.

1. Introduction

The notion of invariant operation has been key to the development of differential geometry and many of its applications. A paradigmatic example is its relevance in the early days of the nascent theory of General Relativity ([23]). As time went by, the theory of these invariant operations evolved and produced significant mathematical results, such as the characterisation of the Pontryagin forms on Riemannian manifolds ([11, 2]) or the proof of the uniqueness of the Chern–Gauss–Bonnet formula ([12]), both found by P. Gilkey during the mid-70s.

In 1993, Kolář-Michor-Slovák ([21]) published the monograph which has become the standard reference in this subject since then. It summarises and enhances the main results and techniques that were known up to that point. However, this book is written with a functorial language that, outside specialists on the field, has certainly not become standard; this has probably motivated that, in recent years, there have appeared various references that rewrite some of its most prominent results ([9, 20, 27]).

Among the invariant operations that can be performed on a manifold, tensor-valued operations are particularly relevant. Also known as *natural tensors*, their description in the easiest possible terms has always been a relevant question. In presence of a linear connection, the main result of the theory describes these spaces of natural tensors as certain finite-dimensional linear representations of a classical Lie group: the linear groups Gl_n or Sl_n when considering natural tensors associated to linear connections ([17], [18], [28]), the orthogonal groups $\mathrm{O}(s_+, s_-)$ or $\mathrm{SO}(s_+, s_-)$ when considering natural tensors associated to

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pseudo-Riemannian metrics ([25], [31]), or the unitary groups U_n or SU_n for the corresponding case of Kähler metrics ([14], [31]). This description permits classical invariant theory to come into play and, in certain cases, to achieve this way an exhaustive computation of the spaces of natural tensors under consideration (see, for example, [2], [13], [17] or [25]).

Nevertheless, in this picture above, the symplectic group was missing; in other words, there was no theorem describing natural tensors associated to the so called Fedosov structures. Fedosov manifolds constitute the skew-symmetric version of Riemannian manifolds: they are defined as a triple (X, ω, ∇), where X is a smooth manifold of even dimension, ω is a symplectic form and ∇ is a symplectic connection, that is, a symmetric linear connection such that $\nabla\omega = 0$. They are named after B. Fedosov, who first constructed a canonical deformation quantization on these manifolds ([6], [7]).

The remedy to this situation started in 1998, when Gelfand-Retakh-Shubin ([10]) proved that any finite order, natural tensor associated to a Fedosov structure is indeed a function of the curvature and its successive derivatives. This nice result, however, still had strong limitations: it did not allow the use of the invariant theory of the symplectic group yet, and it imposed a strong finiteness hypothesis on the order of the local invariants.

In this paper, we overcome this inconvenience and prove a statement (Theorem 2.6) that describes natural tensors associated to Fedosov structures in terms of certain finite-dimensional linear representations of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. Our theorem is completely analogous to the aforementioned results for linear connections or Riemannian metrics; in particular, it imposes no restrictions on the order of the natural tensors and it allows the use of classical invariant theory. We plan to exploit these features in the future, as it is plausible that they will allow the computation of interesting dimensional curvature identities, analogous to those in [13] or [14], as well as another approach to moduli spaces of jets of Fedosov structures, different to that used in [4].

Finally, let us mention that the use of the language of sheaves and ringed spaces, much in the spirit of our previous works [17] and [18], plays in this paper an essential role, especially to get rid of the finite order conditions of other developments.

2. Statement of the Main Theorem

Let X be a smooth manifold of dimension n. Let $\mathrm{Diff}\,(X)$ denote the set of local diffeomorphisms[1] between open subsets of X.

[1]Throughout this text, the term *diffeomorphism* will refer to a local diffeomorphism between two open subsets of a smooth manifold, unless explicitly otherwise stated.

Definition 2.1. Let $\pi\colon F \to X$ be a (fibre) bundle over X. A natural bundle over X is a bundle $F \to X$ together with a map

$$\begin{aligned} \mathrm{Diff}\,(X) &\longrightarrow \mathrm{Diff}\,(F) \\ \tau &\longmapsto \tau_* \qquad , \end{aligned}$$

called lifting of diffeomorphisms, satisfying the following properties [2]:

- If $\tau\colon U \to V$ is a diffeomorphism between open subsets of X, then $\tau_*\colon F_U \to F_V$ is a diffeomorphism covering τ, i.e. it makes the following square commutative:

$$\begin{tikzcd} F_U \arrow[r, "\tau_*", "\sim"'] \arrow[d, "\pi"'] & F_V \arrow[d, "\pi"] \\ U \arrow[r, "\tau", "\sim"'] & V \end{tikzcd} ,$$

  where $F_U := \pi^{-1}(U)$ and $F_V := \pi^{-1}(V)$.
- Functoriality: $\mathrm{Id}_* = \mathrm{Id}$ and $(\tau \circ \tau')_* = (\tau)_* \circ (\tau')_*$.
- Locality: for any diffeomorphism $\tau : U \to V$ and any open subset $U' \subset U$, $(\tau_{|U'})_* = (\tau_*)_{|F_{U'}}$.

Definition 2.2. A natural sheaf $\mathcal{F}$ over X is a subsheaf of the sheaf of smooth sections of a natural bundle $F \to X$ over X such that, for any diffeomorphism $\tau : U \to V$, the morphism

$$\begin{aligned} \tau_*\colon \mathcal{F}(U) &\longrightarrow \mathcal{F}(V) \\ s &\longmapsto \tau_* \circ s \circ \tau^{-1} \end{aligned}$$

is well defined[3].

Examples:

(1) Let $F \to X$ be a natural bundle. It is easy to prove that the sheaf of smooth sections of F is a natural sheaf, using that the lifting covers the lifted diffeomorphism. As such, the sheaf $\mathcal{T}_p^q$ of (p, q)-tensors over X is a natural sheaf.

(2) The Fedosov sheaf, defined on any open subset $U \subseteq X$ as

$$\mathcal{F}(U) := \{(\omega, \nabla) \in (\Lambda^2 \times \mathcal{C}^{\mathrm{sym}})(U) : \nabla\omega = 0\} ,$$

is a natural sheaf, where Λ^2 denotes the sheaf of non-singular 2-forms on X and $\mathcal{C}^{\mathrm{sym}}$ denotes the sheaf of symmetric linear connections on X. Observe that the condition $\nabla\omega = 0$ is natural: if $(\omega, \nabla) \in \mathcal{F}(U)$, then $(\tau_*\nabla)(\tau_*\omega) = 0$.

[2]In the literature, a condition of regularity is added to the definition of natural bundle (the lifting of any smooth family of diffeomorphisms is smooth too). However, this property can be derived from the other two (see [5]).

[3]Observe that we are committing an abuse of notation: we are denoting by τ_* both the lifting of τ to F and the 'action' of τ on $\mathcal{F}$. However, the context will help clarify which morphism we are working with.

Definition 2.3. Let $\mathcal{F}$ and $\mathcal{F}'$ be natural sheaves over X. A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{F}'$ is natural if it is regular[4] and commutes with the action of diffeomorphisms on sections; that is to say, if for any diffeomorphism $\tau : U \to V$, the following square commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi"] \arrow[d, "\tau_*"'] & \mathcal{F}'(U) \arrow[d, "\tau_*"] \\ \mathcal{F}(V) \arrow[r, "\phi"] & \mathcal{F}'(V)\,, \end{tikzcd} \tag{2.1}$$

where $\tau_* \colon \mathcal{F}(U) \to \mathcal{F}(V)$ is defined as follows:

$$\begin{aligned} \tau_* \colon \mathcal{F}(U) &\longrightarrow \ \mathcal{F}(V) \\ s &\longmapsto \ \tau_* \circ s \circ \tau^{-1} \ . \end{aligned}$$

Definition 2.4. A natural morphism of sheaves $\mathcal{F} \to \mathcal{T}$ between the Fedosov sheaf $\mathcal{F}$ and a sheaf of tensors $\mathcal{T}$ over X is called a natural tensor (associated to Fedosov structures).

A condition of homogeneity is required to guarantee that the natural tensors depend on a finite amount of variables only:

Definition 2.5. Let $\delta \in \mathbb{R}$. We say that a natural tensor $T : \mathcal{F} \to \mathcal{T}$ is homogeneous of weight δ if, for all non-zero $\lambda \in \mathbb{R}$, it holds that[5]:

$$T(\lambda^2\omega, \nabla) = \lambda^\delta T(\omega, \nabla) \ .$$

Examples:

- The symplectic form can be understood as a natural $(2,0)$-tensor associated to Fedosov structures whose value on a Fedosov structure (ω, ∇) is ω. It is homogeneous of weight 2.
- The $(4,0)$ curvature operator, defined as a natural $(4,0)$-tensor whose value on a Fedosov structure (ω, ∇) defined on an open set $U \subset X$ is:

  $$R_{(\omega,\nabla)}(D_1, D_2, D_3, D_4) := \omega(D_1, \nabla_{D_3}\nabla_{D_4}D_2 - \nabla_{D_4}\nabla_{D_3}D_2 - \nabla_{[D_3,D_4]}D_2) \ ,$$

  which is an homogeneous tensor of weight 2.

The following result, whose proof will be detailed during Section 6, describes all natural tensors associated to Fedosov structures:

Theorem 2.6. *Let X be a smooth manifold of dimension $2n$, and let $\mathcal{F}$ denote the sheaf of Fedosov structures. Let $\mathcal{T}$ be the sheaf of p-covariant tensors over X. Let $\delta \in \mathbb{Z}$.*

Fixing a point $x_0 \in X$ and a chart $U \simeq \mathbb{R}^{2n}$ around x_0 produces a $\mathbb{R}$-linear isomorphism

[4]The regularity condition is technical in nature, and as such it will be properly defined in Section 3, Definition 3.3.

[5]Observe that if (ω, ∇) is a Fedosov structure, then $(\lambda\omega, \nabla)$ is also a Fedosov structure for any $\lambda \in \mathbb{R}\backslash\{0\}$.

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \\ \textit{homogeneous of weight } \delta \end{array} \right\} = \bigoplus_{d_1,\ldots,d_r} \mathrm{Hom}_{\mathrm{Sp}}(S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r, T_{x_0}) \ ,$$

where $\mathrm{Sp} = \mathrm{Sp}(2n,\mathbb{R})$ *denotes the symplectic group,* T_{x_0} *denotes the vector space of* p*-covariant tensors at* x_0 *and* $d_1,\ldots,d_r$ *run over the non-negative integer solutions of the equation*

$$2d_1 + \ldots + (r+1)d_r = p - \delta \ .$$

The spaces N_m are called spaces of normal tensors of symplectic connections, and they are vector spaces made of tensors which recover the symmetries of the functions $\Gamma_{ijk} := \omega_{il}\Gamma^l_{jk}$[6], where Γ^l_{jk} are the Christoffel symbols of a symplectic connection in normal coordinates at the point x_0. They will be rigorously defined during Section 5.

3. The Peetre-Slovák Theorem

Let us briefly introduce the category of ringed spaces: they generalise smooth manifolds in a way that allows us to consider infinite dimensional spaces or quotients of smooth manifolds by the actions of groups.

Definition 3.1. A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sub-algebra of the sheaf of real-valued continuous functions on X.

A morphism of ringed spaces[7] $\varphi\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ is a continuous map $\varphi\colon X \to Y$ such that composition with φ induces a morphism of sheaves $\varphi^*\colon \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$, that is, for any open set $V \subset Y$ and any function $f \in \mathcal{O}_Y(V)$, the composition $f \circ \varphi$ lies in $\mathcal{O}_X(\varphi^{-1}V)$.

The two main properties of this category that we will make use of are the existence of inverse limits and the existence of quotients by the action of a group. For example, if $F \to X$ is a fibre bundle over a smooth manifold X, then the space $J^\infty F$ of ∞-jets of sections of $F \to X$ is defined as the inverse limit of the sequence of k-jets fibre bundles:

$$\ldots \to J^kF \to J^{k-1}F \to \ldots \to F \to X \ .$$

The spaces J^kF are smooth manifolds, and thus they are ringed spaces, choosing as sheaf the sheaf of real-valued smooth functions. Therefore, the space $J^\infty F$ is canonically imbued with a structure of ringed space. This fact will become of great relevance in the Peetre-Slovák theorem, where natural tensors will be related to morphisms of ringed spaces coming from an ∞-jet space.

Additionally, we will require the following corollary:

[6] During this work, we will follow Einstein summation convention, unless the summation is explicitly stated.

[7] By similarity with the category of smooth manifolds, we will often call morphisms of ringed spaces as *smooth morphisms*.

Corollary 3.2. *Let G be a group acting on two ringed spaces X and Y, and let $H \subseteq G$ be a subgroup that acts trivially on Y.*

Then, the universal property of the quotient restricts to a bijection:

$$\left\{ \begin{array}{c} G\text{-equivariant morphisms} \\ \\ \text{of ringed spaces } X \to Y \end{array} \right\} = \left\{ \begin{array}{c} G/H\text{-equivariant morphisms} \\ \\ \text{of ringed spaces } X/H \longrightarrow Y \end{array} \right\} .$$

Now, let us define a sort of "smoothness" condition for morphisms of sheaves:

Definition 3.3. Let $\mathcal{F}$ and $\mathcal{F}'$ be (sub)sheaves of the sheaves of smooth sections of the fibre bundles $F \to X$ and $F' \to X$, and let T be a smooth manifold. A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{F}'$ is said to be regular if, for any smooth family of sections $\{s_t : U \to F\}_{t\in T}$ such that $U \simeq \mathbb{R}^n$ and $s_t \in \mathcal{F}(U)$ for all $t \in T$, the family $\{\phi(s_t) : U \to F'\}_{t\in T}$ is also smooth.

The Peetre-Slovák Theorem ([21, 26]) assures that any natural morphism of sheaves is a natural differential operator:

Theorem 3.4 (Peetre-Slovák)**.** *Let X be a smooth manifold. Let $F' \to X$ and $F'' \to X$ be natural bundles over X, and let $\mathcal{F}'$ and $\mathcal{F}''$ be their respective sheaves of smooth sections over X.*

The choice of a point $p \in X$ allows to define this bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \\ \phi\colon \mathcal{F}' \longrightarrow \mathcal{F}'' \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_{x_0}\text{-equivariant smooth maps} \\ \\ J^\infty_p F' \longrightarrow F''_p \end{array} \right\} ,$$

where Diff_{x_0} stands for the group of germs of diffeomorphisms τ between open sets of X such that $\tau(p) = p$.

4. Natural Operations on a Fedosov Structure

Let X be a smooth manifold of dimension $2n$. Let $\mathcal{F}_X$ and $\mathcal{T}_X$ be the sheaves of Fedosov structures and p-covariant tensors over X, respectively.

Proposition 4.1. *The choice of a chart $U \subseteq X$ gives a bijection:*

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \\ \mathcal{F}_X \longrightarrow \mathcal{T}_X \end{array} \right\} = \left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \\ \mathcal{F}_{\mathbb{R}^{2n}} \longrightarrow \mathcal{T}_{\mathbb{R}^{2n}} \end{array} \right\} .$$

Proof: Let $U \subseteq X$ be any chart, so that $U \simeq \mathbb{R}^{2n}$. We will prove that there exists a bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \\ \phi : \mathcal{F}_X \longrightarrow \mathcal{T}_X \end{array} \right\} = \left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \\ \phi_U : \mathcal{F}_U \longrightarrow \mathcal{T}_U \end{array} \right\} ,$$

thus obtaining the desired result.

For any natural morphism of sheaves $f : \mathcal{F}_U \to \mathcal{T}_U$, let us construct the corresponding natural morphism of sheaves $\phi_f : \mathcal{F}_X \to \mathcal{T}_X$: for any $s \in \mathcal{F}_X(V)$ and $x \in V$, we must define $\phi_f(s)(x)$.

As $\phi_f(s)(x) = \phi_f(s_{|W})(x)$ for any $W \subseteq V$ containing x, we may suppose that V is also a chart, thus obtaining a local isomorphism $\tau : V \to U$, and so we may define:

$$\phi_f(s)(x) = \tau_*^{-1}(f(\tau_* s))(x)$$

It is trivial to check that this morphism is well defined, natural, regular and the inverse of the map $\phi \to \phi_{|U}$. □

Let $(x_1, y_1, \ldots, x_n, y_n)$ be global coordinates on $\mathbb{R}^{2n}$, and set $\eta = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n$. Let $\mathrm{Conn}_\eta \to \mathbb{R}^{2n}$ be the fibre bundle of symplectic connections for the symplectic form η, which is an affine subbundle of $\mathrm{Conn} \to \mathbb{R}^{2n}$. Let $\mathcal{C}_\eta$ be the sheaf of smooth sections of Conn_η.

Proposition 4.2. *With the previous notations, there exists a bijection:*

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \\ \mathcal{F}_{\mathbb{R}^{2n}} \longrightarrow \mathcal{T}_{\mathbb{R}^{2n}} \end{array} \right\} =\!=\!= \left\{ \begin{array}{c} \mathrm{Aut}(\eta)\textit{-natural morphisms of sheaves} \\ \\ \mathcal{C}_\eta \longrightarrow \mathcal{T}_{\mathbb{R}^{2n}} \end{array} \right\},$$

where a natural morphism of sheaves $\phi : \mathcal{C}_\eta \to \mathcal{T}$ is said to be $\mathrm{Aut}(\eta)$*-natural if it is regular and verifies the naturalness condition for any local diffeomorphism $\tau : U \to V$ between open sets of $\mathbb{R}^{2n}$ such that $\tau \cdot (\eta_{|_U}) = \eta_{|_V}$.*

Proof: Given a natural morphism of sheaves $\phi : \mathcal{F} \to \mathcal{T}$, the corresponding morphism of sheaves $\hat{\phi} : \mathcal{C}_\eta \to \mathcal{T}$ is given, at any open subset $U \subseteq \mathbb{R}^{2n}$, by

$$\hat{\phi}_U(\nabla) := \phi_U(\eta, \nabla) \ ,$$

which is trivially an $\mathrm{Aut}(\eta)$-natural morphism of sheaves.

Let us give the inverse map, that is, to define a natural morphism of sheaves $\tilde{\varphi} : \mathcal{F} \to \mathcal{T}$ from an $\mathrm{Aut}(\eta)$-natural morphism of sheaves $\varphi : \mathcal{C}_\eta \to \mathcal{T}_{\mathbb{R}^{2n}}$. Let $(\omega, \nabla) \in \mathcal{F}_{\mathbb{R}^{2n}}(U)$ and $x \in U$. There exists an open subset $V \subseteq U$ and a diffeomorphism $\tau : V \to V$ such that $x \in V$ and $\tau \cdot (\eta_{|_V}) = \omega_{|_U}$. As the value at x of $\tilde{\varphi}(\omega, \nabla)$ does not depend on the neighbourhood of x chosen, we may assume that $V = U$. Then:

$$\tilde{\varphi}(\omega, \nabla)(x) := \tau \cdot \varphi(\tau^{-1} \cdot \nabla)(x) \ .$$

□

Corollary 4.3. *The choice of a point $x_0 \in \mathbb{R}^{2n}$ produces a bijection:*

$$\left\{ \begin{array}{c} \mathrm{Aut}(\eta)\textit{-natural morphisms of sheaves} \\ \\ \mathcal{C}_\eta \longrightarrow \mathcal{T}_{\mathbb{R}^{2n}} \end{array} \right\} =\!=\!= \left\{ \begin{array}{c} \mathrm{Aut}(\eta)_{x_0}\textit{-equivariant smooth maps} \\ \\ J^\infty_{x_0} Conn_\eta \longrightarrow T_{x_0} \end{array} \right\},$$

where $\mathrm{Aut}(\eta)_{x_0}$ *denotes the group of germs of diffeomorphisms* τ *between open sets of* $\mathbb{R}^{2n}$ *such that* $\tau(p) = p$ *and* $\tau \cdot \eta = \eta$.

Proof: A simple variation of the Peetre-Slovák theorem 3.4, substituting naturalness by $\mathrm{Aut}(\eta)$-naturalness, allows us to conclude. □

However, even though fixing a symplectic form in a neighbourhood of a point allows us to use the Peetre-Slovák Theorem – reducing the computations to the ∞-jet space, the resulting space is difficult to reduce. It is convenient to take a step back, unfixing the symplectic form, in order to advance:

Proposition 4.4. *There exists a bijection:*

$$\left\{ \begin{array}{c} \mathrm{Aut}(\eta)_{x_0}\textit{-equivariant smooth maps} \\ \\ J^\infty_{x_0} Conn_\eta \longrightarrow T_{x_0} \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_{x_0}\textit{-equivariant smooth maps} \\ \\ J^\infty_{x_0}\mathcal{F} \longrightarrow T_{x_0} \end{array} \right\} ,$$

where $J^\infty_{x_0}\mathcal{F} := \{(j^\infty_{x_0}\omega, j^\infty_{x_0}\nabla) : (\omega, \nabla) \in \mathcal{F}_{x_0}\}$.

Proof: The proof of this result is similar to that of Proposition 4.2. □

Later on, only the value of the symplectic form at x_0 will be fixed, as the rest of the ∞-jet will be determined by the compatibility condition with the ∞-jet of a symplectic connection.

Remark 4.5. Observe that $J^\infty_{x_0}\mathcal{F}$ coincides with the set

$$\{(j^\infty_{x_0}\omega, j^\infty_{x_0}\nabla) \in J^\infty_{x_0}\Lambda^2 \times J^\infty_{x_0}\mathrm{Conn}^{\,\mathrm{sym}} : j^\infty_{x_0}(\nabla\omega) = 0\} \ .$$

The reasoning goes as follows: due to the formal version of the Poincaré Lemma, the ∞-jet of a non-singular 2-form ω such that $j^\infty_{x_0}(\nabla\omega) = 0$ verifies that $j^\infty_{x_0}\omega = j^\infty_{x_0}(\mathrm{d}\theta)$, for some 1-form θ defined on a neighbourhood of x_0. Therefore, $j^\infty_{x_0}\omega$ can be extended to a symplectic form at a neighbourhood of x_0 (considering, for example, $\mathrm{d}\theta$). Then, a symplectic connection extending $j^\infty_{x_0}\nabla$ can be chosen, as symplectic connections compatible with a fixed symplectic form constitute a fibre bundle.

5. Invariants of Symplectic Connections

Let $x_0 \in X$, let (ω, ∇) be the germ of a Fedosov structure at x_0, and let $\bar{\nabla}$ be the germ of the flat connection at $x_0 \in X$ corresponding, via the exponential map, to the flat connection of $T_{x_0}X$. Let $\mathbb{T} := C^1_2(\omega \otimes \mathbb{T})$, where C^j_i denotes the tensor contraction of the i-th covariant index with the j-th contravariant index.

Definition 5.1. For any integer $m \geq 0$, the m-th normal tensor of ∇ at x_0 is $\bar{\nabla}^m_{x_0}\mathbb{T}$.

In a system of normal coordinates $(x_1, \ldots, x_n)$ around the point x_0 for ∇, the tensor $\bar{\nabla}^m_{x_0}\mathbb{T}$ is written as

$$\bar{\nabla}^m_{x_0}\mathbb{T} = \sum_{i,j,k,a_1,\ldots,a_m} \Gamma_{ijk,a_1\ldots a_m} \cdot \mathrm{d}_{x_0}x_i \otimes \mathrm{d}_{x_0}x_j \otimes \mathrm{d}_{x_0}x_k \otimes \mathrm{d}_{x_0}x_{a_1} \otimes \ldots \otimes \mathrm{d}_{x_0}x_{a_m} \ ,$$

where $\Gamma^k_{ijk,a_1\dots a_m} := \frac{\partial^m \Gamma_{ijk}}{\partial x_{a_1}\dots\partial x_{a_m}}(x_0)$ and $\Gamma_{ijk} = \sum_{l=1}^{2n}\omega_{il}\Gamma^l_{jk}$.

Remark 5.2. Notice that the sequences $\bar\nabla^1_{x_0}\mathbb{T},\dots,\bar\nabla^m_{x_0}\mathbb{T}$ and $\bar\nabla^1_{x_0}(\nabla-\bar\nabla),\dots,\bar\nabla^m_{x_0}(\nabla-\bar\nabla)$ mutually determine each other, as ω is non-singular. Following the notations above, the tensor $\bar\nabla^m_{x_0}(\nabla-\bar\nabla)$ is written as usual:

$$\bar\nabla^m_{x_0}\mathbb{T} = \sum_{i,j,k,a_1,\dots,a_m} \Gamma^k_{ij,a_1\dots a_m}\cdot\left(\frac{\partial}{\partial x_k}\right)_{x_0}\otimes \mathrm{d}_{x_0}x_i\otimes\mathrm{d}_{x_0}x_j\otimes\mathrm{d}_{x_0}x_{a_1}\otimes\dots\otimes\mathrm{d}_{x_0}x_{a_m}\ ,$$

where $\Gamma^k_{ij,a_1\dots a_m} := \frac{\partial^m\Gamma^k_{ij}}{\partial x_{a_1}\dots\partial x_{a_m}}(x_0)$.

Definition 5.3. The space N_m of normal tensors of order m at $x_0\in X$ is the vector subspace of $(m+3)$-tensors whose elements T verify the following symmetries:

(1) they are symmetric in the second and third indices, and in the last m:

$$T_{ikja_1\dots a_m} = T_{ijka_1\dots a_m},\quad T_{ijka_{\sigma(1)}\dots a_{\sigma(m)}} = T_{ijka_1\dots a_m},\quad \forall\sigma\in S_m\ ;$$

(2) the symmetrization of the last $m+2$ covariant indices is zero:

$$\sum_{\sigma\in S_{m+2}} T_{i\sigma(j)\sigma(k)\sigma(a_1)\dots\sigma(a_m)=0}\ ;$$

(3) the following tensor is symmetric in k and a_1:

$$T_{ikja_1\dots a_m} - T_{jkia_1\dots a_m}\ .$$

Due to its symmetries, it is immediate that $N_0 = 0$.

Normal tensors belong in N_m, that is, $\bar\nabla^m_{x_0}\mathbb{T}\in N_m$, due to its expression in normal coordinates ([10]). As the tensor $\bar\nabla^m_{x_0}\mathbb{T}$ depends only on the value of the m-jet $j^m_{x_0}\nabla$, the following map is well-defined:

$$\begin{aligned}\phi_m\colon J^m_{x_0}\mathcal{F} &\longrightarrow \Lambda_0\times\prod_{i=1}^m N_i\\ (j^{r+1}_{x_0}\omega, j^r_{x_0}\nabla) &\longmapsto (\omega_{x_0},\bar\nabla^1_{x_0}\mathbb{T},\bar\nabla^2_{x_0}\mathbb{T},\dots,\bar\nabla^m_{x_0}\mathbb{T})\,,\end{aligned}$$

where Λ_0 denotes the open set of non-singular 2-forms at x_0.

The maps ϕ_m are Diff_{x_0}-equivariant and compatible, meaning that they commute with the restrictions $J^m_{x_0}\mathcal{F}\to J^{m-1}_{x_0}\mathcal{F}$ and $\Lambda_0\times\prod_{i=1}^m N_i\to\Lambda_0\times\prod_{i=1}^{m-1}N_i$. Therefore there exists a morphism of ringed spaces:

$$\begin{aligned}\phi_\infty\colon J^\infty_{x_0}\mathcal{F} &\longrightarrow \Lambda_0\times\prod_{i=1}^\infty N_i\\ (j^\infty_{x_0}\omega, j^\infty_{x_0}\nabla) &\longmapsto (\omega_{x_0},\bar\nabla^1_{x_0}\mathbb{T},\bar\nabla^2_{x_0}\mathbb{T},\dots)\,.\end{aligned}$$

Reduction Theorem. *The* $\mathrm{Diff}^{m+2}_{x_0}-$*equivariant morphism of ringed spaces*

$$\begin{aligned} \phi_m\colon J^m_{x_0}\mathcal{F} &\longrightarrow \Lambda_0\times\prod_{i=1}^m N_i \\ (j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla) &\longmapsto (\omega_{x_0}, \bar\nabla^1_{x_0}\mathbb{T}, \bar\nabla^2_{x_0}\mathbb{T},\dots,\bar\nabla^m_{x_0}\mathbb{T}\,)\,. \end{aligned}$$

is surjective, its fibres are the orbits of $\mathrm{NDiff}^{m+2}_{x_0}$ *and it admits smooth sections passing through any point of* $J^m_{x_0}\mathcal{F}$.

As a consequence, ϕ_m *induces a* Gl*-equivariant isomorphism of ringed spaces:*

$$(J^m_{x_0}\mathcal{F}\ )/\mathrm{NDiff}^{m+2}_{x_0} = N_1\times\ldots\times N_m\ .$$

Proof: Let us first prove that the fibres of ϕ_m are the orbits of $\mathrm{NDiff}^{m+2}_{x_0}$. Let $(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla)$, $(j^{m+1}_{x_0}\omega', j^m_{x_0}\nabla')$ be two points in the orbit of $\mathrm{NDiff}^{m+2}_{x_0}$, that is, $(j^{m+1}_{x_0}\omega', j^m_{x_0}\nabla') = j^{m+2}_{x_0}\tau\cdot(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla)$ for some $j^{m+2}_{x_0}\tau\in\mathrm{NDiff}^{m+2}_{x_0}$. As $\mathrm{NDiff}^{m+2}_{x_0}$ acts by the identity on $\Lambda_0\times\prod_{i=1}^m N_i$,

$$\phi_m(j^{m+2}_{x_0}\tau\cdot(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla)\,) = j^{m+2}_{x_0}\tau\cdot\phi_m((j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla)\,) = \phi_m((j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla)\,)\ .$$

Let now $(j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla)$, $(j^{m+1}_{x_0}\omega', j^m_{x_0}\nabla')\in J^m_{x_0}\mathcal{F}$ be two points in the same fibre of ϕ_m, that is, $\phi_m((j^{m+1}_{x_0}\omega, j^m_{x_0}\nabla)) = \phi_m((j^{m+1}_{x_0}\omega', j^m_{x_0}\nabla')) = (T_1,\dots,T_r)$. Let us fix a base of $T_{x_0}X$, let $x_1,\dots,x_{2n}$ and $x'_1,\dots,x'_{2n}$ be the systems of normal coordinates induced by the fixed base for $j^m_{x_0}\nabla$ and $j^m_{x_0}\nabla'$, respectively, and let τ be the diffeomorphism that verifies $\tau\cdot x_i = x'_i$ for all $i\in\{1,\dots,2n\}$. As $\mathrm{d}_{x_0}x_i = \mathrm{d}_{x_0}x'_i$ for all $i\in\{1,\dots,2n\}$, it holds that $j^{m+2}_{x_0}\tau\in\mathrm{NDiff}^{m+2}_{x_0}$.

Let us write

$$j^m_{x_0}\nabla = (\,0\,,\Gamma^k_{ij,a_1},\dots,\Gamma^k_{ij,a_1\dots a_m}),\quad j^{m+1}_{x_0}\omega = (\omega_{ij},\omega_{ij,k},\dots,\omega_{ij,ka_1\dots a_m})$$

in the coordinates induced by $x_1,\dots,x_{2n}$ on $J^m_{x_0}\mathcal{F}$. Similarly, in the coordinates induced by $x'_1,\dots,x'_{2n}$ on $J^m_{x_0}\mathcal{F}$, we write

$$\begin{aligned} j^m_{x_0}\nabla' &= (\,0\,,(\Gamma')^k_{ij,a_1},\dots,(\Gamma')^k_{ij,a_1\dots a_m}),\\ j^{m+1}_{x_0}\omega' &= (\omega'_{ij},\omega'_{ij,k},\dots,\omega'_{ij,ka_1\dots a_m}),\\ j^m_{x_0}(\tau\cdot\nabla) &= (\,0\,,(\tau\cdot\Gamma)^k_{ij,a_1},\dots,(\tau\cdot\Gamma)^k_{ij,a_1\dots a_m}),\\ j^{m+1}_{x_0}(\tau\cdot\omega) &= ((\tau\cdot\omega)_{ij},(\tau\cdot\omega)_{ij,k},\dots,(\tau\cdot\omega)_{ij,ka_1\dots a_m}). \end{aligned}$$

For all $r\in\{1,\dots,m\}$, using that $j^{m+2}_{x_0}\tau\in\mathrm{NDiff}^{m+2}_{x_0}$ we obtain the following equalities:

$$\begin{aligned} &\sum_{i,j,k,a_1,\dots,a_r}(\Gamma')_{ijk,a_1\dots a_r}\mathrm{d}_{x_0}x'_i\otimes\mathrm{d}_{x_0}x'_j\otimes\mathrm{d}_{x_0}x'_k\otimes\mathrm{d}_{x_0}x'_{a_1}\otimes\ldots\otimes\mathrm{d}_{x_0}x'_{a_r} = T_r\\ &= \tau\cdot T_r = \sum_{i,j,k,a_1,\dots,a_r}(\tau\cdot\Gamma)_{ijk,a_1\dots a_r}\mathrm{d}_{x_0}x'_i\otimes\mathrm{d}_{x_0}x'_j\otimes\mathrm{d}_{x_0}x'_k\otimes\mathrm{d}_{x_0}x'_{a_1}\otimes\ldots\otimes\mathrm{d}_{x_0}x'_{a_r}\ , \end{aligned}$$

and so $(\tau\cdot\Gamma)_{ijk,a_1\dots a_r} = (\Gamma')_{ijk,a_1\dots a_r}$ for all $r\in\{1,\dots,m\}$.

Thus, by Remark 5.2, it is now enough to check that $j^m_{x_0}(\tau\cdot\omega) = j^m_{x_0}(\omega')$:

$$\sum_{i<j}(\tau\cdot\omega)_{ij}\mathrm{d}_{x_0}x'_i\wedge\mathrm{d}_{x_0}x'_j=(\tau\cdot\omega)_{x_0}=\omega_{x_0}=\omega'_{x_0}=\sum_{i<j}\omega_{ij}\mathrm{d}_{x_0}x'_i\wedge\mathrm{d}_{x_0}x'_j\ ,$$

$$(\tau\cdot\omega)_{ij,k}=(\tau\cdot\Gamma)_{ikj}-(\tau\cdot\Gamma)_{jki}=0=(\Gamma')_{ikj}-(\Gamma')_{jki}=\omega'_{ij,k}\ ,$$

$$\vdots$$

$$(\tau\cdot\omega)_{ij,ka_1\dots a_m}=(\tau\cdot\Gamma)_{ikj,a_1\dots a_m}-(\tau\cdot\Gamma)_{jki,a_1\dots a_m}=(\Gamma')_{ikj,a_1\dots a_m}-(\Gamma')_{jki,a_1\dots a_m}=\omega'_{ij,ka_1\dots a_m}\ .$$

Lastly, let us prove the statement about the existence of smooth sections

$$s:\Lambda_0\times N_1\times\ldots\times N_m\,.$$

Let us fix a system of coordinates $x_1,\dots,x_{2n}$ at x_0, and let $(B_{ij},A_{ijka_1},\dots,A_{ijka_1\dots a_m})\in\Lambda_0\times N_1\times\ldots\times N_m$.

The jet $s((B_{ij},A_{ijka_1},\dots,A_{ijka_1\dots a_m}))=(j^{m+1}_{x_0}\omega,j^m_{x_0}\nabla)$ is defined, in the coordinates induced by the fixed system in $J^m_{x_0}\mathcal{F}$, as follows:

$$\Gamma_{ijk}=0,\Gamma_{ijk,a_1}=A_{ijka_1},\dots,\Gamma_{ijk,a_1\dots a_m}=A_{ijka_1\dots a_m}\ ,$$

$$\omega_{ij}=B_{ij},\omega_{ij,k}=0,\omega_{ij,ka_1}=A_{ikja_1}-A_{jkia_1},\dots,\omega_{ij,ka_1\dots a_m}=A_{ikja_1\dots a_m}-A_{jkia_1\dots a_m}\ ,$$

and so the jet $j^m_{x_0}\nabla=(0,\Gamma^k_{ij,a_1},\dots,\Gamma^k_{ij,a_1\dots a_m})$ is defined. The symmetries of the spaces N_i assure that $(j^{m+1}_{x_0}\omega,j^m_{x_0}\nabla)\in\mathrm{Fed}^m_{x_0}=J^m_{x_0}\mathcal{F}$ and that $x_1,\dots,x_{2n}$ is a system of normal coordinates at x_0 for $j^m_{x_0}\nabla$.

□

Corollary 5.4. *The* $\mathrm{Diff}^\infty_{x_0}$*−equivariant morphism of ringed spaces*

$$\begin{aligned}\phi_\infty\colon J^\infty_{x_0}\mathcal{F}&\longrightarrow\quad\Lambda_0\times\prod_{i=1}^\infty N_i\\(j^\infty_{x_0}\omega,j^\infty_{x_0}\nabla)&\longmapsto\ (\omega_{x_0},\bar\nabla^1_{x_0}\mathbb{T},\bar\nabla^2_{x_0}\mathbb{T},\dots),\end{aligned}$$

induces a Gl*-equivariant isomorphism of ringed spaces:*

$$(J^\infty_{x_0}\mathcal{F})/\mathrm{NDiff}^\infty_{x_0}=\!=\Lambda_0\times\prod_{i=1}^\infty N_i\ .$$

Corollary 5.5. *The choice of a non-singular 2-form η_{x_0} at x_0 produces a bijection:*

$$\left\{\begin{array}{c}\mathrm{Gl}\text{-}\textit{equivariant smooth maps}\\ \Lambda_0\times\prod_{i=1}^\infty N_i\longrightarrow T_{x_0}\end{array}\right\}=\!=\!=\left\{\begin{array}{c}\mathrm{Sp}(2n,\mathbb{R})\text{-}\textit{equivariant smooth maps}\\ \prod_{i=1}^\infty N_i\longrightarrow T_{x_0}\end{array}\right\},$$

where $\mathrm{Sp}(2n,\mathbb{R}):=\{\mathrm{d}_{x_0}\tau\colon\tau\in\mathrm{Aut}(\eta)_{x_0}\}$.

Proof: The proof of this result is similar to Proposition 4.2.

6. Proof of Theorem 2.6

Definition 6.1. Let $\delta \in \mathbb{R}$. We say that a natural tensor $T : \mathcal{F} \to \mathcal{T}$ is homogeneous of weight δ if, for all non-zero $\lambda \in \mathbb{R}$, it holds that[8]:

$$T(\lambda^2\omega, \nabla) = \lambda^\delta T(\omega, \nabla) \ .$$

Observe that, if $T \neq 0$ and $\delta \in \mathbb{Z}$, the weight must be an even number: if T is an homogeneous natural tensor of odd weight δ, then the homogeneity condition for $\lambda = -1$ says:

$$T(\omega, \nabla) = T((-1)^2\omega, \nabla) = (-1)^\delta T(\omega, \nabla) = -T(\omega, \nabla) \ ,$$

obtaining that $T = 0$.

Theorem 2.6. *Let X be a smooth manifold of dimension $2n$, and let $\mathcal{F}$ denote the sheaf of Fedosov structures. Let $\mathcal{T}$ be the sheaf of p-covariant tensors over X. Let $\delta \in \mathbb{Z}$.*

Fixing a point $x_0 \in X$ and a chart $U \simeq \mathbb{R}^{2n}$ around x_0 produces a $\mathbb{R}$-linear isomorphism

$$\left\{ \begin{array}{c} \textit{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \\ \textit{homogeneous of weight } \delta \end{array} \right\} = \bigoplus_{d_1,\ldots,d_r} \mathrm{Hom}_{\mathrm{Sp}}(S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r, T_{x_0}) \ ,$$

where $\mathrm{Sp} = \mathrm{Sp}(2n, \mathbb{R})$ *denotes the symplectic group,* T_{x_0} *denotes the vector space of p-covariant tensors at x_0 and $d_1, \ldots, d_r$ run over the non-negative integer solutions of the equation*

$$2d_1 + \ldots + (r+1)d_r = p - \delta \ . \tag{6.1}$$

Proof: Let us fix a point $x_0 \in X$. Choose a chart $U \simeq \mathbb{R}^{2n}$ around x_0, so that Proposition 4.1 produces a bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F} \longrightarrow \mathcal{T} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F}_{\mathbb{R}^{2n}} \longrightarrow \mathcal{T}_{\mathbb{R}^{2n}} \\ \text{homogeneous of weight } \delta \end{array} \right\},$$

where $\mathcal{F}_{\mathbb{R}^{2n}}$ and $\mathcal{T}_{\mathbb{R}^{2n}}$ denote the sheaves $\mathcal{F}$ and $\mathcal{T}$ restricted to U and passed through the diffeomorphism $U \simeq \mathbb{R}^{2n}$.

Fixing the canonical symplectic form η on $\mathbb{R}^{2n}$ lets us invoke Proposition 4.2 and Proposition 4.3, which gives the bijection:

$$\left\{ \begin{array}{c} \text{Natural morphisms of sheaves} \\ \mathcal{F}_{\mathbb{R}^{2n}} \longrightarrow \mathcal{T}_{\mathbb{R}^{2n}} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Aut}(\eta)_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathrm{Conn}_\eta \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\},$$

where an $\mathrm{Aut}(\eta)_{x_0}$-equivariant smooth map $T : J^\infty_{x_0}\mathrm{Conn}_\omega \to T_{x_0}$ being homogeneous of weight δ means that it verifies the following property:

$$T(h_\lambda \cdot (j^\infty_{x_0}\nabla)) = \lambda^{p-\delta} T(j^\infty_{x_0}\nabla) \ ,$$

[8]Observe that if (ω, ∇) is a Fedosov structure, then $(\lambda\omega, \nabla)$ is also a Fedosov structure for any $\lambda \in \mathbb{R}\backslash\{0\}$.

for any homothety[9] h_λ of ratio $\lambda \neq 0$.

Let us now unfix the symplectic form (recall that diffeomorphisms act transitively on symplectic forms due to the existence of Darboux coordinates):

$$\left\{ \begin{array}{c} \mathrm{Aut}(\eta)_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathrm{Conn}_\eta \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Diff}_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} .$$

As the action of both Diff_{x_0} and $\mathrm{Diff}^\infty_{x_0}$ coincide over $J^\infty_{x_0}\mathcal{F}$ and T_{x_0}, we may consider $\mathrm{Diff}^\infty_{x_0}$-equivariant maps instead in the set above.

For the next step, recall that the following sequence of groups is exact:

$$1 \longrightarrow \mathrm{NDiff}^\infty_{x_0} \longrightarrow \mathrm{Diff}^\infty_{x_0} \longrightarrow \mathrm{Gl} \longrightarrow 1$$

As the subgroup $\mathrm{NDiff}^\infty_{x_0}$ acts by the identity over T_{x_0}, Corollary 3.2 in conjunction with the exact sequence above assures the existence of an isomorphism:

$$\left\{ \begin{array}{c} \mathrm{Diff}^\infty_{x_0}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F}/\mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} .$$

Now, Corollary 5.4 allows us to replace this quotient ringed space via the bijection:

$$\left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ J^\infty_{x_0}\mathcal{F}/\mathrm{NDiff}^\infty_{x_0} \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ \Lambda_0 \times \prod\limits_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} .$$

Fixing the non-singular 2-form η_{x_0} at x_0 allows us to remove the space Λ_0, due to the bijection:

$$\left\{ \begin{array}{c} \mathrm{Gl}\text{-equivariant smooth maps} \\ \Lambda_0 \times \prod\limits_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \left\{ \begin{array}{c} \mathrm{Sp}(2n,\mathbb{R})\text{-equivariant smooth maps} \\ \prod\limits_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} ,$$

where, following the previous bijections, a $\mathrm{Sp}(2n,\mathbb{R})$-equivariant smooth map $T : \prod\limits_{i=1}^{\infty} N_i \to T_{x_0}$ is said to be homogeneous of weight δ if, for any $\lambda \neq 0$, it holds that

$$T(\lambda^2 T_1, \lambda^3 T_2, \ldots) = \lambda^{p-\delta} T(T_1, T_2, \ldots) .$$

Therefore, the homogeneity allows us to make the final reduction by applying the Homogeneous Function Theorem below, producing the isomorphism:

[9]We say that $\tau \in \mathrm{Diff}_{x_0}$ is a homothety of ratio $\lambda \neq 0$ if $\mathrm{d}_{x_0}\tau = \lambda \cdot \mathrm{Id}$.

$$\left\{ \begin{array}{c} \mathrm{Sp}(2n,\mathbb{R})\text{-equivariant smooth maps} \\ \prod\limits_{i=1}^{\infty} N_i \longrightarrow T_{x_0} \\ \text{homogeneous of weight } \delta \end{array} \right\} = \bigoplus_{d_1,\ldots,d_r} \mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}(S^{d_1}N_1 \otimes \ldots \otimes S^{d_r}N_r, T_{x_0}) \ ,$$

where $d_1, \ldots, d_r$ are non-negative integers running over the solutions of the equation

$$2d_1 + \ldots + (r+1)d_r = p - \delta \ .$$

□

Homogeneous Function Theorem. *Let $\{E_i\}_{i\in\mathbb{N}}$ be finite dimensional vector spaces.*

Let $f : \prod_{i=1}^{\infty} E_i \to \mathbb{R}$ be a smooth function such that there exist positive real numbers $a_i > 0$, and $w \in \mathbb{R}$ satisfying:

$$f(\lambda^{a_1}e_1, \ldots, \lambda^{a_i}e_i, \ldots) = \lambda^w f(e_1, \ldots, e_i, \ldots) \tag{6.2}$$

for any positive real number $\lambda > 0$ and any $(e_1, \ldots, e_i, \ldots) \in \prod_{i=1}^{\infty} E_i$.

Then, f depends on a finite number of variables $e_1, \ldots, e_r$ and it is a sum of monomials of degree d_i in e_i satisfying the relation

$$a_1d_1 + \cdots + a_rd_r = w \ . \tag{6.3}$$

If there are no natural numbers $d_1, \ldots, d_r \in \mathbb{N} \cup \{0\}$ satisfying this equation, then f is the zero map.

An immediate corolary of the Main theorem is that, if the left side of Equation 6.1 is either null or negative, there are essentially no natural tensors:

Corollary 6.2. *There are no non-constant homogeneous natural p-tensors associated to Fedosov structures of weight $\delta \geq p$.*

6.1. **An application.** Let V be a real vector space of finite dimension $2n$, let ω be a non-degenerate skew-symmetric bilinear form on V and let $\mathrm{Sp}(2n,\mathbb{R})$ be the real Lie group of $\mathbb{R}$-linear automorphisms that preserve ω.

The First Fundamental Theorem of the symplectic group ([15]) describes the vector space of $\mathrm{Sp}(2n,\mathbb{R})$-invariant linear maps $V \otimes \overset{p}{\ldots} \otimes V \longrightarrow \mathbb{R}$:

First Fundamental Theorem of Sp. *The real vector space* $\mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}(V \otimes \overset{p}{\ldots} \otimes V, \mathbb{R})$ *of invariant linear forms on $V \otimes \ldots \otimes V$ is null if p is odd, whereas if p is even it is spanned by*

$$\omega_\sigma((e_1, \ldots, e_p)) := \omega(e_{\sigma(1)}, e_{\sigma(2)}) \ldots \omega(e_{\sigma(p-1)}, e_{\sigma(p)}) \ ,$$

where $\sigma \in S_p$.

The invariant theory of the symplectic group, along with our Main Theorem, allows us to compute the space of natural functions for weights $w = -2$ and $w = -4$:

Corollary 6.3. *There are no non-constant homogeneous natural functions associated to Fedosov structures of weight $w = -2$, and for $w = -4$ there are three $\mathbb{R}$-linearly independent natural functions.*

Proof: Let us fix $x_0 \in X$ and a non-singular 2-form ω at x_0. Let us invoke the Main Theorem 2.6 for $p = 0$ and $\delta = -2$. The only non-negative integer solution of the equation

$$2d_1 + \ldots + (r+1)d_r = p - \delta = 2$$

is $d_1 = 1$.

Therefore, the problem is reduced to computing $\mathrm{Sp}(2n, \mathbb{R})$-equivariant maps $N_1 \to \mathbb{R}$. As the elements in N_1 are 4-covariant tensors symmetric in the second and third indices, by the First Fundamental theorem of Sp it is sufficient to check that the map

$$T_{ijka} \longrightarrow \omega^{ij}\omega^{ka}T_{ijka}$$

is zero:

$$\omega^{ij}\omega^{ka}T_{ijka} = \frac{1}{2}\omega^{ij}\omega^{ka}(T_{ijka} - T_{jika}) = 0 \ ,$$

as the elements in N_1 verify that

$$T_{ijka} - T_{jika} = T_{ijak} - T_{jiak} \ .$$

Repeating the arguments for $p = 0$ and $w = -4$, we obtain two solutions to the equation above: $d_1 = 2$ and $d_3 = 1$. Let us begin with solution $d_1 = 2$: we need to compute total index contractions of the expression $T_{ijkl}T_{abcd}$. Equivalently, we may replace this expression by applying the Sp-equivariant linear isomorphism

$$\begin{aligned} N_1 &\longrightarrow \mathcal{R} \\ T_{ijkl} &\longmapsto R_{ijkl} = T_{ijlk} - T_{ijkl}, \end{aligned}$$

where $\mathcal{R} \subset S^2T^*_{x_0}X \otimes \Lambda^2T^*_{x_0}X$ is the vector subspace of tensors R that satisfy the Bianchi identity:

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \ .$$

Thus, let us compute the total index contractions of the expression $R_{ijkl}R_{abcd}$. As the contraction of the symmetric pair is zero, the possibilities are:

- $f_1 = R_{ijkl}R^{ijkl}$.
- $f_2 = R_{ijk}{}^{k}R^{ijl}{}_{l}$.
- $R_{ijkl}R^{ikjl}$, which is equal to $f_1/2$, by the Bianchi identity.

For $d_3 = 1$, the last three indices of any tensor in N_3 are symmetric, so there is only one possibility: $f_3 = T_{ijk}{}^{ijk}$.

As for the linear independence of the three functions, by naturalness it is enough to check if they are independent at any given Fedosov manifold. For example, consider the Fedosov manifold $(\mathbb{R}^4, \eta, \nabla)$, where $\eta = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4$ and ∇ is the linear connection with the following Christoffel symbols (with the contravariant index lowered):

- $\Gamma_{ijk} = 1$, for any $\{i, j, k\}$ permutation of $\{1, 1, 2\}$.
- $\Gamma_{ijk} = x_1 x_3 x_4$, for any $\{i, j, k\}$ permutation of $\{2, 3, 4\}$.
- $\Gamma_{ijk} = 0$, for any other combination.

Computing the natural functions in this manifold gives:

- $f_1 = -4x_3^2 x_4^2(-4x_1^2 + 4x_1 + 1)$.
- $f_2 = 2x_3^2 x_4^2(4x_1^2 - 1)$.
- $f_3 = 6$,

which are clearly $\mathbb{R}$-linearly independent. $\square$

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Appendix G

MATLAB code

Listing G.1: Computation of the curvature operator associated to linear connections

```
% Declaration of variables.
x=sym('x', [1 3]);
gamma=sym('gamma%d%d_', [3 3 3]);
R=sym('R%d%d%d_', [3 3 3 3]);

% Definition of the Christoffel symbols of the connection in the fixed
% coordinates.

for i=1:3
    for j=1:3
        for k=1:3
             gamma(i,j,k)= 0;
        end
    end
end
gamma(1,1,1)=x(2)*x(3);
gamma(2,3,2)=x(1)*x(2);
gamma(3,2,2)=x(1)*x(2);

%Computation of the curvature tensor R

sum=0;
for i=1:3
    for j=1:3
        for k=1:3
            for l=1:3
                for m=1:3
                    sum=sum+gamma(j,k,m)*gamma(i,m,l) - gamma(i,k,m)*
                        gamma(j,m,l);
```

```
                end
                R(i,j,k,l)= diff(gamma(j,k,l),x(i)) - diff(gamma(i,k,l),
                    x(j)) + sum;
                sum=0;
            end
        end
    end
end
R
simplify(diferencial(R,gamma,x))
```

LISTING G.2: Computation of the tensors C_1 and C_2 (Lemma 5.16)

```
%Computation of the tensors C_1 and C_2 in the uniqueness theorem of the curvature
%operator associated to linear connections, and to linear connections and orientations.

Ricc=sym('R%d%d', [3 3]);
C1=sym('C1%d%d%d_', [3 3 3 3]);
C2=sym('C2%d%d%d_', [3 3 3 3]);

sum=0;
for i=1:3
    for j=1:3
        for m=1:3
            sum=sum+R(m,i,j,m);
        end
        Ricc(i,j)=sum;
        sum=0;
    end
end

for i=1:3
    for j=1:3
        for k=1:3
            for l=1:3
                C1(i,j,k,l)= (Ricc(i,k)+Ricc(k,i))/2 * eq(j,l) - (Ricc(j,k)+Ricc(k,j))/2 * eq(i,l);
            end
        end
    end
end

for i=1:3
    for j=1:3
        for k=1:3
            for l=1:3
                C2(i,j,k,l)=(Ricc(i,k)-Ricc(k,i))/2*eq(j,l)-(Ricc(j,k)-Ricc(k,j))/2*eq(i,l) + (Ricc(i,j)-Ricc(j,i))*eq(k,l);
            end
        end
```

```
    end
end
C1
C2
simplify(expand(diferencial(C1,gamma,x)))
simplify(expand(diferencial(C2,gamma,x)))
```

LISTING G.3: Computation of the tensor ϕ (Lemma 5.16)

```

%Computation of the tensor phi in the uniqueness theorem of the curvature
%operator associated to linear connections and orientations.

T=sym('T%d%d%d_', [3 3 3 3]);
Tb=sym('Tb%d%d%d_', [3 3 3 3]);
phi=sym('phi%d%d%d_', [3 3 3 3]);
delta=sym('delta%d%d', [3 3 3]);

ind=[1 2 3 1 2 3];
ind2=[1 2 3];
for i=1:3
    for j=1:3
        for k=1:3
            delta(i,j,k)=0;
        end
    end
end
delta(1,2,3)=1;
delta(2,3,1)=1;
delta(3,1,2)=1;
delta(2,1,3)=-1;
delta(3,2,1)=-1;
delta(1,3,2)=-1;

for i=1:3
    for j=1:3
        for k=1:3
            for l=1:3
                eqf=1;
                if(i~=j)
                   eqf=find(ind2~=i & ind2~=j);
                end
                T(i,j,k,l)=delta(i,j,eqf)*(diff(gamma(ind(l+1),k,eqf),x(ind(l+2))) - diff(gamma(ind(l+2),k,eqf),x(ind(l+1))));
                if(i==j)
                    T(i,j,k,l)=0;
                end
            end
```

```
            end
        end
end

sum=0;

for i=1:3
    for j=1:3
        for k=1:3
            for l=1:3
                for m=1:3
                   sum=sum+diff(gamma(ind(l+1),m,m),x(ind(l+2))) − diff(
                       gamma(ind(l+2),m,m),x(ind(l+1)));
                end
                Tb(i,j,k,l)=sum*delta(i,j,k);
                sum=0;
            end
        end
    end
end
T;
Tb;
3*T−Tb
simplify(expand(3*diferencial(T,gamma,x)−diferencial(Tb,gamma,x)))
simplify(expand(−3*diferencial(C1,gamma,x)−diferencial(C2,gamma,x)))
```

LISTING G.4: Computation of the differential

```
%Function that computes the valued differential of an endomorphism-valued
%form in dimension 3.

%T denotes the basis of endomorphisms in the considered coordinates.

function d = diferencial(R,gamma,x)
   T=sym('T%d%d', [3 3]);
   delta=sym('delta%d%d', [3 3 3]);
   eqf=1;
   ind2=[1 2 3];
   d=0;
   for i=1:3
        for j=1:3
            for k=1:3
                delta(i,j,k)=0;
            end
        end
   end
   delta(1,2,3)=1;
   delta(2,3,1)=1;
   delta(3,1,2)=1;
   delta(2,1,3)=-1;
   delta(3,2,1)=-1;
   delta(1,3,2)=-1;

   for i=1:3
       for j=1:3
           for k=1:3
               for l=1:3
                   if(R(i,j,k,l)~=0)
                       eqf=find(ind2~=i & ind2~=j);
                       dcovT=0;
                       for m=1:3
                           dcovT=dcovT+gamma(eqf,l,m)*T(k,m)-gamma(eqf,m,k)*T(m,l);
                       end
                       d=d+delta(i,j,eqf)/2*(diff(R(i,j,k,l),x(eqf))*T(k,l)+R(i,j,k,l)*dcovT);
                   end
```

```
                    end
                end
            end
        end
    end
end
```

LISTING G.5: Computation of the curvature operator associated to Fedosov structures (Proposition 5.19)

```
clear all
tic
sym n;
n=4;
x=sym('x_%d', [1 n]);
w=sym('g%d%d', [n n]);
winv=sym('ginv%d%d', [n n]);
gamma=sym('gamma%d%d_', [n n n]);
tensoraux=sym('gamma%d%d_', [n n n]);
R=sym('R%d%d%d_', [n n n n]);

% Definition of the symplectic form
w(:,:)=0;


w(1,2)=1;
w(2,1)=-w(1,2);
w(3,4)=1;
w(4,3)=-w(3,4);

winv=inv(w);

% Definition of the symplectic connection

gamma(:,:,:)=0;
tensoraux(:,:,:)=0;

v=perms([2 3 4]).';
tensoraux(sub2ind(size(tensoraux),v(1,:),v(2,:),v(3,:)))=x(1)*x(3)*x(4);

v=perms([1 1 2]).';
tensoraux(sub2ind(size(tensoraux),v(1,:),v(2,:),v(3,:)))=1;


for i=1:n
    gamma(i,:,:)=-reshape(tensoraux(i,:,:), [n n])*winv;
end
```

```
for k=1:n
    nablaw(k,:,:)=diff(w,x(k))-reshape(gamma(k,:,:), [n n])*w + (reshape(gamma(k,:,:), [n n])*w).';
end

if nonzeros(~isAlways(nablaw == 0,'Unknown','false'))
    fprintf('Incompatible connection\n')
    nablaw
    return
end

%Computation of the curvature tensor

for i=1:n
    dgamma(:,:,:,i)=diff(gamma,x(i));
end

for i=1:n
    for l=1:n
        R(l,i,:,:)= reshape(dgamma(i,:,l,:),[n n]).' - reshape(dgamma(i,:,l,:),[n n]) - reshape(gamma(i,:,:),[n n])*reshape(gamma(:,:,l),[n n]) + (reshape(gamma(i,:,:),[n n])*reshape(gamma(:,:,l),[n n])).' ;
    end
end

R=simplify(R);

Prueba_funciones


toc
```

LISTING G.6: Computation of scalar invariants of Fedosov structures (Proposition 5.19)

```
Rdddd=sym('R22%d%d%d%d', [n n n n]);
Ruddd=sym('R22%d%d%d%d', [n n n n]);
Ruudd=sym('R22%d%d%d%d', [n n n n]);
Ruudu=sym('R22%d%d%d%d', [n n n n]);
Ruuud=sym('R22%d%d%d%d', [n n n n]);
Ruuuu=sym('R22%d%d%d%d', [n n n n]);
Rduuu=sym('R22%d%d%d%d', [n n n n]);
Ru4=sym('R22%d%d%d%d', [n n n n]);
Ru3=sym('R22%d%d%d%d', [n n n n]);
Ricc=sym('Ricc%d%d', [n n]);
gammau=sym('R22%d%d%d', [n n n]);
gammauu=sym('R22%d%d%d', [n n n]);


% Computation of auxiliary tensors

for i=1:n
    for j=1:n
        Rdddd(:,:,i,j)=w*reshape(R(:,:,i,j), [n n]);
    end
end



for i=1:n
    for j=1:n
        Ruudd(:,:,i,j)=-reshape(R(:,:,i,j), [n n])*winv;
    end
end

for i=1:n
    for j=1:n
        Ruudu(i,j,:,:)=-reshape(Ruudd(i,j,:,:), [n n])*winv;
    end
end

for i=1:n
    for j=1:n
        Ruuud(:,:,i,j)=-Ruudu(:,:,j,i);
    end
```

```
end


for i=1:n
    for j=1:n
        Ruuuu(i,j,:,:)=winv*reshape(Ruudu(i,j,:,:), [n n]);
    end
end

for i=1:n
    for j=1:n
        Rduuu(:,:,i,j)=w*reshape(Ruuuu(:,:,i,j), [n n]);
    end
end


for i=1:n
    for j=1:n
        Ricc(i,j)=trace(reshape(R(:,i,:,j),[n n]));
    end
end

Ricc02=-winv*Ricc*winv;

for i=1:n
    gammau(:,:,i)=winv*reshape(gamma(:,:,i), [n n]);
end

for i=1:n
    gammauu(i,:,:)=winv*reshape(gammau(i,:,:), [n n]);
end


% Computation of the natural functions


for i=1:n
    for j=1:n
        dR(:,:,:,:,i,j)=diff(diff(Ruuud,x(i)),x(j));
    end
end

```

```
for i=1:n
    for j=1:n
        for k=1:n
            dgamma(:,:,:,i,j,k)=diff(diff(diff(gammauu,x(i)),x(j)),x(k));
        end
    end
end

f1=simplify(sum(Rdddd(:).*Ruuuu(:)))

f2=simplify(trace(Ricc02*Ricc))

f3=0;
for i=1:n
    for j=1:n
        for k=1:n
            f3=f3+dR(i,j,k,i,j,k);
        end
    end
end

f4=0;
for i=1:n
    for j=1:n
        for k=1:n
            f4=f4+dgamma(i,j,k,i,j,k);
        end
    end
end

f3=simplify(f3)

f4=simplify(f4)
```

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