EXTRACTA MATHEMATICAE Vol. **19**, Núm. 2, 257 – 260 (2004)

Diagonalization in Banach Algebras and Jordan-Banach Algebras

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(Presented by A. Rodríguez Palacios)

AMS *Subject Class.* (2000): 46H70, 17A15 *Received September* 10, 2003

1. Introduction

Let A be a complex Banach or Jordan-Banach algebra. To study the properties of the spectrum function $x \mapsto \operatorname{Sp} x$ we use the so called Hausdorff distance on compact sets of the complex plane $\mathbb{C}$ defined by

$$\Delta(\sigma_1, \sigma_2) = \max\left\{ \sup_{\lambda\in\sigma_2} \{\operatorname{dist}(\lambda, \sigma_1)\}, \sup_{\lambda\in\sigma_1} \{\operatorname{dist}(\lambda, \sigma_2)\} \right\}$$

where $\operatorname{dist}(\lambda, \sigma) = \inf\{|\lambda - \mu| : \mu \in \sigma\}$ is the distance of the point λ to the compact set σ (see [1, p. 48]). In this paper, we intend to prove that if the spectrum of an element $a \in A$ is finite and the function $x \mapsto \operatorname{Sp} x$ is lipschitzian at a, that is $\Delta\big(\operatorname{Sp}(a + x), \operatorname{Sp}(a)\big) \le M||x||$, then a is diagonalizable; in other words we can write a as a linear combination of projections. B. Aupetit proved an analogous spectral characterization for idempotents in [3], that is, elements $e \in A$ such that $e^2 = e$ (hence in particular $\operatorname{Sp} e = \{0, 1\}$), this is in fact contained in the proof of [3, Theorem 1.1]. Recall that a Banach algebra A is said to be semisimple if $\operatorname{Rad}(A) = \{0\}$, where $\operatorname{Rad}(A)$ is the Jacobson radical of A. In what follows $\rho(x)$ stands for the spectral radius of the element x, in our case we can write $\rho(x) = \max\{|\lambda| : \lambda \in \operatorname{Sp} x\}$. We gather now some well-known results on the spectrum [1, 3].

Proposition 1.1. *Let A be a semisimple Banach algebra and let p be an idempotent element of A. Then pAp is a closed semisimple subalgebra of A*

with identity p such that $\mathrm{Sp}_{pAp}(pxp) \subset \mathrm{Sp}_A(pxp)$ and $\rho_{pAp}(pxp) = \rho_A(pxp)$ for every $x \in A$. Moreover, if $a_1, \ldots, a_n$ are elements of A such that $a_i a_j = 0$ for $i \neq j$, then

$$\mathrm{Sp}(a_1 + \cdots + a_n) \cup \{0\} = \mathrm{Sp}\, a_1 \cup \cdots \cup \mathrm{Sp}\, a_n \cup \{0\}.$$

2. The Banach algebra case

To prove our result we need the following lemma which is taken from [3]. A proof is included since it is short and essential for our theorem.

Lemma 2.1. *Let $q \in A$ be a quasinilpotent element. Suppose there exists two positive constants r and M such that $\rho(x) \leq M||x - q||$ for $||x - q|| < r$, then $q \in \mathrm{Rad}(A)$.*

Proof. Let $y \in A$ be arbitrary. For $|\lambda| > \frac{||y||}{r}$, we have $\rho\big(q + \frac{y}{\lambda}\big) \leq M\frac{||y||}{|\lambda|}$, so $\rho(y + \lambda q) \leq M||y||$. Hence the upper semicontinuous function $\lambda \mapsto \rho(y + \lambda q)$ is bounded on the complex plane $\mathbb{C}$. Being subharmonic [1, Theorem 3.4.7], it is constant by Liouville's theorem for subharmonic functions [1, Theorem A.1.1]. So $\rho(y + \lambda q) = \rho(y)$ for every $y \in A$. By the characterization of the radical [1, Theorem 5.3.1] we have $q \in \mathrm{Rad}(A)$. ∎

Theorem 2.2. *Let A be a semisimple Banach algebra, and let $a \in A$ have finite spectrum, $\mathrm{Sp}\, a = \{\alpha_1, \ldots, \alpha_n\}$. Suppose that the spectral mapping $x \mapsto \mathrm{Sp}_A(x)$ is lipschitzian at a. Then there exists n nonzero orthogonal projectons $p_1, \ldots, p_n$ whose sum is 1 such that $a = \alpha_1 p_1 + \cdots + \alpha_n p_n$.*

Proof. Let $a \in A$ be with $\mathrm{Sp}\, a = \{\alpha_1, \ldots, \alpha_n\}$. By Holomorphic Functional calculus (see [5, Proposition 7.9]), there exist n nonzero orthogonal projections $p_1, \ldots, p_n$ such that $1 = p_1 + \cdots + p_n$, $p_i p_j = 0$ for $i \neq j$, $ap_i = p_i a$ and $\mathrm{Sp}_A(a_i) = \{\alpha_i\}$, for $1 \leq i \leq n$. Set $a_i = p_i a$. Then $a_i \in p_i A p_i$ with $\mathrm{Sp}_{p_i A p_i}(a_i) = \{\alpha_i\}$ by Proposition 1.1. Let us now see that the spectral mapping $x \mapsto \mathrm{Sp}_{p_i A p_i}(x)$ is lipschitzian at $a_i - \alpha_i p_i$ for each $1 \leq i \leq n$. Indeed, take $i = 1$ and let $x_1 \in p_1 A p_1$, we have

$$\begin{aligned} a - \alpha_1 1 + x_1 &= \sum_{i=1}^{n} a_i - \alpha_1 \left(\sum_{i=1}^{n} p_i \right) + x_1 \\ &= (a_1 - \alpha_1 p_1 + x_1) + \sum_{j=2}^{n} (a_j - \alpha_1 p_j). \end{aligned}$$

By the preceding and orthogonality we have

$$\mathrm{Sp}_A(a - \alpha_1 1 + x_1) \cup \{0\} = \mathrm{Sp}_A(a_1 - \alpha_1 p_1 + x_1) \cup \{0, \alpha_2 - \alpha_1, \dots, \alpha_n - \alpha_1\}$$

and

$$\begin{aligned}
\rho_{p_1 A p_1}(a_1 - \alpha_1 p_1 + x_1) &= \rho_A(a_1 - \alpha_1 p_1 + x_1) \\
&= \Delta\big(\mathrm{Sp}_A(a_1 - \alpha_1 p_1 + x_1), \mathrm{Sp}_A(a_1 - \alpha_1 p_1)\big) \\
&= \Delta\big(\mathrm{Sp}_A(a - \alpha_1 p_1 + x_1), \mathrm{Sp}_A(a - \alpha_1 p_1)\big) \\
&= \Delta\big(\mathrm{Sp}_A(a + x_1), \mathrm{Sp}_A(a)\big) \\
&\le M||x_1||.
\end{aligned}$$

Therefore, $x \mapsto \mathrm{Sp}_{p_1 A p_1}(x)$ is lipschitzian at $a_1 - \alpha_1 p_1$, with $\mathrm{Sp}_{p_1 A p_1}(a_1 - \alpha_1 p_1) = 0$. Hence $a_1 - \alpha_1 p_1 = 0$ by Lemma 2.1, which completes the proof. ∎

3. The Jordan-Banach case

We recall that a complex Jordan algebra A is non-associative and the product satisfies the identities $ab = ba$ and $(ab)a^2 = a(ba^2)$, for all a, b in A. A unital Jordan-Banach algebra is a Jordan algebra with a complete submultiplicative norm. An element $a \in A$ is said to be invertible if there exists $b \in A$ such that $ab = 1$ and $a^2 b = a$. The spectrum of $x \in A$ is by definition the set of $\lambda \in \mathbb{C}$ for which $\lambda - x$ is not invertible in A.

Since we consider only complex semisimple Jordan-Banach algebras, the analogue of Proposition 1.1 is valid in the Jordan case if we replace the subalgebra pAp by $U_p A$ and use Propositions 4, 5, 6 and 7 of [4].

Using the characterization of the Jacobson radical for Jordan algebras obtained in [2], Lemma 2.1 is also true for Jordan-Banach algebras.

With all these results and those in [6] we can establish exactly as in the associative case the following analogue of Theorem 2.2.

THEOREM 3.1. *Let A be a semisimple complex Jordan-Banach algebra, and let $a \in A$ have finite spectrum, $\mathrm{Sp}\, a = \{\alpha_1, \dots, \alpha_n\}$. Suppose that the spectral mapping $x \mapsto \mathrm{Sp}_A(x)$ is lipschitzian at a. Then there exists n nonzero orthogonal projectons $p_1, \dots, p_n$ whose sum is 1 such that $a = \alpha_1 p_1 + \dots + \alpha_n p_n$.*

Proof. Almost the same as the proof of Theorem 2.2 except that the unital algebra pAp is replaced by the algebra $U_p A$. Note also that the spectral mapping theorem used here is applied to the full closed subalgebra of A generated by a which is unital and associative (cf. [6, Theorem 2.7]). ∎

Acknowledgements

The autor wishes to express his gratitude to the referee for his valuable comments and suggestions.

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