Special Operators on Classical Spaces of Analytic Functions

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Introduction

In the first part of these notes we survey a number of results on composition operators on Hardy spaces and weighted Bergman spaces on the open unit disc D in $\mathbb{C}$. In a straightforward manner we can identify such operators in the Bergman case as formal identities from the ambient spaces into Lebesgue spaces which are associated with so-called Carleson measures on D and we convince ourselves that the extension to Hardy spaces requires to take into account analogous measures on the closed disc $\overline{D}$. We discuss measures of this kind along with the resulting embeddings into Lebesgue spaces in some detail, and we show how results known e.g. for composition operators can be generalized to such embeddings. Carleson measures depending on certain fixed parameters form a Banach lattice whose identification as the dual of an appropriate function space is among the topics of the final section.

These notes are based on published and forthcoming papers by several authors. Their purpose was to serve as classroom notes distributed among the participants of the 2003 summer course in Laredo.

1. Composition operators on Hardy spaces

1.1. Preliminaries. Throughout, we will apply standard terminology and notation of functional analysis and function theory.

We will work on the the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane, its closure $\overline{D}$, and the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} =$

∂D. It is well-known that, with respect to the topology of local uniform convergence, the analytic functions $f : D \to \mathbb{C}$ form a Fréchet space, $\mathcal{H}(D)$. The topology is given by the seminorms $p_K(f) := \max_{z \in K} |f(z)|$, K varying over the non-empty compact subsets of D. It suffices to look at the p_{K_n}'s for $K_n = r_n \overline{D}$ where (r_n) is any sequence in the interval $(0, 1)$ such that $\lim_n r_n = 1$. Note that the p_{K_n}'s are even norms.

Montel's Theorem asserts that bounded sets in $\mathcal{H}(D)$ are relatively compact. Even more, if X is any Banach space, then every (bounded) operator $[\mathcal{H}(D), p_{K_n}] \to X$ is compact. It is in fact even nuclear, so that $\mathcal{H}(D)$ is a nuclear locally convex space (see e.g. [46], [47]).

The following simple fact will be of some importance: every 'point evaluation'

$$\delta_z : \mathcal{H}(D) \longrightarrow \mathbb{C} : f \mapsto f(z) , \quad z \in D,$$

is a continuous linear form. In particular,

- if a linear subspace E of $\mathcal{H}(D)$ carries a vector topology such that $E \hookrightarrow \mathcal{H}(D)$ is continuous, then E has a separating dual.

1.2. HARDY SPACES.   Normalized Lebesgue measure on $\mathbb{T}$ will be denoted by dm, so $dm(e^{it}) \equiv dt/(2\pi)$. We write $L^p(\mathbb{T})$ for $L^p(m)$, $0 < p \le \infty$. If $f \in \mathcal{H}(D)$ and $0 < r < 1$, then $f_r : \overline{D} \to \mathbb{C} : z \mapsto f(rz)$ is continuous, analytic on D and, for any $0 < p \le \infty$,

$$M_p(f, r) := \|f_r\|_{L^p(\mathbb{T})} < \infty .$$

Each $M_p(\,\cdot\,, r) : \mathcal{H}(D) \to [0, \infty)$ defines a norm on $\mathcal{H}(D)$ if $p \ge 1$, and a p-norm if $0 < p < 1$. The $M_\infty(\,\cdot\,, r)$ are just the above norms $p_{r\overline{D}}$. Each $M_p(\,\cdot\,, r)$ is continuous on $\mathcal{H}(D)$; in fact, $M_p(\,\cdot\,, r) \le M_\infty(\,\cdot\,, r)$.

If $f \in \mathcal{H}(D)$ and $0 < p \le \infty$, then $\|f\|_{H^p} := \sup_{r<1} M_p(f, r)$ exists in $[0, \infty]$. The spaces

$$H^p(D) := \{f \in \mathcal{H}(D) : \|f\|_{H^p} < \infty\}$$

are the classical 'Hardy spaces'. They are (p-) Banach spaces. We extend the scale of these spaces by introducing the Banach space $H^\infty(D)$ of bounded analytic functions, the norm being the usual sup-norm. If $0 < q < p < \infty$, then $H^\infty(D) \hookrightarrow H^p(D) \hookrightarrow H^q(D)$ contractively, and each of these spaces embeds continuously into $\mathcal{H}(D)$.

For any $0 < p \le \infty$ and $z \in D$, $\delta_z : H^p(D) \to \mathbb{C} : f \mapsto f(z)$ is bounded. If $p = 2$, then there is a unique function $K(z, \cdot) \in H^2$ such that $f(z) = \delta_z(f) =$

$(f|K(z, \cdot))_{H^2}$; here $(\cdot|\cdot)_{H^2}$ is the scalar product of H^2. A geometric series argument reveals that

$$K(z, w) = (1 - \overline{z}w)^{-1} \qquad \forall w \in D .$$

$K : D \times D \to \mathbb{C}$ is the 'reproducing kernel' for the Hilbert space $H^2(D)$. It is readily verified that $\|\delta_z\|_{(H^2)^*} = \|K(z, \cdot)\|_{H^2} = (1 - |z|^2)^{-1/2}$.

A well-known consequence of Fatou's Theorem asserts that if f belongs to any $H^p(D)$, then

$$f^*(\zeta) := \lim_{r \to 1} f_r(\zeta)$$

exists for m-almost every $\zeta \in \mathbb{T}$. Moreover, an element $f^* \in L^p(\mathbb{T})$ is generated in this way, and $H^p(D) \to L^p(\mathbb{T}) : f \mapsto f^*$ is an isometric embedding. Its range, $H^p(\mathbb{T})$, is the closure in $L^p(\mathbb{T})$ (weak*-closure if $p = \infty$) of all polynomials in ζ. Therefore $H^p(D)$ and $H^p(\mathbb{T})$ are often identified; the common notation is H^p. It is also customary to simply write f instead of f^*.

The extension $\mathbb{T} \times D : (\zeta, z) \mapsto (1 - \overline{\zeta}z)^{-1}$ of the reproducing kernel $K(\cdot, \cdot)$ is the 'Cauchy kernel'; we denote it also by K. If $q \geq 1$ and $g \in H^p(\mathbb{T})$, then

$$G(z) = \int_{\mathbb{T}} g(\zeta)K(\zeta, z) \, dm(\zeta) \tag{1}$$

belongs to $H^p(D)$, and $G^* = g$ m-a.e.

The orthogonal projection in $L^2(\mathbb{T})$ with range $H^2(\mathbb{T})$ defines a bounded projection P_{-1} in $L^p(\mathbb{T})$ with range $H^p(\mathbb{T})$ whenever $1 < p < \infty$; for $g \in L^p(\mathbb{T})$, the corresponding function $P_{-1}(g) \in H^p(D)$ is given by (1). P_{-1} is the 'Szegö projection'. It is, however, unbounded on both, $L^1(\mathbb{T})$ and $L^\infty(\mathbb{T})$. Even worse: if $s = 1, \infty$, then H^s cannot be complemented in any L^s-space. For a simple reason this is also true for $0 < s < 1$: if μ is a non-atomic measure, then 0 is the only bounded linear form on $L^s(\mu)$.

Related are duality results for Hardy spaces. If $1 < p, q < \infty$ and $(1/p) + (1/q) = 1$ then $(H^p)^*$ and H^q are isomorphic; the dual pairing (traditionally not bilinear) is given by

$$\langle f, g \rangle_{-1} = \int_{\mathbb{T}} f\overline{g} \, dm .$$

The dual of H^1 'is' the space $BMOA$ of analytic functions of bounded mean oscillation. It can be represented as the image of $L^\infty(\mathbb{T})$ under the Szegö projection. H^∞ is properly contained in $BMOA$. The subspace $VMOA$ of $BMOA$ consisting of all analytic functions f of vanishing mean oscillation 'is'

the predual of H^1. It can be represented as the image of $\mathcal{C}(\mathbb{T})$ under the Szegö projection. Details are in standard textbooks, e.g. [19], [22], [35], [65], . . .

We shall need a qualitative version of a result which goes back to W.Blaschke and F.Riesz.

1.2.1. *Every $0 \neq f \in H^p$, $0 < p \leq \infty$, admits a factorisation $f = b \cdot g$, where $g \in H^p$ has no zeros and $b \in H^\infty$ satisfies $|b(\zeta)| = 1$ for m-almost all $\zeta \in \mathbb{T}$. In particular, $\|g\|_{H^p} = \|f\|_{H^p}$.*

See e.g. [19]; the function b can in fact be constructed in a natural way from the zeros of f ('Blaschke product'). Here is a simple application.

1.2.2. *Suppose that $0 < p, q, r < \infty$ satisfy $(1/r) = (1/p) + (1/q)$. If $f \in H^p$ and $g \in H^q$, then $fg \in H^r$, and $\|fg\|_{H^r} \leq \|f\|_{H^p} \cdot \|g\|_{H^q}$. Moreover, every $h \in H^r$ can be written $h = fg$ where $f \in H^p$ and $g \in H^q$ are such that $\|h\|_{H^r} = \|f\|_{H^p} \cdot \|g\|_{H^q}$.*

Thus $(f, g) \mapsto fg$ defines a continuous bilinear map $H^p \times H^q \to H^r$ which is *onto*.

Proof. Hölder's inequality yields the first statement. As for the second, write $h = bh_1$ where $h_1 \in H^r$ has no zeros and $|b| = 1$ m-a.e. on $\mathbb{T}$. Note that $f = bh_1^{r/p}$ and $g = h^{r/q}$ exist in H^p and H^q, respectively. These are the functions we are looking for. ∎

1.3. POINT EVALUATIONS AND COMPOSITION OPERATORS. We know already that the norm of a point evaluation δ_z as a functional on H^2 is $\|\delta_z\|_{(H^2)^*} = \|K(z, \cdot)\|_{H^2} = (1 - |z|^2)^{-1/2}$. Clearly, $\|\delta_z\|_{(H^\infty)^*} = 1$. Suppose now that $f \in B_{H^p}$, $0 < p < \infty$. Write $f = bg$ where $|b(\zeta)| = 1$ m-a.e. and $g \in B_{H^p}$ has no zeros. Let $h \in B_{H^2}$ be such that $h^2 = g^p$. Then $|\langle \delta_z, f \rangle|^p = |b(z)g(z)|^p = |b(z)|^p \cdot |h(z)|^2 \leq |\langle \delta_z, h \rangle|^2 \leq \|\delta_z\|_{H^2}^2$, hence $\|\delta_z\|_{(H^p)^*}^p \leq \|\delta_z\|_{(H^2)^*}^2$. Exchange the rôles of 2 and p to see that in fact $\|\delta_z\|_{(H^p)^*} = (1 - |z|^2)^{-1/p}$.

Let us say that a linear form u on H^p is 'multiplicative' if $u \neq 0$ and $u(f\,g) = u(f)\,u(g)$ for all $f, g \in H^p$ such that $fg \in H^p$. It is clear that point evaluations are multiplicative.

1.3.1. *Multiplicative linear forms on H^p, $0 < p < \infty$, are continuous. In fact, they are precisely the point evaluations.*

The picture for H^∞, however, is very much different (L. Carleson [7]).

Proof. Let $u : H^p \to \mathbb{C}$ be a multiplicative linear form. Then $u(1) = 1$, since $u \neq 0$. Define $z_0 := u(z)$ where $u(z)$ is shorthand for $u(id_D)$. We show $z_0 \in D$ and verify then $u = \delta_{z_0}$.

Suppose that $|z_0| \geq 1$ and consider $g : D \to \mathbb{C} : z \mapsto (z_0 - z)^{-1}$. If $|z_0| > 1$ then $g \in H^\infty(D)$. It is an exercise to show that $g \in H^s \setminus H^1$ for $0 < s < 1$ if $|z_0| = 1$. In any case, g doesn't vanish, and so there are $h \in H^p$ and $N \in \mathbb{N}$ such that $h(z)^N = g(z)$ for all $z \in D$. Surely, $(1/h)(z) = (z_0 - z)^{1/N}$ and $(1/h^N)$ are members of H^∞ and hence of H^p. But $u(1/h) = 0$ since $u(1/h^N) = u(z_0 - z) = u(z_0 \cdot 1) - u(id_D) = z_0 - z_0 = 0$, so $1 = u(h \cdot (1/h)) = 0$, a contradiction.

As for the second part, fix $f \in H^p$ and define $g : D \to \mathbb{C}$ by $g(z) := \dfrac{f(z_0) - f(z)}{z_0 - z}$ if $z \neq z_0$ and by $g(z_0) := f'(z_0)$. Then $g \in H^p$, $u(f) = u(f(z_0)) - u(z_0 - z)u(g)$, and so $u(f) = u(f(z_0)) = f(z_0) = \delta_{z_0}(f)$. ∎

The collection of all analytic self-maps $\varphi : D \to D$ will be denoted by Φ. This is the unit ball of H^∞ from which the constant functions of modulus one have been deleted. For each $\varphi \in \Phi$, the 'composition operator'

$$C_\varphi : \mathcal{H}(D) \longrightarrow \mathcal{H}(D) : f \mapsto f \circ \varphi$$

is well-defined, linear and continuous. It is also clear that $C_\varphi : H^\infty \to H^\infty$ exists and is bounded with $\|C_\varphi\| = 1$. What's about the other Hardy spaces H^p?

If $\varphi(0) = 0$, then 'Littlewood's Subordination Principle' ([19], [56]) asserts that

$$M_p(f \circ \varphi, r) \leq M_p(f, r) \qquad \forall \, 0 < r < 1 \, .$$

So in this case, $C_\varphi : H^p \to H^p$ is well-defined with $\|C_\varphi\| \leq 1$. Even $\|C_\varphi\| = 1$, since $C_\varphi(1) = 1$.

Recall that for each $a \in D$, the 'Möbius transform'

$$\tau_a : z \mapsto \frac{a - z}{1 - \overline{a}z}$$

defines an (analytic) automorphism of D; it exchanges a and 0 and satisfies $\tau_a^{-1} = \tau_a$. Actually, $\{e^{i\theta}\tau_a : a \in D, \, \theta \in \mathbb{R}\}$ is the group of all automorphisms of D. Clearly, $\tau_a \in \Phi$.

If $f \in H^p$ and $a \in D$, then $f \circ \tau_a \in H^p$ and $\|f \circ \tau_a\|_{H^p} \leq (1 + |a|)^{1/p}/(1 - |a|)^{1/p} \cdot \|f\|_{H^p}$. In other words, the operator $C_{\tau_a} : H^p \to H^p : f \mapsto f \circ \tau_a$ is well-defined and bounded with $\|C_{\tau_a}\| \leq (1 + |a|)^{1/p}/(1 - |a|)^{1/p}$.

If now $\varphi \in \Phi$ and $a = \varphi(0) \neq 0$, then $\psi = \tau_a \circ \varphi : D \to D$ is analytic and satisfies $\psi(0) = 0$ and $\varphi = \tau_a^{-1} \circ \psi = \tau_a \circ \psi$. It follows that $C_\varphi = C_{\tau_a \circ \psi} = C_\psi \circ C_{\tau_a} : H^p \to H^p$ is well-defined and bounded with $\|C_\varphi\| \leq (1 + |\varphi(0)|)^{1/p}/(1 - |\varphi(0)|)^{1/p}$. So far, however, apparently nobody has been able to calculate the exact value of $\|C_\varphi\|$ in the general case.

We say that a linear operator $u : H^p \to \mathcal{H}(D)$ is 'multiplicative' if $u \neq 0$ and $u(f \cdot g) = u(f) \cdot u(g)$ for all $f, g \in H^p$ such that $fg \in H^p$. Note that $u(1_D) = 1_D$. Composition operators are multiplicative. That the converse is also true goes back to L. Bers [3]:

1.3.2. *Multiplicative operators* $H^p \to \mathcal{H}(D)$ *are continuous. In fact, they are just the composition operators and so even act boundedly* $H^p \to H^p$.

Proof. Let $u : H^p \to \mathcal{H}(D)$ be multiplicative. For each $z \in D$, $\delta_z \circ u$ is a multiplicative linear form on D: $\delta_z \circ u \neq 0$ since $(\delta_z \circ u)(1_D) = u(1_D)(z) = 1_D(z) = 1$. By 1.3.1, $\delta_z \circ u = \delta_w$ where $w = w_z \in D$ is uniquely determined by z and u. Consider $\varphi : D \to D : z \mapsto w_z$. Then $u(f)(z) = (\delta_z \circ u)(f) = \delta_{\varphi(z)}(f) = f(\varphi(z))$. Now φ is analytic since $u(id_D) \in \mathcal{H}(D)$ and $u(id_D)(z) = id_D(\varphi(z)) = \varphi(z)$ for all $z \in D$. ∎

Given $z_0 \in D$, define $\varphi_{z_0} \in \Phi$ by $\varphi_{z_0}(z) = z_0$ for all $z \in D$. Then, for each $f \in H^p$ and $z \in D$,

$$C_{\varphi_{z_0}}(f)(z) = f(\varphi_{z_0}(z)) = f(z_0) = \langle \delta_{z_0}, f \rangle \ .$$

In other words, point evaluations 'are' special composition operators.

Much more on composition operators on Hardy spaces can be found in the books [12] by C. Cowen and B. MacCluer and [56] by J.H. Shapiro.

1.4. DIFFERENT EXPONENTS. Every composition operator C_φ maps H^p boundedly into H^q when $q \leq p$. We will also consider the question under which conditions C_φ 'improves integrability' in the sense that it maps H^p into H^q for $q > p$. We show first of all that this depends only on the ratio p/q (see [27], [28]):

1.4.1. *Let* $1 \leq \beta \leq \infty$ *and* $\varphi \in \Phi$ *be given. If* $C_\varphi(H^p) \subseteq H^{\beta p}$ *holds for some* $0 < p < \infty$, *then this is true for all of them. Moreover, in such a case,* $\|C_\varphi : H^p \to H^{\beta p}\| = \|C_\varphi : H^1 \to H^\beta\|^{1/p}$.

Proof. Fix $0 < p_0 < \infty$ and $f \in H^{p_0}$. Write $f = bg$ where $g \in H^{p_0}$ doesn't vanish and $|b| = 1$ m-a.e. on $\mathbb{T}$. $g^{p_0/p}$ exists in H^p and $C_\varphi(g^{p_0/p}) = C_\varphi(g)^{p_0/p} \in H^{\beta p}$ has no zeros. Thus $C_\varphi(g) \in H^{\beta p_0}$. Also, $C_\varphi(f) \in H^{\beta p_0}$ since $|C_\varphi(f)| = |C_\varphi(b) \cdot C_\varphi(g)| \le |C_\varphi(g)|$. Moreover,

$$
\left(\int_{\mathbb{T}} |C_\varphi(f)(\zeta)|^{\beta p_0} dm(\zeta) \right)^{1/(\beta p_0)} \le \left(\int_{\mathbb{T}} |C_\varphi(g)|^{\beta p_0} dm \right)^{1/(\beta p_0)}
$$

$$
= \left(\int_{\mathbb{T}} |C_\varphi(g^{p_0/p})|^{\beta p} dm \right)^{\frac{1}{\beta p} \frac{p}{p_0}}
$$

$$
\le \|C_\varphi : H^p \to H^{\beta p}\|^{p/p_0} \cdot \|g^{p_0/p}\|_{H^p}^{p/p_0}
$$

$$
= \|C_\varphi : H^p \to H^{\beta p}\|^{p/p_0} \cdot \left(\int_{\mathbb{T}} |b^*|^{p_0} \cdot |g^*|^{p_0} dm \right)^{1/p_0}
$$

$$
= \|C_\varphi : H^p \to H^{\beta p}\|^{p/p_0} \cdot \|f\|_{H^{p_0}} .
$$

Thus $\|C_\varphi : H^{p_0} \to H^{\beta p_0}\|^{p_0} \le \|C_\varphi : H^p \to H^{\beta p}\|^p$. Exchange the rôles of p and p_0 to obtain equality. ∎

1.4.1 is trivially true for $\beta \le 1$. We will see that this does not extend to compactness and related properties. Nevertheless, we can prove:

1.4.2. *If, for some $\beta > 0$ and $0 < p < \infty$, C_φ is compact as an operator $H^p \to H^{\beta p}$, then it is compact as an operator $H^r \to H^{\beta r}$, for every $0 < r < \infty$.*

Proof. Let $(f_n)_n$ be a sequence in B_{H^r}. As a sequence in $\mathcal{H}(D)$, (f_n) has a subsequence which converges locally uniformly to some $f \in \mathcal{H}(D)$. By Fatou's Lemma, f belongs to H^r. We can therefore assume that (f_n) is a sequence in B_{H^r} which converges pointwise to zero. By 1.2.1, each f_n has the form $f_n = b_n g_n$ where $|b_n| = 1$ m-a.e. on $\mathbb{T}$ and g_n has no zeros. As before there is no loss in assuming that (b_n) and (h_n) converge locally uniformly to some $b \in H^\infty$ and $g \in H^r$, respectively. By a classical result of Hurwitz, either g has no zeros or it vanishes identically. Because of $bg = 0$, the second alternative applies. The $g_n^{r/p}$ belong to B_{H^p} and converge pointwise to zero. Use the hypothesis and pass to another subsequence if necessary in order to obtain $\lim_n \|C_\varphi(g_n^{r/p})\|_{H^{\beta p}} = 0$. This implies $\lim_n \|C_\varphi(f_n)\|_{H^{\beta r}} = 0$. ∎

In the sequel, we will refer to operators as in 1.4.2 as 'β-bounded' and 'β-compact' composition operators, respectively. As was already mentionend, there is no need for parameters $\beta < 1$ as far as boundedness is concerned. This changes, however, if we pass to questions related to compactness.

1.5. COMPLETE CONTINUITY. Let X be a Banach space and Y a quasi-Banach space. Recall that an operator $u : X \to Y$ is labeled 'completely continuous' if it maps weakly null sequences of X into a 'norm' null sequences of Y. Compact operators are completely continuous. The converse is false; there are even infinite dimensional Banach spaces X which enjoy the 'Schur property', i.e., id_X is completely continuous. The most prominent example is the sequence space ℓ^1; see [14] or [15], for example.

What's about complete continuity for a composition operator C_φ? Let $E_\varphi := \{\zeta \in \mathbb{T} : |\varphi^*(\zeta)| = 1\}$ be the set of 'contact points' (of any measurable representative) of φ's Fatou extension φ^*. Write φ^n for the function $z \mapsto \varphi(z)^n$ and note that $\|\varphi^n\|_\beta^\beta = m(E_\varphi) + \int_{\mathbb{T}\setminus E_\varphi} |\varphi^n|^\beta dm$ for each n, whence $m(E_\varphi) = \lim_n \|\varphi^n\|_\beta^\beta$; here $\beta > 0$ is arbitrary. The monomials z^n, $n \geq 0$, form a weak null sequence in H^1, so that $m(E_\varphi) = 0$ whenever C_φ is completely continuous as an operator $H^1 \to H^\beta$. For $\beta = 1$ we arrive at the following extension of a result by J. Cima and A. Matheson [9]:

1.5.1. *For each $\varphi \in \Phi$, the following are equivalent statements:*

(i) $m(E_\varphi) = 0$.

(ii) $C_\varphi : H^1 \to H^1$ *is completely continuous.*

(iii) *For all $0 < q < p \leq \infty$, C_φ is compact as an operator $H^p \to H^q$.*

(iv) *There exist $0 < q < p \leq \infty$ such that C_φ is compact as an operator $H^p \to H^q$.*

Proof. (i)$\Rightarrow$(ii): Let (f_n) be a weak null sequence in H^1. The $C_\varphi f_n$ form a weak null sequence in H^1, and for m-almost all $\zeta \in \mathbb{T}$, $(f_n(\varphi(\zeta)))_n = (\langle \delta_{\varphi(\zeta)}, f_n \rangle)_n$ converges to zero.

Since weakly compact subsets of L^1 are uniformly integrable we can find, given any $\varepsilon > 0$, a $\delta > 0$ such that $\int_B |C_\varphi f_n| \, dm < \varepsilon$ for all Borel sets $B \subseteq \mathbb{T}$ which satisfy $m(B) < \delta$. Egorov's Theorem allows us to select B so that in addition $C_\varphi f_n \to 0$ uniformly on $\mathbb{T} \setminus B$. Let $n_\varepsilon \in \mathbb{N}$ be such that $\int_{\mathbb{T}\setminus B} |C_\varphi f_n| \, dm < \varepsilon$ for $n \geq n_\varepsilon$. Then $\|C_\varphi f_n\|_1 \leq 2\varepsilon$ for $n \geq n_\varepsilon$. – We have shown that $\lim_n \|C_\varphi f_n\|_1 = 0$.

(ii)$\Rightarrow$(iii): Let $1 < p \leq \infty$. If C_φ is completely continuous as an operator $H^1 \to H^1$ then it is compact as an operator $H^p \to H^1$, and we are done by 1.4.2.

(iii)$\Rightarrow$(iv) is trivial, and (iv)$\Rightarrow$(i) follows from the introductory remarks. ∎

Note that (i) is really a statement about the image measure $m_\varphi :=$ $m((\varphi^*)^{-1}(\,\cdot\,))$ of φ's Fatou extension: complete continuity of C_φ means that m_φ's restriction to (the Borel subsets of) $\mathbb{T}$ vanishes.

By (iv), β-compactness of C_φ doesn't depend on the particular choice of β as long as $0 < \beta < 1$. This is much in contrast to what happens if $\beta \geq 1$. In fact, there are completely continuous composition operators on H^1 which are not 1-compact; perhaps the best known example is given by the function $\varphi(z) = (z+1)/2$ (see [56]). Moreover:

> 1.5.2.   (a) *Let* $1 \leq \gamma < \beta$. *Every β-bounded composition operator is* γ-*compact.*

(b) *There are β-bounded composition operators which fail to be β-compact.*

(a) was first observed by H. Hunziker [27], [28]. As for (b) consider e.g. a domain Δ inside D for which $\partial\Delta$ is a polygon with $\partial\Delta \cap \mathbb{T} \neq \emptyset$. Let α be the biggest angle at a contact point, and let φ be a conformal map of D onto Δ. Put $\beta = \pi/\alpha$. Then C_φ is β-bounded but not β-compact. In particular, if Δ is a rectangle, then C_φ defines a non-compact operator $H^1 \to H^2$. Results of this kind were first proved by R. Riedl [49] using probabilistic tools. A later function theoretic proof is due to W. Smith and L. Yang [58].

1.6. ADDITIONAL RESULTS.   We claim that the following holds:

> 1.6.1. *Suppose that $0 < p < \infty$ is given and that C_φ is a composition operator which maps H^1 into H^p. This operator is completely continuous if and only if, regardless of how we choose $1 < s \leq \infty$, the composition $C_\varphi \circ i_s : H^s \to H^p$ is compact. Here i_s is the formal identity $H^s \hookrightarrow H^1$.*

This is straightforward if $p \leq 1$. In fact, if C_φ is completely continuous, then $C_\varphi \circ i_s$ is compact since i_s is weakly compact. Suppose conversely that $C_\varphi \circ i_\infty$ is compact. The monomials z^n form a bounded sequence in $H^\infty(D)$ which converges pointwise to zero, and $C_\varphi \circ i_\infty$ maps each z^n to φ^n. By hypothesis, a subsequence of (φ^n) and so (φ^n) itself (monotonicity) converges to zero in H^p. We have seen before that this implies $m(E_\varphi) = 0$. By 1.5.1, even $C_\varphi : H^1 \to H^1 \hookrightarrow H^p$ is completely continuous.

The case $p > 1$ requires more work. We start by quoting a theorem of J.J. Uhl [60] which has the same flavour as 1.6.1:

> 1.6.2. *Suppose that μ is a finite measure and X is a Banach space. Then complete continuity of an operator $u : L^1(\mu) \to X$ is equivalent to*

compactness of $u \circ j_s : L^s(\mu) \to X$ *for some, and then all,* $1 < s \le \infty$. *Here* j_s *is the formal identity* $L^s(\mu) \hookrightarrow L^1(\mu)$.

The proof is based on measure theoretic tools which do not apply when dealing with analytic functions. Nevertheless, there exists a 'carbon copy' of 1.6.2 for Hardy spaces. Let again i_s be the canonical map $H^s \hookrightarrow H^1$, $s > 1$.

1.6.3. *Let* X *be a quasi-Banach space with a separating dual. An operator* $u : H^1 \to X$ *is completely continuous if and only if* $u \circ i_s : H^s \to X$ *is compact for some, and then all,* $1 < s \le \infty$.

Take note of the fact that 1.6.1 is just a very special case of this result. 1.6.3 can be proved by means of the following decomposition theorem (J. Bourgain [6], see also S.V. Kislyakov [33]):

There is a constant $C > 0$ such that, given $\lambda > 0$ and $f \in H^1$, there are $g, h \in H^1$ satisfying

$$f = g + h, \qquad |g|, |h| \le C \cdot |f|, \qquad |g| \le C \cdot \lambda \quad \text{and}$$

$$\int_{\mathbb{T}} |h| \, dm \le C \cdot \int_{\{|f| > \lambda\}} |f| \, dm \ . \tag{2}$$

Proof of 1.6.3. Up to a small correction, we repeat the proof from [29].

Necessity is obvious. As for sufficiency, we argue contrapositively and assume that there is a non-completely continuous $u \in \mathcal{L}(H^1, X)$ such that $u \circ i_\infty$ is compact. So there are a weak null sequence (f_n) in B_{H^1} and an $\varepsilon > 0$ such that $\|uf_n\| > \varepsilon$ for all n. Of course, we may assume that $\|u\| = 1$.

Let κ be the quasi-norm constant of X, and let C be the constant from (2). Since (f_n) is uniformly integrable, there is a $\delta > 0$ such that $\sup_n \int_B |f_n| \, dm \le \varepsilon/(3C\kappa^2)$ for all Borel sets $B \subseteq \mathbb{T}$ with $m(B) \le \delta$. We apply (2) with $\lambda = 1/\delta$ to find $g_n, h_n \in H^1$ such that, for each n,

$$f_n = g_n + h_n, \qquad |g_n|, |h_n| \le C \cdot |f_n|, \qquad |g_n| \le C \cdot \lambda \quad \text{and}$$

$$\int_{\mathbb{T}} |h_n| \, dm \le C \cdot \int_{\{|f_n| > \lambda\}} |f_n| \, dm \ .$$

Put $E_n := \{\zeta \in \mathbb{T} : |f_n(\zeta)| \le \lambda\}$. Then $1 \ge \int_{\mathbb{T} \setminus E_n} |f_n| \, dm \ge \lambda \cdot m(\mathbb{T} \setminus E_n)$, hence $m(\mathbb{T} \setminus E_n) \le \delta$ and so $\int_{\mathbb{T} \setminus E_n} |f_n| \, dm \le \varepsilon/(3C\kappa^2)$ for all n, whence $\sup_n \|h_n\|_1 \le \varepsilon/(3\kappa^2)$.

(g_n) is bounded in H^∞, and $i_\infty : H^\infty \to H^1$ is weakly compact. Passing to a subsequence if necessary, we may assume that (g_n) converges weakly to some $g \in H^1$. Since (f_n) is a weak null sequence in H^1, $-g$ must be the weak limit of (h_n). In particular, $\|g\|_1 \leq \varepsilon/(3\kappa^2)$. Now we use that $u \circ i_\infty$ is compact. Passing to another subsequence if needed, we may assume that (ug_n) converges in X. Since X has a separating dual, the limit must be ug so that, for n large enough, $\|u(g_n - g)\| \leq \varepsilon/(3\kappa)$. We have reached a contradiction:

$$\varepsilon < \|uf_n\| \leq \kappa \cdot \left(\|u(g_n - g)\| + \kappa \cdot (\|ug\| + \|uh_n\|)\right) \leq \frac{\varepsilon}{3} + \kappa^2 \cdot (\|g\|_1 + \|h_n\|_1) \leq \varepsilon. \quad \blacksquare$$

From 1.6.1 we may conclude:

1.6.4. *Let $\varphi \in \Phi$ and $\beta > 0$ be given.*

(a) *If $\beta > 1$ and C_φ is β-bounded, then it is completely continuous as an operator $H^1 \to H^\beta$.*

(b) *If $\beta < 1$, then complete continuity and compactness of $C_\varphi : H^1 \to H^\beta$ are equivalent.*

Proof. (a) Let $\beta > 1$. By 1.6.1, $C_\varphi : H^1 \to H^\beta$ is completely continuous if and only if $H^\beta \overset{i_\beta}{\hookrightarrow} H^1 \overset{C_\varphi}{\to} H^\beta$ is compact. But β-bounded composition operators are 1-compact by 1.5.2.

(b) Let $C_\varphi : H^1 \to H^\beta$ be completely continuous, $\beta < 1$. By 1.5.2 and 1.4.2, C_φ is compact as an operator $H^{1/\beta} \to H^1$, and so as an operator $H^1 \to H^\beta$. $\blacksquare$

Next we turn to weak compactness. In 1991, D. Sarason [53] has proved:

1.6.5. *A composition operator $H^1 \to H^1$ is compact if and only if it is weakly compact.*

Sarason's proof exploits the duality of *VMOA*, H^1, and *BMOA*. This can be circumvented, and we can even extend 1.6.5 as follows:

1.6.6. *Suppose that $\beta \geq 1$ and C_φ is a β-bounded composition operator. C_φ is compact as an operator $H^{1/\beta} \to H^1$ if and only if it is weakly compact.*

We refer to 3.5.2 and 3.5.3 for even stronger results.

It is well-known that H^1 fails the Dunford-Pettis property: there are weakly compact operators with domain H^1 which are not completely continuous; the classical Paley projection $H^1 \to \ell^2 \ (\subseteq H^1)$ provides an example.

But by 1.6.6, the class of operators with domain H^1 for which the conclusion is valid is still rather big. Is there a characterisation of the class of operators $H^1 \to H^1$ (or $H^{1/\beta} \to H^1$ for $\beta \geq 1$) which are completely continuous (or even compact) once they are weakly compact?

1.7. COMPACTNESS. Other than for $p > q$, we haven't touched upon the question of how to characterize compactness of a composition operator $C_\varphi : H^p \to H^q$ in terms of the generating symbol $\varphi \in \Phi$. For $p \leq q$, this is a delicate topic. The case $p = q$ was settled only in 1987 by J.H. Shapiro [55], see also [56].

It is known that if (a_n) is the sequence of zeros of a function $f \in H^2$, then $\sum_n (1 - |a_n|) < \infty$; it is customary to arrange the $|a_n|$ in increasing order, counting multiplicities. Hence if $\varphi \in \Phi$ then, considering $\varphi^{-1}(w)$ as the sequence of zeros of $\varphi(\,\cdot\,) - w$, we get $\sum_{z \in \varphi^{-1}(w)} (1 - |z|) < \infty$. Now, for any $0 < r < 1$, $1 - |z| \sim \log(1/|z|)$ for $z \in D \setminus rD$, so that the 'Nevanlinna counting function'

$$N_\varphi : \mathbb{C} \longrightarrow [0, \infty] : w \mapsto \begin{cases} 0 & \text{if } w \notin \varphi(D) \\ \sum_{z \in \varphi^{-1}(w)} \log \frac{1}{|z|} & \text{if } w \in \varphi(D), \ w \neq \varphi(0) \\ \infty & \text{if } w = \varphi(0) \end{cases}$$

is well-defined. Also, $N_\varphi(w) = \int_0^1 (n(r, w)/r)\, dr$ where $n(r, w)$ is the number of elements in $\varphi^{-1}(w) \cap (r \cdot \overline{D})$, $0 < r < 1$.

The following is Shapiro's Theorem.

1.7.1. *Let $\varphi : D \to D$ be analytic. Then*

$$\|C_\varphi\|_e = \limsup_{|w| \to 1} \left(\frac{N_\varphi(w)}{\log 1/|w|} \right)^{1/2}.$$

In particular, $C_\varphi : H^2 \to H^2$ is compact if and only if

$$\lim_{|w| \to 1} \frac{N_\varphi(w)}{\log(1/|w|)} = 0.$$

Here $\|C_\varphi\|_e$ is the distance of $C_\varphi \in \mathcal{L}(H^2)$ to the space $\mathcal{K}(H^2)$ of all compact operators on H^2, i.e. the norm of the canonical image of C_φ in the Calkin algebra $\mathcal{L}(H^2)/\mathcal{K}(H^2)$ ('essential norm').

1.7.1 has been extended by R. Riedl [49] and W. Smith [57]:

1.7.2. *If $0 < p \leq q$ then*

(a) $C_\varphi : H^p \to H^q$ *exists as a bounded operator if and only if* $N_\varphi(w) = O\big(\log(1/|w|)]^{2q/p}\big)$ $(|w| \to 1)$.

(b) $C_\varphi : H^p \to H^q$ *exists as a compact operator if and only if* $N_\varphi(w) = o\big([\log(1/|w|)]^{2q/p}\big)$ $(|w| \to 1)$.

2. CLASSICALLY WEIGHTED BERGMAN SPACES

2.1. PRELIMINARIES. We are now going to consider classes of linear subspaces of $\mathcal{H}(D)$ which are bigger than Hardy spaces. The basic measure is now normalized area measure σ on D, so that $d\sigma(z) = (dx\,dy)/\pi$. For each $\alpha > -1$,

$$d\sigma_\alpha(z) := (\alpha + 1)(1 - |z|^2)^\alpha d\sigma(z)$$

is a (Borel) probability measure on D. The spaces

$$\mathcal{A}_\alpha^p := \mathcal{H}(D) \cap L^p(\sigma_\alpha) \qquad (0 < p < \infty)$$

are the (classically) 'weighted Bergman spaces'. The (p-) norm on $L^p(\sigma_\alpha)$ and its subspaces will be denoted by $\| \cdot \|_{\alpha,p}$. Each $\mathcal{A}_\alpha^p$ is a closed subspace of $L^p(\sigma_\alpha)$, and the polynomials form a dense subspace of $\mathcal{A}_\alpha^p$. Moreover, $H^p \hookrightarrow \mathcal{A}_\alpha^p$ with norm 1. Further inclusions will be discussed later.

The function $v_\alpha : D \to [0, \infty) : z \mapsto (\alpha + 1)(1 - |z|^2)^\alpha$ is an example of a 'weight function'. There is an extensive literature on the problem how, for more general weights v, properties of the corresponding 'weighted Bergman spaces' $\mathcal{A}_\alpha^p(v)$ depend on v. For the sake of simplicity, however, we stay with the classical weights v_α.

As we know, $K(z, w) = (1 - \overline{z}w)^{-1}$ is the reproducing kernel of the Hilbert space H^2, so that $f(z) = (f|K(z, \cdot))_{H^2}$ for all $f \in H^2$, $\forall\, z \in D$. A calculation reveals that

$$K^{(\alpha)}(z, w) = K(z, w)^{\alpha+2} \qquad (z, w \in D)$$

is the reproducing kernel for the Hilbert space $\mathcal{A}_\alpha^2$. This is one of the reasons why results on weighted Bergman space can *sometimes* be taken over to Hardy spaces by formally substituting $\alpha = -1$. But, as we shall see, there are also quite a few exceptions.

2.2. PROJECTIONS AND DUALITY. One of the advantages of Bergman spaces over Hardy spaces is that analytic projections are available in abundance. As in the Hardy case ($\alpha = -1$), the orthogonal projection P_α in $L^2(\sigma_\alpha)$ onto $\mathcal{A}_\alpha^2$ ($\alpha > -1$) is obtained by integration against the reproducing kernel:

$$P_\alpha(f)(z) = \int_D f(w)\overline{K^{(\alpha)}(z,w)}\, d\sigma_\alpha(w) = \int_D \frac{f(w)}{(1 - \overline{w}z)^{\alpha+2}}\, d\sigma_\alpha(w) \ .$$

The integral is defined even for each $f \in L^1(\sigma_\alpha)$. For $1 < p < \infty$, P_α defines a projection in $L^p(\sigma_\alpha)$ onto $\mathcal{A}_\alpha^p$. But P_α does not project $L^1(\sigma_\alpha)$ onto $\mathcal{A}_\alpha^1$. However, for any $s > 0$ and $1 \leq p < \infty$, $P_{\alpha+s}$ defines a projection of $L^p(\sigma_\alpha)$ onto $\mathcal{A}_\alpha^p$. The image of $L^\infty(\sigma)$ of P_α is the 'Bloch space' $\mathcal{B}$ which consists of all $f \in \mathcal{H}(D)$ such that $\|f\|_\mathcal{B} := |f(0)| + \sup_{z \in D}(1 - |z|^2)|f'(z)| < \infty$. BMOA embeds boundedly into $\mathcal{B}$. See e.g. [65]. Rather than with $\mathcal{B}$, we shall deal with its isomorphic copies

$$X_s := \{f \in \mathcal{H}(D) : \|f\|_{X_s} = \sup_{z \in D}(1 - |z|^2)^s |f(z)| < \infty\} \qquad (s > 0) \ .$$

Clearly, $X_s \hookrightarrow X_t$ with norm one if $s < t$. We note already at this stage that $\mathcal{A}_\alpha^p$ ($\alpha \geq -1$, $0 < p < \infty$) embeds boundedly into X_s for $s = (\alpha + 2)/p$, and that this choice of s is best-possible.

It follows:

2.2.1. *If $1 < p < \infty$ and $s \geq 0$ then $(\mathcal{A}_\alpha^p)^*$ is isomorphic to $\mathcal{A}_{\alpha+sp^*}^{p^*}$ under the duality pairing $\langle f, g \rangle_{\alpha+s} := \int_D f\,\overline{g}\, d\sigma_{\alpha+s}$. In particular, $(\mathcal{A}_\alpha^p)^*$ is isomorphic to $\mathcal{A}_\alpha^{p^*}$ under $\langle f, g \rangle_\alpha$. Moreover, for any $s > 0$, $(\mathcal{A}_\alpha^1)^*$ is isomorphic to X_s with respect to $\langle f, g \rangle_{\alpha+s}$.*

We refer to D. Békollé [2], T. Domenig [16], U. Kollbrunner [34], D.H. Luecking [38] as well as to the books [65] of K. Zhu and [24] of H. Hedenmalm, B. Korenblum and K. Zhu for more details, generalizations and additional results.

There is no bounded linear form on $L^p(\sigma_\alpha)$ when $0 < p < 1$. But $\mathcal{A}_\alpha^p$ embeds continuously into $\mathcal{H}(D)$, and so has a separating dual. The description is as follows:

2.2.2. *Given $\alpha \geq -1$ and $0 < p < 1$, define $\alpha' > -1$ by $\alpha' + 2 = (\alpha + 2)/p$. Then $\mathcal{A}_\alpha^p$ and $\mathcal{A}_{\alpha'}^1$ have the same dual, namely $X_{(\alpha+2)/p}$.*

In other words, $\mathcal{A}_{\alpha'}^1$ is the 'Banach space envelope' of the non-locally convex space $\mathcal{A}_\alpha^p$. For $\alpha = -1$, 2.2.2 is due to P.L. Duren, B.W. Romberg and A.L. Shields [20]; the general case was solved by J.H. Shapiro [54].

2.3. COMPOSITION OPERATORS. Several results known for the Hardy space case can easily be carried over.

2.3.1. *Let $\alpha \geq -1$ and $0 < p < \infty$ be given.*

(a) *For each $z \in D$, $\delta_z : \mathcal{A}_\alpha^p \to \mathbb{C} : f \mapsto f(z)$ is bounded, with $\|\delta_z\|_{\alpha,p}^* = (1 - |z|^2)^{-(\alpha+2)/p}$.*

(b) *For each $\varphi \in \Phi$, $C_\varphi : \mathcal{A}_\alpha^p \to \mathcal{A}_\alpha^p : f \mapsto f \circ \varphi$ is well-defined with*
$$\|C_\varphi\| \leq \Big(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \Big)^{(\alpha+2)/p}.$$

(a) is obtained by direct verification. To get (b) for $\alpha > -1$, use polar coordinates and apply Littlewood's Subordination Principle in order to settle the case $\varphi(0) = 0$; then proceed as before and pass to the general case via Möbius transforms.

1.3.1 and 1.3.2 can be generalized as well:

2.3.2. (a) *Multiplicative linear forms on $\mathcal{A}_\alpha^p$ are bounded; they are precisely the point evaluations.*

(b) *Multiplicative linear maps $\mathcal{A}_\alpha^p \to \mathcal{H}(D)$ are continuous and take their values in $\mathcal{A}_\alpha^p$; they are precisely the composition operators.*

Now we look at composition operator C_φ between two given weighted Bergman space $\mathcal{A}_\alpha^p$ and $\mathcal{A}_\beta^q$. It is easy to see that $C_\varphi : \mathcal{A}_\alpha^p \to \mathcal{A}_\beta^q$ exists as a bounded (compact) operator if and only if, regardless of $s > 0$, C_φ maps $\mathcal{A}_\alpha^{sp}$ boundedly (compactly) into $\mathcal{A}_\beta^{sq}$. In several cases, we can do better, by investigating the reproducing kernels $K^{(\alpha)}(\cdot, \cdot)$, $\alpha \geq -1$. We have $\|K^{(\alpha)}(z, \cdot)\|_{\alpha,2} = (1 - |z|^2)^{-(2+\alpha)/2}$ for the $\mathcal{A}_\alpha^2$-norms. It follows that the functions
$$k_p^{(\alpha)}(z, w) := \Big(\frac{1 - |z|^2}{(1 - \overline{z}w)^2} \Big)^{(\alpha+2)/p} \qquad (w \in D)$$

have 'norm' one in $\mathcal{A}_\alpha^p$ $(0 < p < \infty)$. It suffices to look at these functions in order to decide whether C_φ is bounded or compact.

2.3.3. *Let $\varphi \in \Phi$, $\alpha, \beta \geq -1$ and $q \geq p > 0$ be given. Then:*

(a) *C_φ exists as a bounded operator $\mathcal{A}_\alpha^p \to \mathcal{A}_\beta^q$ if and only if*
$\sup_{z \in D} \|C_\varphi(k_p^{(\alpha)}(z, \cdot))\|_{\beta,q} < \infty.$

(b) C_φ exists as a compact operator $\mathcal{A}_\alpha^p \rightarrow \mathcal{A}_\beta^q$ if and only if $\lim_{|z|\to 1} \|C_\varphi(k_p^{(\alpha)}(z,\,\cdot\,))\|_{\beta,q} = 0$.

See [18] and [34]. Note that dependence is only on β and $q(\alpha+2)/p$. In particular, when $q \geq p$, we can choose α' such that $\alpha' + 2 = q(\alpha+2)/p$, and then boundedness (compactness) of C_φ as an operator $\mathcal{A}_\alpha^p \rightarrow \mathcal{A}_\beta^q$ is equivalent to boundedness (compactness) of C_φ as a Hilbert space operator $\mathcal{A}_{\alpha'}^2 \rightarrow \mathcal{A}_\beta^2$.

With each $\gamma > 0$ and $\varphi \in \Phi$ we associate the 'generalized Nevanlinna counting function'

$$N_{\varphi,\gamma}(w) := \sum_{z \in \varphi^{-1}(w)} \Big[\log \frac{1}{|z|} \Big]^\gamma, \qquad w \in D \setminus \{\varphi(0)\} \ .$$

These functions were employed by J.H.Shapiro [55] to settle the problem of compactness of composition operators $C_\varphi : \mathcal{A}_\alpha^p \rightarrow \mathcal{A}_\alpha^p$ and later on by W. Smith [57] and W. Smith - L. Yang [58] to characterize existence and of compactness of C_φ as an operator $\mathcal{A}_\alpha^p \rightarrow \mathcal{A}_\beta^q$ for arbitrary α, $\beta \geq -1$ and $0 < p,q < \infty$. See also M.E. Robbins [50]. We will prove by functional analytic tools that if $\alpha \geq -1$, $\beta > -1$ and $p > q$, then composition operators $C_\varphi : \mathcal{A}_\alpha^p \rightarrow \mathcal{A}_\beta^q$ are always compact – provided they are defined.

2.4. ATOMIC DECOMPOSITION.   We will need the following theorem:

2.4.1. *Given $\alpha > -1$ and $0 < p < \infty$, there exists an isomorphism of $\mathcal{A}_\alpha^p$ onto ℓ^p.*

This is in marked contrast with the case $\alpha = -1$: in fact, by a result of R.P. Boas [5], see also S. Kwapień and A. Pełczyński [36], the Hardy space H^p is isomorphic to $L^p[0,1]$ whenever $1 < p < \infty$.

We present a functional analytic proof of 2.4.1 in the Banach space case due to J. Lindenstrauss and A. Pełczyński [37]. We require a well-known technical lemma (see e.g. [15], Lemma 3.3).

2.4.2. *Let μ be any measure and $1 \leq p \leq \infty$. Let $\emptyset \neq M \subseteq L^p(\mu)$ be compact and $\varepsilon > 0$. Then there is a projection $P \in \mathcal{L}(L^p(\mu))$ of finite rank, say n, such that $\|P\| = 1$, $\|Pf - f\| \leq \varepsilon$ for all $f \in M$, and $\mathrm{Im}(P)$ is isometric to ℓ_n^p.*

Proof of 2.4.1. Fix a covering $\emptyset \neq K_1 \subseteq K_2 \subseteq \ldots \subseteq D$ of compact sets. Put $K_0 = \emptyset$ and $M_k = K_{k+1} \setminus K_k$, $k \in \mathbb{N}_0$. Each $R_k : \mathcal{A}_\alpha^p \rightarrow L^p(M_k) =$

$L^p(M_k, \sigma_\alpha) : f \mapsto f|_{M_k}$ is compact (even nuclear): it is the restriction of the map $\mathcal{H}(D) \to L^p(M_k) : f \mapsto f|_{M_k}$. Now $L^p(\sigma_\alpha)$ is isometrically isomorphic to $X := \left(\bigoplus_k L^p(M_k) \right)_{\ell^p}$, so $R : A^p_\alpha \to X : f \mapsto (R_k f)_k$ is isometric.

Let $\varepsilon > 0$ be given. Thanks to 2.4.2 we can find, for each k, a projection $P_k \in \mathcal{L}(L^p(M_k))$ of finite rank, n_k say, so that $\|P_k\| = 1$, $\|P_k g - g\| \leq \varepsilon/2^{(k+1)/p}$ for all $g \in R_k(B_{\mathcal{A}^p_\alpha})$, and $Y_k = \mathrm{Im}\,(P_k)$ is isometric to $\ell^p_{n_k}$. Then $Y = \left(\bigoplus_k Y_k \right)_{\ell^p}$ is isometric to ℓ^p and naturally complemented in X. Define $S : \mathcal{A}^p_\alpha \to Y$ by $f \mapsto (P_k R_k f)_k$. Then $\|Sf\|_Y \leq \|f\|_{\alpha,p}$ for $f \in \mathcal{A}^p_\alpha$ and

$$\|f\|_{\alpha,p} = \|Rf\|_X = \Big(\sum_{k=0}^\infty \|R_k f\|^p_{L^p(M_k)} \Big)^{1/p} \leq \Big(\sum_{k=0}^\infty \|P_k R_k f\|^p_{L^p(M_k)} \Big)^{1/p}$$
$$+ \Big(\sum_{k=0}^\infty \|R_k f - P_k R_k f\|^p_{L^p(M_k)} \Big)^{1/p}$$
$$\leq \|Sf\|_Y + \varepsilon \cdot \|f\|_{\alpha,p} \ .$$

$S(\mathcal{A}^p_\alpha)$ is complemented in Y and so isomorphic to ℓ^p, by a classical result of A. Pełczyński [45]. ∎

The proof even reveals that $\mathcal{A}^p_\alpha$ is 'almost isometric' to ℓ^p. If $p \notin 2\mathbb{N}$ then $\mathcal{A}^p_\alpha$ cannot be isometric to ℓ^p [13]. It is not clear, however, what happens if $p = 4, 6, 8, \ \ldots$.

2.4.2 doesn't apply for $0 < p < 1$ if μ has no atoms. But 2.4.1 is true in this case as well. There is a proof of 2.4.1 due to R.R. Coifman and R. Rochberg [11] for the Banach space case which is based on a close analysis of D's hyperbolic metric (see [65] for $\alpha = 0$). This allows the explicit construction of operators $S : \ell^p \to \mathcal{A}^p_\alpha$ and $T : \mathcal{A}^p_\alpha \to \ell^p$ with $ST = id_{\mathcal{A}^p_\alpha}$. Therefore T maps $\mathcal{A}^p_\alpha$ isomorphically onto an infinite dimensional complemented subspace of ℓ^p which, by the above Pełczyński theorem, is isomorphic to ℓ^p. The construction can be adapted to the case $0 < p < 1$ as well, see N.J. Kalton and D.A. Trautman [32]. That Pełczyński's theorem on complemented subspaces of ℓ^p holds true also in this case was shown by W.J. Stiles [59].

Even more general weights are admitted; compare e.g. D. Békollé [3], T. Domenig [16], [17], U. Kollbrunner [34], D.H. Luecking [41], $\ldots$.

For an immediate consequence of 2.4.1 recall Pitt's Theorem [48]:

2.4.3. *If $0 < q < p < \infty$ then every operator $\ell^p \to \ell^q$ is compact.*

A proof which covers indeed all $0 < q < p < \infty$ is due to E. Oja [43]. Under additional assumptions on (p, q), H.P. Rosenthal [51] has proved a corresponding result for operators $\ell^p \to L^q$ and $L^p \to \ell^q$.

Combing 2.4.3 with 2.4.1 we get

2.4.4. *If $0 < q < p < \infty$ and $\alpha, \beta > -1$, then every operator $\mathcal{A}_\alpha^p \to \mathcal{A}_\beta^q$ is compact.*

The case $\alpha = -1$ can be included e.g. for composition operators C_φ, since then C_φ maps the reflexive space $H^{p/q}$ boundedly into the Schur space $\mathcal{A}_\beta^1$.

3. Carleson measures for weighted Bergman spaces and Hardy spaces

An investigation of multiplication operators on weighted Bergman spaces will produce results which are suspiciously close to those for composition operators. In fact, a 'common denominator' does exist; it is provided by 'Carleson measures'. We investigate such measures first of all on D for spaces $\mathcal{A}_\alpha^p$, $\alpha \geq -1$, $0 < p < \infty$. Pecularities for the Hardy case $\alpha = -1$ will be discussed separately. See [30] and [34] for details.

3.1. The concept. Let $F \subseteq \mathcal{H}(D)$ be a linear subspace, endowed with a 'nice' topology, and let $0 < q < \infty$. Henceforth, all measures on D (or $\overline{D}$, $\mathbb{T}$) will be positive, finite Borel measures. A measure μ on D is a 'q-Carleson measure for F' if the formal identity

$$J_\mu : F \longrightarrow L^q(\mu) : f \mapsto f$$

exists as a continuous operator. J_μ is then said to be a 'Carleson embedding'.

Here F will be one of the space $\mathcal{A}_\alpha^p$, $\alpha \geq -1$, $0 < p < \infty$, and we refer to q-Carleson measures for $\mathcal{A}_\alpha^p$ as '(α, p, q)-Carleson measures'. We say that μ is a 'compact (α, p, q)-Carleson measure' if $J_\mu : \mathcal{A}_\alpha^p \to L^q(\mu)$ exists as a compact operator.

The following connects us with a former topic.

3.1.1. (a) *Let $\varphi : D \to D$ be analytic. The composition operator $C_\varphi : \mathcal{A}_\alpha^p \to \mathcal{A}_\beta^q : f \mapsto f \circ \varphi$ is well-defined if and only if $\sigma_{\beta,\varphi} := \sigma_\beta \circ \varphi^{-1}$ is an (α, p, q)-Carleson measure.*

(b) *$F \in L^q(\sigma_\beta)$ defines a multiplication operator $M_F : \mathcal{A}_\alpha^p \longrightarrow L^q(\sigma_\beta) : f \mapsto f \cdot F$ if and only if $d\mu := |F|^q d\sigma_\beta$ is an (α, p, q)-Carleson measure.*

Weighted composition operators are operators of the form $f \mapsto F \cdot (f \circ \varphi)$. They are obtained by combining (a) and (b). Similar for operators of the form $f \mapsto (F \cdot f) \circ \varphi$.

Note that, in the situation of (a), $C_\varphi f \mapsto J_{\sigma_{\beta,\varphi}} f$ extends to an isometric isomorphism of the closure of the range of C_φ onto the closure of the range of $J_{\sigma_{\beta,\varphi}}$. Properties like compactness, weak compactness, complete continuity, ... for C_φ are therefore equivalent to analogous properties of the corresponding Carleson embedding.

If $\alpha > -1$, then (a) is a special case of (b); in fact, it can be shown that $\sigma_{\beta,\varphi} \ll \sigma$ (M. Vaeth [61]). So in this case the above examples concern σ-absolutely continuous measures. The most important σ-singular Carleson measures are the discrete ones. They come up naturally in topics like 'interpolation and sampling', and in the context of atomic decomposition (see [24]).

3.2. THE CASE $p \leq q$. For each $0 \neq z \in D$ we introduce the arc

$$I(z) := \left\{ e^{it} \cdot \frac{z}{|z|} : \ - (1 - |z|) \leq t < (1 - |z|) \right\}$$

and the 'Carleson box'

$$S(z) := \left\{ w \in D : |z| \leq |w|, \ \frac{w}{|w|} \in I(z) \right\}.$$

(α, p, q)-Carleson measures μ will be characterized in terms of the function

$$H_{\alpha,p,q} : D \longrightarrow [0, \infty) : z \mapsto \frac{\mu(S(z))^{1/q}}{(1 - |z|^2)^{(\alpha+2)/p}} \ .$$

We need to switch from the euclidean metric on D, which suffers from a lack of invariance under analytic automorphisms, to another one without such a defect. We take the 'hyperbolic metric' $\varrho(z, w) := \inf_\gamma \int_\gamma 1/(1 - |\zeta|^2)|d\zeta|$. The infimum extends over all arcs γ joining z and w and is attained in the circular arc which joins z and w and hits $\mathbb{T}$ orthogonally. It can be shown that

$$\varrho(z, w) := \frac{1}{2} \cdot \log \frac{1 + |\tau_z(w)|}{1 - |\tau_z(w)|} \ .$$

Actually, $d(z, w) := |\tau_z(w)|$ also defines a metric on D, known as the 'pseudohyperbolic metric'. ϱ and d are equivalent to the euclidean metric. In fact, every open ϱ-ball $B_r(z) := \{w \in D : \varrho(z, w) < r\}$, $z \in D$, $r > 0$, is also a euclidean ball, with euclidean center $\frac{1 - \tanh^2 r}{1 - |z|^2 \tanh^2 r} \cdot z$ and euclidean radius $\frac{1 - |z|^2}{1 - |z|^2 \tanh^2 r} \cdot \tanh r$. Möbius invariance of d and so of ϱ is a consequence of the Schwarz-Pick Theorem, see e.g. [22].

It is not hard to see that $\sigma_\alpha(B_r(w)) \sim (1 - |w|^2)^{\alpha+2}$ for all $w \in D$, with constants depending only on α and r. It can also be shown $\mu(S(z)) \sim \mu(B_r(z))$ for all $z \in D$, with constants depending only on r. Consequently, in our context, $H_{\alpha,p,q}$ can be replaced by any of the functions

$$D \longrightarrow [0,\infty) : z \mapsto \frac{\mu(B_r(z))^{1/q}}{\sigma_\alpha(B_r(z))^{1/p}} \; ;$$

the choice of r will only influence constants.

3.2.1. *Let $\alpha \geq -1$ and $0 < p \leq q < \infty$. For a measure μ on D, the following are equivalent:*

(i) *μ is an (α, p, q)-Carleson measure.*

(ii) *$H_{\alpha,p,q}$ is bounded on D.*

(iii) *$\sup_{z \in D} \|k_p^{(\alpha)}(z, \cdot)\|_{L^q(\mu)}^q = \sup_{z \in D} \int_D |\tau_z'(w)|^{q(\alpha+2)/p} d\mu(w) < \infty$.*

This result has a long history. In 1962, L. Carleson [8] proved (i) ⟺ (ii) for $\alpha = -1$ and $p = q$. (i) ⟺ (ii) for $\alpha = -1$ and $p \leq q$ was settled by P.L. Duren in 1969; see [19]. The generalisation to $\alpha > -1$ can be found, for example, in V.L. Oleinikov and B.S. Pavlov [44], W.W. Hastings [23], D.H. Luecking, [39], [40]. For $\alpha = -1$, (i) ⟺ (iii) for $\alpha = -1$ is due to R. Aulaskari, D.A. Stegenga and J. Xiao [1]; the extension to weighted Bergman spaces is from R. Zhao [64].

There is a 'compact companion' of 3.2.1 which is obtained by simply replacing the 'O-conditions' in 3.2.1 by the corresponding 'o-conditions':

3.2.2. *Let $\alpha \geq -1$ and $0 < p \leq q < \infty$. For a measure μ on D, the following are equivalent:*

(i) *μ is a compact (α, p, q)-Carleson measure.*

(ii) *$\lim_{|z|\to 1} H_{\alpha,p,q} = 0$.*

(iii) *$\lim_{|z|\to 1} \|k_p^{(\alpha)}(z, \cdot)\|_{L^q(\mu)}^q = 0$.*

In these results, it is only $q(\alpha + 2)/p$ which matters. In particular, this allows a reduction to a Hilbert space setting:

3.2.3. *Let $\alpha, \alpha' \geq -1$ and $0 < p \leq q < \infty$ be such that $\alpha' + 2 = q(\alpha + 2)/p$. Then $J_\mu : \mathcal{A}_\alpha^p \to L^q(\mu)$ exist as a (compact) operator if and only if $J_\mu : \mathcal{A}_{\alpha'}^2 \to L^2(\mu)$ exists (and is compact).*

3.3. THE CASE $p > q$. Given $\zeta \in \mathbb{T}$ and $0 < \lambda < 1$ let $\Gamma(\zeta)$ be the interior of the convex hull of $\{e^{i\theta}\} \cup (\lambda \cdot \overline{D})$ (a 'Stolz domain'). The choice of λ doesn't really matter. The following is due to I.V. Videnskii [63]:

3.3.1. $0 < q < p < \infty$, μ is a $(-1, p, q)$-Carleson measure (a q-Carleson measure for H^p) if and only if $\mathbb{T} \to \mathbb{C} : \zeta \mapsto \int_{\Gamma(\zeta)} 1/(1 - |z|^2) \, d\mu(z)$ belongs to $L^{p/(p-q)}(dm)$.

The corresponding result for $\alpha > -1$ is in D.H. Luecking [41]; see also I.E. Verbitsky [62].

3.3.2. If $\alpha > -1$ and $0 < q < p < \infty$ then μ is (α, p, q)-Carleson if and only if $z \mapsto \mu(S(z))/(1 - |z|^2)^{\alpha+2}$ is in $L^{p/(p-q)}(\sigma_\alpha)$.

The condition is also equivalent to requiring

$$H_{\alpha,p,q} \in L^{pq/(p-q)}(\Lambda)$$

where $d\Lambda(z) := (1 - |z|^2)^{-2} d\sigma(z)$ is the 'Möbius invariant measure' on D.

Dependence is on q/p only, but compared with the former case, we now have less freedom to change parameters.

So far, there has been no mentioning of compactness. There is a good reason for this:

3.3.3. If $\alpha > -1$ and $p > q > 0$ then every (α, p, q)-Carleson measure is compact.

Banach space theory provides a straightforward proof. Suppose that X_0, X, X_1 and Y_0, Y, Y_1 are continuously embedded (quasi-) Banach spaces: $X_0 \hookrightarrow X \hookrightarrow X_1$, $Y_0 \hookrightarrow Y \hookrightarrow Y_1$. Let $T_1 : X_1 \to Y_1$ be an operator which induces operators $T : X \to Y$ and $T_0 : X_0 \to Y_0$. Under suitable assumptions, interpolation theory tells us that T is compact whenever T_0 or T_1 is compact.

Our operators T_1, T, T_0 will be Carleson embeddings $\mathcal{A}_\alpha^{p_1} \to L^{q_1}(\mu)$, $\mathcal{A}_\alpha^p \to L^q(\mu)$, $\mathcal{A}_\alpha^{p_0} \to L^{q_0}(\mu)$, $q_0/p_0 = q/p = q_1/p_1$. Application of the above interpolation result is legitimate and can be started from any of the following two observations. The first one is:

1. If $p > 1$, then every Carleson embedding $\mathcal{A}_\alpha^p \to L^1(\nu)$ is compact.

   It suffices to prove complete continuity ($\mathcal{A}_\alpha^p$ is reflexive). Let $(f_n)_n$ be weakly null in $\mathcal{A}_\alpha^p$. Then $(f_n(z))_n \to 0$ for all $z \in D$. Combine uniform integrability and Egorov's theorem to get $\lim_n \|f_n\|_1 = 0$.

   The second possibility is based on 2.4.1 and 2.4.3:

2. *If $p > 2 \geq q \geq 1$, then every operator $u : \mathcal{A}_\alpha^p \to L^q(\nu)$ is compact.*

In fact, u admits a factorisation $u : \mathcal{A}_\alpha^p \xrightarrow{w} \ell^2 \xrightarrow{v} L^q(\nu)$. By 2.4.1, $\mathcal{A}_\alpha^p \cong \ell^p$, and so w is compact by Pitt's Theorem 2.4.3.

The preceding results generalize corresponding ones obtained by W. Smith [57] and by W. Smith and L. Yang [58] for composition operators through an investigation of generalized Nevanlinna counting functions.

3.4. ORDER BOUNDEDNESS.   Let X be a Banach space and L a Banach lattice. $u \in \mathcal{L}(X, Y)$ is 'order bounded' if $u(B_X)$ is contained in the 'order interval' $J_h := \{g \in L : |g| \leq h\}$ generated by some some $0 \leq h \in L$. The span Z_h of J_h is a Banach lattice, with L's order and (a suitable multiple of) J_h's gauge functional as a norm. In fact, Z_h is an 'abstract M-space' with unit and so, by Kakutani's theorem, isometrically isomorphic (as a Banach lattice) to $\mathcal{C}(K)$ for some compact Hausdorff space K. Consequently, u factorizes $X \xrightarrow{u} Z_h \cong \mathcal{C}(K) \xrightarrow{j} L$, j being the formal identity.

For illustration and orientation we state:

3.4.1. *An operator $u : L^2(\mu) \to L^2(\nu)$ is order bounded if and only if it is Hilbert-Schmidt.*

Let now $s > 0$ be given.

$$\widehat{X}_s := \{f : D \to \mathbb{C} : f \text{ measurable}, \ \sup_{z \in D} \ (1 - |z|^2)^s |f(z)| < \infty\}$$

is a Banach space, and the 'Bloch type space'

$$X_s := \widehat{X}_s \cap \mathcal{H}(D)$$

is a closed subspace. Let $\alpha \geq -1$ and $0 < p < \infty$. We have already noted (in Section 1.) that $\mathcal{A}_\alpha^p \hookrightarrow X_{(\alpha+2)/p}$ (boundedly) and that $(\alpha + 2)/p$ is best-possible. The spaces $\widehat{X}_s$ and X_s have a natural place in our context. With J_μ as before, we have:

3.4.2. *Let $\alpha \geq -1$, $0 < p < \infty$ and $1 \leq q < \infty$ be given. Then, with $s := (\alpha + 2)/p$, the following are equivalent:*

(i) *$J_\mu : \widehat{X}_s \to L^q(\mu)$ exists and is bounded / order bounded.*
(ii) *$J_\mu : X^s \to L^q(\mu)$ exists and is bounded / order bounded.*

(iii) $J_\mu : \mathcal{A}_\alpha^p \to L^q(\mu)$ exists and is order bounded.

(iv) $(1 - |z|^2)^{-s} \in L^q(\mu)$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i) are obvious since bounded operators on $\widehat{X}_s$ and X_s are readily seen to be order bounded.

(iii) $\Rightarrow$ (vi): Let $0 \le h \in L^q(\mu)$ be an order bound for $J_\mu : \mathcal{A}_\alpha^p \to L^q(\mu)$. We know that the functions $w \mapsto k_p^{(\alpha)}(z, w) = (1 - |z|^2)^s (1 - \overline{z}w)^{-2s}$ have $\mathcal{A}_\alpha^p$-norm one and so are dominated by h. In particular, $k_p^{(\alpha)}(z, z) = (1 - |z|^2)^{-s} \le h(z)$ ($z \in D$). ∎

Once more, dependence is only on $q(\alpha + 2)/p$, but now there is no need to distinguish cases $p \le q$ and $p > q$. Analogous to 3.2.3 we may state:

3.4.3. *Let* $\alpha, \alpha' \ge -1$, $p > 0$, $q \ge 1$, $\alpha' + 2 = q \cdot (\alpha + 2)/p > 1$ *and* μ *an* (α, p, q)-*Carleson measure.* $J_\mu : \mathcal{A}_\alpha^p \to L^q(\mu)$ *is order bounded if and only if* $J_\mu : \mathcal{A}_{\alpha'}^2 \to L^2(\mu)$ *is Hilbert-Schmidt.*

It is well-known that if $q \ge 1$ and $u : X \to L^q(\mu)$ is an order bounded operator, then it u is q-summing. The converse is false for general Banach space operators. But:

3.4.4. *Let* $\alpha \ge -1$, $1 < p < \infty$, $p^* < q < \infty$ *and* μ *an* (α, p, q)-*Carleson measure.* $J_\mu : \mathcal{A}_\alpha^p \to L^q(\mu)$ *is order bounded if and only if* J_μ *is* (q, p^*)-*summing.*

We refer to [15] for details an q-summing and (q, p^*)-summing operators.

The question of when a weighted Bergman space $\mathcal{A}_\alpha^p$ embeds into another weighted Bergman space $\mathcal{A}_\beta^q$ and which properties the formal identity $\mathcal{A}_\alpha^p \hookrightarrow \mathcal{A}_\beta^q$ might enjoy in such a case is a question on the Carleson nature of σ_β. Accordingly:

3.4.5. *Let* $\alpha, \beta > -1$ *and* $0 < p, q < \infty$.

(a) $p \le q$: $\mathcal{A}_\alpha^p \hookrightarrow \mathcal{A}_\beta^q \Leftrightarrow (\alpha + 2)/p \le (\beta + 2)/q$.

(b) $p > q$: $\mathcal{A}_\alpha^p \hookrightarrow \mathcal{A}_\beta^q \Leftrightarrow (\alpha + 1)/p < (\beta + 1)/q$.

(c) *The embedding in* (a) *is compact if and only if* $(\alpha + 2)/p < (\beta + 2)/q$. *The embedding in* (b) *is always compact.*

(d) $\mathcal{A}_\alpha^p \hookrightarrow \mathcal{A}_\beta^q \subseteq L^q(\sigma_\beta)$ *order boundedly* $\Leftrightarrow (\alpha + 2)/p < (\beta + 1)/q$.

Of course, in order to prove this there is no need to resort to Carleson measures. In fact, (a) and (b) are 'folklore'; but so far, (d) seems to have escaped undetected. Keep in mind that (d) is equivalent to the factorisation $\mathcal{A}_\alpha^p \hookrightarrow X_{(\alpha+2)/p} \hookrightarrow \mathcal{A}_\beta^q$ of formal identities.

3.4.5 can be used to gather some information about

$$\mathcal{A}_\omega := \bigcap_s X_s \qquad \text{and} \qquad \mathcal{A}^\omega := \bigcup_s X_s \,.$$

In a natural fashion, $\mathcal{A}_\omega$ is a Fréchet space. The canonical inductive limit topology makes $\mathcal{A}^\omega$ a strong dual of such a space. Both, $\mathcal{A}_\omega$ and $\mathcal{A}^\omega$, are algebras with continuous multiplication ($\mathcal{A}^\omega$ is the 'Korenblum algebra'). From the preceding observations (or directly) we can infer that, independent of $\alpha > -1$,

$$\mathcal{A}_\omega = \bigcap_p \mathcal{A}_\alpha^p \qquad \text{and} \qquad \mathcal{A}^\omega = \bigcup_p \mathcal{A}_\alpha^p \,.$$

3.4.6.   (a) $\mathcal{A}^\omega$ *is a nuclear locally convex algebra.*

(b) $\mathcal{A}_\omega$ *is a Fréchet-Schwartz algebra, but not nuclear.*

Proof. (sketch) (a) follows from 3.4.5.(d); in fact, it is not hard to show that $\mathcal{A}^\omega$ is even isomorphic to the strong dual of the space s of rapidly decreasing sequences.

(b) The 'Schwartz part' comes from 3.4.5.(c). Suppose that $\mathcal{A}_\omega$ is nuclear. Then the formal identity $j : W \hookrightarrow \mathcal{A}_0^2$ is nuclear, W being the Wiener algebra. Use the canonical identifications $W \equiv \ell^1$, $\mathcal{A}_0^2 \equiv \ell^2$ to see that j is equivalent to the diagonal operator $D : \ell^1 \to \ell^2$ given by $\left((k+1)^{-1/2}\right)_k$ –which is not a nuclear operator. ∎

Among others, formal identities between weighted Bergman spaces can thus be used to provide counter-examples to natural questions on the compositions of summing operators.

3.5. CARLESON MEASURES FOR HARDY SPACES. By definition, $(-1, p, q)$-Carleson measures are measures μ on D for which $J_\mu : H^p(D) \to L^q(\mu)$ is defined. For any $s > 0$, a (compact) $(-1, p, q)$-Carleson measure is a (compact) $(-1, sp, sq)$-Carleson measure. Also, each $(-1, p, q)$-Carleson measure is a compact $(-1, p, q - \varepsilon)$-Carleson measure, for all $0 < \varepsilon < q$.

Let $\varphi : D \to D$ be analytic. Since the composition operator $C_\varphi : H^p \to H^p : f \mapsto f \circ \varphi$ exists as a bounded operator for any $0 < p < \infty$, it is a fortiori

bounded as a map $H^p \to H^q$ if $0 < q < p$. But as we have seen in Section 1.6, $C_\varphi : H^p \to H^q$ $(q < p)$ is compact if and only if $C_\varphi : H^1 \to H^1$ is completely continuous.

Happily, this does *not* contradict any of the introductory statements. It is only the notion of $(-1, p, q)$-Carleson measure which is not appropriate for investigating composition operators on Hardy spaces. Such measures live on D and need not have any sensitiveness for what happens on $\mathbb{T}$.

Extend φ by radial limits (Fatou) m-a.e. to obtain a measurable function $\varphi^* : \mathbb{T} \to \overline{D}$. By

$$m_\varphi(B) := m((\varphi^*)^{-1}(B))$$

($B \subseteq \mathbb{T}$ a Borel set) we get a measure on $\overline{D}$ which has the 'Carleson property': for any $0 < p < \infty$,

$$J_{m_\varphi} : H^p \longrightarrow L^p(m_\varphi) \hookrightarrow L^q(m_\varphi) : f \mapsto f$$

is well-defined. (Strictly speaking, we are looking at $f \vee f^*$ for $f \in H^p(D)$.) Write $(m_\varphi)_D$ for m_φ's restriction to D; similar for $\mathbb{T}$. Note that $(m_\varphi)_D$ is a $(-1, p, q)$-Carleson measure in the former sense. If $E_\varphi := \{\zeta \in \mathbb{T} : \varphi^*(\zeta) \in \mathbb{T}\}$ is again the set of 'contact points' for φ, then clearly

$$m_\varphi = (m_\varphi)_D \quad \Leftrightarrow \quad (m_\varphi)_\mathbb{T} = 0 \quad \Leftrightarrow \quad m_\varphi(E_\varphi) = 0 \ .$$

By 1.6.1 this is further equivalent to $C_\varphi : H^1 \to H^1$ being completely continuous. More generally:

3.5.1. *Let μ be a measure on $\overline{D}$ such that $f \mapsto f$ defines a bounded operator $J_\mu : H^1 \to L^1(\mu)$. Then $\mu|_\mathbb{T}$ vanishes if and only if J_μ is completely continuous.*

Proof. If J_μ is completely continuous, then $\|z^n\|_{L^1(\mu)} \to 0$ since (z^n) is a weak null sequence in H^1. But by monotone convergence, $\|z^n\|_{L^1(\mu)} = \mu(\mathbb{T}) + \int_D |z^n| d\mu \to \mu(\mathbb{T})$. Conversely, if $\mu(\mathbb{T}) = 0$ and (f_n) is weakly null in H^1, then $f_n(z) \to 0$ for all $z \in D$, and uniform integrability in tandem with Egorov's Theorem yields $\|f_n\|_{L^1(\mu)} \to 0$. ∎

Also, Sarason's result 1.6.5 can be extended:

3.5.2. *Let μ be a measure on $\overline{D}$ such that the Carleson embeddings $J_\mu : H^1 \to L^1(\mu)$ exists and is weakly compact. Then J_μ is compact.*

Proof. (Sketch) The strategy is still Sarason's, but there is no need to appeal to H^1's duality relations:

(1) It is standard to show that if μ is a measure on $\overline{D}$ for which $J_\mu : H^1 \to L^1(\mu) : f \mapsto f$ exists, then $\mu_{\mathbb{T}} = F\,dm$ with $F \in L^\infty(m)$. So, if $P : L^1(\mu) \to L^1(\mu_{\mathbb{T}})$ is the canonical projection, then $P \circ J_\mu$ 'is' the multiplication operator $M_F : H^1(\mathbb{T}) \to L^1(m) : f \mapsto f \cdot F$.

(2) M_F is weakly compact since J_μ is. The key is to show that this forces $F = 0$ a.e.

One proof can be based on Lebesgue's Differentiation Theorem, another one on Szegö's Theorem, and a third one can be derived from a factorisation theorem due to M. Marsallis and G. West [42] according to which, given $0 < p < \infty$ and $\varepsilon > 0$, every $f \in L^p(\mathbb{T})$ has the form $f = g \cdot h$ where g is in the unit ball of $L^\infty(\mathbb{T})$ and h belongs to $H^p(\mathbb{T})$, doesn't vanish, and has the property that $1/h$ is bounded and that $\|h\|_{H^p} \le (1 + \varepsilon)\|f\|_{L^p(\mathbb{T})}$. (In particular, the bilinear map $L^\infty(\mathbb{T}) \times H^p(\mathbb{T}) \to L^p(\mathbb{T}) : (g, h) \to gh$ is onto.) In our case, this can also be derived from Szegö's Theorem, but it can also be given an independent proof which is then valid even in a setting of von Neumann algebras.

(3) Once we know that $\mu = \mu_D$, compactness of $J_\mu : H^1 \to L^1(\mu)$ is obtained from weak compactness as before, applying the usual trick to combine uniform integrabilty and Egorov's Theorem. ∎

We can even do a little better (compare with 1.6.6).

3.5.3. *If $0 < p \le 1$ and $J_\mu : H^p \to L^1(\mu)$ exists and is weakly compact, then it is compact.*

Proof. Our assumption implies that J_μ is weakly compact as an operator $H^1 \to L^1(\mu)$; it is therefore compact, by 3.5.2. In particular, $\mu|_{\mathbb{T}} = 0$.

From here we get to compactness of $J_\mu : H^p \to L^1(\mu)$ by meanwhile familiar arguments. Let (f_n) be a bounded sequence in H^p. By Montel's Theorem, some subsequence of (f_n) converges locally uniformly to some $f \in \mathcal{H}(D)$, and by Fatou's Lemma, f belongs to H^p. Therefore it suffices to look at bounded sequences (f_n) in H^p which converge pointwise to zero. By hypothesis and since μ vanishes on $\mathbb{T}$, (f_n) is uniformly integrable in $L^1(\mu)$. Since $f_n \to 0$ pointwise on D, Egorov's Theorem yields $\lim_n \|f_n\|_{L^1(\mu)} = 0$. ∎

More on these and related topics will be contained in [4], [21] and [31].

3.6. ON BANACH SPACES OF CARLESON MEASURES. Let $\Omega \subseteq \mathbb{T}$ be an open set. The 'tent' over Ω is $\Theta(\Omega) := \Omega \cup \{z \in D : I(z) \subseteq \Omega\}$ and the 'Stolz domain' (rather: 'Stolz-like domain') associated with $\zeta \in \mathbb{T}$ is $\Gamma(\zeta) := \{z \in D : \zeta \in I(z)\}$. We recommend to draw some pictures. The terminology is reminiscent of the situation in $\mathcal{H}$, the upper half plane (R.R. Coifman, Y. Meyer and E.M. Stein [10]). There the 'tent' over an open set $U \subseteq \mathbb{R}$ is $\Theta(U) = \{x + iy \in \mathcal{H} : (x - y, x + y) \subseteq U\}$, and the 'Stolz domain' given by $t \in \mathbb{R}$ is $\Gamma(t) = \{x + iy \in \mathcal{H} : |t - x| < y\}$.

H. Heiming [25], [26] has investigated in detail so-called 'β-Carleson measures' for $\beta > 0$; these are members μ of $\mathcal{M}(\overline{D}) = \mathcal{C}(\overline{D})^*$ for which there is a constant $C \geq 0$ such that $|\mu|(\Theta(\Omega)) \leq C \cdot |\Omega|^\beta$ for all open sets $\Omega \subseteq \mathbb{T}$. In this final chapter we report briefly about some parts of this work.

The collection $\mathcal{M}_\beta(\overline{D})$ of β-Carleson measures μ is a Banach lattice with $\|\mu\|_\beta := \inf\ C$ (C from above) as a norm. If $\beta \geq 1$ then we can replace tents $\Theta(\Omega)$ by Carleson boxes $S(z)$ and get back to the measures discussed in 3.5. But in case $\beta < 1$ the traditional definition will produce too many measures for a satisfying duality theory to hold.

Say that $\mu \in \mathcal{M}(\overline{D})$ is a 'vanishing β-Carleson measure', $\mu \in \mathcal{M}_\beta^0(\overline{D})$, if $\lim_{|\Omega| \to 0} |\mu|(\Theta(\Omega))/|\Omega|^\beta = 0$. Clearly, $\mathcal{M}_\beta(\overline{D}) \subseteq \mathcal{M}_\gamma^0(\overline{D})$ for $0 < \gamma < \beta$. Moreover, $\mathcal{M}_\beta^0(\overline{D})$ is a closed sublattice of $\mathcal{M}_\beta(\overline{D})$, and if $\beta \geq 1$ then $\mu \in \mathcal{M}(\overline{D})$ belongs to $\mathcal{M}_\beta^0(\overline{D})$ if and only if, regardless of $0 < q < \infty$, $J_\mu : H^q \to L^{\beta q}(\mu)$ exists as a compact operator.

There are various similarities between spaces $\mathcal{M}_\beta(\overline{D})$ and $\mathcal{M}(\overline{D})$. For example, $\mathcal{M}_\beta(\overline{D})$ has a predual which resembles a $\mathcal{C}(K)$-space and in which an Ascoli-Arzelà type characterisation of compactness is available. Moreover, recall the following consequence of the Bartle-Dunford-Schwartz Theorem: if $W \subseteq \mathcal{M}(K)$ is weakly compact, then there is a $0 \leq \lambda \in \mathcal{M}(K)$ such that $\mu \ll \lambda$ for all $\mu \in W$. Moreover W 'is' weakly compact in $L^1(\lambda)$.

There is a perfect analogue for β-Carleson measures. The rôle of $L^1(\lambda)$ is taken by the 'Carleson function space' $M_\beta^1(\lambda) := \{f \in L^0(\lambda) : f\, d\lambda \in \mathcal{M}_\beta(\overline{D})\}$ which is a closed sublattice of $\mathcal{M}_\beta(\overline{D})$:

3.6.1. *Let $\beta > 0$. If $W \subseteq \mathcal{M}_\beta(\overline{D})$ is weakly compact, then there is a $0 \leq \lambda \in \mathcal{M}_\beta(\overline{D})$ such that $\mu \ll \lambda$ for all $\mu \in W$. Moreover W 'is' weakly compact in $M_\beta^1(\lambda)$.*

Let now $\mathcal{N}f(\zeta) := \sup_{z \in \Gamma(\zeta)} |f(z)|$ be the 'non-tangential supremum' (in $[0, \infty]$) of a given function $f : D \to \mathbb{C}$. Then $\mathcal{N}f : \mathbb{T} \to [0, \infty]$ is measurable,

so that $\|f\|_{T^q} := \left(\int_{\mathbb{T}} |\mathcal{N}f|^q dm \right)^{1/q}$ exists (in $[0, \infty]$) for any $0 < q < \infty$. This defines an 'extended q-norm' on $\mathbb{C}^D$, and $\mathcal{T}^q(D) := \{f : D \to \mathbb{C} : \|f\|_{T^q} < \infty\}$ is a (q-) Banach lattice with (q-)norm $\|\cdot\|_{T^q}$.

Think of $\mathcal{C}(\overline{D})$ as the space of all uniformly continuous functions on D. By the very definition, each $f \in \mathcal{C}(\overline{D})$ (rather, $f|_D$) belongs to $\mathcal{T}^q(D)$, and the resulting map $\mathcal{C}(\overline{D}) \to \mathcal{T}^q(D) : f \mapsto f$ is a contractive linear injection. Define the 'tent space' $T^q(D)$ to be the closure of $\mathcal{C}(\overline{D})$ in $\mathcal{T}^q(D)$, and the 'little tent space' $t^q(D)$ to be the closure of $\mathcal{C}_0(D)$ in $\mathcal{T}^q(D)$. It is readily seen that both, $T^q(D)$ and $t^q(D)$, are separable (q)-Banach lattices. But these spaces are by no means reflexive!

It is not hard to see that $|f(z)| \leq \|f\|_{T^q} \cdot |I(z)|^{-1/q} = \|f\|_{T^q} \cdot (1 - |z|^2)^{-1/q}$ holds for all f in $T^q(D)$ and z in D. Point evaluations $\delta_z : T^q(D) \to \mathbb{C} : f \mapsto f(z)$, $z \in D$, are thus bounded linear forms on T^q, with $\|\delta_z\|_{T^q}^* = (1-|z|^2)^{-1/q}$. In particular, $T^q(D)$ and $t^q(D)$ do have separating duals.

By the above estimate, if $K \subseteq D$ is compact, then even $\sup_{z \in K} |f(z)| \leq c_K \cdot \|f\|_{T^q}$ for $f \in T^q(D)$ where $c_K := \max_{z \in K}(1 - |z|^2)^{-1/q}$. It follows that $T^q(D) \hookrightarrow \mathcal{C}(D) : f \mapsto f$ is well-defined and continuous if we endow $\mathcal{C}(D)$ with the topology of local uniform convergence.

We thus may write $\mathcal{C}(\overline{D}) \hookrightarrow T^q(D) \hookrightarrow \mathcal{C}(D)$. Uniformly continuous functions $D \to \mathbb{C}$ have boundary values everywhere; continuous functions $D \to \mathbb{C}$ need not have any boundary values. How do functions in $T^q(D)$ behave in this respect?

If $f \in T^q(D)$ then it may happen that $(f(z_n))$ is unbounded for some sequence (z_n) in D such that $|z_n| \to 1$. But the degree of 'unboundedness' is under control: $f(z) \mapsto (1 - |z|^2)^{1/q} f(z)$ defines a bounded linear injection $T^q(D) \to \mathcal{C}_0(D)$. In particular, given $0 < r < 1$, $\mathcal{C}_0(rD)$ is canonically isomorphic to a closed subspace of $t^q(D)$. In fact, if $f \in \mathcal{C}_0(rD)$ then $(1 - r^2)^{1/q} \cdot \|f\|_\infty \leq \|f\|_{T^q} \leq \|f\|_\infty$. To see this, extend $f \in \mathcal{C}_0(rD)$ to $f \in \mathcal{C}_0(D)$ by setting $f(z) := 0$ for $|z| \geq r$ and use the preceding observations to get $(1 - |z|^2)^{1/q} \cdot \|f\|_\infty \leq \|f\|_{T^q} \leq \|f\|_\infty$. The statement follows.

Given $\varepsilon > 0$, choose $0 < r < 1$ with $(1 - r^2)^{-1/q} \leq 1 + \varepsilon$. Then $(1 + \varepsilon)^{-1} \cdot \|f\|_\infty \leq \|f\|_{T^q} \leq \|f\|_\infty$. It is routine to construct an isometric isomorphism $\mathcal{C}[0, 1] \to \mathcal{C}_0(rD)$. Since every separable Banach space is isometric to a subspace of $\mathcal{C}[0, 1]$, we may conclude that, regardless of $0 < q < \infty$ and $\varepsilon > 0$, every separable Banach space is $(1 + \varepsilon)$-isomorphic to a subspace of $t^q(D)$. Together with the version of the Ascoli-Arzelà Theorem alluded to earlier this allows to show that tent spaces $T^q(D)$ and $t^q(D)$ do have the approximation property. However, it is open if they even have the metric (or

bounded) approximation property.

The Stolz domains $\Gamma(\zeta) = \{z \in D : \zeta \in I(z)\}$ are special 'non-tangential approach regions' for functions $f \in H^q(D)$. In fact, by Fatou's Theorem, $f^*(\zeta) = \lim\limits_{z \in \Gamma(\zeta), z \to \zeta} f(\zeta)$ m-a.e. on $\mathbb{T}$. These domains play the same rôle for functions in tent spaces:

3.6.2. *If $f \in T^q(D)$ then the above non-tangential limits exist for m-almost all $\zeta \in \mathbb{T}$, and they generate an element $f^* \in L^q(\mathbb{T})$. The resulting map $T^q(D) \to L^q(\mathbb{T}) : f \mapsto f^*$ is linear, has norm one, and is onto. Its kernel is $t^q(D)$. Moreover, $\|f\|_{L^q} = \mathrm{dist}\,(f, t^q(D))$ holds for all $f \in T^q(D)$.*

Of course, now relations to Hardy spaces call for investigation. According to (a variant of) the Burkholder-Gundy-Silverstein Theorem (see P. Koosis [35]) there is a function $K : (0, \infty) \to (0, \infty)$ such that $\|\mathcal{N}f\|_{L^q} \leq K(q) \cdot \|f\|_{H^q}$ for all $f \in H^q$ and $0 < q < \infty$. As a consequence, $H^q(D)$ is isomorphic to a subspace of $T^q(D)$; in fact, $H^q(D) = \{f \in T^q(D) : f \text{ analytic}\}$. If $1 < q < \infty$, then even $T^q(D) = t^q(D) \oplus h^q(D)$, where $h^q(D)$ is the 'harmonic Hardy space' of exponent q.

The solution of the duality problem for spaces of Carleson measures is in form of a Riesz type Representation Theorem:

3.6.3. *For every $0 < q < \infty$, $T^q(D)^* \cong \mathcal{M}_{1/q}(\overline{D})$ isometrically.*

The first ingredient for this is an inequality: if $0 < \beta, q < \infty$, $f \in \mathcal{C}(\overline{D})$ and $\mu \in \mathcal{M}_\beta(\overline{D})$, then $\|f\|_{L^{\beta q}(|\mu|)} \leq \|\mu\|_\beta^{1/q} \cdot \|f\|_{T^q}$. Moreover, $\|\mu\|_\beta = \sup\left\{\|f\|_{L^{\beta q}(|\mu|)} : f \in \mathcal{C}(\overline{D}), \ \|f\|_{T^q} \leq 1\right\}$.

The second ingredient reads as follows: Given $0 < q < \infty$, put $\beta := 1/q$. If $\Phi : \mathcal{C}(\overline{D}) \to \mathbb{C}$ is a $\|\cdot\|_{T^q}$-bounded linear form, then there is a measure $\mu \in \mathcal{M}_\beta(\overline{D})$ such that $\|\mu\|_\beta \leq \|\Phi\|_{T^q}^*$ and $\langle \Phi, f \rangle = \int_{\overline{D}} f(z) d\mu(z)$ for each $f \in \mathcal{C}(\overline{D})$. Of course, $\int_{\overline{D}} f(z) d\mu(z)$ is $\int_D f(z)\, d\mu|_D(z) + \int_{\mathbb{T}} f^*(z)\, d\mu|_{\mathbb{T}}(z)$.

It is more difficult to describe the dual of $t^q(D)$. We have already mentioned that $T^q(D)^* = t^q(D)^* \oplus h^q(D)^*$ when $1 < q < \infty$. If we replace, in the definition of tent spaces, $L^q(m)$ by a Lorentz space $L^{q,r}(m)$, then we arrive at tent spaces $T^{q,r}(D)$ and $t^{q,r}(D)$. It can then be shown that, for $q \geq 1$, $t^{q,1}(D)^*$ is isomorphic to $\{\mu \in \mathcal{M}^{1/q}(\overline{D}) : \mu|_{\mathbb{T}} = 0\}$.

Out of further interesting results related to Carleson embeddings we mention:

3.6.4. *Let $\mu \in \mathcal{M}(\overline{D})$ and $\beta > 0$. Then μ is in $\mathcal{M}_\beta(\overline{D})$ if and only if $J_\mu : T^q(D) \to L^{\beta q}(\mu) : f \mapsto f$ exists as a bounded operator, for some (and then all) $0 < q < \infty$.*

Recall that, for $\beta \geq 1$, $\mu \in \mathcal{M}_\beta^0(\overline{D})$ is equivalent to compactness of $J_\mu : H^q \to L^{\beta q}(\mu)$ for some, and then all, $0 < q < \infty$. In particular, compactness of $J_\mu : T^q(D) \to L^{\beta q}(\mu)$ implies $\mu \in \mathcal{M}_\beta^0(\overline{D})$. The converse fails. But:

3.6.5. *If $\beta \geq 1$ and $\mu \in \mathcal{M}_\beta^0(\overline{D})$, then $J_\mu : T^q(D) \to L^{\beta q}(\mu)$ is the uniform limit of (βq)-integral operators $T^q(D) \to L^{\beta q}(\mu)$.*

A Banach space operator $u : X \to Y$ is called 'absolutely continuous' if there are a Banach space Z containing Y as a subspace (embedding $j : Y \hookrightarrow Z$) and a sequence of s-integral (equivalently, s-summing) operators $v_n : X \to Z$ such that $\lim_n \|j \circ u - v_n\| = 0$. The choice of s doesn't matter, but the enlargement Z cannot be avoided. Weakly compact operators with domain a space $\mathcal{C}(K)$ are always absolutely continuous which leads to another proof of H.P. Rosenthal's result [52] according to which every reflexive quotient of $\mathcal{C}(K)$ is super-reflexive and even a quotient of some $L^q(\mu)$, $2 \leq q < \infty$. We refer to Ch.15 of [15] for details and additional references.

3.6.6. *If $\beta = q = 1$, or if $\beta \geq 1$ and $q > 1$, then $\mu \in \mathcal{M}_\beta(\overline{D})$ is a vanishing β-Carleson measure if and only if $J_\mu : T^q(D) \to L^{\beta q}(\mu)$ exists and is absolutely continuous. In such a case, J_μ is even the uniform limit of a sequence of (βq)-integral operators $T^q(D) \to L^{\beta q}(\mu)$.*

If we try to avoid the passage to uniform limits, then we arrive at a characterisation which we are familiar with from 3.4.2:

3.6.7. *Suppose that $\beta > 0$ and that $\mu \in \mathcal{M}(\overline{D})$ has no mass on $\mathbb{T}$. The following are equivalent:*

(i) *$J_\mu : T^q(D) \to L^{\beta q}$ is (βq)-integral for some/all $q \geq \beta^{-1}$.*

(ii) *$J_\mu : T^q(D) \to L^{\beta q}$ is (βq)-summing for some/all $q \geq \beta^{-1}$.*

(iii) *For some/all $q \geq \beta^{-1}$ there exists $0 \leq g \in L^{\beta q}(\mu)$ such that $|f| \leq g \cdot \|f\|_{T^q}$ for all $f \in T^q(D)$.*

(iv) *The function $z \mapsto (1 - |z|^2)^{-1}$ belongs to $L^\beta(\mu)$.*

(v) *$J_\mu : T^q(D) \to L^{\beta q}(\mu)$ is order bounded for some/all $q \geq \beta^{-1}$.*

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