

Ordinal Indices in Banach Spaces

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0. INTRODUCTION

The use of ordinal indices in Banach space theory dates back to the origins of the subject and can be found in S. Banach's famous book [14]. Interest in such indices was rejuvenated just over 35 years ago with a paper by Szlenk [50]. Since then a number of ordinal indices have been constructed which measure the complexity of various aspects of the structure of a separable Banach space X . Typically a property (P) is considered and X has (P) corresponds to the index of X is ω_1 , the first uncountable ordinal, while if X fails (P) then its index is an ordinal $\alpha < \omega_1$. One advantage to this approach is that X can be indirectly shown to have (P) by showing that its index exceeds every $\alpha < \omega_1$. Another advantage is that, presuming the index to be an isomorphic invariant, one can often construct an uncountable number of nonisomorphic spaces (failing (P)) by showing that for all $\alpha < \omega_1$ there exists X_α of countable index exceeding α . Other results are of the type that if the index of X is large enough then X admits a substructure of a certain degree of complexity and this may yield other consequences.

In section 2 we discuss Bourgain's ℓ_1 index, $I(X)$. $I(X) = \omega_1$ iff X contains an isomorph of ℓ_1 . This index generalizes readily to other bases besides the unit vector basis of ℓ_1 and leads to a number of results of this type: Let Y be a fixed Banach space and let C be a certain class of separable Banach spaces. If X contains isomorphs of all spaces in C then X contains Y . If $Y = C(\Delta)$ we conclude that X is *universal*, i.e., every separable Banach space embeds isomorphically into X .

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In section 3 we begin by presenting Szlenk's index $\eta(X)$. One has that $\eta(X) < \omega_1$ iff X^* is separable. We also present some more results on indices like those in section 2 and some variants of those indices. We explain the relationship of the Szlenk index with one of these, the ℓ_1^+ -weakly null index.

Section 4 is devoted to certain Baire-1 indices. If X is not reflexive then there exists $x^{**} \in X^{**}$ so that $x^{**} : K \rightarrow \mathbb{R}$ is not continuous where K is the compact metric space (B_{X^*}, ω^*) . The topological nature of this function yields certain information about the subspace structure of X .

In section 5 we define the Schreier classes $(S_\alpha)_{\alpha < \omega_1}$ and give some applications. These are classes of compact hereditary families of finite subsets of \mathbb{N} of increasing complexity. One can measure the behavior of weakly null sequences against these sets and produce conclusions about their structure.

More results and references on ordinal indices can be found in [30].

1. PRELIMINARIES

We will generally use X, Y, Z, \dots to denote separable infinite dimensional real Banach spaces. In this section we present some background material. For more detail we suggest the reader consult [39] or [22] (see also [41]). The two books [33] and [34] are excellent more advanced references. We assume the reader has a solid background in functional analysis.

$X \subseteq Y$ will denote that X is a closed linear subspace of Y . $B_X = \{x \in X : \|x\| \leq 1\}$ is the unit ball of X and $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of X . The ω^* topology (weak* topology) restricted to B_{X^*} makes the dual unit ball into a compact metrizable space K . Recall that a base for the ω^* topology on X^* is given by all sets of the form $(x^* \in X^*, F \subseteq X \text{ is a finite set, } \varepsilon > 0)$

$$N(x^*, F, \varepsilon) = \{y^* \in X^* : |y^*(x) - x^*(x)| < \varepsilon \text{ for all } x \in F\}.$$

$K = (B_{X^*}, \omega^*)$ is always compact and the separability of X yields that it is metrizable. Every element $x \in X$ or $x^{**} \in X^{**}$ can be regarded as a bounded function on K (for $x \in X$, $x(x^*) = x^*(x)$) and $x^{**} \in X$ (i.e., there exists $x \in X$ with $x(x^*) = x^{**}(x^*)$ for all $x^* \in K$) iff $x^{**}|_K \in C(K)$, the space of continuous real valued functions on K . Thus X is reflexive iff every $x^{**} \in X^{**}$ is continuous on K . Also X is reflexive iff B_X is weakly compact. Again we recall that the weak topology on X is generated by the open base

$$N(x, F, \varepsilon) = \{y \in X : |x^*(x) - x^*(y)| < \varepsilon \text{ for all } x^* \in F\}$$

where $x \in X$, $F \subseteq X^*$ is finite and $\varepsilon > 0$. (B_X, ω) is metrizable iff X^* is separable. By the Eberlein-Smulian theorem, B_X is weakly compact iff for all $(x_n) \subseteq B_X$, some subsequence is weakly convergent (necessarily to an element in B_X). Thus X can fail to be reflexive in one of two ways.

- (a) There exists $(x_n) \subseteq S_X$ with no weak Cauchy subsequence (for all subsequences $(y_n) \subseteq (x_n)$, (y_n) is not pointwise convergent on K) or
- (b) There exists $(x_n) \subseteq S_X$ which is weak Cauchy but converges pointwise to a discontinuous function on K .

In case (a) a remarkable theorem of H. Rosenthal yields that a subsequence of (x_n) is equivalent to the unit vector basis of ℓ_1 (see below for the definition) [45]. In case (b) there exists $x^{**} \in X^{**} \setminus X$ so that $x_n \rightarrow x^{**}$ in the ω^* topology of X^{**} . Thus $x^{**}|_K$ is Baire-1 (the pointwise limit of a sequence of continuous functions).

X and Y are *isomorphic* if there exists a bounded linear 1-1 onto operator $T : X \rightarrow Y$. We write $X \sim Y$. If in addition $\|Tx\| = \|x\|$ for all $x \in X$ we say X and Y are *isometric*, $X \cong Y$. If X is *isomorphic to a subspace* of Y we write $X \hookrightarrow Y$ and say X *embeds* into Y or Y *contains* X . If $T : X \rightarrow Y$ is an isomorphic embedding with $\|T\| \|T^{-1}|_{T(X)}\| \leq K$ we write $X \xrightarrow{K} Y$ and say that X *K-embeds* into Y . If T is onto we write $X \xrightarrow{K} Y$. This terminology holds for finite dimensional spaces as well. Thus if we say that X *contains* ℓ_1^n 's *uniformly* then we mean that for some $K < \infty$ and all $n \in \mathbb{N}$, $\ell_1^n \xrightarrow{K} X$. (Actually in this case by a theorem of James [31], $\ell_1^n \xrightarrow{\lambda} X$ for all n and $\lambda > 1$).

$(x_i)_{i=1}^\infty$ is a *basis* for X if for all $x \in X$ there exist unique scalars $(a_i) \subseteq \mathbb{R}$ with $x = \sum_{i=1}^\infty a_i x_i$. $(x_i)_1^\infty \subseteq X$ is *basic* if $(x_i)_{i=1}^\infty$ is a basis for $[(x_i)_{i=1}^\infty] \equiv$ closed linear span of $(x_i)_{i=1}^\infty$. This is known to be equivalent to: $x_i \neq 0$ for all i and there exists $K < \infty$ so that $\|\sum_{i=1}^n a_i x_i\| \leq K \|\sum_{i=1}^m a_i x_i\|$ for all $n < m$ and $(a_i)_1^m \subseteq \mathbb{R}$. The smallest such K is called the *basis constant* of (x_i) and (x_i) is said to be *K-basic* if its basis constant does not exceed K . A 1-basic sequence is called *monotone*. These definitions also hold for finite sequences $(x_i)_{i=1}^n$. A finite sequence is basic iff it is linearly independent.

(x_i) is *K-unconditional basic* if $x_i \neq 0$ for all i and for all m and $(a_i)_1^m \subseteq \mathbb{R}$, $\varepsilon_i = \pm 1$,

$$\left\| \sum_{i=1}^m \varepsilon_i a_i x_i \right\| \leq K \left\| \sum_{i=1}^m a_i x_i \right\|.$$

This is equivalent to: for all $x \in [(x_i)]$ there exist unique $(a_i) \subseteq \mathbb{R}$ so that

$x = \sum_{i=1}^{\infty} a_{\pi(i)} x_{\pi(i)}$ for all permutations π of \mathbb{N} . If (x_i) is K -unconditional basic it is also K -basic.

A sequence $(x_i) \subseteq X$ is *normalized* if $\|x_i\| = 1$ for all i . It is *seminormalized* if $0 < \inf_i \|x_i\| \leq \sup_i \|x_i\| < \infty$. Basic sequences (x_i) and (y_i) are *K -equivalent* if there exist $0 < c, C < \infty$ with $c^{-1}C \leq K$ and

$$c \left\| \sum a_i x_i \right\| \leq \left\| \sum a_i y_i \right\| \leq C \left\| \sum a_i x_i \right\|$$

for all scalars $(a_i) \subseteq \mathbb{R}$. This just says that there is an isomorphism $T : [(x_i)] \rightarrow [(y_i)]$ with $Tx_i = y_i$ and $\|T\| \leq C, \|T^{-1}\| \leq c^{-1}$.

We assume the reader is familiar with the classical Banach spaces c_0, ℓ_p ($1 \leq p \leq \infty$), $C(K), L_p[0, 1]$ ($1 \leq p \leq \infty$). Δ denotes the Cantor set. $C(\Delta)$ is 1-universal: For all $X, X \xrightarrow{1} C(\Delta)$. $C(\Delta) \sim C[0, 1]$ and $C[0, 1]$ is also 1-universal. Indeed, $K = (B_{X^*}, \omega^*)$ is compact metric. Thus there exists a continuous onto map $\phi : \Delta \rightarrow K$. Then $X \xrightarrow{1} C(\Delta)$ via $x \mapsto x \circ \phi$. Also it is easy to then show that $C(\Delta) \xrightarrow{1} C[0, 1]$.

c_{00} is the linear space of all finitely supported sequences of scalars on \mathbb{N} . The *unit vector basis* is $(e_i)_{i=1}^{\infty}$ where $e_i = (0, 0, \dots, 0, 1, 0, \dots)$, the 1 occurring in the i^{th} place. (e_i) is a linear basis for c_{00} but naturally lives in c_0 and ℓ_p ($1 \leq p \leq \infty$) and forms a 1-unconditional basis for these spaces ($p < \infty$). More generally, if (x_i) is a normalized basis for X we may regard $X = \overline{(c_{00}, \|\cdot\|)}$, the completion of $(c_{00}, \|\cdot\|)$ under $\|(a_i)\| = \left\| \sum a_i x_i \right\|$ and then (e_i) is a normalized basis for X . Many Banach spaces are constructed in this way. One selects, for example, a certain $\mathcal{F} \subseteq c_{00}$ with $e_i \in \mathcal{F}$ for all i , $(b_i) \in \mathcal{F}$ implies $|b_i| \leq 1$ for all i and $(b_1, \dots, b_n, 0, 0, \dots) \in \mathcal{F}$ for all n . Then

$$\|(a_i)\|_{\mathcal{F}} \equiv \sup \left\{ \left| \sum a_i b_i \right| : (b_i) \in \mathcal{F} \right\}$$

yields a norm on c_{00} that makes (e_i) into a normalized monotone basis for the completion $(c_{00}, \|\cdot\|_{\mathcal{F}})$.

If (X_i) is a sequence of Banach spaces then for $1 \leq p < \infty$

$$\left(\sum_{i=1}^{\infty} X_i \right)_p \equiv \left\{ (x_i)_{i=1}^{\infty} : \|(x_i)_{i=1}^{\infty}\| \equiv \left(\sum_{i=1}^{\infty} \|x_i\|^p \right)^{1/p} < \infty \text{ and } x_i \in X_i \text{ for all } i \right\}.$$

$(\sum_{i=1}^{\infty} X_i)_{c_0}$ is defined similarly. These are Banach spaces under the indicated norm. If $1 < p < \infty$ and each X_i is reflexive then $(\sum X_i)_p$ is also reflexive.

If (x_i) is K -basic, a *block basis* of (x_i) is a sequence (y_i) given by $y_i = \sum_{j=p_{i-1}+1}^{p_i} a_j x_j$ for some $p_0 < p_1 < \dots$, $(a_i) \subseteq \mathbb{R}$ with $y_i \neq 0$ for all i . (y_i) is

then also K -basic (and maybe better). If $a_j \geq 0$ for all j and $\sum_{j=p_{i-1}+1}^{p_i} a_j = 1$ for all i then (y_i) is a *convex block basis* of (x_i) .

Let (e_i) be the unit vector basis for c_{00} and define for $(a_i) \in c_{00}$

$$\left\| \sum a_i e_i \right\| = \sup_n \left| \sum_{i=1}^n a_i \right|.$$

Then (e_i) is a basis for the completion X of c_{00} under this norm. We call this the *summing basis* and denote it by (s_i) . X is isomorphic to c_0 and (s_i) is a *conditional* basis (not unconditional) for X .

Every X contains a $1 + \varepsilon$ -basic sequence for all $\varepsilon > 0$. If $(x_n) \subseteq S_X$ is weakly null (such a sequence exists in X by Rosenthal's ℓ_1 theorem if $\ell_1 \not\hookrightarrow X$) then some subsequence is $1 + \varepsilon$ -basic. If $(x_n) \subseteq S_X$ converges ω^* in X^{**} to $x^{**} \in X^{**} \setminus X$ then some subsequence (y_n) is basic and *dominates* the summing basis: this means that for some $c > 0$,

$$\left\| \sum a_i y_i \right\| \geq c \left\| \sum a_i s_i \right\|$$

for all scalars (a_i) .

Let (x_i) be a normalized basic sequence in X . Using Ramsey theory (see e.g. [41] or [15]), given $\varepsilon_n \downarrow 0$, one can extract a subsequence (y_i) satisfying:

for all $n \in \mathbb{N}$, $(a_i)_1^n \in [-1, 1]^n$ and $n \leq i_1 < \dots < i_n$, $n \leq j_1 < \dots < j_n$

$$\left| \left\| \sum_{k=1}^n a_k y_{i_k} \right\| - \left\| \sum_{k=1}^n a_k y_{j_k} \right\| \right| < \varepsilon_n.$$

Thus we can define a norm on c_{00} by

$$\lim_{i_1 \rightarrow \infty} \dots \lim_{i_n \rightarrow \infty} \left\| \sum_{k=1}^n a_k y_{i_k} \right\| \equiv \left\| \sum_{k=1}^n a_k e_k \right\|.$$

E = the completion of $(c_{00}, \|\cdot\|)$ is a *spreading model* of X generated by (y_i) [21]. (e_i) is a basis for E . If (y_i) is weakly null then (e_i) is 2-unconditional.

We also recall a few facts about perturbations. If (x_i) is a normalized C -basic sequence and $\varepsilon > 0$ then there exist $\varepsilon_n \downarrow 0$ (depending solely upon C and ε) so that if $(y_i) \subseteq X$ satisfies $\|y_i - x_i\| < \varepsilon_i$ for all i , then (y_i) is basic and $1 + \varepsilon$ -equivalent to (x_i) . Thus if X has a basis (e_i) and $Y \subseteq X$ then for all $\varepsilon > 0$ there exists $(y_i) \subseteq S_Y$ which is a suitable perturbation of a block basis of (e_i) and hence is $1 + \varepsilon$ -equivalent to this block basis. Similarly if $(y_i) \subseteq S_Y$ is weakly null then some subsequence is a perturbation of a block basis of (e_i) .

We will also assume some familiarity with the ordinals up to ω_1 , the first uncountable ordinal (see e.g. [28]). Listing these in increasing order we have

$$(0, 1, 2, 3, \dots, \omega, \omega + 1, \omega + 2, \dots, \omega \cdot 2, \omega \cdot 2 + 1, \dots, \omega \cdot 3, \\ \dots, \omega^2, \omega^2 + 1, \dots, \omega^3, \dots, \omega^n, \dots, \omega^\omega, \omega^\omega + 1, \dots, \omega^\omega + \omega, \\ \dots, \omega^{\omega \cdot 2}, \dots, \omega^{(\omega^2)}, \dots, \omega_1)$$

These form a well ordered set and thus we can use induction to define and prove things. One usually will use different definitions/arguments for successor and limit ordinals. A *limit ordinal* is an ordinal not of the form $\alpha + 1$ (often written $\alpha+$), the latter being a *successor ordinal*. Every ordinal $0 \leq \alpha < \omega_1$, can be written in *Cantor normal form* as

$$\omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \cdot n_2 + \dots + \omega^{\alpha_k} \cdot n_k + n_{k+1}$$

where $k \geq 0$, $n_{k+1} \geq 0$, $(n_i)_1^k \subseteq \mathbb{N}$ and $\omega_1 > \alpha_1 > \alpha_2 > \dots > \alpha_k$.

There are uncountably many limit ordinals $< \omega_1$ and uncountably many successor ordinal $< \omega_1$.

2. BOURGAIN'S INDEX AND SOME VARIATIONS

The definition of this index depends upon trees or more accurately on trees on Banach spaces.

DEFINITION 2.1. By a *tree* we shall mean a nonempty partially ordered set (T, \leq) for which the set $\{y \in T : y \leq x\}$ is linearly ordered and finite for each $x \in T$. The elements of T are called *nodes*. The *predecessor* node of x is the maximal element of $\{y \in T : y < x\}$. An *immediate successor* of x is any node y such that x is the predecessor node of y . The *initial nodes* of T are the minimal elements of T (the nodes without predecessors). The *terminal nodes* of T are the maximal elements of T (the nodes without successors). A *branch* of T is a maximal linearly ordered subset. T is *well founded* if all of its branches are finite. A *subtree* τ of T is a subset of T with the inherited partial order from T .

If X is a set, by a *tree on X* we shall mean a subset $T \subseteq \bigcup_{n=1}^{\infty} X^n$ such that for $(x_1, \dots, x_m), (y_1, \dots, y_n) \in T$, $(x_1, \dots, x_m) \leq (y_1, \dots, y_n)$ iff $m \leq n$ and $x_i = y_i$ for $i \leq m$. Clearly a tree on X is a tree (as defined above).

DEFINITION 2.2. If T is a tree we set $D(T) = \{x \in T : x < y \text{ for some } y \in T\}$. (Thus $D(T) = T \setminus \{\text{terminal nodes of } T\}$.)

We next define the *order* or *height* of T , $o(T)$. We inductively define $T^0 = T$, $T^{\alpha+1} = D(T_\alpha)$ for $\alpha < \omega_1$ and $T^\alpha = \bigcap_{\beta < \alpha} T^\beta$ if $\alpha < \omega_1$ is a limit ordinal. We set $o(T) = \inf\{\alpha < \omega_1 : T^\alpha = \emptyset\}$ if such an α exists and $o(T) = \omega_1$ otherwise. (Of course these notions could all be defined for larger ordinals but this suffices for our purposes.)

To help understand this we describe certain canonical trees T_α of order α . T_1 is just a single node. Given T_α we select a new node $z \notin T_\alpha$ and add this as a new initial node to form $T_{\alpha+1}$. Thus $T_{\alpha+1} = \{z\} \cup T_\alpha$ with $z < x$ for all $x \in T_\alpha$. If β is a limit ordinal we let T_β be the disjoint union of $\{T_\alpha : \alpha < \beta\}$ ordered by $x \leq y$ iff there exists $\alpha < \beta$ with $x, y \in T_\alpha$ and $x \leq y$ in T_α . It is an easy exercise [36] to show by induction on α that T_α is then a *minimal tree of order α* : this means that if T is well founded with $o(T) \geq \alpha$ then there exists a subtree τ of T which is order isomorphic to T_α .

If T is a tree on a topological space X , we say that T is *closed* if $T \cap X^n$ is closed for all $n \in \mathbb{N}$ (X^n is given the product topology).

PROPOSITION 2.3. *If T is a closed well founded tree on a Polish space X then $o(T) < \omega_1$.*

Remark 2.4. This is the only result we require below. One can actually prove more general results. For example the conclusion holds if we only assume (see e.g. [37, section 31]) that T is a well founded *analytic tree* on X ($T \cap X^n$ is an analytic set for each $n \in \mathbb{N}$). A *Polish space* is a complete separable metric space.

Proof of Proposition 2.3. Let d be the metric on X . If the result is false then for each $\alpha < \omega_1$, T contains a subtree \tilde{T}_α isomorphic to T_α . Let $S = \{\alpha < \omega_1 : \alpha \text{ is a successor ordinal}\}$. For $\alpha \in S$, \tilde{T}_α has a unique initial node $x^\alpha(1) \equiv (x_i^\alpha)_{i=1}^{n_\alpha}$. Let $\varepsilon_n \downarrow 0$. For uncountably many $\alpha \in S$, $n_\alpha = n_1$ for some fixed $n_1 \in \mathbb{N}$. Also, since X is separable, for uncountably many of these α 's, say for some uncountable $U_1 \subseteq S$,

$$d(x_i^\alpha, x_i^\beta) < \varepsilon_1 \text{ if } \alpha, \beta \in U_1, \text{ and } i \leq n_1.$$

For each $\alpha \in U_1$, let $\tilde{T}_\alpha(x^\alpha(1)) = \{y \in \tilde{T}_\alpha : y > x^\alpha(1)\}$. Thus

$$\sup_{\alpha \in U_1} o(\tilde{T}_\alpha(x^\alpha(1))) = \omega_1.$$

It follows that for each $\alpha \in S$ we can extract a subtree of $\tilde{T}_\beta(x^\beta(1))$ for some $\beta > \alpha$ with $\beta \in U_1$ which is isomorphic to T_α , with initial node $x^\alpha(2)$ and

repeat the process for ε_2 . We obtain an uncountable $U_2 \subseteq S$, $n_2 \in \mathbb{N}$, $n_2 > n_1$, so that if $\alpha \in U_2$ then $x^\alpha(2) \in X^{n_2} \cap T$ and

- (i) $\sup_{\alpha \in U_2} o\{y \in \tilde{T}_\alpha(x^\alpha(2)) : y > x^\alpha(2)\} = \omega_1$
- (ii) if $\alpha, \beta \in U_2$ and $x^\alpha(2) = (x_i)_1^{n_2}$ and $x^\beta(2) = (y_i)_1^{n_2}$ then $d(x_i, y_i) < \varepsilon_2$ for $i \leq n_2$.

Continuing as above we obtain, using that X is complete and T is closed, a sequence $(x_i)_1^\infty \subseteq X$ and $n_1 < n_2 < \dots$ so that $(x_i)_1^{n_j} \in T$ for all j . Thus T is not well founded. ■

We are now ready to define Bourgain's ℓ_1 -index [19]. Let X be a separable Banach space and $K < \infty$. A tree T on X is an $\ell_1 - K$ tree if $\forall (x_i)_1^n \in T$, $(x_i)_1^n \subseteq S_X$ and $(x_i)_1^n$ is K -equivalent to the unit vector basis of ℓ_1^n (thus for all $(a_i)_1^n \subseteq \mathbb{R}$, $\|\sum_1^n a_i x_i\| \geq K^{-1} \sum_1^n |a_i|$). Set $I(X, K) = \sup\{o(T) : T \text{ is an } \ell_1 - K \text{ tree on } X\}$. Clearly

$$I(X, K) = o(T(X, \ell_1, K))$$

where

$$T(X, \ell_1, K) = \{(x_i)_1^n \subseteq S_X : n \in \mathbb{N} \text{ and } (x_i)_1^n \text{ is } K\text{-equivalent to the unit vector basis of } \ell_1^n\}.$$

This latter tree is easily seen to be closed and so by Proposition 2.3

$$I(X) \equiv \sup\{I(X, K) : K < \infty\} = \omega_1 \text{ iff } \ell_1 \hookrightarrow X.$$

Indeed $\ell_1 \hookrightarrow X$ iff there exists a sequence $(x_i) \subseteq X$ which is equivalent to the unit vector basis of ℓ_1 . But then $(x_i/\|x_i\|)$ is K -equivalent to the unit vector basis of ℓ_1 for some K . Hence $\ell_1 \hookrightarrow X$ iff $T(X, \ell_1, K)$ has an infinite branch for some $K < \infty$.

$I(X)$ measures the complexity of the ℓ_1^n 's inside X . $I(X, K) \geq \omega$ for some $K > 1$ iff X contains ℓ_1^n 's uniformly iff (see [31]) for all $K > 1$, $I(X, K) \geq \omega$. If X does not contain ℓ_1^n 's uniformly then $I(X) = \omega$. Also it can be seen that $I(X) = I(Y)$ if $Y \sim X$.

This definition easily generalizes to an index for any semi-normalized basic sequence. In this more general setting we can no longer in general restrict to normalized sequences only. If (x_i) is such we say a tree T on X is a $(x_i) - K$ tree if for all $(a_i)_1^n \subseteq \mathbb{R}$ and $(y_i)_1^n \in T$, $K^{-1} \leq \|y_i\| \leq K$ for

$i \leq n$ and $c^{-1} \|\sum_1^n a_i x_i\| \leq \|\sum_1^n a_i y_i\| \leq C \|\sum_1^n a_i x_i\|$ for some $cC \leq K$ (i.e., $(x_i)_1^n \stackrel{K}{\sim} (y_i)_1^n$). Let

$$I(X, (x_i), K) = \sup\{o(T) : T \text{ is a } (x_i) - K \text{ tree on } X\}.$$

Alternatively, $I(X, (x_i)) = \sup\{I(X, (x_i), K) : K < \infty\}$, where $I(X, (x_i), K)$ is the maximal such tree.

As in the case of the ℓ_1 index we have

PROPOSITION 2.5. *Let (x_i) be a semi-normalized basic sequence. Let X be a separable Banach space. Then X contains a basic sequence equivalent to (x_i) iff $I(X, (x_i)) = \omega_1$. Also $I(X, (x_i)) = I(Y, (x_i))$ if $X \sim Y$.*

DEFINITION 2.6. A Banach space X is *universal* for a class of Banach spaces C if for all $Y \in C$, $Y \hookrightarrow X$.

THEOREM 2.7. ([18], [50]) *There does not exist a separable reflexive space X which is universal for the class of all separable reflexive spaces. Moreover if X is universal for all reflexive spaces then X is universal (i.e., $C(\Delta)$ embeds into X).*

Proof. To better illustrate the proof we first show that if a separable space X is universal for all separable reflexive spaces then X contains ℓ_1 (and hence is not reflexive). We achieve this by producing for all $\alpha < \omega_1$ a reflexive space X_α with $I(X_\alpha, 1) \geq \alpha$. Each X_α will have a normalized basis indexed by the nodes of the canonical trees T_α defined above and they are defined inductively much like the T_α 's.

Let $X_1 = \mathbb{R}$. If X_α has been defined set $X_{\alpha+1} = (\mathbb{R} \oplus X_\alpha)_1$. If X_β has been defined for $\beta < \alpha$, where α is a limit ordinal, $X_\alpha = (\oplus \sum_{\beta < \alpha} X_\beta)_2$. The desired properties are easily checked. Each X_α is reflexive and has a 1-unconditional basis $(e_\gamma)_{\gamma \in T_\alpha}$ so that if $\beta = (\gamma_i)_1^n$ is a branch in T_α then $(e_{\gamma_i})_1^n \stackrel{1}{\sim} (e_i)_1^n$, the unit vector basis of ℓ_1^n .

(Also the X_α 's are easily shown to be ℓ_2 -saturated (for all $Y \subseteq X_\alpha$, $\ell_2 \hookrightarrow Y$), but we do not need this here).

The proof of the moreover statement is similar. We prove by an inductive construction on $\alpha < \omega_1$ that if (x_i) is any normalized bimonotone basis then there exists a reflexive space X_α with $I(X_\alpha, (x_i), 1) \geq \alpha$ by modifying the above construction slightly. If the result is proved for α and $(x_i)_1^\infty$ is given we consider the space Y_α produced for α and $(x_i)_2^\infty$ by the inductive construction.

Y_α has a bimonotone basis naturally written as $(e_\gamma)_{\gamma \in T_\alpha}$ where for any branch $(\gamma_i)_{i=1}^n$ of T_α , $(e_{\gamma_i})_{i=1}^n$ is 1-equivalent to $(x_i)_{i=2}^{n+1}$. Set $X_{\alpha+1} = \langle e \rangle \oplus Y_\alpha$ where the norm is given for $x = ae + y$ by

$$\|x\| = \sup_{\beta} \left\{ \left\| ax_1 + \sum_{i=2}^n y(\gamma_i)x_i \right\| : \beta = (\gamma_i)_{i=2}^n \text{ is a branch in } T_\alpha \right\} \vee \|y\|.$$

At limit ordinals we use

$$X_\alpha = \left(\sum_{\beta < \alpha} X_\beta \right)_2$$

as before, where X_β is constructed for $(x_i)_{i=1}^\infty$.

In particular $C(\Delta)$ embeds into X , since as is well known, $C(\Delta)$ has a monotone basis which can be renormed to be bimonotone. ■

Remark 2.8. The moreover part of Theorem 2.7 is due to Bourgain [18]. The first part is due to Szlenk [50] (see section 2 below) who proved that X^* is nonseparable if X is universal for all separable reflexive spaces. A number of these results and more are in the very nice expositions [46] and [9]. A stronger theorem than 2.7 is due to S. Argyros [5]. A Banach space X is HI (*hereditarily indecomposable*) if whenever $X = Y \oplus Z$ (i.e., Y is the range of a projection on X with null space Z) then Y or Z must be finite dimensional. HI spaces were first produced by Gowers and Maurey in their famous paper [26]. Since then they have been shown to be ubiquitous (e.g., [7], [13], [6], [10]). Argyros, using index arguments, proved that if X is universal for all reflexive HI spaces then X is universal.

Many theorems of the type of Theorem 2.7 can be proved using these ordinal indices. Here are some more.

DEFINITION 2.9. Let $K > 1$. X has the *K -Strong Schur property* if for all $\varepsilon > 0$ and $(x_n) \subseteq S_X$ with $\|x_n - x_m\| \geq \varepsilon$ if $n \neq m$ there exists a subsequence of (x_n) which is K/ε -equivalent to the unit vector basis of ℓ_1 . X has the *Strong Schur property* if it has the K -strong Schur property for some K .

It is easy to check that ℓ_1 has the $2 + \delta$ -Strong Schur property for all $\delta > 0$.

PROPOSITION 2.10. *Let X be a separable Banach space which is universal for all Strong Schur spaces. Then X is universal.*

The proof is similar to the proof of Theorem 2.7, taking ℓ_1 sums in place of ℓ_2 sums at limit ordinals. One needs to check that if X_α has the 2-Strong Schur property for $\alpha < \beta$ then $(\sum_{\alpha < \beta} X_\alpha)_1$, also has the 2-Strong Schur property.

DEFINITION 2.11. Let $1 < p < \infty$. X has property $\omega\text{-}\ell_p$ if for all $\varepsilon > 0$ and all normalized weakly null sequences $(x_i) \subseteq X$, some subsequences of (x_i) is $1 + \varepsilon$ -equivalent to the unit vector basis of ℓ_p . The property $\omega\text{-}c_0$ is defined similarly.

PROPOSITION 2.12. *Let X be a separable Banach space which is $\omega\text{-}\ell_p$ for some $1 < p < \infty$ or $\omega\text{-}c_0$. Then X is universal.*

Again the proof is nearly the same as that of Theorem 2.7. In the $\omega\text{-}\ell_p$ case at limit ordinals we can take ℓ_p sums. In the $\omega\text{-}c_0$ case we take ℓ_1 sums at limit ordinals.

We recall that if $X = C(K)$ where K is countable compact metric then X is isomorphic to $C(\alpha)$ for some compact (successor) ordinal $\alpha < \omega_1$. Here $\alpha = \{\beta : \beta < \alpha\}$ is given the order topology: i.e., a base for the open sets is all order intervals (γ, δ) . For example $c_0 \sim c(\omega+)$. The next result is also due to Bourgain (see also [47] for a different proof from Bourgain's or the one below).

PROPOSITION 2.13. *If X is universal for the class of spaces $C(K)$, where K is countable compact metric, then X is universal.*

Some preliminaries are needed. Let T be a countable tree with a unique initial node or finitely many initial nodes. We define a Banach space $S(T)$ as follows. Let $x : T \rightarrow \mathbb{R}$ be finitely supported (i.e., $x \in c_{00}(T)$). Set $\|x\| = \sup\{|\sum_{e \in \beta} x(e)| : \beta \text{ is any branch or an initial segment of a branch in } T\}$. $S(T)$ is the completion of $c_{00}(T)$ under this norm.

Note that if we define for such a β , $f_\beta(x) = \sum_{e \in \beta} x(e)$ then $\{f_\beta : \beta \text{ is a branch or a finite initial segment of a branch in } T\}$ is a compact metric space K under the topology of pointwise convergence. Each $f_\beta \in 2^T$ which is topologized by the product of the discrete metric on T . Hence each $S(T)$ is isometric to a subspace of $C(K)$ for some compact metric space K under the isometry $x \rightarrow \{f_\beta(x) : \beta \in K\}$. In fact using the Stone-Weierstrass theorem it is not hard to see that $S(T) \cong C(K)$. If T is well founded then K is countable.

Let \mathcal{D} denote the infinite dyadic tree. Precisely we may take $\mathcal{D} = \{(\varepsilon_i)_{i=1}^n : n \geq 0 \text{ and } \varepsilon_i \in \{0, 1\} \text{ for all } i\}$ ordered as usual by extension. Thus the empty

sequence \emptyset is the unique initial node of \mathcal{D} and every node has two successors. It can be shown that $S(\mathcal{D})$ is isomorphic to $C(\Delta)$.

Each $S(T)$ has a monotone node basis $(e_\alpha)_{\alpha \in T}$ where $e_\alpha(\gamma) = \delta_{\alpha,\gamma}$ for $\alpha, \gamma \in T$ when linearly ordered as (e_i) to be compatible with the tree order: if $e_i = e_{t_i}$ and $e_j = e_{t_j}$ where $i < j$, then $t_i < t_j$ in T or t_i and t_j are incomparable. One can show using similar arguments to those above that for all $\alpha < \omega_1$ if (e_i) is a compatibly ordered node basis for $S(\mathcal{D})$ then there exists a well founded tree T' so that the node basis for $S(T')$ compatibly ordered as $(u_i)_{i=1}^\infty$ has the property that the tree

$$S = \left\{ (u_{n_i})_{i=1}^j : j \in \mathbb{N}, \text{ the } n_i\text{'s distinct and} \right. \\ \left. (u_{n_i})_{i=1}^j \text{ is 1-equivalent to } (e_i)_{i=1}^j \right\}$$

when ordered by extension has order at least α .

Thus by proposition 2.5, $C(\Delta)$ embeds into X . ■

The main unsolved problem for $C(\Delta)$ is to classify all of its complemented subspace. Each $C(\alpha)$ for $\alpha < \omega_1$ is complemented in $C(\Delta)$ and it is suspected that these spaces along with $C(\Delta)$ are a complete list (up to isomorphism). Many partial results are known (see [46], [23] and many of these make use of the Szlenk index (section 3).

Using index theory and the Haar basis, Bourgain, Rosenthal and Schechtman [20] were able to construct an uncountable number of mutually non isomorphic complemented subspaces of $L_p[0, 1]$ ($1 < p < \infty$, $p \neq 2$). The hard part here is to do the constructions of higher index in L_p so that the resulting space is complemented in L_p .

Nicole Tomczak-Jaegermann [51] proved the following beautiful result using an ordinal index for unconditionality. The K -unconditional index of X is defined by considering trees on X with nodes $(x_i)_1^n$ being K -unconditional.

Let $|\cdot|$ be an *equivalent norm* on $(X, \|\cdot\|)$. This means that the identity on X is an isomorphism between $(X, \|\cdot\|)$ and $(X, |\cdot|)$. X is of *bounded distortion* if there exists $D < \infty$ so that for all equivalent norms $|\cdot|$ on X

$$\sup_{Z \subset X} \inf_{Y \subseteq Z} \left\{ \frac{|x|}{|y|} : x, y \in S_{Y, \|\cdot\|} \right\} \leq D.$$

THEOREM 2.14. ([51]) *Let X be of bounded distortion. Then X contains an unconditional basic sequence.*

Very recently W.B. Johnson and the author proved the following theorem. Before stating the result we need some notation. If X and Y are isomorphic,

$d(X, Y) = \inf\{K : X \overset{K}{\sim} Y\}$, denotes the *Banach-Mazur distance* between X and Y [14]. (Actually $\log d(\cdot, \cdot)$ gives the metric where X and Z are identified if $d(X, Z) = 1$.) We set $D(X) = \sup\{d(Y, Z) : Y \sim X, Z \sim X\}$. The following is known: There exists $c < \infty$ so that if dimension $F = n$ then $D(F) \geq cn$ [27].

THEOREM 2.15. ([35]) *Let X be a separable infinite dimensional Banach space. Then $D(X) = \infty$.*

A key step in the proof is the following result. We say that X is *elastic* if there exists $K < \infty$ so that if $Y \hookrightarrow X$ then $Y \overset{K}{\hookrightarrow} X$. If X failed the conclusion of Theorem 2.15 then X would necessarily be elastic.

THEOREM 2.16. ([35]) *If X is elastic then $c_0 \hookrightarrow X$.*

The proof uses an index argument based upon the unit vector basis of c_0 . The argument is a bit more tricky than the ones we have presented. Of course any X containing $C[0, 1]$ must be elastic and it seems likely that the converse is true. This problem is raised in [35].

OPEN PROBLEM 2.17. Let X be elastic. Does $C[0, 1] \hookrightarrow X$?

3. SZLENK'S AND OTHER INDICES

The *Szlenk index* [50] of a separable Banach space X can be defined as follows. Given $\varepsilon > 0$ and a ω^* closed subset P of B_{X^*} we let $P'_\varepsilon = \{x^* \in P : \text{for all } \omega^* \text{ neighborhoods } U \text{ of } x^*, \text{diam}(U \cap P) > \varepsilon\}$. In this definition "diam" refers to the norm diameter of $U \cap P$. It is easy to check that P'_ε is a ω^* closed subset of P . We inductively define $P_\alpha(X, \varepsilon)$ for $\alpha < \omega_1$ by setting $P_0(X, \varepsilon) = B_{X^*}$, $P_{\alpha+1}(X, \varepsilon) = (P_\alpha(X, \varepsilon))'_\varepsilon$ and if α is a limit ordinal, $P_\alpha(X, \varepsilon) = \bigcap_{\beta < \alpha} P_\beta(X, \varepsilon)$.

We next set

$$\eta(X, \varepsilon) = \sup\{\alpha : P_\alpha(X, \varepsilon) \neq \emptyset, \alpha < \omega_1\}$$

and set the Szlenk index of X to be

$$\eta(X) = \sup\{\eta(X, \varepsilon) : \varepsilon > 0\}.$$

For example $\eta(\ell_1) = \omega_1$. Indeed consider

$$A = \{x^* \in \ell_1^* = \ell_\infty : x^* = (\varepsilon_i)_{i=1}^\infty \text{ with } \varepsilon_i = \pm 1 \text{ for all } i\}.$$

Then if $x^* \in A$ and U is any ω^* neighborhood of x^* there exists $\delta > 0$ and $n \in \mathbb{N}$ so that $U \cap B_{\ell_\infty} \supseteq \{y^* \in B_{\ell_\infty} : |y^*(i) - x^*(i)| < \delta \text{ for } i \leq n\}$. Clearly this contains $y^* \in A$ with $y^* \neq x^*$ and so $\|y^* - x^*\| = 2$. Hence $P_\alpha(\ell_1, \varepsilon) \supseteq A$ for all $\alpha < \omega_1$, $0 < \varepsilon < 2$.

On the other hand $\eta(\ell_2) = \omega$. Indeed let $x^* = (a_i) \in P_1(\ell_2, \varepsilon)$ where $0 < \varepsilon < 1$. Let $\delta > 0$ be small and choose n_0 so that $(\sum_{i=1}^{n_0} a_i^2)^{1/2} > \|x^*\| - \delta$. We can choose a ω^* open set U containing x^* by prescribing that the coordinates of $y^* = (b_i) \in U$ satisfy $|b_i - a_i| < \delta_1$ for $i \leq n_0$ (where $\delta_1 > 0$ is arbitrarily small). In particular we can insure that if also $z^* = (c_i) \in U$ then by the triangle inequality

$$\begin{aligned} \|y^* - z^*\| &= \left(\sum_{i=1}^{n_0} |b_i - c_i|^2 + \sum_{i>n_0} |b_i - c_i|^2 \right)^{1/2} \\ &< \delta + \left(\sum_{i>n_0} b_i^2 \right)^{1/2} + \left(\sum_{i>n_0} c_i^2 \right)^{1/2}. \end{aligned}$$

Now by taking δ_1 sufficiently small we can insure also that $(\sum_{i=1}^{n_0} b_i^2)^{1/2} > \|x^*\| - \delta$ and so

$$1 \geq \sum_{i \leq n_0} b_i^2 + \sum_{i > n_0} b_i^2 > (\|x^*\| - \delta)^2 + \sum_{i > n_0} b_i^2.$$

Thus $(\sum_{i>n_0} b_i^2)^{1/2} < (1 - (\|x^*\| - \delta)^2)^{1/2}$. A similar estimate holds for z^* and so $\|y^* - z^*\| < \delta + 2((1 - (\|x^*\| - \delta)^2))^{1/2}$. Since $\delta > 0$ is arbitrary we obtain $\varepsilon \leq 2(1 - \|x^*\|^2)^{1/2}$ and so $\|x^*\| \leq (1 - \frac{\varepsilon^2}{4})^{1/2}$. Thus

$$P_1(\ell_2, \varepsilon) \subseteq \left(1 - \frac{\varepsilon^2}{4}\right)^{1/2} B_{X^*}.$$

This argument repeats until we ultimately obtain for some n , $P_n(\ell_2, \varepsilon) = \emptyset$. Thus $\eta(\ell_2, \varepsilon) < \omega$ and the same sort of considerations yield $\eta(\ell_2) = \omega$.

THEOREM 3.1. $\eta(X) < \omega_1$ iff X^* is separable.

Proof. If X^* is not separable then there exists $\varepsilon > 0$ and an uncountable set $A \subseteq B_{X^*}$ with $\|x^* - y^*\| > \varepsilon$ for all $x^* \neq y^* \in A$. Since (B_{X^*}, ω^*) is compact metric there exist an uncountable $V \subseteq A$ which is ω^* -dense in itself. Every $x^* \in V$ is the ω^* -limit of a sequence in $V \setminus \{x^*\}$. But then $V \subseteq P_\alpha(X, \varepsilon)$ for all $\alpha < \omega_1$ so $\eta(X) = \omega_1$.

Suppose that X^* is separable. We need only show that if $P_\alpha(X, \varepsilon) \neq \emptyset$ then $P_{\alpha+1}(X, \varepsilon) \subsetneq P_\alpha(X, \varepsilon)$. Since a compact metric space does not admit a transfinite strictly decreasing family of closed sets, eventually $P_\alpha(X, \varepsilon) = \emptyset$. Thus $\eta(X, \varepsilon) < \omega_1$ for all $\varepsilon > 0$ hence $\eta(X) = \sup_n \eta(X, \frac{1}{n}) < \omega_1$.

To show that $P_{\alpha+1}(X, \varepsilon) \neq P_\alpha(X, \varepsilon)$ when $P_\alpha(X, \varepsilon) \neq \emptyset$, we will use the Baire Category theorem. First we observe that if $x^* \in P_{\alpha+1}(X, \varepsilon)$ then there exists $(x_n^*) \subseteq P_\alpha(X, \varepsilon)$ with $\omega^* - \lim x_n^* = x^*$ and $\|x_n^* - x^*\| \geq \varepsilon/2$. For each n choose $x_n \in S_X$ with $(x_n^* - x^*)(x_n) \geq \varepsilon/2$. Since X^* is separable by passing to a subsequence we may assume that (x_n) is weak Cauchy.

Since $(x_n^* - x^*)$ is ω^* null by passing to a further subsequence we may assume that for $m < n$, $|(x_n^* - x^*)(x_m)| < \varepsilon/4$ and so $(x_n^* - x^*)(x_n - x_m) > \varepsilon/4$ for all $m < n$. Setting $y_n^* = x_{2n}^*$ and $y_n = (x_{2n} - x_{2n-1})/2$ we obtain a weakly null sequence $(y_n) \subseteq B_X$ and $(y_n^*) \subseteq P_\alpha(X, \varepsilon)$ with $y_n^* \rightarrow x^*(\omega^*)$ and $(y_n^* - x^*)(y_n) > \varepsilon/8$ for all n . Since $x^*(y_n) \rightarrow 0$, passing to one more subsequence we may assume that for all n , $y_n^*(y_n) > \varepsilon/9$.

Now suppose that $P_{\alpha+1}(X, \varepsilon) = P_\alpha(X, \varepsilon) \neq \emptyset$. Let $(x^n)_{n=1}^\infty$ be dense in $P_\alpha(X, \varepsilon)$. Using the observations above we construct for $n \in \mathbb{N}$, $(y_m^n)_{m \in \mathbb{N}} \subseteq P_\alpha(X, \varepsilon)$ converging ω^* to x^n and a weakly null sequence $(y_m^n)_{m \in \mathbb{N}} \subseteq B_X$ with $y_m^n(y_m^n) > \varepsilon/9$ for all m . By a standard diagonal argument (using X^* separable implies (B_X, ω) is metrizable) we may choose a weakly null $(x_m) \subseteq \{y_m^n : n, m \in \mathbb{N}\}$ so that $\{x_m\}_{m \in \mathbb{N}} \cap \{y_m^n\}_{m \in \mathbb{N}}$ is infinite for each n . Set $Q = \{x^* \in P_\alpha(X, \varepsilon) : \text{there exists in } P_\alpha(X, \varepsilon) \text{ a sequence } (y_n^*) \text{ converging } \omega^* \text{ to } x^* \text{ and satisfying } \overline{\lim}_n y_n^*(x_n) \geq \varepsilon/9\}$. Then $Q = P_\alpha(X, \varepsilon)$ from our construction. Now (x_m) may be regarded as a sequence of continuous functions in the unit ball of $C(P_\alpha(X, \varepsilon), \omega^*)$ which is pointwise null. For $N \in \mathbb{N}$ set

$$A_N = \{x^* \in P_\alpha(X, \varepsilon) : |x^*(x_m)| \leq \varepsilon/10 \text{ for all } m \geq N\}.$$

Each A_N is closed and $\bigcup_{N=1}^\infty A_N = P_\alpha(X, \varepsilon)$. Thus by the Baire Category theorem some A_N has non empty (ω^*) interior. But this contradicts $Q = P_\alpha(X, \varepsilon)$.

Remark 3.2. Szlenk's original definition of his index varies slightly with ours (see [50]) but the two definitions yield the same index. As mentioned in the previous section every $C(K)$ where K is countable compact metric is isomorphic to $C(\alpha)$ for some $\alpha < \omega_1$. In fact α can be taken to be $\omega^{\omega^\beta} +$ for some $\beta < \omega_1$ [16]. $\eta(C(\omega^{\omega^\beta} +)) = \omega^{\beta+1}$ [3] and it is open if given X complemented in $C(\Delta)$ with $\eta(X) = \omega^{\beta+1}$, must $C(\omega^{\omega^\beta} +) \hookrightarrow X$? (Yes, if $\beta = 1$ [1].) For more on this problem we refer the reader to [46] and to [23]

which, in addition to new results, contains a discussion, with many references, of this and related problems. The first part of Theorem 2.7 can be proved using Szlenk's index. Indeed if the X_α 's are as constructed in our proof of 2.7 (to show $\ell_1 \hookrightarrow X$) then $\eta(X_\alpha) \geq \alpha$ for all $\alpha < \omega_1$.

We shortly will discuss some more indices and eventually will relate $\eta(X)$ to one of these indices. First let $K < \infty$. We defined the ℓ_1 index $I(X, K)$ earlier. One might wonder what ordinal values $I(X)$ might take. We discuss this in a broader context. For $K < \infty$ let $P(K)$ be a property that $(x_i)_1^n \subseteq S_X$ might satisfy (e.g., being K -equivalent to the unit vector basis of ℓ_1). A tree on X has property $P(K)$ if each node has $P(K)$.

We can define

$$I_P(X, K) = \sup\{o(T) : T \text{ is a tree on } X \text{ with property } P(K)\}$$

and

$$I_P(X) = \sup\{I_P(X, K) : K < \infty\}.$$

PROPOSITION 3.3. *Let X be a Banach space. For each $K \geq 1$ let $P(K)$ be a property satisfying the following (in X):*

- (i) *If $(x_i)_1^m \in P(K)$ then $(x_i)_1^m$ is normalized and K -basic.*
- (ii) *Given $L, C \geq 1$ there exists $K' = K'(K, L, C)$ such that if $(x_i)_1^m \in P(K)$ and $(y_i)_1^n \in P(L)$ and $\max(\|x\|, \|y\|) \leq C\|x + y\|$ for all $x \in \langle x_i \rangle_1^m$, $y \in \langle y_i \rangle_1^n$ then $(x_1, \dots, x_m, y_1, \dots, y_n) \in P(K')$.*
- (iii) *There exists $L = L(K) \geq 1$ such that for every $(x_i)_1^m \in P(K)$ and any $1 \leq k \leq \ell \leq m$, $(x_i)_k^\ell \in P(L)$.*
- (iv) *There exists $K'' = K''(K) \geq 1$ such that the closure of $X^n \cap P(K)$ in the product topology of X^n is contained in $X^n \cap P(K'')$ for all n .*

Then either $I_P(X) = \omega_1$ and there exists $(x_i)_1^\infty \subseteq S_X$ such that $(x_i)_1^m \in P(K)$ for all $m \geq 1$ or else $I_P(X) = \omega^\alpha$ for some $\alpha < \omega_1$.

The proof [4] is a technical argument involving trees and we omit it here.

We now define four indices each of which are defined by properties $P(K)$ that can be shown to satisfy the hypothesis of Proposition 3.3. We include the already defined ℓ_1 index.

- (1) $I(X)$: $(x_i)_1^m \in P(K)$ if $(x_i) \subseteq S_X$ is K -equivalent to the unit vector basis of ℓ_1^m .

- (2) $I^+(X)$: the ℓ_1^+ -index defined by $(x_i)_1^m \in P(K)$ if $(x_i)_1^m \subseteq S_X$ is a K -basic ℓ_1^+ - K sequence; i.e., $\|\sum_1^m a_i x_i\| \geq K^{-1} \sum_1^m a_i$ whenever $(a_i)_1^m \subseteq [0, \infty)$.

(Note: By the geometric form of the Hahn Banach theorem a K -basic $(x_i)_1^m$ is an ℓ_1^+ - K sequence iff there exists $x^* \in S_{X^*}$ with $x^*(x_i) \geq K^{-1}$ for $i \leq m$.)

- (3) $J(X)$: This is the ℓ_∞ index defined by $(x_i)_1^m \in P(K)$ if $(x_i)_1^m \subseteq S_X$ is K -basic and there exist $0 < c, C < \infty$ with $c^{-1}C \leq K$ and

$$c \max_{i \leq m} |a_i| \leq \left\| \sum_1^m a_i x_i \right\| \leq C \max_{i \leq m} |a_i|$$

for all $(a_i)_1^m \subseteq \mathbb{R}$.

- (4) $J^+(X)$: The ℓ_∞^+ -index defined by $(x_i)_1^m \in P(K)$ if $(x_i)_1^m \subseteq S_X$ is K -basic and there exists $(a_i)_1^m \subseteq [K^{-1}, 1]$ such that

$$\left\| \sum_{i=1}^m a_i x_i \right\| \leq K.$$

We should note that Rosenthal [47] studied ℓ_1^+ and ℓ_∞^+ sequences under the names wide-(s) and wide-(c) sequences with different quantifications.

Each of these four indices has a companion *block basis index* where the corresponding property has the additional requirement that X has an understood fixed basis (e_i) and $(x_i)_1^m \in P(K)$ requires also that $(x_i)_1^m$ be a normalized block basis of (e_i) . We will use the subscript “b” to denote this index: $I_b(X)$, $I_b^+(X)$, $J_b(X)$ and $J_b^+(X)$.

These indices can depend on the particular basis and more properly one might write $I_b(X, (e_i))$. One can extend Proposition 3.3 to this case.

PROPOSITION 3.4. ([4]) *Let P be a property satisfying the conditions of proposition 3.3 and let X have a basis (e_i) . Let $I_b^P(X)$ be the block basis index defined via P and (e_i) . Then either $I_b^P(X) = \omega_1$ and there exists $K < \infty$ and a block basis $(x_i)_1^\infty$ of (e_i) with*

$$(x_i)_1^m \in P(K) \text{ for all } m \text{ or else } I_b^P(X) = \omega^\alpha \text{ for some } \alpha < \omega_1.$$

DEFINITION 3.5. We define trees $T(\alpha, s)$ for each $\alpha < \omega_1$. The construction is similar to that of the minimal trees T_α except that at each stage an infinite sequence of nodes is added instead of a single node. Let $s = (z^1, z^2, \dots)$ be an infinite sequence of incomparable nodes and let $T(1, s) = s$. To construct $T(\alpha + 1, s)$ we begin with a copy of s and after each node place a tree isomorphic to $T(\alpha, s)$. For example $T(n, s)$ is a countable infinitely branching tree of “ n levels”. If α is a limit ordinal we let $T(\alpha, s)$ be the disjoint union of $T(\beta, s)$, $\beta < \alpha$.

Each $T(\alpha, s)$ has order α , an infinite sequence of initial nodes and if z is not a terminal node, then z has infinitely many immediate successors.

DEFINITION 3.6. If T is a tree on X we shall say that T is isomorphic to $T(\alpha, s)$ if they are isomorphic as trees and also require that if $(x_1, \dots, x_k) \in T$ then $(x_1, \dots, x_{k-1}) \in T$. If $(x_1, \dots, x_k, y_i) \in T$ for all i are the immediate successors of (x_1, \dots, x_k) we call (y_i) an s -sequence of T . The initial nodes of T are also called an s -sequence.

DEFINITION 3.7. For $K \geq 1$ a tree T on X is an ℓ_1^+ - K -weakly null tree if each node $(x_i)_1^m \in T$ is an ℓ_1^+ - K sequence and T is a weakly null tree: this means that T is isomorphic to $T(\alpha, s)$ for some $\alpha < \omega_1$ and each s -sequence of T is weakly null. $I_\omega^+(X) = \sup\{o(T) : T \text{ is an } \ell_1^+\text{-}K\text{-weakly null tree on } X \text{ for some } K < \infty\}$.

We gather together our results above and include some new ones in the following theorem. In any statement involving a block basis, X is assumed to have a basis (e_i) .

THEOREM 3.8. *Let X be a separable Banach space. Then*

- (i) $I(X) < \omega_1$ iff $\ell_1 \not\hookrightarrow X$
- (ii) If $I(X) \geq \omega^\omega$ then $I(X) = I_b(X)$
- (iii) If $I(X) = \omega^n$ for some $n \in \mathbb{N}$ then $I_b(X) = \omega^m$ where $m \in \{n, n - 1\}$.
- (iv) $I^+(X) < \omega_1$ iff X is reflexive
- (v) $I^+(X) = \omega$ iff X is superreflexive
- (vi) $I^+(X) = J^+(X)$
- (vii) $I_b^+(X, (e_i)) < \omega_1$ iff (e_i) is a shrinking basis for X
- (viii) $J_b^+(X, (e_i)) < \omega_1$ iff (e_i) is a boundedly complete basis for X
- (ix) If $\ell_1 \not\hookrightarrow X$ then $I_\omega^+(X) < \omega_1$ iff X^* is separable.

(i) is Bourgain's result discussed earlier. (ii) and (iii) are in [36]. (iv) and (v) are ℓ_1^+ -index restatements of old results of James [32] and Milman and Milman [40]. The rest are proved in [4].

(ix) suggests that the $I_\omega^+(X)$ index might be related to the Szlenk index for spaces not containing ℓ_1 . This is true [4].

THEOREM 3.9. *If X is a separable Banach space not containing ℓ_1 then $\eta(X) = I_\omega^+(X)$.*

The smallest possible Szlenk index is ω . Spaces X with $\eta(X) = \omega$ were studied in [38] (see also [25]) where it was shown that such an X could be renormed to have the ω^* uniform Kadec-Klee property: For all $\varepsilon > 0$ there exists $\delta > 0$ so that if $(x_n^*) \subseteq B_{X^*}$, $\|x_n^* - x\| \geq \varepsilon$ for all n and $\omega^* - \lim_n x_n^* = x^*$ then $\|x^*\| \leq 1 - \delta$. In fact for some $p < \infty$ and $c > 0$

$$\delta(\varepsilon) \geq c\varepsilon^p.$$

As we saw earlier this case was modeled by ℓ_2 .

4. BAIRE-1 INDICES

We shall define an ordinal index for a bounded Baire-1 function $F : K \rightarrow \mathbb{R}$ where K is a compact metric space. But first let us see how this might pertain to Banach space theory. Let X be a separable Banach space and $K = (B_{X^*}, \omega^*)$, so K is compact metric. Then as we remarked earlier, we may regard $X \subseteq C(K)$ via $x(x^*) = x^*(x)$.

In fact X can be identified with all continuous functions f on K which are affine and satisfy $f(0) = 0$. Similarly X^{**} can be identified with bounded affine functions f on K satisfying $f(0) = 0$ and then we have that $f \in X$ iff $f \in C(K)$. Goldstine's theorem yields that B_X is ω^* dense in $B_{X^{**}}$ so for $x^{**} \in B_{X^{**}}$, all finite $G \subseteq K$ and $\varepsilon > 0$ there exists $x \in B_X$ with $|x(g) - x^{**}(g)| < \varepsilon$ for $g \in G$. $F : K \rightarrow \mathbb{R}$ is a *bounded Baire-1 function* (denoted $F \in B_1(K)$) if F is bounded and there exists $(f_n) \subseteq C(K)$ which converge pointwise on K to F . In [42] it is shown that if $F = x^{**}|_K$ for some $x^{**} \in X^{**}$ then $F \in B_1(K)$ iff F is the pointwise limit of $(x_n) \subseteq B_X$. Let us recall an important theorem of Baire.

THEOREM 4.1. *Let K be compact metric, $F : K \rightarrow \mathbb{R}$ a bounded function. Then $F \in B_1(K)$ iff for all non-empty closed $K_0 \in K$, $F|_{K_0}$ has a point of continuity.*

If X^* is separable it is easy to see from Goldstine's theorem that $X^{**} \subseteq B_1(K)$ and more generally we have

THEOREM 4.2. ([42]) $X^{**} \subseteq B_1(K)$ iff $\ell_1 \not\hookrightarrow X$.

The main direction to be proved is that if for some $x^{**} \in X^{**}$, $x^{**}|_K \equiv F \notin B_1(K)$ then whenever (f_n) is a bounded sequence in $C(K)$ which converges pointwise to F , some subsequence of (f_n) is equivalent to the unit vector basis of ℓ_1 . This is achieved via Theorem 4.1 and the Baire category theorem. Assume $\|F\| = 1$. If $F|_{K_0}$ has no point of continuity, set $K_n = \{k \in K_0 : \text{for all neighborhoods } U \text{ of } k \text{ in } K_0, \text{diam } F(U) > 1/n\}$. Then each K_n is closed and $\bigcup_{n \in \mathbb{N}} K_n = K_0$ so by Baire category at least one has non empty interior, say $\text{int}(K_{n_0}) \neq \emptyset$. Let $L = \overline{\text{int}(K_{n_0})}$. Let $(r_n)_{n \in \mathbb{N}}$ be dense in $[-1, 1]$ and for all n let (taking $\varepsilon = 1/2n_0$)

$$L_n = \{k \in L : \text{for all neighborhoods } U \text{ of } k \text{ in } L, \text{there exists } k_0, k_1 \in U \text{ with } F(k_0) > r_n + \varepsilon \text{ and } F(k_1) < r_n\}.$$

Then L_n is again closed and $\bigcup_{n \in \mathbb{N}} L_n = L$ so some $\text{int}(L_{m_0}) \neq \emptyset$. We thus obtain a $P = \overline{\text{int}(L_{m_0})} \neq \emptyset$ so that for all non empty open neighborhoods U in P there exist $k_0, k_1 \in P$ with $F(k_0) > r + \varepsilon$ and $F(k_1) < r$ (taking $r = r_{m_0}$).

Using this we can produce Rademacher-like behavior amongst a subsequence of (f_n) . We can find $f_{n_1}(U_0) > r + \varepsilon$ and $f_{n_1}(U_1) < r$ for two disjoint open sets U_0 and U_1 in P by approximating F by f_{n_1} at two points where its value is larger than $r + \varepsilon$ and smaller than r , and then using the continuity of f_{n_1} . Continuing thusly we produce a dyadic tree of non-empty open sets $U_{\vec{\varepsilon}}$ (where $\vec{\varepsilon}$ is a finite sequence of 0's and 1's) and $(f_{n_i}) \subseteq (f_n)$ so that $U_{\vec{\varepsilon}} \supseteq U_{(\vec{\varepsilon}, 0)} \cup U_{(\vec{\varepsilon}, 1)}$ and $f_{n_i}(U_{\vec{\varepsilon}, 0}) > r + \varepsilon$ while $f_{n_i}(U_{\vec{\varepsilon}, 1}) < r$ if the length of the sequences $(\vec{\varepsilon}, 0)$ and $(\vec{\varepsilon}, 1)$ is i .

In particular if we set $B_i = \{k \in K : f_{n_i}(k) < r\}$ and $A_i = \{k \in K : f_{n_i}(k) > r + \varepsilon\}$ then $(A_i, B_i)_{i=1}^\infty$ is Boolean independent [45]. This means that for all finite $F, G \subseteq \mathbb{N}$ with $F \cap G = \emptyset$ we have $\bigcap_{i \in F} A_i \cap \bigcap_{i \in G} B_i \neq \emptyset$. From this we can prove that $\|\sum a_i f_{n_i}\| \geq \frac{\varepsilon}{2} \sum |a_i|$ for all $(a_i) \subseteq \mathbb{R}$. Indeed if $(a_i)_1^k \subseteq \mathbb{R}$ with $\sum_{i=1}^k |a_i| = 1$ let $F = \{i \leq k : a_i \geq 0\}$ and $G = \{i \leq k : a_i < 0\}$. Let $k_1 \in \bigcap_{i \in F} A_i \cap \bigcap_{i \in G} B_i$ and $k_2 \in \bigcap_{i \in G} A_i \cap \bigcap_{i \in F} B_i$. Then

$$\begin{aligned}
2\left\|\sum_1^k a_i f_{n_i}\right\| &\geq \sum_{i=1}^k a_i f_{n_i}(k_1) - \sum_{i=1}^k a_i f_{n_i}(k_2) \\
&= \sum_{i \in F} a_i \left[f_{n_i}(k_1) - f_{n_i}(k_2) \right] - \sum_{i \in G} a_i \left[f_{n_i}(k_2) - f_{n_i}(k_1) \right] \\
&> \sum |a_i|(r + \varepsilon - r) = \varepsilon.
\end{aligned}$$

Thus $\left\|\sum_{i=1}^k a_i f_{n_i}\right\| \geq \frac{\varepsilon}{2} \sum_{i=1}^k |a_i|$. ■

An important subclass of $B_1(K)$ are the *difference of bounded semicontinuous functions*, $DBSC(K)$. $F : K \rightarrow \mathbb{R}$ is in $DBSC(K)$ if there exist $(f_n)_{n=0}^\infty \subseteq C(K)$ and $C < \infty$ so that $f_0 = 0$, (f_n) converges pointwise to F and for all $k \in K$, $\sum_{n=0}^\infty |f_{n+1}(k) - f_n(k)| \leq C$. The terminology comes from the fact that in this case $F = F_1 - F_2$ where

$$F_1(k) = \sum_{n=0}^\infty (f_{n+1} - f_n)^+(k) \quad \text{and} \quad F_2(k) = \sum_{n=0}^\infty (f_{n+1} - f_n)^-(k)$$

and both of these are lower semicontinuous. The converse is equally trivial. If $F = F_1 - F_2$ where F_1 and F_2 are both bounded and semicontinuous then such a sequence $(f_n)_0^\infty$ exists. A famous theorem of Bessaga and Pełczyński [17] can be stated as follows (as usual $K = (B_{X^*}, \omega^*)$).

THEOREM 4.3. $c_0 \hookrightarrow X$ iff $[X^{**} \cap DBSC(K)] \setminus C(K) \neq \emptyset$.

We indicate the proof of the “if” implication.

Let $x^{**} \in [X^{**} \cap DBSC(K)] \setminus C(K)$ and $F = x^{**}|_K$. It is easy to check that there exists $C < \infty$ and $(x_n)_{n=0}^\infty \subseteq X$ with $x_0 = 0$, (x_n) converges pointwise to F and

$$(1) \quad \sum_{n=0}^\infty |x^*(x_{n+1}) - x^*(x_n)| \leq C \quad \text{for all } \|x^*\| \leq 1.$$

In other words the sequence (f_n) above can be taken from X . Since F is not continuous some subsequence of (x_n) is seminormalized basic and dominates the summing basis. Moreover (1) hold for this subsequence so we may relabel the subsequence as (x_n) . But (1) easily yields that (x_n) is equivalent to the summing basis, so $c_0 \hookrightarrow X$.

In both theorems we see that the topological nature of $F = x^{**}|_K$ affects the subspace structure of X . Our ordinal index for $\beta_1(K)$ will localize these theorems.

Let K be any compact metric space and let $F \in DBSC(K)$. We define $|F|_D$ to be the inf of those C 's so that there exists $(f_n)_0^\infty \subseteq C(K)$, $f_0 = 0$, converging pointwise to F with $\sum_0^\infty |f_{n+1}(k) - f_n(k)| \leq C$ for all $k \in K$. It is easy to show that $(DBSC(K), |\cdot|_D)$ is a Banach space using the series criterion for completeness. $\|F\|_\infty \leq |F|_D$ but the norms are not generally equivalent. This leads to two subclasses of $B_1(K)$. We note first that one can show that $B_1(K)$ is closed under uniform limits.

$$B_{1/2}(K) = \{F \in B_1(K) : \text{there exists } (F_n) \subseteq DBSC(K) \text{ with } (F_n) \\ \text{converging uniformly to } F\}$$

$$B_{1/4}(K) = \{F \in B_1(K) : \text{there exists } (F_n) \subseteq DBSC(K) \text{ with } (F_n) \\ \text{converging uniformly to } F \text{ and } \sup_n |F_n|_D < \infty\}$$

$B_{1/4}(K)$ can be shown to be a Banach space as well under the “inf $|F_n|_D$ ” norm — the inf taken over all such (F_n) . Clearly $C(K) \subseteq B_{1/4}(K) \subseteq B_{1/2}(K) \subseteq B_1(K) \subseteq \ell_\infty(K)$ and all the inclusions are proper if K is large enough [29].

We are now ready to define our ordinal index for $B_1(K)$. If $F : K \rightarrow \mathbb{R}$ is bounded, $L \subseteq K$ is a closed subset and $\ell \in L$ we define the oscillation of $F|_L$ at ℓ to be

$$\text{osc}_L(F, \ell) = \limsup_{\varepsilon \downarrow 0} \{F(\ell_1) - F(\ell_2) : \ell_i \in L \text{ and } d(\ell_i, \ell) < \varepsilon \text{ for } i = 1, 2\}$$

Let $\delta > 0$. Set $K_0(F, \delta) = K$ and if $\alpha < \omega_1$ let $K_{\alpha+1}(F, \delta) = \{k \in K_\alpha(F, \delta) : \text{osc}_{K_\alpha(F, \delta)}(F, k) \geq \delta\}$. If α is a limit ordinal we take

$$K_\alpha(F, \delta) = \bigcap_{\beta < \alpha} K_\beta(F, \delta).$$

It is easy to see that $K_\alpha(F, \delta)$ is closed for all $\alpha < \omega_1$ and we set $\beta(F, \delta) = \inf\{\alpha < \omega_1 : K_\alpha(F, \delta) = \emptyset\}$ provided that $K_\alpha(F, \delta) = \emptyset$ for some $\alpha < \omega_1$ and set $\beta(F, \delta) = \omega_1$ otherwise. Since K is separable the transfinite decreasing sequence of closed sets must stabilize so there exists $\beta < \omega_1$ with $K_\alpha(F, \delta) = K_\beta(F, \delta)$ for all $\alpha \geq \beta$. Baire's theorem (4.1) yields that $\beta(F, \delta) < \omega_1$ for all $\delta > 0$ iff $F \in \beta_1(K)$. More characterizations are given in the next theorem [29]. \mathcal{A} is the algebra of *ambiguous* (simultaneously F_σ and G_δ subsets of K).

PROPOSITION 4.4. *Let $F : K \rightarrow \mathbb{R}$ be bounded where K is a compact metric space. The following are equivalent.*

- (1) $F \in \beta_1(K)$
- (2) $\beta(F, \delta) < \omega_1$ for all $\delta > 0$
- (3) If $a, b \in \mathbb{R}$ then $[F \leq a]$ and $[F \geq b]$ are both G_δ subsets of K .
- (4) If $a < b$ then $[F \leq a]$ and $[F \geq b]$ may be separated by disjoint sets in \mathcal{A} .
- (5) F is the uniform limit of \mathcal{A} -simple functions (i.e., \mathcal{A} -measurable functions on K with finite range).

So if $\ell_1 \not\hookrightarrow X$ then $\beta(x^{**}|_K) < \omega_1$ for all $x^{**} \in X^{**}$ (with $K = (B_{X^*}, \omega^*)$). In fact this is uniformly so by a result of Bourgain [19].

THEOREM 4.5. *Let X be a separable Banach space not containing ℓ_1 and $K = (B_{X^*}, \omega^*)$. Then $\sup\{\beta(x^{**}|_K, \delta) : x^{**} \in X^{**}, \delta > 0\} < \omega_1$.*

We will not give the proof. The argument sketched above for Theorem 4.2 is localized by Bourgain to show if the “sup” were ω_1 then $I(X) = \omega_1$ and hence $\ell_1 \hookrightarrow X$.

We next characterize $B_{1/2}(K)$. \mathcal{D} is the subalgebra of \mathcal{A} consisting of all finite unions of differences of closed sets.

PROPOSITION 4.6. *Let $F : K \rightarrow \mathbb{R}$ be a bounded function on the compact metric space K . The following are equivalent.*

- (1) $F \in B_{1/2}(K)$
- (2) F is the uniform limit of \mathcal{D} -simple functions
- (3) $\beta(F, \delta) < \omega$ for all $\delta > 0$
- (4) $\alpha(F; a, b) < \omega$ for all $a < b$.

The index in (4) requires definition. Set $K_0(F; a, b) = K$ and

$$K_{\alpha+1}(F; a, b) = \{k \in K_\alpha(F; a, b) : \text{for all } \varepsilon > 0 \text{ and } i = 1, 2 \text{ there exist } k_i \in K_\alpha(F; a, b) \text{ with } d(k_i, k) \leq \varepsilon, F(k_1) \geq b \text{ and } F(k_2) \leq a\}.$$

If α is a limit ordinal, $K_\alpha(F; a, b) = \bigcap_{\beta < \alpha} K_\beta(F; a, b)$. Then we define

$$\alpha(F; a, b) = \inf\{\gamma < \omega_1 : K_\gamma(F; a, b) = \emptyset\}$$

if such a γ exists and let it be ω_1 if this set is empty. For the proof see [29]. One can also make index type observations about $B_{1/4}$ functions (see [29]).

Our interest in $B_{1/4}$ and $B_{1/2}$ comes from the following theorem which can be regarded as a localization of Theorems 4.2 and 4.3 using the indices above.

THEOREM 4.7. *Let X be a separable Banach space and $x^{**} \in B_{X^{**} \setminus X}$. Let $F = x^{**}|_K$ where $K = (B_{X^*}, \omega^*)$ and assume $F \in B_1(K)$ so there exists $(x_n) \subseteq B_X$ converging pointwise to F on K .*

- (a) *If $F \notin B_{1/2}(K)$ then (x_n) has a subsequence with spreading model equivalent to the unit vector basis of ℓ_1 .*
- (b) *If $F \in B_{1/4}(K)$ then (x_n) admits a convex block basis (y_n) whose spreading model is equivalent to the summing basis.*

The spreading model statements actually characterize $B_{1/2}(K)$ and $B_{1/4}(K)$ (see [9]).

We sketch the proof of (a). Let $f_n = x_n|_K$. We may assume $\|f_n\| = 1$, (f_n) is basic and has a spreading model. Indeed since $F \notin C(K)$, (f_n) admits a basic subsequence and a further subsequence has a spreading model. We shall show that there is $\varepsilon > 0$ so that for all m we can extract a subsequence $(g_n)_{n=1}^m$ of (f_n) satisfying $\|\sum_{i=1}^m a_i g_i\| \geq \varepsilon \sum_{i=1}^m |a_i|$ for all $(a_i)_1^m \subseteq \mathbb{R}$. This will complete the proof of a) by a diagonal argument since (f_n) has a spreading model.

By Proposition 4.6 there exists $a < b$ with $\alpha(F; a, b) \geq \omega$. Thus $K_m(F; a, b) \neq \emptyset$ for $m \in \mathbb{N}$. The proof will follow from the following lemma.

LEMMA 4.8. *Let $K_m(F; a, b) \neq \emptyset$ and let $a < a' < b' < b$. Then there exists a subsequence of length m , $(f_{n_i})_{i=1}^m$, so that $(A_i, B_i)_{i=1}^m$ are Boolean independent where*

$$A_i = \{s \in K : f_{n_i}(s) > b'\} \quad \text{and} \quad B_i = \{s \in K : f_{n_i}(s) < a'\}.$$

Indeed the proof of Theorem 4.2 yields that $\|\sum_{i=1}^m a_i f_{n_i}\| > \varepsilon \sum_{i=1}^m |a_i|$ for all $(a_i)_1^m \subseteq \mathbb{R}$ where $\varepsilon = \frac{b'-a'}{2}$.

To prove the lemma let $k_\phi \in K_k(F; a, b)$. Thus there exist k_0 and k_1 in $K_{k-1}(F; a, b)$ with $F(k_0) \leq a$ and $F(k_1) \geq b$. Choose n_1 and neighborhoods U_0 and U_1 of k_0 and k_1 , respectively, so that $f(U_0) < a'$ and $f(U_1) > b'$. Let $k_{\varepsilon_1, \varepsilon_2} \in U_{\varepsilon_1} \cap K_{k-2}(F; a, b)$ for $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$ with $F(k_{\varepsilon_1, 0}) \leq a$ and $F(k_{\varepsilon_1, 1}) \geq b$ for $\varepsilon_1 = 0$ or 1 . Choose $n_2 > n_1$ and neighborhoods $U_{\varepsilon_1, \varepsilon_2}$ of $k_{\varepsilon_1, \varepsilon_2}$ so that $f_{n_2}(U_{\varepsilon_1, 0}) < a'$ and $f_{n_2}(U_{\varepsilon_1, 1}) > b'$. Continue up to f_{n_m} . ■

Rosenthal [48] proved a beautiful c_0 theorem (dual to his famous ℓ_1 theorem) using an index for DBSC functions. Let $(f_n) \subseteq C(K)$. (f_n) is called *strongly summing* (s.s.) if (f_n) is weak Cauchy and basic so that whenever $(c_j) \subseteq \mathbb{R}$ satisfies $\sup_n \|\sum_1^n c_j f_j\| < \infty$, the series $\sum_{j=1}^\infty c_j$ converges.

THEOREM 4.9. *Let $F : K \rightarrow \mathbb{R}$ be a bounded discontinuous function on the compact metric space K . Let $(f_n) \subseteq C(K)$ be a uniformly bounded sequence converging pointwise to F .*

(a) (f_n) admits a convex block basis equivalent to the summing basis iff $f \in DBSC(K)$.

(b) (f_n) has an (s.s.) subsequence iff $f \notin DBSC(K)$.

Part (a) is essentially discussed above. The hard part of the argument is “if” in (b). We give the index involved in the proof. If $g : K \rightarrow \mathbb{R}$ the *upper semicontinuous envelope* of g is given by

$$U_g(x) = \overline{\lim}_{y \rightarrow x} g(y).$$

DEFINITION 4.10. Given a bounded function $F : K \rightarrow \mathbb{R}$, set $\text{osc}_0 f = 0$, and if $\beta = \alpha + 1$ define for $x \in K$,

$$\widetilde{\text{osc}}_\beta f(x) = \overline{\lim}_{y \rightarrow x} |F(y) - F(x)| + \text{osc}_\alpha F(y).$$

If β is a limit ordinal set

$$\widetilde{\text{osc}}_\beta F(x) = \sup_{\alpha < \beta} \text{osc}_\alpha F(x)$$

and finally let

$$\text{osc}_\beta F = U(\widetilde{\text{osc}}_\beta F).$$

Rosenthal proves that there exist a (smallest) $\alpha < \omega_1$ with $\text{osc}_\alpha F = \text{osc}_\beta F$ for $\beta > \alpha$ and then $F \in DBSC(K)$ iff $\text{osc}_\alpha f$ is bounded and then $\|F\|_D = \| |f| + \text{osc}_\alpha f \|_\infty$.

For more on these Baire 1 indices we recommend the paper by S. Argyros and V. Kanellopoulos [11].

5. SCHREIER SETS AND AN APPLICATION

The discovery of the Schreier sets $(S_\alpha)_{\alpha < \omega_1}$ [2] has had an important impact on Banach space theory in the last decade. These are certain families of finite subsets of \mathbb{N} .

DEFINITION 5.1. $S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$. Let $\alpha < \omega_1$. If S_α has been defined then

$$S_{\alpha+1} = \left\{ \bigcup_{i=1}^n F_i : n \geq 1, n \leq F_1 < \cdots < F_n \text{ and } F_i \in S_\alpha \text{ for } 1 \leq i \leq n \right\} \cup \{\emptyset\}.$$

If α is a limit ordinal, choose $\alpha_n \uparrow \alpha$.

$$S_\alpha = \{F : \text{for some } n \in \mathbb{N}, F \in S_{\alpha_n} \text{ and } n \leq F\} \cup \{\emptyset\}.$$

Note: $n \leq F$ means $n \leq \min F$ and $F_1 < F_2$ means $\max F_1 < \min F_2$.

The Schreier sets $(S_\alpha)_{\alpha < \omega_1}$ are natural collections of finite subsets of \mathbb{N} of increasing complexity. For example let $T(S_\alpha)$ be the tree

$$\{(n_i)_{i=1}^m : n_1 < \cdots < n_m \text{ and } \{n_i\}_1^m \in S_\alpha\}$$

ordered by extension. It is not hard to prove by induction that $o(T(S_\alpha)) = \omega^\alpha$. Thus the Schreier sets can be used as a collection of measures against which we can examine the behavior of the subsequences of a given sequence (x_n) . For example consider the properties (P) given by either $(x_i)_{i \in F}$ is K -equivalent to the unit vector basis of ℓ_1 or there exists $x^* \in B_{X^*}$ with $|x^*(x_i)| \geq \varepsilon$ for $i \in F$ (for some fixed $\varepsilon > 0$). Suppose that for all $\alpha < \omega_1$ there exists $(y_i) \subseteq (x_i)$ so that $\forall F \in S_\alpha$, $(y_i)_{i \in F}$ has (P) . Then some subsequence of (x_i) must have (P) . Indeed otherwise the tree $T = \{(x_{n_i})_{i=1}^m \subseteq (x_i) : (x_{n_i})_{i=1}^m \text{ has } (P)\}$ is well founded and thus, being countable, must have countable order.

Hence from the second property we have the following.

PROPOSITION 5.2. *Let (x_n) be a normalized weakly null sequence in a Banach space X . Then there exists $\alpha < \omega_1$ so that for all $\varepsilon > 0$ and all subsequences $M = (m_i)$ of \mathbb{N}*

$$\left\{ (n_i)_{i=1}^j : n_1 < \cdots < n_j \text{ and there exists } x^* \in B_{X^*} \right. \\ \left. \text{with } |x^*(x_{m_{n_i}})| \geq \varepsilon \text{ for } 1 \leq i \leq j \right\}$$

does not contain S_α .

Our goal in this section is to present a sketch of one particular application of the Schreier sets. Namely if (x_n) is a normalized weakly null sequence we will show how to explicitly construct a norm null convex block basis of (x_n) [12]. This will require much more than Proposition 5.2. One might question whether the Schreier sets are too special to be of general use for the study of subsequences (y_n) of (x_n) with respect to such properties (P) as above but this is not the case.

First let us note that each class S_α is *hereditary* (if $F \subseteq G \in S_\alpha$ then $F \in S_\alpha$) and *closed* in the topology of pointwise convergence in $2^\mathbb{N}$: If $M \subseteq \mathbb{N}$ and $(F_n) \subseteq S_\alpha$ with $\mathbf{1}_{F_n} \rightarrow \mathbf{1}_M$ pointwise then $M \in S_\alpha$. In particular M is finite.

Let \mathcal{F} be a collection of finite subsets of \mathbb{N} . If $M = (m_i)$ is a subsequence of \mathbb{N} we define

$$\mathcal{F}(M) = \{(m_i)_{i \in F} : F \in \mathcal{F}\} \text{ and } \mathcal{F}[M] = \mathcal{F} \cap [M]^{<\omega}.$$

$[M]^{<\omega}$ denotes the finite subsets of M .

The following remarkable theorem is due to I. Gasparis [24].

THEOREM 5.3. *Let \mathcal{F} and \mathcal{G} be hereditary families of finite subsets of \mathbb{N} . For all subsequences N of \mathbb{N} there exists a subsequence M of N so that either $\mathcal{G}[M] \subseteq \mathcal{F}$ or $\mathcal{F}[M] \subseteq \mathcal{G}$.*

As a consequence we have an improvement of Proposition 5.2.

PROPOSITION 5.4. *Let (x_n) be a normalized weakly null sequence in a Banach space. For $\varepsilon > 0$ let*

$$\mathcal{F}_\varepsilon = \{(n_i)_1^j : \text{there exists } x^* \in B_{X^*} \text{ with } |x^*(x_{n_i})| \geq \varepsilon \text{ for } i \leq j\}.$$

Then there exists $\alpha < \omega_1$ so that for all subsequences N of \mathbb{N} there exists a subsequence M of \mathbb{N} with $\mathcal{F}_\varepsilon[M] \subseteq S_\alpha$.

Proof. If not then by Theorem 5.3 for all $\alpha < \omega_1$ there exists M_α and $\varepsilon_\alpha > 0$ with $\mathcal{F}_{\varepsilon_\alpha} \supseteq S_\alpha[M_\alpha]$. It follows that for some $\varepsilon > 0$, $\mathcal{F}_\varepsilon \supseteq S_\alpha[M_\alpha]$ for uncountably many α 's. Thus \mathcal{F}_ε , regarded as a tree as above, has order ω_1 (one can generalize the remark earlier to show that $o(T(S_\alpha[M_\alpha])) = \omega^\alpha$) so \mathcal{F}_ε must contain an infinite branch. Hence (x_n) is not weakly null.

The Schreier sets are named for Schreier's space X_1 (see [49]) which can be described as the completion of c_{00} under the norm $\|x\|_1 = \sup\{|\sum_F x(i)| :$

$F \in S_1\}$. This space has a normalized basis (e_i) , the unit vector basis, which is weakly null but no subsequence Cesaro sums to 0. One could also define generalized Schreier spaces X_α for $\alpha < \omega_1$ using “ $F \in S_\alpha$ ” in place of “ $F \in S_1$ ”. (e_i) is still a normalized weakly null basis for X_α and so by Mazur’s theorem some convex block basis is norm null. Indeed by Mazur’s theorem for all n the closed convex hull of $(x_i)_{i \geq n}$ is weakly closed and hence contains 0. Thus given $\varepsilon_i \downarrow 0$ we can produce $n_1 < n_2 < \dots$ and $y_i \in$ convex hull $(x_j : n_{i-1} < j \leq n_i)$ with $\|y_i\| < \varepsilon_i$. But as α increases the complexity of this block basis must also increase.

Indeed if $F \in S_\alpha$ then $(x_i)_{i \in F}$ is 2-equivalent to the unit vector basis of $\ell_1^{|F|}$ and so the *supports* of the y_n ’s cannot be sets in S_α ; $\text{supp}(\sum a_i x_i) = \{i : a_i \neq 0\}$. It turns out that one can prove that such y_n ’s can be chosen with $\text{supp}(y_n) \in S_{\alpha+1}$. Moreover as we shall see this can be carried out in general in a certain constructive manner.

It is perhaps also worth mentioning some other properties of the spaces X_α . First each X_α embeds into a $C(K)$ space for some countable compact metric space K . Indeed let S_α be given the pointwise topology so it is countable compact and metrizable. Then $X_\alpha \xrightarrow{1} C(S_\alpha)$ via $x(F) = \sum_{i \in F} x_i$. Hence each X_α is c_0 saturated and in particular $\ell_1 \not\hookrightarrow X_\alpha$. But the unit vector basis (e_i) for X_α is an ℓ_1 - S_α -spreading model with constant 2.

DEFINITION 5.5. A normalized basic sequence (x_i) is an ℓ_1 - S_α -spreading model (with constant K) if for all $F \in S_\alpha$ and $(a_i)_{i \in F} \subseteq \mathbb{R}$,

$$\left\| \sum_{i \in F} a_i x_i \right\| \geq K^{-1} \sum_{i \in F} |a_i|.$$

The definition of the S_α ’s may appear a bit arbitrary in that the choice of $\alpha_n \uparrow \alpha$ is arbitrary but the particular choice, while sometimes a matter of convenience, has no impact on the theory as witnessed by Theorem 5.3 and Proposition 5.4.

Let’s return to the problem above of constructing a norm null convex block basis of a normalized weakly null sequence (x_n) . From our work above we know that there exists $\alpha < \omega_1$ so that for $\varepsilon > 0$ if

$$\mathcal{F}_\varepsilon = \{F \subseteq \mathbb{N} : \text{there exists } x^* \in B_{X^*} \text{ with } |x^*(x_i)| \geq \varepsilon \text{ for } i \in F\}$$

then $\mathcal{F}_\varepsilon[M]$ is contained in S_α for some $M \subseteq \mathbb{N}$. But this does not reveal enough to solve our problem. From this we could, with some work, construct a block basis of convex combinations of (x_i) whose supports were in $S_{\alpha+1}$ and

which were norm null. But even in this case we would not have a constructive way of accomplishing this. We need a deeper analysis to proceed. We will localize Ptak's theorem [44] in a precise manner. $S_{\ell_1^+} = \{(a_i) \in S_{\ell_1} : a_i \geq 0 \text{ for all } i\}$.

THEOREM 5.6. *Let $\varepsilon > 0$ and let \mathcal{F} be a closed hereditary family of subsets of \mathbb{N} with the following property. For all $(a_i) \in S_{\ell_1^+}$ there exists $F \in \mathcal{F}$ with $\sum_{i \in F} a_i > \varepsilon$. Then there exists a subsequence M of \mathbb{N} so that all finite subsets of M are in \mathcal{F} .*

Proof. Define a norm on c_{00} by

$$\|(a_i)\|_{\mathcal{F}} = \sup \left\{ \left| \sum_F a_i \right| : F \in \mathcal{F} \right\}.$$

The hypothesis yields that the unit vector basis (e_i) is an ℓ_1 basis for $(c_{00}, \|\cdot\|_{\mathcal{F}})$. On the other hand this space embeds into $C(\mathcal{F})$ via $(a_i) \rightarrow (\sum_{i \in F} a_i)_{F \in \mathcal{F}}$. If the theorem is false then \mathcal{F} is countable and so we have an embedding of ℓ_1 into some $C(\alpha)$, $\alpha < \omega_1$. But these spaces are c_0 -saturated so we have a contradiction since $c_0 \not\hookrightarrow \ell_1$. So \mathcal{F} is uncountable and thus contains an infinite set M .

We will localize this theorem using the Schreier sets and the following *repeated averages hierarchy* [12]. Let (e_i) be the unit vector basis for c_{00} . If $f = \sum a_i e_i \in c_{00}$, $\text{supp}(f) = \{i : a_i \neq 0\}$.

For $\alpha < \omega_1$ and an infinite subsequence $M = (m_i)$ of \mathbb{N} we define a convex block basis $(f_i^{M,\alpha})_{i=1}^{\infty}$ of $(e_i)_{i \in M}$ so that

- (i) $\bigcup_i \text{supp}(f_i^{M,\alpha}) = M$ and $\text{supp } f_i^{M,\alpha} \in S_{\alpha}$ for $i \in \mathbb{N}$.
- (ii) If $M = (m_i)$, $N = (n_i)$ and $M' = \bigcup_i \text{supp}(\alpha_{n_i}^M)$ then $f_{n_k}^{M,\alpha} = f_k^{M',\alpha}$.

The definition is inductive. $f_i^{M,0} = e_{m_i}$. If $(f_n^{M,\alpha})_n$ has been defined we let

$$f_1^{M,\alpha+1} = \frac{1}{m_1} \sum_{i=1}^{m_1} f_i^{M,\alpha}$$

and inductively

$$f_{n+1}^{M,\alpha+1} = \frac{1}{m_{k_n}} \sum_{i=k_n}^{k_n+m_{k_n}-1} f_i^{M,\alpha}$$

where $k_n = \min\{k \in \mathbb{N} : m_k > \text{supp}(f_n^{M,\alpha})\}$. At limit ordinals

$$f_1^{M,\alpha} = f_{m_1}^{M,\alpha_{m_1}} \quad \text{and} \quad f_k^{M,\alpha} = f_1^{M_k,\alpha_{n_k}} \quad \text{for } k > 1,$$

where

$$M_k = \{m \in M : m > \text{supp } f_{k-1}^{M,\alpha}\} \quad \text{and} \quad n_k = \min M_k.$$

In words, to get for example $(f_i^{M,\alpha+1})$ we take the largest averages of $(f_i^{M,\alpha})$ we can to have (i) above and continue thusly. Property (ii) is easily verified.

DEFINITION 5.7. Let \mathcal{F} be a closed hereditary collection of finite subsets of \mathbb{N} . Let M be a subsequence of \mathbb{N} , $\varepsilon > 0$ and $\alpha < \omega_1$. \mathcal{F} is (M, α, ε) large if for all subsequences N of M and $n \in \mathbb{N}$,

$$\sup_{F \in \mathcal{F}} \langle f_n^{N,\alpha}, \mathbf{1}_F \rangle \geq \varepsilon.$$

THEOREM 5.8. ([12]) *If \mathcal{F} is (M, α, ε) large then there exists a subsequence N of M with*

$$\mathcal{F} \supseteq S_\alpha(N).$$

THEOREM 5.9. *Let (e_n) be normalized and weakly null in X . Then there exists $\alpha < \omega_1$ and $N \subseteq \mathbb{N}$ so that the convex block basis of repeated averages, $(f_i^{N,\alpha})$, is norm null.*

Proof. Given $\varepsilon > 0$ let \mathcal{F}_ε be the closed hereditary collection of finite sets,

$$\mathcal{F}_\varepsilon = \{F \subseteq \mathbb{N} : \text{there exists } x^* \in S_{X^*} \text{ with } |x^*(e_i)| \geq \varepsilon \text{ for } i \in F\}.$$

Let $M \subseteq \mathbb{N}$ and $\alpha < \omega_1$. If \mathcal{F}_ε is (M, α, ε) large then there exists $N \subseteq M$ with $\mathcal{F}_\varepsilon \supseteq S_\alpha(N)$. Thus there exists $\alpha < \omega_1$ so that $\forall \varepsilon > 0 \forall M \subseteq \mathbb{N}$, \mathcal{F}_ε is not (M, α, ε) large.

Thus there exists $\alpha < \omega_1$ such that for all $M \subseteq \mathbb{N}$ and $\varepsilon > 0$ there exists $N \subseteq M$ and $n \in \mathbb{N}$ so that

$$\sup_{F \in \mathcal{F}_\varepsilon} \langle f_n^{N,\alpha}, \mathbf{1}_F \rangle < \varepsilon.$$

Hence if $\|x^*\| = 1$ then writing $f_n^{N,\alpha} = \sum b_i e_i$ we have

$$|x^*(f_n^{N,\alpha})| \leq \left| \sum_{\{i: |x^*(e_i)| \geq \varepsilon\}} b_i x^*(e_i) \right| + \left| \sum_{\{i: |x^*(e_i)| < \varepsilon\}} b_i x^*(e_i) \right| < 2\varepsilon.$$

The corollary follows applying this for successive $\varepsilon_i \downarrow 0$ and $M_i = \mathbb{N} \setminus \{1, 2, \dots, \max \text{supp } f_{i-1}^{\alpha, N}\}$.

Actually more precise statements can be made (see [12], [8]) which specify exactly when one can get $\|f_i^{\alpha, M}\| \rightarrow 0$ versus an ℓ_1 - S_α -spreading model $(e_i)_{i \in M}$.

Schreier sets and indices have been used in numerous other places. For example in [43] various ordinal indices involving the Schreier sets are introduced to measure the proximity of asymptotic ℓ_1 spaces to ℓ_1 .

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