EXTRACTA MATHEMATICAE Vol. **19**, Núm. 1, 141 – 144 (2004)
V *Curso Espacios de Banach y Operadores.*
Laredo, Agosto de 2003.

Renorming and Operators

Sebastián Lajara, Antonio J. Pallarés

Dpto. de Matemáticas, Universidad de Castilla La Mancha, Escuela Politécnica Superior
Avda. de España s/n, 02071 *Albacete, Spain*
Dpto. de Matemáticas, Universidad de Murcia, Campus de Espinardo
30100 *Espinardo* (*Murcia*)*, Spain*

e-mail: Sebastian.Lajara@uclm.es, apall@um.es

AMS *Subject Class.* (2000): 46B03, 46B06

This communication presented in the summer course "Espacios de Banach y operadores" help in Laredo (Spain), august 2003, is an annoucement of some results about MLUR renorming of Banach spaces. These results will appear in [6].

Let us start by recalling some convexity properties of norms. Let $(X, \| \, \|)$ be a Banach space. We say that X (or the norm of X) is:

(1) *locally uniformly rotund* (LUR for short) if, for every x and every sequence $(x_n)_n$ in X such that $\|x_n + x\| \to 2\|x\|$ and $\|x_n\| \to \|x\|$, we have $\|x_n - x\| \to 0$;

(2) *midpoint locally uniformly rotund* (MLUR for short) if, for every x and every sequence $(x_n)_n$ in X such that $\|x_n + x\| \to \|x\|$ and $\|x_n - x\| \to \|x\|$, we have $\|x_n\| \to 0$;

(3) *strictly convex or rotund* (R for short) if $x = y$ whenever x and y are points of X such that $\|x\| = \|y\| = \|\frac{x+y}{2}\|$, i.e., if the unit sphere of X does not contain any nondegenerate segment.

It is clear that LUR $\Rightarrow$ MLUR and that MLUR $\Rightarrow$ R. The converse implications are not true in general, even under renormings: as dual of a separable space, ℓ_∞ has an equivalent (dual) rotund norm, but it does not admit MLUR renorming [2]. In the paper [5], Haydon showed the first example of MLUR space with no equivalent LUR norm.

Banach spaces with equivalent MLUR norms were characterized in [8], in terms of countable decompositions of such spaces, involving the following

Supported by MCYT and FEDER BFM 2002-01719

Definition 1. Let A be a subset of a Banach space $(X, \| \ \|)$. A point $x \in A$ is said to be a ϵ-strongly extreme point of A if there is $\delta > 0$ such that $\|u - v\| < \epsilon$ whenever u and v are points in A with $\left\|x - \frac{u+v}{2}\right\| < \delta$.

It is easy to see that X is MLUR if and only if every point of the unit sphere is a ϵ-strongly extreme point of the unit ball, for every $\epsilon > 0$. The characterization of MLUR spaces mentioned above is given by the following

Theorem 1. ([8], Theorem 1) *A Banach space X admits an equivalent MLUR norm if, and only if, for every $\epsilon > 0$ we have a countable decomposition*

$$X = \bigcup_{n=1}^{\infty} X_{n,\epsilon}$$

in such a way that every $x \in X_{n,\epsilon}$ is a ϵ-strongly extreme point of the convex envelope $\operatorname{co}(X_{n,\epsilon})$.

A similar result was proved for LUR renormability in [7] and [10], where roughly speaking, ϵ-strong extremality is replaced by ϵ-dentability.

Theorem 2. ([7], Main Theorem) *A Banach space X has an equivalent LUR norm if, and only if, for every $\epsilon > 0$ we have a countable decomposition*

$$X = \bigcup_{n=1}^{\infty} X_{n,\epsilon}$$

in such a way that for every $n \in \mathbb{N}$ and every $x \in X_{n,\epsilon}$ there is an open half space $H \subset X$ such that $x \in H$ and $\operatorname{diam}(H \cap X_{n,\epsilon}) < \epsilon$. Recall that an open half space of X is a set of the form $H = f^{-1}(\alpha, \infty)$, with $f \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$.

This result has motivated the following notion, introduced and extensively studied by Moltó, Orihuela, Troyanski and Valdivia in their recent memoir [8], where a non linear transfer method for LUR renormability is provided.

Definition 2. Let X and Y be Banach spaces, and let A be a subset of X. A map $\Psi : A \to Y$ is said to be σ-slicely continuous if for every $\epsilon > 0$ we may write

$$A = \bigcup_{n} A_{n,\epsilon}$$

in such a way that for every $x \in A_{n,\epsilon}$ there exists an open half space H such that $x \in H$ and $\operatorname{diam} \Psi(H \cap A_{n,\epsilon}) < \epsilon$.

We are going to combine the covering characterization of Theorem 1 and some properties of σ-slicely continuous maps to get some results about MLUR renormability on Banach spaces. Our first theorem contains, as a particular case, a version of the three space property for MLUR norms.

THEOREM 3. *Let X be a Banach space. Suppose that there exist a closed MLUR renormable subspace Y of X and a σ-slicely continuous map $\Phi : X \to X$ such that $x - \Phi x \in Y$ for all $x \in X$. Then X admits an equivalent MLUR norm.*

The basic idea to prove this result is to get ϵ-MLUR decompositions on X from ϵ-MLUR decompositions of Y via the operator $Id - \Phi$. The map $\Phi : X \to X$ given by $\Phi = g \circ Q$, where $Q : X \to X/Y$ is the quotient map and X/Y is LUR renormable, and $g : X/Y \to X$ is a continuous selector, is σ-slicely continuous. If moreover Y has an MLUR renorming, we obtain the following result Alexandrov [1] (see also [3, p. 181]).

COROLLARY 1. *Let X be a Banach space. Suppose that there exists a closed subspace Y of X with an equivalent MLUR norm and such that the quotient X/Y is LUR renormable. Then X is MLUR renormable.*

Let us recall that MLUR is not a three space property. In the paper [5] Haydon provided an example of Banach space X with a closed subspace Y sucht that Y and X/Y admit a LUR norm and a MLUR norm, respectively, while X does not have any equivalent rotund norm.

As another application of our technique we get a partial generalization of a result of Haydon ([5, Proposition 5.3]), which is the main tool for the construction of MLUR norms in $C(\Upsilon)$ spaces, Υ a tree.

THEOREM 4. *Let K be a locally compact space. Suppose that there exist a σ-slicely continuous map $\Psi : C_0(K) \to c_0(\Gamma)$ and a family $\{K_\gamma\}_{\gamma\in\Gamma}$ of closed and open subsets of K with the following properties:*

(1) *for each $\gamma \in \Gamma$, $C_0(K_\gamma)$ is MLUR renormable;*

(2) *for each $x \in C_0(K)$, $x \neq 0$, $\operatorname{supp}(x) \subset \bigcup\{K_\gamma : \Psi x(\gamma) \neq 0\}$.*

Then $C_0(K)$ admits an equivalent MLUR norm.

The idea now to obtain the ϵ-MLUR decompositions in $C_0(K)$ is to use the σ-slicely continuity of Ψ and condition (2) to get a first decomposition where the functions x can be approximated by its restriction on some K_γ, and to transfer the MLUR decompositions of the spaces $C_0(K_\gamma)$.

References

[1] Alexandrov, G., On the three space problem for MLUR renorming of Banach spaces, *C.R. Bulg. Acad. Sci.*, **42** (1989), 17–20.

[2] Alexandrov, G., Dimitrov, I., On equivalent weakly midpoint locally uniformly rotund renormings of the space ℓ_∞, Math. and Math. Education, Proc. 14th Spring Conference of the Union of Bulg. Mathematicians, Sunny Beach, 1985, 189–191 (russian).

[3] Castillo, J.M., González, M., "Three Space Problems in Banach Space Theory", Lecture Notes in Math. 1667, Springer, Berlin, 1997.

[4] Deville, D., Godefroy, G., Zizler, V., "Smoothness and Renormings in Banach Spaces", Pitman Monographs and Surveys in Pure and Appl. Math. 64, Longman Scientific and Technical, Longman House, Burnt Mill, Harlow, 1993.

[5] Haydon, R., Trees in renorming theory, *Proc. London Math. Soc.* **78** (1999), 541–585.

[6] Lajara, S., Pallarés, A.J., MLUR renormings of Banach spaces and σ-slicely continuous maps (preprint).

[7] Moltó, A., Orihuela, J., Troyanski, S., Locally uniformly rotund renorming and fragmentability, *Proc. London Math. Soc.* **75** (1997), 619–640.

[8] Moltó, A., Orihuela, J., Troyanski, S., Valdivia, M., Midpoint locally uniformly rotundity and a decomposition method for renorming, *Quartely Journal of Mathematics* **52** (2001), 181–193.

[9] Moltó, A., Orihuela, J., Troyanski, S., Valdivia, M., A non linear transfer technique (preprint).

[10] Raja, M., On locally uniformly rotund norms, *Mathematika* **46** (1999), 343–358.