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Note on a Banach Space Having Equal Linear Dimension with its Second Dual

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In the paper [4] it is stated that

(E) *there exists a Banach space X whose bidual X^{**} is isometric to a subspace of X, but not isomorphic to X.*

Since X is isometric to a subspace of X^{**}, (E) gives another solution to a Banach problem, posed in [1], p.193 (and solved for the first time in [2]): *Let a Banach space X be isomorphic to a subspace of a Banach space Y and let Y be isomorphic to a subspace of X. Are then X and Y isomorphic?* Let us note that this problem becomes essential (and difficult) if one assumes additionally that X and Y are isomorphic to *complemented* subspaces of each other (the Schroeder-Bernstein problem, solved in the negative by Gowers [5]).

The construction in [4] is as follows: one takes $Z = Z^{(0)} = L_1(0,1)$ and its successive even duals Z^{**}, $Z^{(4)}, \ldots$, and X is defined as $(\sum_{n=0}^{\infty} Z^{(2n)})_{\ell_2}$. Of course, X^{**} is isometric to a subspace of X. Next, the author attempts to prove that X and X^{**} are non-isomorphic; however, Ezrohi's arguments at this point are difficult to follow. In the proof he refers to his Ph. D. result which, in our opinion, is incorrect. On the other hand, it is known that L_1 is complemented in L_1^{**} (see e.g.[8], Proposition 8.3 (v), p. 113; cf. [9]). It follows that $L_1 \oplus L_1^{**}$ is isomorphic to L_1^{**}, and hence, X *is* isomorphic to X^{**} and the construction is improper. Nevertheless, this space X is a simple example of a nonreflexive Banach *lattice* isomorphic to its bidual. (Recall, that the famous James space [6] appears to be a solution to another problem of Banach: whether there is a nonreflexive Banach space which is isomorphic to its bidual?)

In this note we show that the assertion (E) holds true for X constructed as above with the basic space c_0 ($=Z$) instead of $L_1(0,1)$.

We recall that a Banach space Z is a Grothendieck space if the weak and weak* convergence of sequences in Z^* coincide. A compact Hausdorff space K is called Stonian [quasi-Stonian, resp.] if each open [and F_σ, resp.] subset of K has an open closure. The Čech-Stone compactification $\beta\mathbb{N}$ of the discrete space $\mathbb{N}$, of all positive integers, is a sample Stonian space. Every $C(K)$-space, with K quasi-Stonian, is a Grothendieck space; in particular, $\ell_\infty = C(\beta\mathbb{N})$ is of this type (see [8], pp. 131-132; or [7], pp.111 and 348-360). Moreover, from the Kakutani-Krein theorem (see [8]; Corollary 1, p. 104) it follows that both $L_\infty(\mu)$-space and the biduals of $C(K)$-spaces, for K arbitrary compact Hausdorff (being isometric to $C(L)$-spaces for some L Stonian) are Grothendieck spaces ([8], p. 121).

It easy to check that the class of Grothendieck spaces is closed with respect to quotient operations and linear isomorphisms. It is known that *separable* quotients of Grothendieck spaces are reflexive ([7], Proposition 5.3.2); in particular

no complemented subspace of a Grothendieck space is isomorphic to c_0. (1)

We note that the class of Grothendieck spaces is also closed with respect to ℓ_p-sums, $1 < p < \infty$, and we apply this property to construct a Banach space X such that X^{**} is isometric to a complemented subspace of X, with X and X^{**} non-isomorphic. The details follow.

Lemma 1. *Let (Z_n) be a sequence of Grothendieck spaces, and let $1 < p < \infty$. Then $Z := (\sum_{n=1}^\infty Z_n)_{\ell_p}$ is a Grothendieck space.*

Theorem 1. *Put $Z_0 = c_0$, and $Z_{n+1} = (Z_n)^*$, $n \geq 0$. Then for the space $X := (\sum_{n=0}^\infty Z_{2n})_{\ell_p}$, $1 < p < \infty$, we have: X^{**} is isometric to a complemented subspace of X but X and X^{**} are non-isomorphic.*

The proof of Lemma 1. This lemma is rather known (at least if $Z_1 = Z_2 = \ldots$ [3]) and we present the proof for sake of completeness. We apply the following corollary of the Banach-Steinhaus theorem:

($*$) *Let (T_n) be a uniformly bounded sequence of linear operators defined on a Banach space X and let D be a dense subset of X. If $\|T_n x\| \to 0$ as $n \to \infty$ for every $x \in D$, then $\|T_n x\| \to 0$ for every $x \in X$.*

Notice first that $Z^* = (\sum_{n=1}^{\infty} Z_n^*)_{\ell_q}$, with $1/p + 1/q = 1$, and hence $Z^{**} = (\sum_{n=1}^{\infty} Z_n^{**})_{\ell_p}$. Thus every $f \in Z^*$ is of the form $f(z) = \sum_{n=1}^{\infty} f_n(z_n)$, where $z = (z_n) \in Z$ and $f_n \in Z_n^*$, $n \geq 1$, with $\sum_{n=1}^{\infty} \|f_n\|^q < \infty$.

Let $(f^{(m)})$ be a weak*-null sequence in Z^*:

$$\sum_{n=1}^{\infty} f_n^{(m)}(z_n) \to 0, \quad \text{as} \quad m \to \infty,$$

for every sequence $(z_n) \in Z$ with $z_n \in Z_n$; in particular,

$$\lim_{m \to \infty} f_n^{(m)}(z_n) = 0 \quad \text{for every} \quad n. \tag{2}$$

Since Z_n's are Grothendieck, from (2) we obtain that for every $F_n \in Z_n^{**}$, $n = 1, 2, \ldots$, we have

$$\lim_{m \to \infty} F_n(f_n^{(m)}) = 0. \tag{3}$$

Now let D denote the dense in Z^{**} set of the elements F of the form $F = \sum_{n=1}^{k} F_n$, where $F_n \in Z_n^{**}$, and $k = 1, 2, \ldots$, and consider the linear functionals z_m^{***}, $m \geq 1$, on Z^{**} of the form $z_m^{***}(F) := F(f^{(m)})$. By (3), these functionals fulfil the assumptions of (*), and therefore (z_m^{***}) is weak*-null in Z^{***} or, equivalently, $(f^{(m)})$ is weak-null in Z^*. ∎

The proof of Theorem 1. We have that $Z_{2n} = C(K_n)$, where $n \geq 1$ with $K_1 = \beta\mathbb{N}$, and the remaining K_n's are "gigantic" Stonian spaces (see the remarks preceding Lemma 1). Therefore, by Lemma 1, X^{**} is a Grothendieck space, which evidently is isometric to a complemented subspace of X and which, by property (1), cannot be isomorphic to X. ∎

Remark. Of course, the space X in Theorem 1 is not isomorphic to a complemented subspace of X^{**}, and hence the following question seems to be quite natural. Let X be isomorphic to a complemented subspace of X^{**} and let X^{**} be isomorphic to a complemented subspace of X. Are then X and X^{**} isomorphic? Professor Galego has remarked that this question is connected with the following one: Let X be complemented in X^{**}. Is X^{**} isomorphic to $X \oplus X^{**}$? In particular, is X^{***} isomorphic to $X^* \oplus X^{***}$? Evidently, it is true if X is isomorphic to its square.

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