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On Banach Spaces Containing Complemented and Uncomplemented Subspaces Isomorphic to $\mathbf{c_0}$

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1. Introduction

In this short note we give a negative answer to a question of Argyros, Castillo, Granero, Jiménez and Moreno concerning to Banach spaces which contain complemented and uncomplemented subspaces isomorphic to c_0. To be more precise, let X be a Banach space, following [1] we say that X is *separably Sobczyk* if every subspace of X isomorphic to c_0 is complemented in X.

We also say that X is *hereditarily separably Sobczyk* space if every subspace Y of X which is isomorphic to c_0, contains a subspace Z isomorphic to c_0 and complemented in X. In [1, page 754] was posed the following question:

Question 1.1. Suppose that a Banach space X is hereditarily separably Sobczyk. Does it follow that X is separably Sobczyk space?

Next we will present several examples of Banach spaces which give negative answers to this question. The first one that we obtained is constructed from some n-Sobczyk spaces introduced in [9]. Then, we realized that even for $C(K)$ spaces, where K is dispersed compact, the answer to above question is negative. Finally, we show that every Banach space which contains no subspace isomorphic to l_1 is hereditarily separably Sobczyk. In particular, the well known Johnson-Lindenstraus space JL [6, Example 1, page 222] is also a counterexample to Question 1.1.

2. The first example

Let us recall that a Banach space is n-Sobczyk if every K-isomorphic copy of c_0 therein is the range of a projection with norm at most nK. In [9] was introduced examples of Banach spaces X for which $\inf\{\lambda : X \text{ is } \lambda\text{-Sobczyk}\}$ is arbitrarily large. Moreover, each one of these spaces X contains a subspace isometric to c_0.

In order to present a solution to Question 1.1, we fix by this previous result, a sequence $(X_n)_{n\in\mathbb{N}}$ of separably Sobczyk spaces having the following property:

(1) There is a subspace X_n^0 of X_n isometric to c_0, such that every projection of X_n onto X_n^0 has norm greater or equal than n.

Theorem 2.1. *Let $X = (\sum_1^\infty X_n)_{c_0}$ be the c_0-sum of $(X_n)_{n\in\mathbb{N}}$. Then*

(a) *X is hereditarily separably Sobczyk.*
(b) *X is not separably Sobczyk.*

Proof. (a) Assume that Y is a subspace of X isomorphic to c_0. Denote by d_n the distance $\operatorname{dist}(S(Y), [(X_i)_{i=n}^\infty])$ between the unit sphere $S(Y)$ of Y and the closed subspace $[(X_i)_{i=n}^\infty]$ of X generated by $(X_i)_{i=n}^\infty$. We distinguish two possible cases:

First case: $d_n = 0$ for every $n \in \mathbb{N}$.

Therefore for $0 < \varepsilon < 1$ there exist a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and sequences $(y_k)_{k\in\mathbb{N}}$ in $S(Y)$ and $(z_k)_{k\in\mathbb{N}}$ in $S([(X_i)_{i=n_k+1}^{n_{k+1}}])$ with

$$\|y_k - z_k\| < \varepsilon 2^{-k}\ , \quad k = 1, 2, \dots \tag{2}$$

Obviously, $(z_k)_{k\in\mathbb{N}}$ is equivalent to the unit vector basis of c_0 and there is a projection $P : X \to [(z_k)_{k\in\mathbb{N}}]$ with $\|P\| = 1$. So by (2) we deduce that

$$\operatorname{dist}(S([(y_k)_{k\in\mathbb{N}}]), \ker P) > 1 - \varepsilon\ .$$

Since $[(y_k)_{k\in\mathbb{N}}] \oplus \ker P = X$, it follows that there is a projection $Q : X \to [(y_k)_{k\in\mathbb{N}}]$ with $\|Q\| < (1-\varepsilon)^{-1}$. By (2), we conclude that the subspace $[y_k]_1^\infty$ is isomorphic to c_0.

Second case: $d_{n+1} > 0$ for some $n \in \mathbb{N}$.

In this case, the natural projection $Q_n : X \to [(X_k)_{k=1}^n]$, restricted to Y, is an isomorphism onto the subspace Q_nY which is isomorphic to c_0. Moreover, it easy to see by (1) that there is a bounded projection $R : [X_k]_{k=1}^n \to Q_nY$. Thus $Q_n^{-1}RQ_n$ is also a bounded projection of X onto Y.

(b) Let $X_0 = \left(\sum_1^\infty X_n^0\right)_{c_0}$ be the c_0-sum of $(X_n^0)_{n\in\mathbb{N}}$, where X_n^0 is the Banach space mentioned in (1).

Clearly X_0 is isometric to c_0. Now suppose that there exists a bounded projection $P : X \to X_0$ and take $m \in \mathbb{N}$ satisfying $\|P\| < m$. Thus for the natural projection $P_n : X \to X_n$ the operator $P_n P|_{X_n}$ would be a projection of X_n onto X_n^0 with the norm less or equal than m, which gives a contradiction for $n > m$. ∎

3. An example from the $C(K)$ spaces

In [7, Theorem 11] Lotz, Peck and Porta proved that a compact space K is scattered if and only if every infinite dimensional subspace of $C(K)$ contains a subspace isomorphic to c_0 and complemented in $C(K)$. Therefore, in the case where K is a dispersed compact, $C(K)$ is hereditarily separably Sobczyk space.

However Moltó [8] has constructed a scattered compact K such that $C(K)$ has a subspace isometric to c_0 which is not complemented in $C(K)$. Thus, this $C(K)$ is not separably Sobczyk space.

4. Some more hereditarily separably Sobczyk spaces

In [2, Theorem 2.1] was observed that the following reformulation of a result of Hagler and Johnson [5, Theorem 1.a] is true.

Theorem 4.1. *Let X be a real Banach space and $(x_n^*)_{n\in\mathbb{N}}$ a sequence in X^* equivalent to the unit vector basis of l_1. If no normalized l_1-block of $(x_n^*)_{n\in\mathbb{N}}$ is weak* null sequence, then X contains a subspace isomorphic to l_1.*

As a consequence of this reformulation, Diaz and Fernández proved:

Theorem 4.2. *Let X be a real Banach space that does not contain a subspace isomorphic of l_1. If X contains a subspace isomorphic to c_0, then X contains a complemented subspace Z isomorphic to c_0. Moreover, for every $\epsilon > 0$, we can find the subspace Z so that there is a projection $P : X \to Z$ with $\|P\| < 1 + \epsilon$.*

In [4, Proposition 2.1] was noted that one can slightly modify the proof of Theorem 4.2 to show that every Banach space which contains no subspace isomorphic to l_1 is hereditarily separably Sobczyk space. Since no proof was presented in [4], for sake of completeness, we will prove Theorem 4.3.

Theorem 4.3. *Let X be a real Banach space that does not contain a subspace isomorphic to l_1. If Y is a closed subspace of X that contains a subspace isomorphic to c_0, then Y contains a subspace Z isomorphic to c_0 which is complemented in X. Moreover, for every $\epsilon > 0$, we can find the subspace Z so that there is a projection $P : X \to Z$ with $\|P\| < 1 + \epsilon$.*

Proof. Fix $\varepsilon > 0$. Since Y contains a subspace isomorphic to c_0, by James distortion theorem [3, Theorem 1, XIV], taking $\delta = \epsilon/(1+\epsilon)$, there is a sequence $(y_n)_{n\in\mathbb{N}}$ in the unit ball of Y such that

$$(1-\delta)\sup_k |a_k| \le \Big\| \sum_{k=1}^n a_k y_k \Big\| \le \sup_k |a_k| \tag{3}$$

for all scalars $(a_k)_{k=1}^n$ and all $n \in \mathbb{N}$.

For each $m \in \mathbb{N}$ define y_m^* by

$$y_m^*\Big(\sum_{k=1}^n a_k y_k\Big) = a_m\ .$$

By the Hahn-Banach theorem we can extend y_m^* to an element of X^*, that we still denote y_m^*, so that

$$y_m^*(y_n) = \delta_{mn} \quad \text{and} \quad \|y_m^*\| \le \frac{1}{1-\delta} = 1+\epsilon, \tag{4}$$

for all $m, n = 1, 2, \ldots$, where δ_{mn} denotes the Kronecker delta.

The sequence $(y_n^*)_{n\in\mathbb{N}}$ is equivalent to the unit vector basis of l_1 in X^*. In fact, we have that

$$\sum_{k=1}^n |a_k| \le \Big\| \sum_{k=1}^n a_k y_k^* \Big\| \le \frac{1}{1-\delta} \sum_{k=1}^n |a_k|,$$

for all scalars $(a_k)_{k=1}^n$ and for all $n \in \mathbb{N}$.

Notice that if $a_1, a_2, \ldots, a_n$ are real scalars, we can consider numbers ϵ_k with $\epsilon_k a_k = |a_k|$, $k = 1, 2, \ldots$. Then

$$\Big\| \sum_{k=1}^n a_k x_k^* \Big\| \ge \Big| \sum_{k=1}^n a_k x_k^* (\sum_{k=1}^n \epsilon_k x_k) \Big| = \Big| \sum_{k=1}^n \epsilon_k a_k \Big| = \sum_{k=1}^n |a_k|.$$

According to Theorem 4.1, there is a weak* null sequence $(z_n^*)_{n\in\mathbb{N}}$ which is a normalized l_1-block of $(y_n^*)_{n\in\mathbb{N}}$. It has the following form

$$z_n^* = \sum_{k\in A_n} a_k y_k^*,$$

where$(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint finite subsets of $\mathbb{N}$, and $\sum_{k\in A_n}|a_k| = 1$, for every $n \in \mathbb{N}$. Put

$$z_n = \sum_{k\in A_n} \epsilon_k y_k,$$

where $\epsilon_k = \text{sign} a_k$, for all $k \in A_n$ and $n \in \mathbb{N}$. Thus, we have constructed sequences $(z_n)_{n\in\mathbb{N}}$ in Y and $(z_n^*)_{n\in\mathbb{N}}$ in X^* such that

(5) (3) remains unchanged for $(z_n)_{n\in\mathbb{N}}$.

(6) $(z_n^*)_{n\in\mathbb{N}}$ is a weak* null sequence in X^* that is equivalent to the unit vector basis of l_1, and (4) remains unchanged for $(z_n^*)_{n\in\mathbb{N}}$.

Let us consider the mapping P from X onto the closed subspace Z generated by $(z_n)_{n\in\mathbb{N}}$

$$P(x) = \sum_1^\infty z_n^*(x) z_n \ .$$

Now from (5) and (6), it is easy to verify that Z is isomorphic to c_0 and that P is the projection from X onto Z with

$$\|P\| \le \frac{1}{1-\delta} = 1 + \epsilon \ .$$

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