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Variations on the Banach-Stone Theorem

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1. Introduction

The classical Banach-Stone theorem asserts that, for a compact space K, the linear metric structure of $C(K)$ (endowed with the sup-norm) determines the topology of K. This result has found a large number of extensions, generalizations and variants in many different contexts, and these notes are intended to be a guided tour around some of them. We shall review some classical results, as well as very recent contributions on the area, but of course we are not trying to be exhaustive. The main theme along the paper will be the kind of results asserting that, if X is a topological space (maybe endowed with some richer structure, such as metric space, or Banach space, or smooth manifold), the topology of X (respectively, its metric, or linear, or smooth structure) is characterized in terms of a given algebraic or topological-algebraic structure on $C(X)$ or on a subfamily of $C(X)$.

The contents of the paper are as follows. Section 2 is devoted to the proof of the classical Banach-Stone theorem. We shall obtain that if K and L are compact spaces, then a linear isometry $T : C(L) \to C(K)$ must be a weighted composition map of the form $(Tf) = a \cdot (f \circ h)$, where a is a continuous scalar function with $|a| = 1$ and h is a homeomorphism from K onto L. Further, the original Banach proof for compact metric spaces is discussed. We also include the Mazur-Ulam theorem about linearity of isometries between normed spaces.

In Section 3 we review several results concerning isometries between Banach function spaces of continuous functions. We consider here the space $C_0(X)$, of all continuous functions vanishing at infinity on a locally compact space X, endowed with the sup-norm. We shall see that if X and Y are locally compact spaces, then the existence of a linear isometry between $C_0(X)$ and $C_0(Y)$ (or between certain special subspaces $E \subset C_0(X)$ and $F \subset C_0(Y)$) leads into a homeomorphism between X and Y (or between some related subspaces of X and Y). A representation of isometries as weighted composition maps is also achieved in this case. In fact, the existence of just an "approximate isometry" between $C_0(X)$ and $C_0(Y)$ (that is, a linear isomorphism T with $\|T\| \cdot \|T^{-1}\| < 2$) is enough to obtain a homeomorphism between X and Y. The case of isometries between spaces of vector-valued continuous functions is briefly considered in this Section. Finally, some results about bounded uniformly continuous and bounded Lipschitz functions are included.

Section 4 is devoted to algebra isomorphisms. We shall see that the purely algebraic structure of $C(X)$ determines the topology of a realcompact space X. In fact, if X and Y are realcompact, every algebra isomorphism $T : C(Y) \to C(X)$ is a composition map of the form $Tf = f \circ h$, where h is a homeomorphism from X to Y. Analogous results are obtained for isomorphisms between special subalgebras $A \subset C(X)$ and $B \subset C(Y)$, and some applications are given. For example, if E and F are Banach spaces (respectively, M and N are smooth manifolds), we give conditions under which the existence of an algebra isomorphism between $C^\infty(E)$ and $C^\infty(F)$ (respectively, between $C^\infty(M)$ and $C^\infty(N)$) implies that E and F are isomorphic (respectively, M and N are C^∞−diffeomorphic). Finally, we shall see that some of the previous results about algebra isomorphisms can be extended to the more general setting of biseparating maps between function spaces.

In Section 5 we are concerned with vector lattice isomorphisms. Our main goal is to characterize the uniform structure and the Lipschitz structure of a metric space X in terms of the families $U(X)$ and $Lip(X)$ of uniformly continuous and Lipschitz functions on X. Since these families do not have, in general, algebra structure, we are lead to consider its natural unital vector lattice structure. Then we shall see that, if X and Y are complete metric spaces, every unital vector lattice isomorphism $T : U(Y) \to U(X)$ (respectively, $T : Lip(Y) \to Lip(X)$) is a composition map of the form $Tf = f \circ h$, where $h : X \to Y$ is a uniform homeomorphism (respectively, a Lipschitz homeomorphism). Furthermore, the respective cases of bounded functions are considered.

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2. Classical results

For a compact topological space K (which will always be assumed to be Haussdorf), we denote as usual by $C(K)$ the Banach space of all continuous real functions on K endowed with the sup-norm:

$$\|f\|_\infty = \sup\{|f(x)| : x \in K\}.$$

In his fundamental work *Théorie des Opérations Linéaires* (1932), Banach [11] considered the problem of when two spaces of type $C(K)$ are isometric. He solved this problem for the case of compact metric spaces K, also giving a description of such isometries. In 1937, Stone [60] extended this result to general compact spaces K, in what is known as the classical Banach-Stone theorem. Before giving the precise statement, some comments about isometries are in order.

If E and F are normed spaces, by an *isometry* we mean a bijection $u : E \to F$ preserving the distance, that is, such that $\|u(x)-u(y)\| = \|x-y\|$, for every $x, y \in E$. We say that $u : E \to F$ is a *linear isometry* when u is an isometry which is also linear. This is equivalent to say that u is a linear isomorphism and $\|u(x)\| = \|x\|$, for every $x \in E$. Concerning this, we recall the following nice result due to Mazur and Ulam [49]. Note that, as a consequence, we have that two normed spaces are isometric if, and only if, they are linearly isometric.

Theorem 1. (Mazur and Ulam 1932) *Let E and F be normed spaces and let $u : E \to F$ an isometry such that $u(0) = 0$. Then u is a linear isometry.*

Proof. The key is to prove that u preserve the midpoints, i.e. for every $x, y \in E$:

$$u\left(\frac{x+y}{2}\right) = \frac{u(x)+u(y)}{2} \qquad (*)$$

Indeed, from $(*)$ it follows that u is additive, since

$$u(x) = u\left(\frac{2x+0}{2}\right) = \frac{u(2x)+u(0)}{2} = \frac{u(2x)}{2}$$

implies that $u(2x) = 2u(x)$, and then

$$u(x+y) = u\left(\frac{2x+2y}{2}\right) = \frac{u(2x)+u(2y)}{2} = \frac{2u(x)+2u(y)}{2} = u(x)+u(y).$$

Now it is easy to check that $u(qx) = qu(x)$, for every $q \in \mathbb{Q}$, and the continuity of u implies that $u(\lambda x) = \lambda u(x)$ for every $\lambda \in \mathbb{R}$.

Thus, in order to prove $(*)$, we consider the sets

$$H_1 = \left\{p \in E : \|x-p\| = \|y-p\| = \frac{1}{2}\|x-y\|\right\}$$

and, for $n > 1$,

$$H_n = \left\{p \in H_{n-1} : \|p-q\| \le \frac{\operatorname{diam}(H_{n-1})}{2}, \text{ for all } q \in H_{n-1}\right\}.$$

Claim 1. If $p \in H_n$ then $\tilde{p} = x+y-p \in H_n$.

This can be derived easily by induction.

Claim 2. The midpoint $(x+y)/2$ belongs to H_n, for every n.

Indeed, it is clear that $(x+y)/2 \in H_1$. For $n > 1$ and for all $q \in H_{n-1}$, we have

$$2\left\|\frac{x+y}{2} - q\right\| = \|x+y-2q\| = \|\tilde{q}-q\| \le \operatorname{diam}(H_{n-1}).$$

On the other hand, we have $H_1 \supset H_2 \supset \cdots$ and

$$\operatorname{diam}(H_n) \le \frac{1}{2}\operatorname{diam}(H_{n-1}) \le \cdots \le \frac{1}{2^{n-1}}\operatorname{diam}(H_1).$$

Therefore

$$\bigcap_{n=1}^{\infty} H_n = \left\{\frac{x+y}{2}\right\}.$$

Since the sets H_n are defined only in terms of the distance, the midpoint $(x+y)/2$ is then characterized in terms of the distance, so in particular it is preserved by isometries. This proves $(*)$. ∎

Now, we state the classical Banach-Stone theorem:

THEOREM 2. (Banach 1932 and Stone 1937) *Let K and L be compact spaces. Then, $C(K)$ is isometric to $C(L)$ if, and only if, K and L are homeomorphic. Moreover, every linear isometry $T : C(L) \to C(K)$ is of the form*

$$(Tf)(k) = a(k) \cdot (f \circ h)(k), \quad k \in K$$

where $h : K \to L$ is a homeomorphism and $a : K \to \mathbb{R}$ is a continuous function with $|a(k)| = 1$, for every $k \in K$.

The proof we are going to give here is quite standard (see, e.g. Dunford and Schwartz [24] or Roy [57]), and is originally due to Arens and Kelley [7]. Before proceed with the details, we need to recall some notions and previous results. We first see how a compact space K can be embedded into $C(K)^*$, the dual of $C(K)$, by means of the evaluation functionals δ_k, where $\delta_k(f) = f(k)$ for every $f \in C(K)$.

LEMMA 3. *The map $\delta : k \in K \rightsquigarrow \delta_k \in C(K)^*$ is a topological embedding from K into $C(K)^*$ endowed with the weak-star topology w^*.*

Proof. Since, by Urysohn's Lemma, $C(K)$ separates the points of K, we have that δ is one-to-one. Now if k_α is a net converging to k in K, then for every $f \in C(K)$ we have that $f(k_\alpha)$ converges to $f(k)$, that is, δ_{k_α} converges to δ_k in the w^* topology. This shows the continuity of δ. Finally, the compactness of K assures that δ is a homeomorphism onto its image. ∎

It is immediate to check that every evaluation functional δ_k is in fact in $B_{C(K)^*}$, the unit ball of $C(K)^*$. But, how to recognize in a metric way, those functionals in $B_{C(K)^*}$ that are given by evaluation at some point of K? The next Theorem, due to Arens and Kelley [7], shows that the evaluation functionals and their negatives are precisely the extreme points of $B_{C(K)^*}$. For a proof of this result we refer to Arens and Kelley [7] or Dunford and Schwartz [24]. Recall that an *extreme point* of a convex subset C of a linear space E is a point of C that is not in the interior of any segment contained in C. That is, $x \in C$ belongs to $ext(C)$ if whenever, $x = \lambda y + (1 - \lambda)z$ with $0 < \lambda < 1$ and $y, z \in C$, we have $x = y = z$. This notion was introduced by Minkowski [51] in 1911 for finite dimensional spaces, and then (1940) studied by Krein and Milman [45] in general. A detailed study concerning extreme points can be seen in the nice survey by Roy [57].

THEOREM 4. (Arens and Kelley 1947) *Let K be a compact space. Then*

$$ext(B_{C(K)^*}) = \{\pm\, \delta_k : k \in K\}.$$

Now we are ready for the proof of Banach-Stone theorem:

Proof of the Banach-Stone theorem (Theorem 2). If $h : K \to L$ is a homeomorphism and $a : K \to \mathbb{R}$ is continuous with $|a| = 1$, it is clear that $Tf = a \cdot (f \circ h)$ defines a linear isometry from $C(L)$ onto $C(K)$.

Conversely, let $T : C(L) \to C(K)$ be a linear isometry. It is easy to see that the adjoint map $T^* : C(K)^* \to C(L)^*$ is also a linear isometry, and in addition it is w^*-to-w^* continuous. As a consequence, T^* defines a bijection from $ext(B_{C(K)^*})$ onto $ext(B_{C(L)^*})$.

For each $k \in K$ consider $\delta_k \in ext(B_{C(K)^*})$. There exists a unique $h(k) \in L$ and a unique scalar $a(k) = \pm 1$ such that $T^*(\delta_k) = a(k)\delta_{h(k)}$. In this way we have defined a map $h : K \to L$ and a map $a : K \to \mathbb{R}$ with $|a| = 1$.

If k_α is a net converging to k in K, we have that δ_{k_α} converges to δ_k in the w^* topology and then $T^*(\delta_{k_\alpha}) = a(k_\alpha)\delta_{h(k_\alpha)}$ converges to $T^*(\delta_k) = a(k)\delta_{h(k)}$ in the w^* topology. Choosing $g \equiv 1 \in C(L)$, we see that $a(k_\alpha)$ converges to $a(k)$. This shows that a is continuous and also δ_{k_α} converges to δ_{k_α}. That is, $h(k_\alpha)$ converges to $h(k)$, which gives the continuity of h.

Now it is easy to see that h is bijective and h^{-1} is continuous. On the other hand, note that $(Tf)(k) = (\delta_k \circ T)(f) = (T^*\delta_k)(f) = a(k) \cdot \delta_{h(k)}(f) = a(k) \cdot (f \circ h)(k)$. ∎

The proof of Stone in [60] is different. Instead of considering the evaluation functionals in $C(K)^*$, he uses a somewhat dual method, working with the "flat faces" of the unit ball of $C(K)$. A similar proof can be seen in Behrends [12]. On the other hand, the original proof of Banach [11] for metric spaces uses the differentiability of the sup-norm in $C(K)$ in order to characterize the evaluation functionals. We are going to give an sketch of this proof, which depends on the following Lemma.

Lemma 5. *Let $(E, \|\cdot\|)$ be a Banach space, and $x \in E$. The following are equivalent:*

(1) *The norm $\|\cdot\|$ is Gâteaux-differentiable at x, i.e., there exists $x^* \in E^*$ such that for every $v \in E$*

$$x^*(v) = \lim_{t \to 0} \frac{\|x + tv\| - \|x\|}{t}.$$

(2) *There exists a unique supporting functional x^* for x, i.e., there exists a unique $x^* \in E^*$ with $\|x^*\| = 1$ and $x^*(x) = \|x\|$. (In this case, x^* is the Gâteaux-differential of the norm at x.)*

If now $(E, \|\cdot\|) = (C(K), \|\cdot\|_\infty)$ *and* $x = f$*, these conditions are also equivalent to:*

(3) *The set* $M_f = \{k \in K : \|f\|_\infty = |f(k)|\}$ *is a singleton* $\{k_0\}$*. (In this case,* $x^* = sig(f(k_0)) \cdot \delta_{k_0}$ *is the supporting functional for* f*.)*

Proof. To see that (1) implies (2), let x^* be the Gâteaux-differential of the norm at x. It is easily seen that x^* is a supporting functional for x. Assume now that y^* is also a supporting functional for x. Since

$$\lim_{t\to 0} \frac{\|x+tv\| + \|x-tv\| - 2\|x\|}{t} = x^*(v) + x^*(-v) = 0$$

we have that for each $\varepsilon > 0$ there is some $\delta > 0$ such that $\|x+tv\|+\|x-tv\| \leq 2\|x\| + \varepsilon t$, when $0 < t < \delta$. Thus,

$$\begin{aligned} 2\|x\| + t(x^* - y^*)(v) &= x^*(x+tv) + y^*(x-tv) \\ &\leq \|x+tv\| + \|x-tv\| \leq 2|x\| + \varepsilon t \end{aligned}$$

and therefore $(x^* - y^*)(v) \leq \varepsilon$. This shows that $x^* = y^*$.

The fact that (2) implies (3) is clear, since for every $k \in M_f$ we have that $sig(f(k)) \cdot \delta_s$ is a supporting functional for f.

In order to see that (3) implies (1), let $M_f = \{k_0\}$, and assume for example that $\|f\|_\infty = f(k_0)$ (the case $\|f\|_\infty = -f(k_0)$ is analogous). Consider $v \in C(K)$. Now for each $t \in \mathbb{R}$ there exists some $k_t \in K$ with $|f + tv\|_\infty = |f(k_t) + tv(k_t)|$. Then,

$$\begin{aligned} f(k_0) \leq |f(k_0) + tv(k_0)| + |tv(k_0)| &\leq |f(k_t) + tv(k_t)| + |tv(k_0)| \\ &\leq |f(k_t)| + |t|(|v(k_0)| + |v(k_t)|. \end{aligned}$$

And therefore

$$0 \leq f(k_0) - |f(k_t)| \leq 2|t| \cdot \|v\|_\infty.$$

As a consequence, $\lim_{t\to 0} |f(k_t)| = f(k_0)$, and from the compactness of K we deduce that $\lim_{t\to 0} f(k_t) = f(k_0) > 0$. Now, for small $|t|$, we have

$$tv(k_0) \leq \|f + tv\|_\infty - \|f\|_\infty = f(k_t) + tf(k_t) - f(k_0) \leq tv(k_t),$$

and from this we obtain that

$$\lim_{t\to 0} \frac{\|f+tv\|_\infty - \|f\|_\infty}{t} = v(k_0).$$

Finally, for a proof of (2) implies (1), we refer to [23], Corollary I.1.5, where it is proved using Smulyan test. ∎

Note that the characterization (3) of the above Lemma holds for arbitrary compact spaces K. Now the proof of the Theorem goes as follows:

Banach's proof for metric spaces. If $T : C(L) \to C(K)$ is a linear isometry, it is is easy to see that $\| \cdot \|_\infty$ is Gâteaux-differentiable at $f \in C(L)$ if, and only if, $\| \cdot \|_\infty$ is Gâteaux-differentiable at $Tf \in C(K)$; furthermore x^* is a supporting functional for Tf if, and only if, $T^*(x*)$ is a supporting functional for f.

Now since K is a metric space, for each $k \in K$ we can choose a peak-function $g_k \in C(K)$, that is, $\|g_k\|_\infty = g_k(k)$ and $M_{g_k} = \{k\}$. Thus δ_k is a supporting functional for g_k, and therefore $T^{-1}(g_k)$ has a supporting functional of the form $\pm\delta_{h(k)} = T^*(\delta_k)$. This means that there exist a unique $h(k) \in L$ and a unique $a(k) \in \mathbb{R}$ with $|a(k)| = 1$ such that $T^*(\delta_k) = a(k) \cdot \delta_{h(k)}$. At this point, the proof continues in the same way as before. ∎

As we have seen, metrizability of base-spaces is used in order to assure the existence of continuous peak-functions. Nevertheless, these kind of functions exist in more general compact spaces, and the same proof works in this case. See Eilenberg [27] for further results in this direction.

3. Isometries

To start this Section, we would like to recall that the classical Banach-Stone theorem can be extended to the space of functions vanishing at infinity on a locally compact space, with essentially the same proof (see e.g. Behrends [12]). More precisely, if X is a locally compact space (which will be always assumed to be Hausdorff), let $C_0(X)$ denote the Banach space of all continuous real functions on X which vanish at infinity, endowed with the sup-norm. Then we have:

Theorem 6. *Let X and Y be locally compact spaces. Then, $C_0(X)$ is isometric to $C_0(Y)$ if, and only if, X and Y are homeomorphic. Moreover, every linear isometry $T : C_0(Y) \to C_0(X)$ is of the form*

$$(Tf)(x) = a(x) \cdot (f \circ h)(x), \quad x \in X$$

where $h : X \to Y$ is a homeomorphism and $a : X \to \mathbb{R}$ is a continuous function with $|a(x)| = 1$, for every $x \in X$.

One of the lines of research which developed from the Banach-Stone theorem, is the study of isometries between different Banach function spaces, and how the existence of one such isometry leads into topological connections between the base-spaces or related subspaces. In this way, a large number of extensions of the Theorem have been obtained, and in this Section we are going to review some of them, without proofs. The earliest result in this direction is due to Myers [52], who considered special vector subspaces of $C(X)$, for a compact space X. This has been extended by Araujo and Font [6] to the same kind of subspaces of $C_0(X)$, where X is a locally compact space. More precisely, a closed vector subspace $E \subset C_0(X)$ is said to be *completely regular* if, for each $x_0 \in X$ and each neighborhood U of x_0 in X, there exists some $f \in E$ such that $\|f\| = |f(x_0)|$ and $|f(x)| < \|f\|$ for every $x \in X \setminus U$.

Theorem 7. (Myers 1948, Araujo and Font 1997) *Let X and Y be locally compact spaces, and let $E \subset C_0(X)$, $F \subset C_0(Y)$ be completely regular subspaces. If E and F are isometric, then X and Y are homeomorphic. Moreover, every linear isometry $T : F \to E$ is of the form*

$$(Tf)(x) = a(x) \cdot (f \circ h)(x), \quad x \in X$$

where $h : X \to Y$ is a homeomorphism and $a : X \to \mathbb{R}$ is a continuous function with $|a(x)| = 1$, for every $x \in X$.

For further extensions along this line, we need the concept of boundary of a subspace. Recall that a subset Ω of a locally compact space X is said to be a *boundary* of a vector subspace E of $C_0(X)$ if, for each $f \in E$, there exists some $x \in \Omega$ with $|f(x)| = \|f\|$. When E admits a unique minimal closed boundary, it is called the *Shilov boundary* of E and is denoted by ∂E. On the other hand, the *Choquet boundary* of E, denoted $Ch(E)$, is defined as the set of all $x \in X$ such that δ_x is an extreme point of the unit ball of E^*. Now, modifying slightly the terminology of Araujo and Font [6], we say that the vector subspace $E \subset C_0(X)$ is *strongly separating* if the following two conditions are satisfied:

(i) For each $x_1 \neq x_2 \in X$, there exits $f \in E$ such that $|f(x_1)| \neq |f(x_2)|$.

(ii) For each $x \in X$, there exists $f \in E$ such that $f(x) \neq 0$.

In this case, we define σE as the set of all $x_0 \in X$ such that, for each neighborhood U of x_0 in X, there exists $f \in E$ with $|f(x)| < \|f\|$, for every $x \in X \setminus U$. The connection with Shilov and Choquet boundaries is given in Araujo and Font [5]:

THEOREM 8. (Araujo and Font 1997) *Let X be a locally compact space, and let $E \subset C_0(X)$ a strongly separating subspace. Then*

(1) $\sigma E = \partial E$.

(2) $\sigma E = \overline{ChE}$.

Note that if the subspace $E \subset C_0(X)$ is completely regular, then it is strongly separating and $\sigma E = X$. In particular, the following result from [6] extends Theorem 7.

THEOREM 9. (Araujo and Font 1997) *Let X and Y be locally compact spaces, let $E \subset C_0(X), F \subset C_0(Y)$ be strongly separating subspaces, and let $T : F \to E$ be a linear isometry. Then there exist a homeomorphism $h : \sigma E \to \sigma F$ and a continuous function $a : \sigma E \to \mathbb{R}$ with $|a(x)| = 1$, for every $x \in \sigma E$, such that*

$$(Tf)(x) = a(x) \cdot (f \circ h)(x),$$

for every $x \in \sigma E$ and every $f \in F$. Moreover, $h(ChE) = Ch(F)$.

In the hypotheses of the above Theorem we have that, if the vector subspaces E and F are isometric, then the Shilov boundary ∂E is homeomorphic to ∂F, and also the Choquet boundary ChE is homeomorphic to ChF. Nevertheless, we cannot deduce in general that X and Y are homeomorphic, as the following example shows (see Araujo and Font [6]). Let Y be the open real interval $(0,1)$, let $X = (0,1) \cup (1,2)$ and consider $T : C_0(Y) \to C_0(X)$ defined by:

$$(Tf)(t) = \begin{cases} f(t) & \text{if} \quad t \in (0,1) \\ \frac{1}{2} f(t-1) & \text{if} \quad t \in (1,2). \end{cases}$$

Now if $E = C_0(Y)$ and $F = T(E)$, then $\sigma E = (0,1) = \sigma F$, and T is an isometry from E onto F, but X is not homeomorphic to Y.

Into isometries. Also starting from the Banach-Stone theorem, several related results have been obtained for *into isometries* (that is, not necessarily surjective). In this sense, we would like to mention the theorem due to Holsztyński [38]:

THEOREM 10. (Holsztyński 1966) *Let X and Y be compact spaces, and let $T : C(Y) \to C(X)$ be an into linear isometry. Then there exist a closed* *subset X_0 of X, a continuous function h from X_0 onto Y and a continuous function $a : X_0 \to \mathbb{R}$ with $|a(x)| = 1$, for every $x \in X_0$, such that*

$$(Tf)(x) = a(x) \cdot (f \circ h)(x),$$

for every $x \in X_0$ and every $f \in C(Y)$.

Now the following extension has been obtained in Araujo and Font [6].

Theorem 11. (Araujo and Font 1997) *Let X and Y be locally compact spaces, let $F \subset C_0(Y)$ be a strongly separating subspace, and let $T : F \to C_0(X)$ be an into linear isometry. Then there exist a subset X_0 of X which is a boundary for $T(F)$, a continuous function h from X_0 onto σF, and a continuous function $a : X_0 \to \mathbb{R}$ with $|a(x)| = 1$, for every $x \in X_0$, such that*

$$(Tf)(x) = a(x) \cdot (f \circ h)(x),$$

for every $x \in X_0$ and every $f \in F$. Moreover, $h(ChT(F)) = Ch(F)$. In addition, if σF is closed then X_0 is also closed.

Approximate isometries. Next we pass to a different kind of variant of the Banach-Stone theorem, by considering approximate isometries. If X and Y are locally compact spaces, and we weaken the connection between $C_0(X)$ and $C_0(Y)$, for example asking them to be merely isomorphic, then X and Y need not be homeomorphic. Indeed, Milutin [50] proved that if X and Y are both uncountable compact metric spaces, then $C(X)$ and $C(Y)$ are always linearly isomorphic. Thus, for example, $C[0, 1]$ is linearly isomorphic to $C([0, 1] \times [0, 1])$. Nevertheless, the situation is different if we consider isomorphisms with certain bound on the norm, also called *approximate isometries*. In this sense, the following result was obtained independently by Amir [1] and Cambern [19], [20].

Theorem 12. (Amir 1966, Cambern 1966) *Let X and Y be locally compact spaces, and assume that there exists a linear isomorphism $T : C_0(X) \to C_0(Y)$ with $\|T\| \cdot \|T^{-1}\| < 2$. Then X and Y are homeomorphic.*

The following example, due to Cambern [21], shows that, in general, 2 is the best constant. Let $X = \{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and

$Y = \{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \cup \{n : n \in \mathbb{N}\}$, and consider the isomorphism $T : C_0(Y) \to C_0(X) = C(X)$ defined by:

$$Tg(x) = \begin{cases} g(0) & \text{if} \quad x = 0, \\ g(-1/n) + g(n) & \text{if} \quad x = 1/n, \\ g(-1/n) - g(n) & \text{if} \quad x = -1/n. \end{cases}$$

In this case we have $\|T\| \cdot \|T^{-1}\| = 2$, but obviously X is not homeomorphic to Y.

The case of approximate isometries between subspaces was considered by Cengiz [22], extending the above result. He defined a closed vector subspace E of $C_0(X)$ to be *extremely regular* if, for each $x_0 \in X$, each neighborhood U of x_0 in X and each $0 < \varepsilon < 1$, there exists some $f \in E$ such that $\|f\| = |f(x_0)| = 1$ and $|f(x)| < \varepsilon$ for every $x \in X \setminus U$.

Theorem 13. (Cengiz 1973) *Let X and Y be locally compact spaces and let $E \subset C_0(X)$ and $F \subset C_0(Y)$ be extremely regular subspaces. Assume that there exists a linear isomorphism $T : E \to F$ with $\|T\| \cdot \|T^{-1}\| < 2$. Then X and Y are homeomorphic.*

Spaces of vector-valued functions. Another line of research motivated by the Banach-Stone theorem has been the study of similar problems for spaces of vector-valued functions. There is a large number of results in this direction, which naturally connects with the Geometry of Banach Spaces and Operator Theory. We refer to the book by Behrends [12] for a survey in this area. Just to give the flavor, we would like to mention some basic results. For a locally compact space X and a Banach space E, we denote as usual by $C_0(X, E)$ the Banach space of all continuous functions on X with values in E, vanishing at infinity, endowed with the sup-norm:

$$\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}.$$

We say that the Banach space E has the *Banach-Stone property* if, for any locally compact spaces X and Y, the existence of an isometry from $C_0(X, E)$ onto $C_0(Y, E)$ implies that X and Y are homeomorphic. In this way, the classical Banach-Stone theorem asserts that the real line $\mathbb{R}$ has the Banach-Stone property. On the other hand, both the bidimensional space $E = (\mathbb{R}^2, \|\cdot\|_\infty)$ and the space $E = (C[0,1], \|\cdot\|_\infty)$ fail the Banach-Stone property (see,

e. g. Behrends [12], pg. 143). To give positive results, we need the following definitions. A linear operator $S : E \to E$ is said to be a *multiplier* if every $x^* \in ext(B_{E^*})$ is an eigenvector of the adjoint S^*, that is, $S^*(x^*) = a_S(x^*) \cdot x^*$, where a_S is a scalar-valued function on $ext(B_{E^*})$. The algebra of all multipliers of E is denoted by $Mult(E)$. This concept can also be defined for a complex Banach space E, in which case the *centralizer* of E, denoted by $Z(E)$, is defined as the maximal self-adjoint subalgebra of $Mult(E)$. For a real Banach space E the centralizer $Z(E)$ coincides with $Mult(E)$. We say that $Z(E)$ *is trivial* if it contains only multiples of the identity. Now we can state the following result, due to Behrends [12].

THEOREM 14. (Behrends 1978) *A Banach space with trivial centralizer has the Banach-Stone property.*

The class of Banach spaces with trivial centralizer includes all strictly convex spaces and all smooth spaces (see Behrends [12]), so the above Theorem covers a wide range of spaces.

As a variant, we say that the Banach space E has the *isomorphic Banach-Stone property* if there exists some $\varepsilon > 0$ such that, for any locally compact spaces X and Y, the existence of a linear isomorphism $T : C_0(X, E) \to C_0(Y, E)$ with $\|T\| \cdot \|T^{-1}\| < 1 + \varepsilon$, implies that X and Y are homeomorphic. It was shown by Behrends and Cambern [13], and by Jarosz [40] respectively, that uniformly smooth and uniformly convex spaces have the isomorphic Banach-Stone property. For further developments in this direction and related results, we refer to the survey by Jarosz and Pathak [41] and references therein.

Bounded uniformly continuous and Lipschitz functions. We would like to recall now some results concerning isometries between spaces of bounded continuous and uniformly continuous functions on a complete metric space. More precisely, if X is a metric space and E is a Banach space, let $C^*(X, E)$ (respectively $U^*(X, E)$) denote the space of all bounded continuous (resp. uniformly continuous) functions from X to E endowed with the sup-norm. Using techniques of differentiability, Bachir has recently obtained in [10] the following.

THEOREM 15. (Bachir 2001) *Let X, Y be complete metric spaces, and let E, F be smooth Banach spaces. If $C^*(X, E)$ is isometric to $C^*(Y, F)$ (respectively, $U^*(X, E)$ is isometric to $U^*(Y, F)$), then X and Y are homeomorphic. Moreover, every linear isometry $T : C^*(Y, F) \to C^*(X, E)$ (respectively,*

$T : U^*(X,E) \to U^*(Y,F)$) is of the form

$$(Tf)(x) = U(x) \cdot (f \circ h)(x), \quad x \in X$$

where $h : X \to Y$ is a homeomorphism and $U : X \to Isom(F,E)$ is a continuous map from X into the space of linear isometries between F and E.

In fact, in the case of uniformly continuous functions, we get a closer connection between X and Y. Namely, the following result has been proved by Araujo in [3]. The scalar-valued case has been independently obtained by Hernández in [36].

Theorem 16. (Araujo 2001) *Let X, Y be complete metric spaces, and let E, F be Banach spaces with trivial centralizer. If $U^*(X,E)$ is isometric to $U^*(Y,F)$, then X and Y are uniformly homeomorphic. Moreover, every linear isometry $T : U^*(X,E) \to U^*(Y,F)$ is of the form*

$$(Tf)(x) = U(x) \cdot (f \circ h)(x), \quad x \in X$$

where $h : X \to Y$ is a uniform homeomorphism and $U : X \to Isom(F,E)$ is a continuous map from X into the space of linear isometries between F and E.

On the other hand, let $Lip^*(X,E)$ denote the space of all bounded Lipschitz functions from a complete metric space (X,d) into a Banach space E, endowed with the norm $\|f\|_{Lip^*} = \sup\{\|f\|_\infty, L(f)\}$, where $L(f)$ is the Lipschitz constant of f:

$$L(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x,y)}.$$

Furthermore, if X and E are Banach spaces, let $C_b^m(X,E)$ denote the space of m-times continuously Fréchet-differentiable functions from X to E, such that the function and its m derivatives are bounded on X, endowed with the norm

$$\|f\|_m = \sup\{\|f\|_\infty, \|f'\|_\infty, \dots, \|f^{(m)}\|_\infty\}.$$

When we consider this space, we always assume that there exists in $C_b^m(X,\mathbb{R})$ a *bump function*, that is, a function with bounded nonempty support. In [9] and [10] Bachir develops a general treatment which can also be applied to these function spaces. In particular he obtains that, if E and F have a Fréchet differentiable norm, then a linear isomorphism $T : Lip^*(Y,F) \to Lip^*(X,E)$

(respectively, $T : C_b^m(Y,F) \to C_b^m(X,E)$) is of the canonical form $Tf = U \cdot (f \circ h)$ if, and only if, T is an isometry for the sup-norm.

Finally, concerning isometries between spaces of real bounded Lipschitz functions, we have the following result due to Weaver [61]. First, recall that a metric space is said to be *1-connected* if it cannot be decomposed into two disjoint sets whose distance is ≥ 1.

THEOREM 17. (Weaver 1995) *Let X, Y be 1-connected complete metric spaces with diameter ≤ 2. If $Lip^*(X)$ is isometric to $Lip^*(Y)$, then X and Y are isometric. Moreover, every linear isometry $T : Lip^*(Y) \to Lip^*(X)$ is of the form*

$$(Tf)(x) = a \cdot (f \circ h)(x), \quad x \in X$$

where $h : X \to Y$ is an isometry and $|a| = 1$.

4. Algebra isomorphisms

Another line of research which started with the Banach-Stone theorem stresses the link between algebraic properties of $C(X)$ and the topology of X. Here X will be a (Hausdorff) completely regular space and we shall consider on $C(X)$ its algebra structure or (equivalently in this context) its ring structure. The first topological versus algebraic result was given by Gelfand and Kolmogoroff [33] in 1939 for compact spaces. For that, they considered the space of maximal ideals of $C(X)$ endowed with the Stone topology. Incidentally, this theorem follows from the Banach-Stone theorem and the fact that algebra isomorphism implies isometry (see Gillman and Jerison [34] 1J.6). Nevertheless their ideas were used by Hewitt [37] in order to obtain his well known generalization for the class of realcompact spaces (Theorem 20 below).

THEOREM 18. (Gelfand and Kolmogoroff 1939) *Let X and Y be compact spaces. Then, $C(X)$ and $C(Y)$ are isomorphic as algebras if, and only if, X and Y are homeomorphic. Moreover, every algebra isomorphism $T : C(Y) \to C(X)$ is of the form $Tf = f \circ h$ where $h : X \to Y$ is a homeomorphism.*

Before doing the proof, we need to establish a useful Lemma due to Stone [60] (see also Dunford and Schwartz [24]). We shall denote by $Z(f) = \{x : f(x) \neq 0\}$ the zero-set of the function $f \in C(X)$.

Lemma 19. (Stone 1937) *Let X be a compact space. For every nonzero multiplicative functional $\varphi : C(X) \to \mathbb{R}$ there exists a unique $x \in X$ such that $\varphi = \delta_x$, that is, $\varphi(f) = f(x)$, for all $f \in C(X)$.*

Proof. Firstly we shall see that $\varphi(1) = 1$. Indeed, since $\varphi(1) = \varphi(1 \cdot 1) = \varphi(1) \cdot \varphi(1)$ and $\varphi \neq 0$, then $\varphi(1) = 1$. As consequence $\varphi(\lambda) = \lambda$, for every $\lambda \in \mathbb{R}$. Moreover $\varphi(g) \neq 0$ when $Z(g) = \emptyset$, since in this case $1/g \in C(X)$ and $1 = \varphi(1) = \varphi(g) \cdot \varphi(1/g)$.

On the other hand, for every $f \in C(X)$ there exists $x \in X$ such that $\varphi(f) = f(x)$. Otherwise, for some function $f \in C(X)$ we have $g(x) = f(x) - \varphi(f) \neq 0$ for all $x \in X$. Thus $Z(g) = \emptyset$ and $\varphi(g) = 0$, which is a contradiction.

Now it is easy to check that the family $\{Z(f - \varphi(f))\}_{f \in C(X)}$ of closed subsets of X has the finite intersection property. So, by compactness, there exists $x \in X$ with $x \in \bigcap Z(f - \varphi(f))$, i.e. $\varphi = \delta_x$. Finally, note that this x is unique since $C(X)$ separates the points of X. ∎

Proof of Gelfand-Kolmogoroff's theorem. It is clear that when $h : X \to Y$ is a homeomorphism, then $T : C(Y) \to C(X)$ defined by $Tf = f \circ h$ is an algebra isomorphism.

Conversely, if $T : C(Y) \to C(X)$ is an algebra isomorphism then, for each $x \in X$, $\delta_x \circ T : C(Y) \to \mathbb{R}$ is a nonzero multiplicative functional. So, by Lemma 19, there exists a unique $y = h(x) \in Y$ such that $\delta_x \circ T = \delta_{h(x)}$, i.e. $Tf(x) = f(h(x))$, for every $f \in C(Y)$. Thus, the map $h : X \to Y$ satisfies $Tf = f \circ h \in C(X)$, for every $f \in C(Y)$. Hence h is continuous, since $C(Y)$ separates points and closed subsets of Y and this implies that Y is endowed with the weak topology generated by $C(Y)$. Finally, considering T^{-1} we get that h is a homeomorphism. ∎

Note that the above proof relies on the following two key facts:
(i) The topology of a compact space X is the weak topology given by $C(X)$.
(ii) Every nonzero algebra homomorphism $\varphi : C(X) \to \mathbb{R}$ (X compact) is an evaluation.

In fact the same proof works in the class of spaces X satisfying these two conditions. This was observed by Hewitt in [37], where he defined a topological space X to be *realcompact* if X is completely regular and every nonzero real algebra homomorphism on $C(X)$ is given by evaluation at some point of X. In this way he obtained the following theorem.

Theorem 20. (Hewitt 1948) *Let X and Y be realcompact spaces. Then, $C(X)$ and $C(Y)$ are isomorphic as algebras if, and only if, X and Y are* *homeomorphic. Moreover, every algebra isomorphism $T : C(Y) \to C(X)$ is of the form $Tf = f \circ h$ where $h : X \to Y$ is a homeomorphism.*

It is clear that every compact space is realcompact. More generally, it is not difficult to see that every Lindelöf space is realcompact. For further information on realcompactness we refer to Gillman and Jerison [34] and Engelking [26]. On the other hand, we would like to point out that realcompactness of a metric space is a set-theoretic notion. In fact, a metric space is realcompact if, and only if, every discrete closed subset has nonmeasurable cardinal. Recall that a set J is said to have *measurable cardinal* if there exists a non-trivial two-valued measure defined on the power set 2^J. We refer to Jech [42] for further information about measurable cardinals. Just note that the existence of measurable cardinals is not provable from the usual axioms of set theory (ZFC). These cardinals, whether they exist at all, must be extremely big; in fact, bigger than $\aleph_0$, $2^{\aleph_0}$, $2^{2^{\aleph_0}}$, ...

Isomorphisms between subalgebras. Now, we shall see how to generalize last result by Hewitt for unital subalgebras A of $C(X)$. We denote by $Hom(A)$ the set of all nonzero algebra homomorphisms $\varphi : A \to \mathbb{R}$. We say that $Hom(A) = X$ whenever for every $\varphi \in Hom(A)$ there exists some $x \in X$, such that $\varphi = \delta_x$. Note that this x will be unique when A separates the points of X. Thus, with the same proof of Theorem 18 or Theorem 20, the following general result can be obtained.

THEOREM 21. *Let X and Y be completely regular spaces. Let $A \subset C(X)$ and $B \subset C(Y)$ unital subalgebras, such that:*

(i) *A separates points and closed sets of X and $Hom(A) = X$.*

(ii) *B separates points and closed sets of Y and $Hom(B) = Y$.*

If A and B are isomorphic as algebras, then X and Y are homeomorphic. Moreover, every algebra isomorphism $T : B \to A$ is of the form $Tf = f \circ h$ where $h : X \to Y$ is a homeomorphism.

Here, the problem arises to determine conditions under which $Hom(A) = X$. This has been studied by several authors, mainly in the case of algebras of smooth functions, see for instance Kriegl and Michor [45] (Chapter IV) or Garrido, Gómez and Jaramillo [28], and references therein. One of the more general results in this respect is Theorem 23 below. First we need an extension of Lemma 19 for unital subalgebras which are inverse-closed. Recall that a subalgebra $A \subset C(X)$ is said to be *inverse-closed* if for every $f \in A$ such that

$Z(f) = \emptyset$, the function $1/f \in A$. As is customary, βX denotes the Stone-Čech compactification of X and $f^\beta : \beta X \to \mathbb{R} \cup \{\infty\}$ the continuous extension of $f \in C(X)$.

Lemma 22. (Garrido, Gómez and Jaramillo 1994) *Let X be a completely regular space, and let A be a unital inverse-closed subalgebra of $C(X)$. Then for every $\varphi \in Hom(A)$ there exists some $\xi \in \beta X$ such that $f^\beta(\xi) \neq \infty$ and $\varphi(f) = f^\beta(\xi)$, for every $f \in A$.*

Note that the above Lemma, whose proof can be seen in [28], describes every nonzero homomorphism on A as an evaluation at some point of βX. As a consequence, the condition $Hom(A) = X$, also called $A-$*realcompactness*, can be characterized as follows.

Theorem 23. (Garrido, Gómez and Jaramillo 1994) *Let X be a completely regular space and let $A \subset C(X)$ a unital inverse-closed subalgebra that separates points and closed subsets of X. Then, the following conditions are equivalent:*

(a) $Hom(A) = X$.

(b) *For every $\xi \in \beta X \setminus X$ there is a function $f \in A$ such that $f^\beta(\xi) = \infty$.*

This characterization looks quite abstract, but it can be checked directly in many cases. For example, if $A \subset C(\mathbb{R}^n)$ is any unital inverse-closed subalgebra containing the projection maps $\pi_j : \mathbb{R}^n \to \mathbb{R}$, for $j = 1, \ldots, n$, then it is easy to see that A separates points and closed sets, and it follows that $\mathbb{R}^n$ is $A-$realcompact. Indeed, in this case the function $f = \pi_1^2 + \cdots + \pi_n^2 \in A$ satisfies $f^\beta(\xi) = \infty$, for every $\xi \in \beta\mathbb{R}^n \setminus \mathbb{R}^n$. In particular, A could be the algebra of all rational functions, or all real-analytic functions, or all $C^\infty-$functions on $\mathbb{R}^n$.

Another consequence of Theorem 23 is the following.

Corollary 24. (Garrido, Gómez and Jaramillo 1994) *Let X be a realcompact space and let $A \subset C(X)$ a unital inverse-closed subalgebra. If A is uniformly dense in $C(X)$ then $Hom(A) = X$.*

This can be applied to the algebra $C^\infty(M)$ of smooth real functions on a smooth manifold M of finite dimension (which will be assumed to be Hausdorff and second countable). Indeed, in this case, M is realcompact (since it is Lindelöf) and $C^\infty(M)$ is uniformly dense in $C(M)$. Then, from Corollary 24, we have $Hom(C^\infty(M)) = M$. Now we can derive the following result, which is due to Myers [53] for compact manifolds, and Pursell [56] for the general case.

THEOREM 25. (Myers 1954 and Pursell 1955) *Let M and N be smooth manifolds of finite dimension. Then $C^\infty(M)$ and $C^\infty(N)$ are isomorphic as algebras if, and only if, M and N are $C^\infty-$diffeomorphic.*

Proof. Note that if $T: C^\infty(N) \to C^\infty(M)$ is an algebra isomorphism we obtain, from Theorem 21, a homeomorphism $h: M \to N$ such that $Tf = f \circ h$. Now, we only need to use the fact that if $f \circ h$ is $C^\infty-$smooth for every $f \in C^\infty(N)$, then h is $C^\infty-$smooth. ∎

Functions on Banach spaces. Next, we turn our attention to the case of algebras of smooth functions on a Banach space E. As usual, $P(E)$ denotes the algebra of all continuous real polynomials on E, and $R(E)$ the algebra of all rational functions of E, that is, the functions of the form P/Q where $P, Q \in P(E)$ and $Q(x) \neq 0$ for all $x \in E$. Recall that a real function f defined on an open set $U \subset E$ is said to be real-analytic on U if, for every $x \in U$ there exist a neighborhood W of 0 and a sequence (P_j), with each $P_j \in P(E)$ being j-homogeneous, such that $f(x+h) = \sum_{j=0}^{\infty} P_j(h)$, for every $h \in W$. Note that this series can be assumed to be uniformly convergent on W (see Bochnak and Siciak [15], Propositions 5.1 and 5.2, or Kriegl and Michor [45], Lemma 7.14). We denote by $C^\omega(U)$ the algebra of all real-analytic functions on U. It is clear that $R(E) \subset C^\omega(E) \subset C^\infty(E)$, where $C^\infty(E)$ is the algebra of infinitely differentiable real functions on E, in the usual Fréchet sense.

The following result, due to Biström and Lindström [14], provides a general condition under which every homomorphism on $C^\infty(E)$ is given by evaluation at some point of E.

THEOREM 26. (Biström and Lidström 1993) *Let E be a Banach space which injects linear and continuously into $c_0(\Gamma)$, for some index set Γ with nonmeasurable cardinal. Then $Hom(C^\infty(E)) = E$.*

As a consequence, and using Theorem 21, we obtain:

THEOREM 27. (Garrido, Jaramillo and Prieto 2000) *Let E and F be Banach spaces which inject linear and continuously into $c_0(\Gamma)$, for some index set Γ with nonmeasurable cardinal. The following are equivalent:*

(a) *The algebras $C^\infty(E)$ and $C^\infty(F)$ are isomorphic.*

(b) *The spaces E and F are isomorphic.*

Proof. Again, as in Theorem 21, if $T : C^\infty(F) \to C^\infty(E)$ is an algebra isomorphism we obtain a bijection $h : E \to F$ such that $Tf = f \circ h \in C^\infty(E)$, for every $f \in C^\infty(F)$. In particular, for every $x^* \in F^*$, we have that $x^* \circ h \in C^\infty(E)$, and therefore by [35], Corollary 3.8, h is $C^\infty-$smooth. Since the same is true for h^{-1}, h is a diffeomorphism. Hence, the differential of h at every point is an isomorphism between E and F. ∎

This Theorem can be applied to a wide class of spaces. Recall that a Banach space E is said to be Weakly Compactly Generated (WCG) if there is a weakly compact subset whose linear span is dense in E. In particular, every separable and every reflexive space are WCG. Now, if either E or E^* is WCG (with nonmeasurable cardinal), then E injects linear and continuously into $c_0(\Gamma)$, for some Γ (with nonmeasurable cardinal) (see e.g. Deville, Godefroy and Zizler [23], p. 247). For further spaces satisfying the hypothesis of Theorem 27, we refer to Kriegl and Michor [45], 53.20 and 53.21.

In the case of algebras of rational and real-analytic functions, we have the following.

Theorem 28. (Garrido, Gómez and Jaramillo 1994) *Let E be a Banach space which inject linear and continuously into $\ell_p(\Gamma)$, for some $1 < p < \infty$, and some index set Γ with nonmeasurable cardinal. Then $Hom(R(E)) = Hom(C^\omega(E)) = Hom(C^\infty(E)) = E$.*

And as a consequence,

Theorem 29. (Garrido, Jaramillo and Prieto 2000) *Let E and F be Banach spaces which inject linear and continuously into $\ell_p(\Gamma)$, for some $1 < p < \infty$ and some index set Γ with nonmeasurable cardinal. The following are equivalent:*

(a) *The algebras $C^\infty(E)$ and $C^\infty(F)$ are isomorphic.*
(b) *The algebras $C^\omega(E)$ and $C^\omega(F)$ are isomorphic.*
(c) *The algebras $R(E)$ and $R(F)$ are isomorphic.*
(d) *The spaces E and F are isomorphic.*

We point out that the class of spaces satisfying the hypothesis in Theorem above includes separable spaces and superreflexive spaces with nonmeasurable cardinal (see [43]). On the other hand, the equivalence between (c) and (d) should be compared with the results by Cabello, Castillo and García in [18] (see also Lassalle and Zalduendo [48]). They provide examples of separable non-isomorphic spaces E and F for which the algebras $P(E)$ and $P(F)$ do verify to be isomorphic. For instance, this is the case if $E = c_0$ and F is another predual of ℓ_1, not isomorphic to c_0.

Structure space. Our previous results are based on Theorem 21, and therefore depend on the fact that the homomorphisms of the corresponding subalgebras are given by evaluation at some point of the base-space. In the case that this is not longer true, or when we do not know whether it is true, we can consider the so-called structure space associated to a subalgebra. If X is a completely regular space and $A \subset C(X)$ is a subalgebra with unit, then the set $Hom(A)$ endowed with the topology of pointwise convergence (that is, considered as a topological subspace of $\mathbb{R}^A$ with the product topology) is called the *structure space* of A. The following fundamental properties of the structure space were obtained by Isbell [39].

THEOREM 30. (Isbell 1958) *Let X be a completely regular space, and let $A \subset C(X)$ a unital inverse-closed subalgebra, which separates points and closed sets of X. Then the map $\delta : x \in X \rightsquigarrow \delta_x \in Hom(A)$ is a topological embedding from X into a dense subspace of $Hom(A)$. Moreover, each function $f \in A$ admits a (unique) continuous extension $\hat{f}$ to $Hom(A)$, given by $\hat{f}(\varphi) = \varphi(f)$.*

Functions on Banach manifolds. Now, as an example of application of the structure space, we focus on smooth and real-analytic functions on Banach manifolds. We refer to Lang [47] or Kriegl and Michor [45] for an account on this topic. Recall that a Banach space E is said to be $C^\infty-$smooth if there exists a $C^\infty-$real function on E with nonempty bounded support. If M is a Banach manifold modeled on $C^\infty-$smooth Banach spaces, it is easy to see that $C^\infty(M)$ separates points and closed sets of M, that is, M carries the initial topology for the family $C^\infty(M)$ (see e.g. Bonic and Frampton [16]).

THEOREM 31. (Garrido, Jaramillo and Prieto 2000) *Let M and N be paracompact Banach manifolds, modeled on $C^\infty-$smooth Banach spaces. The following are equivalent:*

(a) *The algebras $C^\infty(M)$ and $C^\infty(N)$ are isomorphic.*
(b) *M and N are $C^\infty-$diffeomorphic.*

Proof. That (b) implies (a) is clear. To the converse, we consider the structure space $Hom(C^\infty(M))$. As we mentioned before, $C^\infty(M)$ separates points and closed sets of M, and then, by Theorem 30, M can be embedded as a dense subspace of $Hom(C^\infty(M))$ by identifying each $x \in M$ with the point evaluation δ_x. Moreover, each $f \in C^\infty(M)$ admits a continuous extension $\hat{f}$ to $Hom(C^\infty(M))$, given by $\hat{f}(\varphi) = \varphi(f)$, for each $\varphi \in Hom(C^\infty(M))$. The same can be said for N.

Now assume that $T : C^\infty(N) \to C^\infty(M)$ is an algebra isomorphism. Then the map $h : Hom(C^\infty(M)) \to Hom(C^\infty(N))$ defined by $h(\varphi) = T \circ \varphi$ is a bijection, and in fact h is a homeomorphism since $\pi_g \circ h = \pi_{T(g)}$, for each $g \in C^\infty(N)$ (where π denotes the corresponding projection on the product space). Next we are going to see that $h(M) = N$. This will be a consequence of the following:

CLAIM. *A point $\varphi \in Hom(C^\infty(M))$ has a countable neighborhood basis in $Hom(C^\infty(M))$ if, and only if, $\varphi \in M$.*

In order to prove the Claim, first note that every paracompact Banach manifold is completely metrizable (see Palais [55], Theorem 3), so there exists a complete metric d on M which gives the topology.

Assume first that $\varphi \in Hom(C^\infty(M)) \setminus M$ has a countable neighborhood basis. Since M is dense in $Hom(C^\infty(M))$, we can choose a sequence (x_n) in M converging to φ. The completeness of the metric d implies that (x_n) has no d-Cauchy subsequence, and therefore there exist $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $d(x_{n_k}, x_{n_j}) \geq \varepsilon$ for $k \neq j$. For each k, consider the d-ball $B_k = B(x_{n_k}, \frac{\varepsilon}{4})$, and a function $f_k \in C^\infty(M)$ such that the support of f_k is contained in B_k and $f_k(x_{n_k}) = 1$. Note that for every $x \in M$ the d-ball $B(x, \frac{\varepsilon}{4})$ meets at most one B_k. Thus the sum $f = \sum_{k=0}^{\infty} f_{2k}$ is locally finite and $f \in C^\infty(M)$. In addition, $f(x_{n_{2k}}) = 1$ and $f(x_{n_{2k+1}}) = 0$ for every k. But this is a contradiction since f extends continuously to $Hom(C^\infty(M))$ and φ is in the closure of the sets $A = \{x_{n_{2k}}\}$ and $B = \{x_{n_{2k+1}}\}$.

Conversely, given $x \in M$, consider for each n the open d-ball $B_n = B(x, \frac{1}{n})$, and let W_n be the closure of B_n in $Hom(C^\infty(M))$. It is easy to check that the family $\{W_n\}$ is a countable neighborhood basis of x in $Hom(C^\infty(M))$. This completes the Claim.

In this way we obtain that $h : M \to N$ is a homeomorphism and $T(g) = g \circ h \in C^\infty(M)$, for every $g \in C^\infty(N)$. This implies that h is $C^\infty-$smooth. Indeed, let $x_0 \in M$ and consider $\phi : U \to E$ and $\psi : V \to F$ charts around x_0 and $h(x_0)$, respectively. Using that F is a $C^\infty-$smooth space, we can find open neighborhoods $U_0 \subset U$ and $V_0 \subset V$, of x_0 and $h(x_0)$ respectively, and $\theta \in C^\infty(N)$ such that $h(U_0) \subset V_0$, $\theta = 1$ on V_0, and the support of θ is contained in V. In order to see that $\psi \circ h \circ \phi^{-1}$ is C^∞ on $\phi(U_0)$, it is enough to check that $v^* \circ \psi \circ h \circ \phi^{-1}$ is C^∞ for every functional $v^* \in F^*$ (see [35], Corollary 3.8). Now, given $v^* \in F$, define $g : V \to \mathbb{R}$ by

$$g(y) = \begin{cases} \theta(y) \cdot v^*(\psi(y)) & \text{if} \quad y \in V \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in C^\infty(N)$, $g \circ h \in C^\infty(M)$ and $v^* \circ \psi \circ h \circ \phi^{-1} = g \circ h \circ \phi^{-1}$ is C^∞ on $\phi(U_0)$.

Working with h^{-1} we obtain that $h : M \to N$ is a C^∞−diffeomorphism. ∎

In the real-analytic setting, we restrict ourselves to submanifolds of certain Banach spaces. In this context we give the following two results whose proofs can be seen in [32].

Theorem 32. (Garrido, Jaramillo and Prieto 2000) *Let E and F be Banach spaces which admit a continuous linear injection into $\ell_p(\Gamma)$, for some $1 < p < \infty$ and some index set Γ with nonmeasurable cardinal. Let M and N be real-analytic submanifolds of E and F, respectively. The following are equivalent:*

(a) *The algebras $C^\omega(M)$ and $C^\omega(N)$ are isomorphic.*

(b) *M and N are real-analytically isomorphic.*

Next proposition reveals that Theorem 32 does not hold for arbitrary Banach manifolds. In particular, this shows that there is no general real-analytic Banach–Stone theorem.

Proposition 33. (Garrido, Jaramillo and Prieto 2000) *Consider the Banach space $c_0(\Gamma)$, where Γ is an uncountable set. Let $M = c_0(\Gamma)$ and $N = c_0(\Gamma) \setminus \{0\}$. Then,*

(i) *The algebras $C^\omega(M)$ and $C^\omega(N)$ are isomorphic.*

(ii) *M and N are not real-analytically isomorphic.*

Biseparating maps. Some of the results of Banach-Stone type for algebra isomorphisms, which we have seen at the beginning of this Section, can be extended to the more general context of biseparating maps between function spaces. Recall that a map $T : C(Y) \to C(X)$ is said to be *separating* if it is additive and $f \cdot g = 0$ implies that $(Tf) \cdot (Tg) = 0$. Then T is said to be *biseparating* if it is bijective and both T and T^{-1} are separating. Of course, every algebra isomorphism is biseparating, but the converse is not true. For example, every map $T : C(Y) \to C(X)$ of the form $(Tf)(x) = a(x) \cdot (f \circ h)(x)$, where $h : X \to Y$ is a homeomorphism and $a : X \to \mathbb{R}$ is continuous with $a(x) \neq 0$ for every $x \in X$, is biseparating. Now we can state the following result of Araujo, Beckenstein and Narici [4].

Theorem 34. (Araujo, Beckenstein and Narici 1995) *Let X and Y be realcompact spaces. If there exists a biseparating map $T : C(Y) \to C(X)$, then X and Y are homeomorphic.*

If, in addition, we assume that the biseparating map is linear, something stronger is obtained in [4].

Theorem 35. (Araujo, Beckenstein and Narici 1995) *Let X and Y be realcompact spaces, and assume that there exists a linear biseparating map $T : C(Y) \to C(X)$. Then X and Y are homeomorphic and T is of the form*

$$(Tf)(x) = a(x) \cdot (f \circ h)(x), \quad x \in X$$

where $h : X \to Y$ is a homeomorphism and $a : X \to \mathbb{R}$ is a continuous function with $a(x) \neq 0$ for every $x \in X$.

These results have been recently extended by Araujo in [2] and [3] to the case of spaces of vector-valued functions. First, it is necessary to extend the concept of biseparating map to this case. Thus, for a function $f \in C(X, E)$, where X is a topological space and E is a Banach space, we denote by $coz(f)$ the *cozero-set* of f, that is, $coz(f) = \{x \in X : f(x) \neq 0\}$. Now if Y is also a topological space and F a Banach space, we say that a map $T : C(Y, F) \to C(X, E)$ is *separating* if it is additive and $coz(f) \cap coz(g) = \emptyset$ implies that $coz(Tf) \cap coz(Tg) = \emptyset$. As before, we say that T is biseparating if it is bijective and both T and T^{-1} are separating. Then the following is obtained in [2].

Theorem 36. (Araujo 2002) *Let X, Y be realcompact spaces, and let E, F be Banach spaces. If there exists a biseparating map $T : C(Y, F) \to C(X, E)$, then X and Y are homeomorphic.*

In order to obtain, as before, a representation of biseparating linear maps, we have to consider the set $b(F, E)$ of all (not necessarily continuous) bijective linear maps from F to E. Then the following is proved in [3].

THEOREM 37. (Araujo 2002) *Let X, Y be realcompact spaces, let E, F be Banach spaces, and assume that there exists a linear biseparating map $T : C(Y, F) \to C(X, E)$. Then X and Y are homeomorphic and T is of the form*

$$(Tf)(x) = a(x) \cdot (f \circ h)(x), \quad x \in X$$

where $h : X \to Y$ is a homeomorphism, and a is a map from X into $b(F, E)$.

5. Vector lattice isomorphisms

Following the line which connects algebraic properties of $C(X)$ with the topology of X, different algebraic structures on $C(X)$ have been considered, such as the lattice structure. The first result in this context was given by Kaplansky [44] who proved that the topology of a compact space X is determined by the lattice structure of $C(X)$. This was extended by Shirota [59] to the class of realcompact spaces.

In this Section, we shall be interested in the vector lattice structure. Our motivation is to characterize the uniform and the Lipschitz structure of a metric space X in terms of the families $U(X)$ and $Lip(X)$ of uniformly continuous functions and Lipschitz functions on X. For that reason we start with an arbitrary vector sublattice $\mathcal{L} \subset C(X)$, where X is a completely regular space, and we define its associated structure space. On the other hand, we point out that a general study for this kind of sublattices can be carried out much in the same way as in the case of subalgebras, and we refer to [30] and [31] for details.

Structure space. Let X be a completely regular space, and let $\mathcal{L}$ be a unital vector sublattice of $C(X)$ that separates points and closed subsets of X. We say that $\xi : \mathcal{L} \to \mathbb{R}$ is a *lattice homomorphism* whenever it satisfies:

(i) $\xi(\lambda f + \mu g) = \lambda \xi(f) + \mu \xi(g)$, for all f, $g \in \mathcal{L}$ and λ, $\mu \in \mathbb{R}$.

(ii) $\xi(|f|) = |\xi(f)|$, for all $f \in \mathcal{L}$.

(iii) $\xi(1) = 1$.

Thus, we define the *structure space of* $\mathcal{L}$ as the set $H(\mathcal{L})$ of all lattice homomorphisms on $\mathcal{L}$, considered as a topological subspace of $\mathbb{R}^{\mathcal{L}}$ endowed with the product topology. Now we have the following result which is analogous to Theorem 30.

Theorem 38. (Garrido and Jaramillo 2000) *Let X be a completely regular space, and let $\mathcal{L} \subset C(X)$ a unital vector sublattice which separates points and closed sets of X. Then the map $\delta : x \in X \rightsquigarrow \delta_x \in H(\mathcal{L})$ is a topological embedding from X into a dense subspace of $H(\mathcal{L})$. Moreover, each function $f \in \mathcal{L}$ admits a (unique) continuous extension $\hat{f}$ to $H(\mathcal{L})$, given by $\hat{f}(\xi) = \xi(f)$.*

Uniformly continuous functions. Let (X, d) be a metric space and let $\mathcal{L} \subset C(X)$ be a unital vector lattice. We say that $\mathcal{L}$ is *uniformly separating* if for every pair of subsets A and B of X with $d(A, B) > 0$ there exists some $f \in \mathcal{L}$ such that $\overline{f(A)} \cap \overline{f(B)} = \emptyset$. Typical examples are $\mathcal{L} = U(X)$, the lattice of all uniformly continuous real functions on X, as well as $\mathcal{L} = U^*(X)$ the lattice of bounded elements in $U(X)$.

Theorem 39. (Garrido and Jaramillo 2000) *Let (X, d_X) and (Y, d_Y) be complete metric spaces. Let $\mathcal{L}_X \subset U(X)$ and $\mathcal{L}_Y \subset U(Y)$ be unital vector lattices which are uniformly separating. If $\mathcal{L}_X$ is isomorphic to $\mathcal{L}_Y$ as unital vector lattices, then X is uniformly homeomorphic to Y.*

Proof. Assume that $T : \mathcal{L}_Y \to \mathcal{L}_X$ is an isomorphism of unital vector lattices. As usual, we define $h : H(\mathcal{L}_X) \to H(\mathcal{L}_Y)$ by $h(\xi) = \xi \circ T$, and it is clear that h is a homeomorphism. In a similar way as in the Claim of Theorem 31 (see also [29], Lemma 1), it is easy to see that a point $\xi \in H(\mathcal{L}_X)$ has a countable neighborhood basis in $H(\mathcal{L}_X)$ if, and only if, $\xi \in X$. The same is true for the points in $H(\mathcal{L}_Y)$, and therefore we obtain that $h(X) = Y$.

We are going to see that $h_{|X} : X \to Y$ is uniformly continuous. First note that

$$d_X(A, B) = 0 \Rightarrow d_Y(h_{|X}(A), h_{|X}(B)) = 0.$$

Otherwise, there exist $A, B \subset X$ such that $d_X(A, B) = 0$ and $d_Y(h_{|X}(A), h_{|X}(B)) > 0$. Since $\mathcal{L}_Y$ is uniformly separating, there exists $g \in \mathcal{L}_Y$ with $\overline{g(h_{|X}(A))} \cap \overline{g(h_{|X}(B))} = \emptyset$. But this is impossible because $g \circ h_{|X} = T(g) \in \mathcal{L}_X$ is a uniformly continuous function, and $d_X(A, B) = 0$.

Now applying a classical result due to Efremovich [25] (see also Engelking [26], pg. 573), it follows that $h_{|X}$ is uniformly continuous. The same holds for $(h_{|X})^{-1}$, and the proof is completed. ∎

From the above Theorem, it follows at once a Banach-Stone type result for uniformly continuous functions. An analogous result for the lattice $U^*(X)^+$, of nonnegative bounded uniformly continuous functions, was given by Nagata in [54].

Theorem 40. (Garrido and Jaramillo 2000) *Let X and Y be complete metric spaces. The following are equivalent:*

(i) *$U(X)$ is isomorphic to $U(Y)$ as unital vector lattices.*

(ii) *$U^*(X)$ is isomorphic to $U^*(Y)$ as unital vector lattices.*

(iii) *X is uniformly homeomorphic to Y.*

Note that if X is a metric space and $\tilde{X}$ denotes its completion, then both metric spaces have the same uniformly continuous real functions. Therefore, completeness cannot be avoided in the above results. On the other hand, note that Theorem 40 do not extend to the general class of uniform spaces. Indeed, let (X, d) be an uncountable discrete metric space, and let μ be the weak uniformity on X generated by the set of all real functions on X. It is easy to see that μ is not metrizable. On the other hand, if X has non-measurable cardinal then (X, μ) is complete (see e.g. Gillman-Jerison [34]). In this case, (X, d) and (X, μ) are complete uniform spaces which are not uniformly homeomorphic, but nevertheless they have the same uniformly continuous real functions.

Lipschitz functions. Recall that for a Lipschitz function between metric spaces $h : (X, d_X) \to (Y, d_Y)$ its *Lipschitz constant* $L(h)$ is defined by

$$L(h) = \sup_{x \neq y} \frac{d_Y(h(x), h(y))}{d_X(x, y)} < \infty.$$

If we now fix a point $x_0 \in X$, it is easy to see that $Lip\,(X)$ is a Banach space when endowed with the *Lipschitz norm* defined by

$$\|f\|_{Lip} = \sup\{|f(x_0)|, L(f)\},$$

for every $f \in Lip\,(X)$. Note that, if we change the base-point $x_0 \in X$ we obtain a different, but equivalent, norm in $Lip\,(X)$. In general, this norm is not compatible with the lattice structure, that is, $|f| \leq |g|$ does not imply that $\|f\|_{Lip} \leq \|g\|_{Lip}$. Nevertheless, we are going to see how the continuity properties of lattice homomorphisms with respect to the Lipschitz norm are the key to obtain a Banach-Stone type theorem for $Lip\,(X)$.

Theorem 41. (Garrido and Jaramillo 2001) *Let (X, d_X) and (Y, d_Y) be metric spaces. Then, every unital vector lattice homomorphism $T : Lip\,(Y) \to Lip\,(X)$ is automatically continuous for the respective Lipschitz norms.*

As a consequence of last Theorem, whose proof can be seen in [31], we obtain the following result concerning composition of Lipschitz maps, which is reminiscent of analogous results by Efremovich [25] or Lacruz and Llavona [46], for uniformly continuous functions. In the case of compact metric spaces, this was obtained by Sherbert in [27].

Theorem 42. (Garrido and Jaramillo 2001) *Let (X, d_X) and (Y, d_Y) be metric spaces, and let $h : X \to Y$. Assume that $f \circ h \in Lip\,(X)$ for each $f \in Lip\,(Y)$. Then, h is Lipschitz.*

Proof. First fix $x_0 \in X$ and $y_0 \in Y$ to define the Lipschitz norms on $Lip\,(X)$ and $Lip\,(Y)$, respectively. The map h induces, by composition, the unital vector lattice homomorphism $T : Lip\,(Y) \to Lip\,(X)$ given by $T(f) = f \circ h$, for every $f \in Lip\,(Y)$. Note that, by Theorem above, T is continuous. We shall see that h is K-Lipschitz, where $K = \|T\|$.

For $y_1, y_2 \in Y$, we have

$$d_Y(y_1, y_2) = \sup\left\{\frac{|f(y_1) - f(y_2)|}{L(f)} : f \in Lip\,(Y),\ L(f) \neq 0,\ f(y_0) = 0\right\}.$$

Indeed, if we choose $f = d_Y(\cdot, y_1) - d_Y(y_0, y_1)$, then $f \in Lip\,(Y)$, $L(f) = 1$ (if Y is not a singleton), $f(y_0) = 0$ and $|f(y_1) - f(y_2)| = d_Y(y_1, y_2)$. This shows that $d_Y(y_1, y_2)$ is not greater that the supremum on the right hand side. The converse inequality is clear.

Thus, for each $f \in Lip\,(Y)$ with $f(y_0) = 0$, we have $\|f\|_{Lip} = L(f)$. Now, if we denote $g = T(f) = f \circ h$ then, by the continuity of T, it follows that

$$L(g) \leq \|g\|_{Lip} \leq K \cdot \|f\|_{Lip} = K \cdot L(f).$$

In this way, for $x_1, x_2 \in X$ we obtain

$$\begin{aligned} d_Y(h(x_1), h(x_2)) &= \sup\left\{\frac{|f(h(x_1)) - f(h(x_2))|}{L(f)} : f \in Lip\,(Y),\ L(f) \neq 0,\ f(y_0) = 0\right\} \\ &\leq \sup\left\{K \cdot \frac{|g(x_1) - g(x_2)|}{L(g)} : g = f \circ h,\ f \in Lip\,(Y),\ L(g) \neq 0\right\} \\ &\leq K \cdot d_X(x_1, x_2). \end{aligned}$$

∎

Next we show that the vector lattice structure of $Lip\,(X)$ determines the Lipschitzian structure of a complete metric space X. Again, completeness is necessary here, since every metric space has the same Lipschitz functions as its completion. We say that two metric spaces X and Y are *Lipschitz homeomorphic* if there exists a bi-Lipschitz bijection $h : X \to Y$, that is, both h and h^{-1} are Lipschitz.

THEOREM 43. (Garrido and Jaramillo 2001) *Let (X, d_X) and (Y, d_Y) be complete metric spaces. Then, $Lip\,(X)$ is isomorphic to $Lip\,(Y)$ as unital vector lattices if, and only if, X is Lipschitz homeomorphic to Y. Moreover, every unital vector lattice isomorphism $T : Lip\,(Y) \to Lip\,(X)$ is of the form $T(f) = f \circ h$, where $h : X \to Y$ is a Lipschitz homeomorphism.*

Proof. As in Theorem 39, if $T : Lip\,(Y) \to Lip\,(X)$ is a unital vector lattice isomorphism we obtain a (uniform) homeomorphism $h : X \to Y$ such that $Tf = f \circ h$, for every $f \in Lip\,(Y)$. Now, by Theorem 42, we have that in fact h is a Lipschitz homeomorphism. ∎

Note that, from the proofs of Theorems 42 and 43, it follows at once that if there exists a unital vector lattice isomorphism $T : Lip\,(Y) \to Lip\,(X)$ which is an isometry for the Lipschitz norms (that is, $\|T\| = \|T^{-1}\| = 1$), then X and Y are in fact isometric.

It is well-known (see e.g. Gillman and Jerison [34], 9.8) that two metric spaces X and Y are homeomorphic if, and only if, $C^*(X)$ and $C^*(Y)$ are isomorphic as algebras, or equivalently in this case, as unital vector lattices. As usual we denote by $C^*(X)$ the set of all bounded and continuous real functions on X. Moreover, Theorem 40 says that an analogous result works for the family $U^*(X)$ when X is a complete metric space. Now the question arises whether, for complete metric spaces, there is a theorem of this kind for Lip^*. Of course this is true when both X and Y have finite diameter, since in this case $Lip = Lip^*$. But the answer is in general negative as simple examples show. Indeed, consider (X, d) with infinite diameter and let $d' = \min\{1, d\}$. In this case (X, d) and (X, d') are uniformly homeomorphic (hence (X, d') is complete whenever (X, d) is), but they are not Lipschitz homeomorphic. On the other hand, it is clear that $Lip^*(X, d) = Lip^*(X, d')$.

Nevertheless, in [31] we obtain a positive answer to the above question in the class of *length spaces* or, more generally, of *quasiconvex spaces*. Recall that the length of a path $\sigma : [a,b] \to X$ in a metric space (X,d) is defined as

$$\ell(\sigma) = \sup \sum_{i=1}^{n} d(\sigma(t_i), \sigma(t_{i-1})),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$. Now, (X,d) is said to be a *length space* if for every $x, y \in X$, the distance $d(x,y)$ coincides with the infimum of all lengths of continuous paths in X from x to y. Typical examples are Banach spaces as well as Riemannian manifolds endowed with its geodesic distance, but the class of length spaces also includes many other "singular" spaces. We refer to the book by Bridson and Haefliger [17] and references therein for an account on this topic. On the other hand, a metric space (X,d) is said to be *quasiconvex* if there is a constant $C > 0$ so that every pair of points $x, y \in X$ can be joined by a continuous path σ whose length satisfies $\ell(\sigma) \leq Cd(x,y)$. It is easily seen that a metric space is quasiconvex if, and only if, it is Lipschitz homeomorphic to some length space.

As a first step, we prove in [31] the following:

Theorem 44. (Garrido and Jaramillo 2001) *Let (X, d_X) and (Y, d_Y) be quasiconvex spaces, and let $h : X \to Y$. Assume that $f \circ h \in Lip^*(X)$ for each $f \in Lip^*(Y)$. Then, h is Lipschitz.*

Finally, with a proof similar to that of Theorem 43 we obtain the following result.

Theorem 45. (Garrido and Jaramillo 2001) *Let (X, d_X) and (Y, d_Y) be complete quasiconvex metric spaces. Then, $Lip^*(X)$ is isomorphic to $Lip^*(Y)$ as unital vector lattices if, and only if, X is Lipschitz homeomorphic to Y. Moreover, every such isomorphism $T : Lip^*(Y) \to Lip^*(X)$ is of the form $T(f) = f \circ h$, where $h : X \to Y$ is a Lipschitz homeomorphism.*

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