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König-Witstock Quasi-norms on Quasi-Banach Spaces

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1. Introduction

In [9], König and Wittstock considered non-equivalent norms on Banach spaces making continuous some prescribed linear functional. Our purpose is to relate their results with the theory of extensions of Banach spaces, and then to extend the constructions to quasi-Banach spaces. To this end, we define the general form of what we shall call a König-Wittstock quasi-norm and to obtain a characterization of when such quasi-norms are equivalent, isomorphic or isometric. We then consider what occurs when one wants to make continuous a prescribed functional but keeping continuous all the already continuous functionals. We show that the completion of these norms are all isometric to a minimal extension of X, although most of these norms are not even isomorphic.

If X is a Banach space we denote by X' the space of all linear functional and by X^* the space of all linear continuous functionals on X. Let $f \in X'$ and let $p \in X$ such that $f(p) = 1$. After [9] we call

$$\|x\|_f = |f(x)| + \inf_{\lambda\in\mathbb{R}}\|x - \lambda p\|$$

the König-Wittstock norm induced by f. In [9] it is shown that $\|\cdot\|_f$ is a complete norm on X that makes f continuous. Nevertheless, since two nonequivalent complete norms cannot have a common set of continuous functionals that separates points, some $\|\cdot\|$-continuous functional has ceased to be $\|\cdot\|_f$-continuous: precisely the $\|\cdot\|$-continuous projection onto $< p >$. Let us observe this from a different point of view: since $X =< p > \times \mathrm{Ker} f$ as vector spaces, although not as normed spaces, we can endow $\mathrm{Ker} f$ with the quotient norm and obtain the necessarily non-equivalent norm on X:

$$\|x\|_f = |f(x)| + \operatorname{dist}(x, < p >),$$

which is precisely the König-Wittstock norm. We set $X \simeq Z$ to mean that X is isomorphic to Z. The norm $\|\cdot\|_f$ is complete since $(X, \|\cdot\|_f) \simeq \mathbb{R} \oplus (X/ < p >)$. Moreover, since all closed hyperplanes of a Banach space are isomorphic (if H and G are two such closed hyperplanes then $H \simeq \mathbb{R} \oplus (H \cap G) \simeq G$) then all the quotients $X/ < p >$ are isomorphic and thus all the norms $\|\cdot\|_f$ are isomorphic for all choices of functionals $f \in X'$; and, of course, isomorphic to the original norm.

Let us briefly recall the basic facts about the theory of twisted sums of quasi-Banach spaces as developed by Kalton and Peck [6, 7]. A comprehensive description can be found in the monograph [2]. Exact sequences $0 \to Y \to X \to Z \to 0$ of quasi-Banach spaces correspond to homogeneous (usually non-linear and non-continuous) maps $F : Z \to Y$ with the property that here exists a constant K such that for each two points $x, y \in Z$

$$\|F(x+y) - F(x) - F(y)\| \leq K(\|x\| + \|y\|).$$

Such maps are called *quasi-linear*. This correspondence associates to an exact sequence $0 \to Y \to X \to Z \to 0$ a quasi-linear map $F : Z \to Y$ obtained by taking a homogeneous bounded selection $b : Z \to X$ for the quotient map and a linear (non-necessarily continuous) selection $l : Z \to X$ for the quotient map, and making their difference $F = b - l$. Conversely, given a quasi-linear map $F : Z \to Y$ then endowing the product space $Y \times Z$ with the quasi-norm

$$\|(y, z)\| = \|y - Fz\| + \|z\|,$$

one obtains a quasi-Banach space denoted $Y \oplus_F Z$ for which there exists an exact sequence $0 \to Y \to Y \oplus_F Z \to Z \to 0$. The quasi-Banach space $Y \oplus_F Z$ is called a *twisted sum of* Y *and* Z or an *extension of* Z *by* Y. Recall that two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be equivalent if there exists an arrow $T : X \to X_1$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \to & Y & \to & X_1 & \to & Z & \to & 0 \end{array}$$

On the other hand, two quasi-linear maps F and G, defined between the same spaces $Z \to Y$, are said to be equivalent if $F = G + B + L$, where

$B : Z \to Y$ is a bounded (homogeneous) map $B : Z \to Y$ and $L : Z \to Y$ is a linear (not necessarily continuous) map. It turns out that equivalent sequences correspond to equivalent quasi-linear maps. An exact sequence is said to be trivial (or that it splits) if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$; a quasi-linear map $Z \to Y$ is therefore said to be trivial if it is equivalent to the 0 map.

2. Twisted König-Wittstock quasi-norms

Some of the arguments used by König and Wittstock do no longer work on nonlocally convex quasi-Banach spaces since the (implicit) argument that copies of $\mathbb{R}$ are complemented fails for quasi-Banach spaces. For instance, no copy of $\mathbb{R}$ is complemented in $L_p(0,1)$ for $0 < p < 1$. Thus, it may happen that the quotients $X/ < p >$ and $X/ < p' >$ are not isomorphic for different p, p'. Such is the case in Ribe's example [10] of a nontrivial exact sequence

$$0 \to < p > \to E \to l_1 \to 0$$

in which the space E is not locally convex although the quotient $E/ < p >$ is a Banach space. Nevertheless, a different line $< p' > \subset E$ produces a non-locally convex quotient $E/ < q >$. Therefore, given $f \in E' - E^*$, it may happen that the König-Wittstock quasi-norms corresponding to different points and functionals are no longer isomorphic.

To clarify this situation the natural way to proceed is to replace the direct sum implicit in the definition by a twisted sum. Let X be a quasi-Banach space and let $0 \neq p \in X$. Given a linear functional $f \in X'$ such that $f(p) = 1$ we also understand that $f : X \to < p >$ is a linear (not necessarily continuous) projection. By $X/ < p >$ we understand the quotient space endowed with the quotient quasi-norm; the quotient map shall be denoted $q : X \to X/ < p >$ and F shall denote a quasi-linear map $F : X/ < p > \to \mathbb{R}$. We define the quasi-norm

$$\|x\|_{f,F} = |f(x) - Fqx| + \|qx\|.$$

The original König-Wittstock quasi-norms are $\|\cdot\|_{f,0}$. It may also be worth to remark that the functional f does not have to be $\|\cdot\|_{f,F}$-continuous.

We introduce some notation. An isomorphism (resp. isometry) $T : X \to Y$ is said to be a p-isomorphism (resp. p-isometry) if $Tp \in < p >$. Given two quasi-norms on X, the symbol $=$ means they are equal; $\sim$ means they are equivalent (the formal identity is an isomorphism) while $\simeq_p$ shall mean that they are p-isomorphic (there exists a p-isomorphism, not necessarily the identity). We shall write $\equiv_p$ to indicate they are p-isometric. Observe that while 'equivalent' and 'p-equivalent' are the same notion, the example [3, Prop. 3.2] shows that 'isomorphic' and 'p-isomorphic' are not the same. The main result of this section determines when the quasi-norms $\|\cdot\|_{f,F}$ and $\|\cdot\|_{g,G}$ are equivalent, p-isomorphic or p-isometric.

Theorem. *Let us say that two functions A and B are proportional if for some constants a and b one has $aA(t) \leq B(t) \leq bA(t)$. Let $p \in X$ be a point and let $f, g \in X'$ be functionals with $f(p) = g(p) = 1$. One has*

1. *The quasi-norms $\|\cdot\|_{f,F}$ and $\|\cdot\|_{g,G}$ are p-isomorphic if and only if there exists an isomorphism S of $X/<p>$ such that $FS - G$ is trivial.*
2. *The quasi-norms $\|\cdot\|_{f,F}$ and $\|\cdot\|_{g,G}$ are equivalent if and only if F and G are equivalent and the functions $|f-g|$ and $|Fq - Gq|$ are proportional.*
3. *The quasi-norms $\|\cdot\|_{f,F}$ and $\|\cdot\|_{g,G}$ are p-isometric if and only if there exists an isometry S of $X/<p>$ such that $FS - G$ is linear.*

Proof. Observe first that f and q provide the algebraic representation of X as $\mathbb{R} \times X/<p>$ in the form $x \to (fx, qx)$. This correspondence is an isometry between $(X, \|\cdot\|)_{f,F}$ and $\mathbb{R} \oplus_F X/<p>$ that we shall call I_f.

Proof of 1. If $\|\cdot\|_{f,F} \simeq_p \|\cdot\|_{g,G}$ then there exist isomorphisms T and S making commutative the diagram

$$\begin{array}{ccccccccc}
0 & \to & <p> & \to & <p> \oplus_F X/<p> & \to & X/<p> & \to & 0 \\
 & & | & & T \downarrow & & S \downarrow & & \\
0 & \to & <p> & \to & <p> \oplus_G X/<p> & \to & X/<p> & \to & 0,
\end{array}$$

Hence, also the diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{R} & \to & \mathbb{R} \oplus_F X/<p> & \to & X/<p> & \to & 0 \\
 & f & \uparrow & & I_f \uparrow & & \| & & \\
0 & \to & <p> & \to & (X, \|\cdot\|_{f,F}) & \to & X/<p> & \to & 0 \\
 & & | & & T \downarrow & & S \downarrow & & \\
0 & \to & <p> & \to & (X, \|\cdot\|_{g,G}) & \to & X/<p> & \to & 0 \\
 & g & \uparrow & & I_g \downarrow & & \| & & \\
0 & \to & \mathbb{R} & \to & \mathbb{R} \oplus_F X/<p> & \to & X/<p> & \to & 0
\end{array}$$

is commutative. Applying [3, Prop. 3.1] GT_q and F are equivalent.

Proof of 2. If in the above proof., $T = id_X$ then $S = id_{X/<p>}$ and one easily gets that F and G are equivalent. Since $\|\cdot\|_{f,F} \sim \|\cdot\|_{g,G}$ then for all $x \in X$ the values $|fx - Fqx|$ and $|gx - Gqx|$ are proportional. Replacing x by $x - (fx - Gqx)p$ one gets that also the functions $|Gq - Fq|$ and $|g - f|$ are proportional. For the converse implication, we observe that

$$\|\cdot\|_{g,F+B+L} = \|\cdot\|_{g-Lq,F+B} \sim \|\cdot\|_{g-Lq,F}\ .$$

Also, observe that the functions $|Gq - Fq|$ and $|g - f|$ are proportional and thus the hypothesis that $F - G$ is trivial yields that the functions $|Lq| = |Bq + Lq| = |Gq - Fq|$ and $|g - f|$ are proportional, and hence $f - g = \pm Lq$. Therefore, if F and $G = F + B + L$ are equivalent quasi-linear maps and f and g are linear functionals such that the functions $|Gq - Fq|$ and $|g - f|$ are proportional then

$$\|\cdot\|_{g,G} = \|\cdot\|_{g,F+B+L} = \|\cdot\|_{g-Lq,F+B} = \|\cdot\|_{f,F+B} \sim \|\cdot\|_{f,F}.$$

Proof of 3. If $T : (X, \|\cdot\|_{g,F}) \to (X, \|\cdot\|_{g,G})$ is a p-isometry, the diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{R} & \to & <p> \oplus_F X/<p> & \to & X/<p> & \to & 0 \\
 & & | & & T\downarrow & & S\downarrow & & \\
0 & \to & \mathbb{R} & \to & <p> \oplus_G X/<p> & \to & X/<p> & \to & 0
\end{array}$$

is commutative and the two sequences are isometrically equivalent. Hence, using the isometric version of [3, prop. 3.1], $GTq - F$ (and thus $G - F(T_q)^{-1}$) is linear. As for the converse, observe first that it always happens that $\|\cdot\|_{f,F} \equiv_p \|\cdot\|_{g,F}$ because the linear map $u : X \to X$ such that $gu = f$ given by $ux = x + (fx - gx)p$, is a p-isometry acting from $\|\cdot\|_{f,F}$ to $\|\cdot\|_{g,F}$. Hence, if $\|\cdot\|_{f,F} \sim_p \|\cdot\|_{g,G}$ then $\|\cdot\|_{g,F} \sim_p \|\cdot\|_{g,G}$. Now, if there is some isometry $S : X/<p> \to X/<p>$ such that $G - FS^{-1} = L$ is linear then the linear map $u : (X, \|\cdot\|_{g,F}) \to (X, \|\cdot\|_{g,G})$ given by $ux = x + (LSqx)p$ is a p-isometry since

$$\begin{aligned}
\|u(x)\|_{g,G} =& \|x + (LSqx)p\|_{g,G}\| \\
=& \|gx + LSqx - Gqx\| + \|qx\| \\
=& \|gx + Fqx\| + \|qx\| \\
=& \|x\|_{g,F}.
\end{aligned}$$

∎

Let us remark that from 1) it follows, in particular, that F is trivial if and only if G is trivial. The result [9, Corollaries 4] that if $f, g \in X'$ with $g(p) = 1 = f(p)$ then $\|\cdot\|_f$ and $\|\cdot\|_g$ are equivalent if and only if $f - g \in X^*$ can be deduced from 2) by taking $F = 0 = G$.

3. Incomplete norms induced by functionals

A different situation appears if one wants to make continuous a new functional $f \in X' - X^*$ but keeping continuous all the elements of X^*. In this case the simplest norm to be put on X is

$$|x|^f = \|x\| + |f(x)|.$$

Being non-equivalent and finer than $\|\cdot\|$, this norm cannot be complete. However, one has.

Proposition. *The completion of $(X, |\cdot|^f)$ is isometric to the twisted sum $\mathbb{R} \oplus_f X$, in turn isometric to $\mathbb{R} \oplus X$.*

Proof. Extending the formal identity map $(X, |\cdot|^f) \xrightarrow{id} X$ to the completion one gets an exact sequence

$$0 \to \mathrm{Ker}\widehat{id} \to \widehat{(X, |\cdot|^f)} \to X \to 0.$$

On the other hand, the elements of $\widehat{(X, |\cdot|^f)}$ are equivalence classes of $|\cdot|^f$-Cauchy sequences of elements of X. This means, $\|\cdot\|$-convergent sequences (since X is complete) that also are $|f(\cdot)|$-convergent. Thus, an element $[(x_n)] \in \widehat{(X, |\cdot|^f)}$ comes described by a couple $(r, x) \in \mathbb{R} \times X$ where $x = \lim x_n$ while $r = \lim f(x_n)$. In this terms,

$$|(r, x)|^f = |[x_n]|^f = \lim |x_n|^f = \lim \|x_n\| + \lim |f(x_n)|,$$

hence the map

$$L([x_n]) = (\lim f(x_n), \lim x_n)$$

defines a linear isometry between the completion $\widehat{(X, |\cdot|^f)}$ and $\mathbb{R} \times X$. The restriction of L to $\mathrm{Ker}\,\widehat{id}$ (which is formed by the equivalence classes of $\|\cdot\|$-null sequences that are $f(\cdot)$-convergent) is $L([(x_n)]) = (\lim f(x_n), 0)$. That $L_{|_{\mathrm{Ker}\,\widehat{id}}}$ is injective is clear, as well as its surjectivity. In conclusion, $\mathrm{Ker}\,\widehat{id} = \mathbb{R}$. ∎

But one is not completely satisfied: the isometry we set was, in some sense, a 'twist'. Now that we know that the elements of $\widehat{(X, |\cdot|^f)}$ are couples $(r,x) \in \mathbb{R} \times X$ probably one would prefer that the identification would respect the fact that the points of X have the form $(0,x)$. To do that we need a norm on $\mathbb{R} \times X$ so that $\|(0,x)\| = |f(x)| + \|x\|$. We only have to untwist the isometry at the cost of twisting the norm. The twisted norm

$$L([x_n]) = (\lim f(x_n), \lim x_n)$$

is a complete norm on $\mathbb{R} \times X$ that coincides with what we wanted on $0 \times X$: that is everything one needs since $0 \times X$ is dense in $\mathbb{R} \oplus_f X$.

We show now that a separable normed space admits uncountable non-isomorphic norms having isometric completions.

Proposition. *If X is separable most norms $|\cdot|^f$ are not isomorphic.*

Proof. The proof is simple after observing that two functionals f and g give equivalent norms if and only if $f - g \in X^*$. Since X is complete and separable $\dim X = c$ and thus $\dim X' = 2^c$; while the separability yields $\dim X^* = c$. Hence there exist 2^c equivalence classes in X'/X^*. In what follows we assume that different f and g have been picked from different equivalence classes.

Being $|\cdot|^f$ and $|\cdot|^g$ nonequivalent, it still remains the question of proving that they are not isomorphic. But since X is separable, it admits a dense countable set, and this set determines the values of a possible isomorphism. Hence there are c possible isomorphisms. That different f and h could be associated to g by the same isomorphism T is impossible since then $id = TT^{-1}$ would be an isomorphism between f and h, and they would be equivalent. ∎

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