EXTRACTA MATHEMATICAE Vol. **17**, Núm. 2, 281 – 288 (2002)

Non-Abelian Tensor Product of Lie Algebras and its Derived Functors

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(Research paper presented by A. Martínez Cegarra)

AMS *Subject Class.* (2000): 18G10, 18G50 *Received December* 12, 2001

In [7, 6] using the non-abelian tensor product of groups (see [1]) and its non-abelian derived functors the non-abelian homology of groups is constructed and studied, generalizing the classical Eilenberg-MacLane homology of groups and extending the non-abelian homology introduced in [4]. Now simmilar theory for Lie algebras will be provided. In [3] Ellis introduced and investigated the non-abelian tensor product of Lie algebras. Applying this tensor product, in [5] Guin defined the low-dimensional H_0 and H_1 non-abelian homologies of Lie algebras with coefficients in crossed modules and gave applications to cyclic homology and Milnor's additive K-theory for non-commutative associative algebras. The main goal of this note is to construct non-abelian homology of Lie algebras with coefficients in any Lie algebra in any dimensions as the non-abelian derived functors of the tensor product of Lie algebras, generalizing the classical homology of Lie algebras and extending Guin's non-abelian homology of Lie algebras. Some properties of the non-abelian tensor product and the non-abelian homology of Lie algebras are established.

Let Λ be a commutative ring with identity. We shall use the term Lie algebra to mean a Lie algebra over Λ and $[\ ,\ ]$ to denote the Lie bracket. We denote the category of Lie algebras over Λ by $\mathcal{L}ie$.

Let P and M be two Lie algebras. By an action of P on M we mean a Λ-bilinear map $P \times M \to M$, $(p, m) \mapsto {}^{p}m$ satisfying the following conditions:

$$ {}^{[p,p']}m = {}^{p}({}^{p'}m) - {}^{p'}({}^{p}m), \qquad {}^{p}[m,m'] = [{}^{p}m, m'] + [m, {}^{p}m'] $$

for all $m,\ m' \in M$ and $p,\ p' \in P$. For example, if P is a subalgebra of some Lie algebra Q, and if M is an ideal in Q, then Lie multiplication in Q yields an action of P on M.

Let M and N be two Lie algebras acting on each other. The tensor product $M \otimes N$ of the Lie algebras M and N is the Lie algebra generated by the symbols $m \otimes n$ ($m \in M$, $n \in N$) subject to the following relations:

(i) $\lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n$,

(ii) $(m + m') \otimes n = m \otimes n + m' \otimes n$,
$m \otimes (n + n') = m \otimes n + m \otimes n'$,

(iii) $[m, m'] \otimes n = m \otimes ({}^{m'}n) - m' \otimes ({}^{m}n)$,
$m \otimes [n, n'] = ({}^{n'}m) \otimes n - ({}^{n}m) \otimes n'$,

(iv) $[(m \otimes n), (m' \otimes n')] = -({}^{n}m) \otimes ({}^{m'}n')$

for all $\lambda \in \Lambda$, $m, m' \in M$, $n, n' \in N$.

Suppose $\phi : M \to A$, $\psi : N \to B$ are Lie homomorphisms, A, B act on each other, and ϕ, ψ preserve the actions in the following sense:

$$\phi({}^{n}m) = {}^{\psi(n)}\phi(m)\ , \qquad \psi({}^{m}n) = {}^{\phi(m)}\psi(n)$$

for all $m \in M$ and $n \in N$. Then by [3] there is a unique homomorphism $\phi \otimes \psi : M \otimes N \to A \otimes B$ such that $(\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \psi(n)$ for all $m \in M$, $n \in N$. Furthermore, if ϕ, ψ are onto, so also is $\phi \otimes \psi$. The tensor product of Lie algebras is symmetric in the sense of the isomorphism $M \otimes N \to N \otimes M$ given by $m \otimes n \mapsto -n \otimes m$ [3].

A crossed P-module (M, f) is a Lie homomorphism $f : M \to P$ together with an action of P on M satisfying: $f({}^{p}m) = [p, f(m)]$ and ${}^{f(m)}m' = [m, m']$ for all $m, m' \in M$, $p \in P$. Note that the image of f is necessarily an ideal in P and the kernel of f is a P-invariant ideal in the center of M. Moreover the action of P on $\operatorname{Ker} f$ induces an action of $P/\operatorname{Im} f$ on $\operatorname{Ker} f$, making $\operatorname{Ker} f$ a $P/\operatorname{Im} f$-module.

The following property from [2] (see also [5]) plays a crucial role for introducing non-abelian homology of Lie algebras. Let M be free as a Λ-module and N a crossed M-module with the trivial homomorphism, then there is a canonical isomorphism $M \otimes N \approx I(M) \otimes_{U(M)} N$, where $U(M)$ is the universal enveloping algebra and $I(M)$ is the augmentation ideal.

In [3] the results on the tensor product $M \otimes N$ are obtained assuming the actions of M and N on each other compatible, i.e.

$$ {}^{({}^{n}m)}n' = [n', {}^{m}n] \qquad \text{and} \qquad {}^{({}^{m}n)}m' = [m', {}^{n}m] \tag{1}$$

for all $m, m' \in M$ and $n, n' \in N$. This is the case, for example, if (M, f) and (N, g) are crossed P-modules, M and M' act on each other via the action of P, then these actions are compatible. These compatibility conditions are not assumed to hold.

Now we establish for the tensor product of Lie algebras the properties of compatibility with the direct limits of Lie algebras and the right exactness.

Proposition 1. *Let $\{M_\alpha,\ \phi_\alpha^\beta,\ \alpha \le \beta\}$ be a direct system of Lie algebras. Let N be a Lie algebra and for every α the Lie algebras M_α, N act on each other and the homomorphisms ϕ_α^β preserve the actions. Then there is a natural isomorphism of Lie algebras*

$$\left(\varinjlim_{\alpha}\{M_\alpha\}\right) \otimes N \approx \varinjlim_{\alpha}\{M_\alpha \otimes N\}\,.$$

Proposition 2. *Suppose $0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$ is a short exact sequence of Lie algebras, N is an arbitrary Lie algebra acting on M', M and M''; the Lie algebras M', M, M'' act on N and ϕ, ψ preserve these actions. Then there is an exact sequence of Lie algebras*

$$M' \otimes N \xrightarrow{\phi \otimes 1_N} M \otimes N \xrightarrow{\psi \otimes 1_N} M'' \otimes N \longrightarrow 0\,.$$

Now we define non-abelian derived functors of the tensor product of Lie algebras.

Let Q be a Λ-module. Let $\mathcal{A}_1(Q) = Q$, $\mathcal{A}_k(Q) = \sum_{0<i<k} \mathcal{A}_i(Q) \otimes_\Lambda \mathcal{A}_{k-i}(Q)$ and $\mathcal{A}(Q) = \sum_{0<k} \mathcal{A}_k(Q)$. The inclusion maps $\mathcal{A}_i(Q) \otimes_\Lambda \mathcal{A}_k(Q) \to \mathcal{A}_{i+k}(Q)$ give rise to a non-associative multiplication on $\mathcal{A}(Q)$ turning it into an algebra over Λ.

Let $\mathcal{B}(Q)$ be the two-sided ideal of $\mathcal{A}(Q)$ generated by the elements

$$xx \qquad \text{and} \qquad x(yz) + y(zx) + z(xy)\ ,$$

for all $x, y, z \in \mathcal{A}(Q)$.

We obtain the Lie algebra $\mathcal{F}(Q) = \mathcal{A}(Q) / \mathcal{B}(Q)$, which is the free Lie algebra on the Λ-module Q satisfying the following universal property: there is a natural Λ-module homomorphism $i : Q \to \mathcal{F}(Q)$ such that for any Lie algebra L and a Λ-module homomorphism $\alpha : Q \to L$ there exists a unique Lie homomorphism $\kappa : \mathcal{F}(Q) \to L$ such that $\kappa i = \alpha$.

Let N be a Lie algebra and $\alpha : Q \to \mathrm{Der}(N)$ be a Λ-module homomorphism then there exists a unique Lie homomorphism $\kappa : \mathcal{F}(Q) \to \mathrm{Der}(N)$ such that $\kappa i = \alpha$, i.e. the action of the Lie algebra $\mathcal{F}(Q)$ on the Lie algebra N.

Now if in addition Q is an N-module, then the module action of N on Q yields an N-module stucture on $\mathcal{A}_k(Q)$: if $x \otimes y \in \mathcal{A}_i(Q) \otimes_\Lambda \mathcal{A}_{k-i}(Q)$ and $n \in N$ then, inductively, we define

$$n(x \otimes y) = nx \otimes y + x \otimes ny\,,$$

and this extends linearly to an action of n on an arbitrary element of $\mathcal{A}_k(Q)$. The action of N on $\mathcal{A}_k(Q)$ extends linearly to an action of N on $\mathcal{A}(Q)$, making $\mathcal{A}(Q)$ into an N-module. Since $\mathcal{B}(Q)$ is N-invariant, the action of N on $\mathcal{A}(Q)$ induces a Lie action of N on $\mathcal{F}(Q)$.

Let $\mathfrak{A}_N$ denote, for a fixed Lie algebra N, the category whose objects are all Lie algebras M together with an action of M on N by derivations of N and an action of N on M by derivations of M. Morphisms in the category $\mathfrak{A}_N$ are all Lie homomorphisms $\phi : M \to M'$ preserving the actions, namely $\phi({}^n m) = {}^n\phi(m)$ and ${}^m n = {}^{\phi(m)}n$ for all $m \in M$, $n \in N$.

Let $\mathcal{F} : \mathfrak{A}_N \to \mathfrak{A}_N$ be the endofunctor defined as follows: for an object M of $\mathfrak{A}_N$, let $\mathcal{F}(G)$ denote the free Lie algebra on the underlying Λ-module M with the above-mentioned actions of N on $\mathcal{F}(M)$ and $\mathcal{F}(M)$ on N; for a morphism $\phi : M \to M'$ of $\mathfrak{A}_N$, let $\mathcal{F}(\phi)$ be the canonical Lie homomorphism from $\mathcal{F}(M)$ to $\mathcal{F}(M')$ induced by ϕ.

Let $\tau : \mathcal{F} \to 1_{\mathfrak{A}_N}$ be the obvious natural transformation and let $\delta : \mathcal{F} \to \mathcal{F}^2$ be the natural transformation induced for every $M \in \mathrm{ob}\,\mathfrak{A}_N$ by the natural Λ-module inclusion $M \to \mathcal{F}(M)$. We obtain a cotriple $\mathbb{F} = (\mathcal{F},\ \tau,\ \delta)$. Let us consider the cotriple resolution $\mathcal{F}_*(M) \xrightarrow{d_0^0} M$ of an object M of the category $\mathfrak{A}_N$, where

$$\mathcal{F}_*(M) \;\equiv\; \cdots \;\; \begin{array}{c}\longrightarrow\\ \vdots\\ \longrightarrow\end{array} \;\; \mathcal{F}_k(M) \;\; \begin{array}{c}\xrightarrow{d_0^k}\\ \vdots\\ \xrightarrow[d_k^k]{}\end{array} \;\; \cdots \;\; \begin{array}{c}\xrightarrow{d_0^2}\\ \longrightarrow\\ \xrightarrow[d_2^2]{}\end{array} \;\; \mathcal{F}_1(M) \;\; \begin{array}{c}\xrightarrow{d_0^1}\\ \xrightarrow[d_1^1]{}\end{array} \;\; \mathcal{F}_0(M)\,,$$

$$\mathcal{F}_k(M) = \mathcal{F}^{k+1}(M) = \mathcal{F}(\mathcal{F}^k(M)),\ d_i^k = \mathcal{F}^i\tau\mathcal{F}^{k-i},\ s_i^k = \mathcal{F}^i\delta\mathcal{F}^{k-i},\ 0 \le i \le k\,.$$

Let $\mathcal{T} : \mathfrak{A}_N \to \mathcal{L}ie$ be any functor. Applying $\mathcal{T}$ dimension-wise to the simplicial Lie algebra $\mathcal{F}_*(M)$ yields a simplicial Lie algebra $\mathcal{T}\mathcal{F}_*(M)$. Define k-th derived functor $\mathcal{L}_k^{\mathbb{F}}\mathcal{T} : \mathfrak{A}_N \to \mathcal{L}ie$, $k \ge 0$, of the functor $\mathcal{T}$ relative to the cotriple $\mathbb{F}$ as the k-th homotopy of $\mathcal{T}\mathcal{F}_*(-)$ (see also [3]). Note that $\mathcal{L}_k^{\mathbb{F}}\mathcal{T}(M)$, $k \ge 1$ is an abelian Lie algebra, i.e. only a Λ-module.

The non-abelian tensor product of Lie algebras defines a covariant functor $-\otimes N : \mathfrak{A}_N \to \mathcal{L}ie$. Let us denote by $\mathcal{L}_k^{\mathbb{F}}(-\otimes N)$, $k \geq 0$, the derived functors of the functor $-\otimes N$ relative to the cotriple $\mathbb{F}$. Using Proposition 2 one can easily show that there is a natural isomorphism $\mathcal{L}_0^{\mathbb{F}}(-\otimes N) \approx -\otimes N$.

Theorem 3. *Let M be a Lie algebra free as a Λ-module and N a module over the Lie algebra G, then there are natural isomorphisms*

$$\begin{gathered}\mathcal{L}_k^{\mathbb{F}}(-\otimes N)(M) \approx H_{k+1}(M,N)\,, \qquad k \geq 1\,,\\ \operatorname{Ker}\nu \approx H_1(M,N)\,,\\ \operatorname{Coker}\nu \approx H_0(M,N)\,,\end{gathered}$$

where N is thought as an abelian Lie algebra acting trivially on M and $\nu : M\otimes N \to N$, $\nu(m\otimes n) = {}^m n$ for all $m \in M$, $n \in N$.

This assertion gives us the possibility to state the following

Definition 4. Let M and N be Lie algebras acting on each other. The non-abelian homology of M with coefficients in N is defined by

$$\begin{aligned}&H_k(M,N) = \mathcal{L}_{k-1}^{\mathbb{F}}(-\otimes N)(M)\,, \qquad k \geq 2\,,\\ &H_1(M,N) = \operatorname{Ker}\nu\,,\\ &H_0(M,N) = \operatorname{Coker}\nu\,,\end{aligned}$$

where $\nu : M\otimes N \to N/H$, $\nu(m\otimes n) = |{}^m n|$ (here $|\ |$ denotes the coset of the quotient Lie algebra), and H is the ideal of the Lie algebra N generated by the elements ${}^{({}^n m)}n' - [n', {}^m n]$ for all $m \in M$, $n, n' \in N$.

Finally we give some properties of non-abelian homology of Lie algebras.

Theorem 5. *There is a natural isomorphism*

$$H_k(-,N) \approx \mathcal{L}_{k-1}^{\mathbb{F}}(H_1(-,N))\,, \qquad k \geq 1\,,$$

where $\mathcal{L}_{k-1}^{\mathbb{F}}(H_1(-,N))$ is the $k-1$-th derived functor of the functor $H_1(-,N) : \mathfrak{A}_N \to \mathcal{L}ie$ relative to the cotriple $\mathbb{F}$.

Proposition 6. *Let $\{M_\alpha,\ \phi_\alpha^\beta,\ \alpha \leq \beta\}$ and $\{N_\alpha,\ \psi_\alpha^\beta,\ \alpha \leq \beta\}$ be direct systems of Lie algebras. Let M and N be Lie algebras and for every α the*

Lie algebras M_α, N and M, N_α act on each other and the homomorphisms ϕ_α^β, ψ_α^β preserve the actions. Then there are natural isomorphism

$$H_k\left(\lim_{\overrightarrow{\alpha}}\{M_\alpha\}, N\right) \approx \lim_{\overrightarrow{\alpha}}\{H_k(M_\alpha, N)\}\,, \qquad k \geq 0\,,$$
$$H_k\left(M, \lim_{\overrightarrow{\alpha}}\{N_\alpha\}\right) \approx \lim_{\overrightarrow{\alpha}}\{H_k(M, N_\alpha)\}\,, \qquad k \geq 0\,.$$

THEOREM 7. *Let $\alpha : N \to N'$ be a surjective Lie homomorphism, M an arbitrary Lie algebra acting on N and N', which act on M and α preserve the actions. Then there is a long exact sequence of non-abelian homology*

$$\begin{aligned}\cdots \to H_3(M, N') \to H_2(M, N, N') \to H_2(M, N) \to H_2(M, N')\\ \to H_1(M, N, N') \to H_1(M, N) \to H_1(M, N')\\ \to H_0(M, N, N') \to H_0(M, N) \to H_0(M, N') \to 0\,,\end{aligned} \tag{2}$$

where

$$\begin{aligned} H_k(M, N, N') &= \pi_{k-1}(\mathrm{Ker}(1_{\mathcal{F}_*(M)} \otimes \alpha))\,, \qquad k \geq 2\,,\\ H_1(M, N, N') &= \frac{\{\mathrm{Ker}(1_{\mathcal{F}^0(M)} \otimes \alpha) \cap (d_0^0 \otimes 1_A)^{-1}(\mathrm{Ker}(1_M \otimes \alpha) \cap \mathrm{Ker}\,\nu)\}}{(d_1^1 \otimes 1_A)(\mathrm{Ker}(1_{\mathcal{F}^1(M)} \otimes \alpha) \cap \mathrm{Ker}(d_0^1 \otimes 1_A))}\,,\\ H_0(M, N, N') &= \mathrm{Ker}\,\widetilde{\alpha}/\nu(\mathrm{Ker}(1_M \otimes \alpha))\,,\end{aligned}$$

$\mathcal{F}_*(M) \xrightarrow{d_0^0} M$ is the $\mathbb{F}$ cotriple resolution of the object M of the category $\mathfrak{A}_N$ and $\widetilde{\alpha} : N/H \to N'/H'$ is the natural homomorphism induced by α.

Remarks. (i) If the actions of M and N satisfy the compatibility conditions (1) (in this case M and N' act on each other compatibly), then $H_0(M, N, N') = H_0(M, N'')$, where $N'' = \mathrm{Ker}\,\alpha$.

(ii) Let $0 \to (N'', 0) \to (N, f) \to (N', g) \to 0$ be an exact sequence of crossed M-modules. Thanks to the result in [5] there is a six term exact non-abelian homology sequence

$$\begin{aligned} H_1(M, N'') \to H_1(M, N) \to H_1(M, N')\\ \to H_0(M, N'') \to H_0(M, N) \to H_0(M, N') \to 0\,.\end{aligned} \tag{3}$$

The first five terms of the sequence (3) coincide with the first five terms of the sequense (2) and we have a natural homomorphism $H_1(M, N'') \to H_1(M, N, N')$.

Let us recall the definition of the Milnor's additive K-functor $K_2^{M\,add}$ ([9, 10]). For a given associative Λ-algebra A with identity, $K_2^{M\,add}(A)$ is a Λ-module generated by the symbols $\langle a,b\rangle$, where $a,b\in A$, subject to the relations

$$\begin{aligned}\langle a,b\rangle + \langle b,a\rangle &= 0\,,\\ \lambda\langle a,b\rangle + \lambda'\langle a',b\rangle &= \langle \lambda a+\lambda' a',b\rangle\,,\\ \langle ab,c\rangle - \langle a,bc\rangle + \langle ca,b\rangle &= 0\,,\end{aligned}$$

where $a,a',b,c\in A$, $\lambda,\lambda'\in\Lambda$.

I f A is commutative than $K_2^{M\,add}(A)$ coincides with the first cyclic homology $HC_1(A)$ (for the definition see [10]).

Suppose A is a non-commutative Λ-algebra. We let A be considered with the usual induced Lie structure ($[a,b]=ab-ba$, $a,b\in A$). From [5] we know that there exists a Lie algebra $V(A)$ defined as the quotient of the tensor product Lie algebra $A\otimes A$ by the ideal generated by the elements

$$a\otimes b+b\otimes a\,,\qquad ab\otimes c-a\otimes bc+ca\otimes b$$

for all $a,b,c\in A$, such that there is a short exact sequence of crossed A-modules

$$0\to HC_1(A)\to V(A)\to [A,A]\to 0\,,$$

where $[A,A]$ is the additive commutator of A. It is clear that $H_0(A,[A,A])=[A,A]/[A,[A,A]]$. Furthermore, it is shown in [5] that $H_0(A,HC_1(A))=HC_1(A)$ and $H_0(A,V(A))\approx K_2^{M\,add}(A)$. Then Theorem 7 and Remarks (i) yield the following

Theorem 8. *Let A be a non-commutative associative Λ-algebra with identity. Then there is an exact sequnce of Λ-modules*

$$\begin{aligned}\cdots\to H_2(A,V(A),[A,A])\to H_2(A,V(A))\to H_2(A,[A,A])\\ \to H_1(A,V(A),[A,A])\to H_1(A,V(A))\to H_1(A,[A,A])\\ \to HC_1(A)\to K_2^{M\,add}(A)\to [A,A]/[A,[A,A]]\to 0\,.\end{aligned}$$

Proofs are based on [7], [6], [2], [11], [12] and will be given in [8].

Acknowledgements

The first author would like to thank the University of Santiago de Compostela for its hospitality at the origin of this paper. The authors were partially supported by INTAS Georgia grant No 213.

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