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Graph Topologies and Uniform Convergence in Quasi-Uniform Spaces

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1. Introduction

As usual, the family of continuous real-valued functions on a topological space X, is denoted by $C(X)$. The family of lower semicontinuous real-valued functions on X will be denoted by $SC(X)$.

In [14] S. Naimpally proved that for a metric space (X, d), the Hausdorff metric topology induced by the product metric on $X \times \mathbb{R}$ agrees with the topology of uniform convergence on the family of uniformly continuous real-valued functions, where functions are identified with their graphs. Later on, G. Beer ([2]) improved Naimpally's theorem as follows: Given a metric space (X, d), the Hausdorff metric topology induced by the product metric on $X \times \mathbb{R}$ agrees with the topology of uniform convergence on $C(X)$ if and only if every member of $C(X)$ is uniformly continuous. In [3] Beer proved the following variant of the above theorem, which reconciles proximal convergence with uniform convergence: Given a metric space (X, d), the proximal topology induced by the product metric on $X \times \mathbb{R}$ agrees with the topology of uniform convergence on $C(X)$ if and only if every member of $C(X)$ is uniformly continuous.

Motivated by the recent applications of hyperspaces and function spaces on quasi-uniform and quasi-pseudo-metric spaces to theoretical computer science (see, for instance, [16, 17, 18, 19, 20]) Rodríguez-López and Romaguera [15] have recently extended Naimpally's theorem and Beer's theorems cited above to quasi-pseudo-metric spaces. In particular, it was proved that if (X, d) is a quasi-pseudo-metric space then every function in $SC(X)$ is quasi-uniformly continuous if and only if the upper Hausdorff quasi-pseudo-metric topology induced by $d^{-1} \times \ell$ agrees with the topology of uniform convergence on $SC(X)$, where ℓ denotes the lower quasi-pseudo-metric.

On the other hand, it is well known that every continuous function from a topological space X to a topological space Y is quasi-uniformly continuous if both X and Y are endowed with the Pervin quasi-uniformity (resp. the fine quasi-uniformity, the semi-continuous quasi-uniformity, etc.). This interesting fact suggests the question of generalizing the results of [15] to the quasi-uniform setting. In this paper, we obtain such generalizations not only for real-valued functions but for the more general case of functions with values in a Scott quasi-uniform semigroup (see Section 3 for definitions).

In fact, when one tries to obtain these generalizations for $SC(X)$, notices that the partial order on $\mathbb{R}$ which induces the topology $\mathcal{T}(\ell)$, plays a crucial role. Hence, it seems natural to think that the techniques that one can use in the real case also work on certain suitable structures of partial order, i.e. lattices. Furthermore, the theory of lattices and quasi-uniform (topological) spaces is very interconnected, because under some assumptions (see [9, Chapter II, Theorem 3.8]) a topological space is a complete lattice considering the specialization order (see Corollary 1 below).

On the other hand, the theory of lattices arises in several contexts. One of them is the theory of computation. A useful tool in this area is to apply an algorithm successively to obtaining better approximations to the desired result. We can assign to each stage of the process a subset where the result lies. The smaller set, the better the approximation. These facts suggest the use of an order relation.

Our basic references for quasi-uniform and quasi-pseudo-metric spaces are [8] and [10]. Terms and undefined concepts may be found in such references.

A quasi-uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ which satisfies: $i)$ $\Delta \subseteq U$ for all $U \in \mathcal{U}$ and $ii)$ given $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where $\Delta = \{(x, x) : x \in X\}$ and $V^2 = \{(x, z) \in X \times X :$ there exists $y \in X$ such that $(x, y) \in V, (y, z) \in V\}$. The elements of the filter $\mathcal{U}$ are called entourages.

The filter $\mathcal{U}^{-1}$, formed by all sets of the form $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$ where $U \in \mathcal{U}$, is a quasi-uniformity on X called the conjugate of $\mathcal{U}$.

If $\mathcal{U}$ is a quasi-uniformity on X, then $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$ is a uniformity.

Every quasi-uniformity $\mathcal{U}$ on X generates a topology $\mathcal{T}(\mathcal{U})$. A neighborhood base for each point $x \in X$ is given by $\{U(x) : U \in \mathcal{U}\}$ where $U(x) = \{y \in X : (x, y) \in U\}$.

Similarly, the topology generated on a set X by a quasi-pseudo-metric d, is denoted by $\mathcal{T}(d)$.

In the sequel, we shall denote by ℓ the quasi-pseudo-metric on $\mathbb{R}$ defined by $\ell(x, y) = (x - y) \vee 0$ for all $x, y \in \mathbb{R}$.

For any nonempty set X, we denote by $\mathcal{P}_0(X)$ the family of all nonempty subsets of X.

2. Preliminaries

Given two quasi-uniform spaces $(X,\mathcal{U})$ and $(Y, \mathcal{V})$, a function f from X to Y is said to be *quasi-uniformly continuous* if for every $V \in \mathcal{V}$ we can find $U \in \mathcal{U}$ such that if $(x, y) \in U$ then $(f(x), f(y)) \in V$.

The set of all quasi-uniformly continuous functions from X to Y is denoted by $UC(X, Y)$.

Definition 1. Let $(X,\mathcal{U})$ be a quasi-uniform space. Following [5] and [11], we define

$$U_H^+ = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A)\}$$

$$U_H^- = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq U^{-1}(B)\}$$

for all $U \in \mathcal{U}$. Then $\{U_H^+ : U \in \mathcal{U}\}$ is a base for the upper Hausdorff quasi-uniformity $H_{\mathcal{U}}^+$ of $\mathcal{U}$ and $\{U_H^- : U \in \mathcal{U}\}$ is a base for the lower Hausdorff quasi-uniformity $H_{\mathcal{U}}^-$ of $\mathcal{U}$. The quasi-uniformity $H_{\mathcal{U}}$ defined as the supremum of the lower and upper Hausdorff quasi-uniformities is called the Hausdorff quasi-uniformity of $\mathcal{U}$.

Given two quasi-uniform spaces $(X,\mathcal{U})$ and $(Y, \mathcal{V})$ we denote by $C(X, Y)$ the family of continuous functions from $(X, \mathcal{T}(\mathcal{U}))$ to $(Y, \mathcal{T}(\mathcal{V}))$. If $f \in C(X, Y)$ we denote the graph of f by $\operatorname{Gr} f$, i.e. $\operatorname{Gr} f = \{(x, f(x)) : x \in X\}$.

On the set $C(X, Y)$ we consider the upper Hausdorff quasi-uniformity $H^+_{\mathcal{U}^{-1}\times\mathcal{V}}$ induced by the product quasi-uniformity $\mathcal{U}^{-1} \times \mathcal{V}$ when we identify each continuous function with its graph. Then, each basic entourage of $H^+_{\mathcal{U}^{-1}\times\mathcal{V}}$ is of the form $(U^{-1} \times V)^+_H = \{(f, g) \in C(X, Y) \times C(X, Y) : (U \times V^{-1})(x, g(x)) \cap \operatorname{Gr} f \neq \varnothing$ for all $x \in X\}$, where $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Similarly, the lower Hausdorff quasi-uniformity $H^-_{\mathcal{U}^{-1}\times\mathcal{V}}$ on $C(X,Y)$ has basic entourages of the form $(U^{-1}\times V)^-_H = \{(f,g)\in C(X,Y)\times C(X,Y):(U^{-1}\times V)(x,f(x))\cap \mathrm{Gr}\, g\neq\varnothing$ for all $x\in X\}$, where $U\in\mathcal{U}$ and $V\in\mathcal{V}$.

The Hausdorff quasi-uniformity $H_{\mathcal{U}^{-1}\times\mathcal{V}}$ induced on $C(X,Y)$ by the product quasi-uniformity $\mathcal{U}^{-1}\times\mathcal{V}$ is the supremum of $H^+_{\mathcal{U}^{-1}\times\mathcal{V}}$ and $H^-_{\mathcal{U}^{-1}\times\mathcal{V}}$.

Definition 2. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two quasi-uniform spaces. The topology generated on $C(X,Y)$ by the sets of the form $G^-=\{f\in C(X,Y):\mathrm{Gr}\, f\cap G\neq\varnothing\}$ where G is a $\mathcal{T}(\mathcal{U}^{-1}\times\mathcal{V})$-open set is called the lower proximal topology and is denoted by $\mathcal{T}^-(\delta_{\mathcal{U}^{-1}\times\mathcal{V}})$.

The topology generated on $C(X,Y)$ by all sets of the form $G^{++}=\{f\in C(X,Y):$ there exists $U\in\mathcal{U}$ and $V\in\mathcal{V}$ such that $(U^{-1}\times V)(\mathrm{Gr}\, f)\subseteq G\}$ where G is a $\mathcal{T}(\mathcal{U}^{-1}\times\mathcal{V})$-open set is called the upper proximal topology and is denoted by $\mathcal{T}^+(\delta_{\mathcal{U}^{-1}\times\mathcal{V}})$.

The topology $\mathcal{T}(\delta_{\mathcal{U}^{-1}\times\mathcal{V}})=\mathcal{T}^+(\delta_{\mathcal{U}^{-1}\times\mathcal{V}})\vee\mathcal{T}^-(\delta_{\mathcal{U}^{-1}\times\mathcal{V}})$ is called the proximal topology.

Let us recall that the proximal topology was essentially introduced in [13], although the term “proximal (hit-and-miss) topology” was firstly used in [4], [6] and [12], where the relationship between the proximal topology and other known hyperspace topologies was studied. In particular, the proximal topology is compatible with *Fisher convergence* [7] of sequences of sets, as considered in [1].

3. The results

Proposition 1. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two quasi-uniform spaces. Then $\mathcal{T}(\delta_{\mathcal{U}^{-1}\times\mathcal{V}})=\mathcal{T}(H^+_{\mathcal{U}^{-1}\times\mathcal{V}})$ on $C(X,Y)$.*

Proof. Let us consider the set G^{++} where G is a $\mathcal{T}(\mathcal{U}^{-1}\times\mathcal{V})$-open set and $A\in G^{++}$. Therefore, we can find $U\in\mathcal{U}$ and $V\in\mathcal{V}$ such that $(U^{-1}\times V)(A)\subseteq G$. Let us consider the set $(U^{-1}\times V)^+_H(A)$. It is evident that $A\in(U^{-1}\times V)^+_H(A)\subseteq G^{++}$.

Furthermore, if we have that $\mathrm{Gr}\, f\in G^-$ where G is a $\mathcal{T}(\mathcal{U}^{-1}\times\mathcal{V})$-open set and $f\in C(X,Y)$, let $(x,f(x))\in\mathrm{Gr}\, f\cap G$. Since $(x,f(x))\in G$ we can find $U\in\mathcal{U}$ and $V\in\mathcal{V}$ verifying $(U^{-1}\times V)((x,f(x)))\subseteq G$. Let $W\in\mathcal{V}$ such that $W^2\subseteq V$. Since $f\in C(X,Y)$ we can find $U'\in\mathcal{U}$ such that $(f(x),f(y))\in W$ whenever $(x,y)\in U'$. We claim that $(U'^{-1}\times W)^+_H(\mathrm{Gr}\, f)\subseteq G^-$. Let $\mathrm{Gr}\, g\in(U'^{-1}\times W)^+_H(\mathrm{Gr}\, f)$. Therefore, we can find $(y,f(y))\in\mathrm{Gr}\, f$ such that

$(x,y) \in U'$ and $(f(y), g(x)) \in W$. Furthermore, by the above observation we obtain that $(f(x), f(y)) \in W$, so $(f(x), g(x)) \in W^2 \subseteq V$. Consequently, $(x, g(x)) \in (U^{-1} \times V)(x, f(x)) \subseteq G$, i.e. $\mathrm{Gr}\, g \cap G \neq \varnothing$.

For the other inclusion, let us consider the set $(U^{-1} \times V)^+_H(A)$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. It is easy to prove that $(int_{\mathcal{T}(\mathcal{U}^{-1}\times\mathcal{V})}((U^{-1} \times V)(A)))^{++} \subseteq (U^{-1} \times V)^+_H(A)$. ∎

If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are two quasi-uniform spaces, the *quasi-uniformity of uniform convergence* on $C(X,Y)$ induced by $\mathcal{V}$ is the quasi-uniformity $\mathcal{V}_X$ on $C(X,Y)$ whose elements V_X are defined by

$$V_X = \{(f,g) \in C(X,Y) \times C(X,Y) : (f(x), g(x)) \in V \text{ for all } x \in X\}.$$

The topology generated by $\mathcal{V}_X$ is called, simply, *the topology of uniform convergence* on $C(X,Y)$ and is denoted by $\mathcal{T}(\mathcal{V}_X)$.

Proposition 2. *Let* $(X,\mathcal{U})$ *and* $(Y,\mathcal{V})$ *be two quasi-uniform spaces. Then* $\mathcal{T}(H_{\mathcal{U}^{-1}\times\mathcal{V}}) \subseteq \mathcal{T}(\mathcal{V}_X)$ *on* $C(X,Y)$.

Proof. Let us suppose that $\{f_\lambda\}_{\lambda\in\Lambda}$ is a net in $C(X,Y)$ which is $\mathcal{T}(\mathcal{V}_X)$-convergent to $f \in C(X,Y)$. Consider the set $(U^{-1} \times V)_H(\mathrm{Gr}\, f)$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Therefore, we can find $\lambda_0 \in \Lambda$ such that $f_\lambda \in V_X(f)$ for all $\lambda \geq \lambda_0$. It is evident that $\mathrm{Gr}\, f_\lambda \in (U^{-1} \times V)_H(\mathrm{Gr}\, f)$ for all $\lambda \geq \lambda_0$. ∎

The following proposition extends Naimpally's theorem to the quasi-uniform setting.

Proposition 3. *Let* $(X,\mathcal{U})$ *and* $(Y,\mathcal{V})$ *be two quasi-uniform spaces. Then* $\mathcal{T}(\delta_{\mathcal{U}^{-1}\times\mathcal{V}}) = \mathcal{T}(H^+_{\mathcal{U}^{-1}\times\mathcal{V}}) = \mathcal{T}(H_{\mathcal{U}^{-1}\times\mathcal{V}}) = \mathcal{T}(\mathcal{V}_X)$ *on* $UC(X,Y)$.

Proof. By the above results, we only have to show that $\mathcal{T}(\mathcal{V}_X) \subseteq \mathcal{T}(H^+_{\mathcal{U}^{-1}\times\mathcal{V}})$.

Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a net in $UC(X,Y)$ such that it is $\mathcal{T}(H^+_{\mathcal{U}^{-1}\times\mathcal{V}})$-convergent to a function $f \in UC(X,Y)$. Fix $W \in \mathcal{V}$. Therefore, we can find $U \in \mathcal{U}$ such that if $(x,y) \in U$ then $(f(x), f(y)) \in W'$ where $W' \in \mathcal{V}$ and $W'^2 \subseteq W$. Furthermore, there exists $\lambda_0 \in \Lambda$ such that $\mathrm{Gr}\, f_\lambda \in (U^{-1} \times W')^+_H(\mathrm{Gr}\, f)$ for all $\lambda \geq \lambda_0$.

Given $x \in X$ and $\lambda \geq \lambda_0$ we can find $y \in X$ such that $(x,y) \in U$ and $(f(y), f_\lambda(x)) \in W'$. Therefore, $(f(x), f_\lambda(x)) \in W'^2 \subseteq W$, i.e. $f \in W_X(f_\lambda)$ for all $\lambda \geq \lambda_0$. ∎

Now, we present some examples where we can apply this result.

Example 1. As we observed at the introduction, if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a continuous function and $\mathcal{U}$ denotes the Pervin (resp. point finite, locally finite, semi-continuous, fine transitive, fine) quasi-uniformity on X (see [8] for the corresponding definitions) and $\mathcal{V}$ the corresponding quasi-uniformity on Y then $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is quasi-uniformly continuous (see [8, Proposition 2.17]). Therefore, we deduce that $\mathcal{T}(H_{\mathcal{U}^{-1}\times\mathcal{V}}) = \mathcal{T}(\mathcal{V}_X)$ on $C(X, Y)$.

Now, we recall some definitions about the theory of lattices. Our main reference is [9].

A *partially ordered set* (or a poset) is a nonempty set L with a reflexive, transitive and antisymmetric relation $\leq$. A *lattice* is a poset where every nonempty finite subset has infimum and supremum.

Given a lattice L and $O \subseteq L$, we denote $\uparrow O = \{l \in L :$ there exists $o \in O$ such that $o \leq l\}$.

A lattice is said to be *complete* if every nonempty subset has infimum and supremum.

Let L be a complete lattice. We say that x *is way below* y, and we denote it by $x \ll y$, if for all directed subsets D of L the relation $y \leq \sup D$ implies the existence of $d \in D$ such that $x \leq d$, where D is called directed set if given $p, q \in D$ we can find $l \in D$ such that $p \leq l$ and $q \leq l$.

A *continuous lattice* is a complete lattice L such that it satisfies the axiom of approximation:

$$x = \sup\{u \in L : u \ll x\}$$

for all $x \in L$.

Let L be a complete lattice. The *Scott topology* $\sigma(L)$ is formed by all subsets O of L which satisfy:

i) $O = \uparrow O$
ii) $\sup D \in O$ implies $D \cap O \neq \varnothing$ for all directed sets $D \subseteq L$.

We recall that a *semigroup* is a pair $(X, \cdot)$ such that $\cdot$ is an associative internal law or operation on X for which exists an identity element e.

Definition 3. If $(X, \mathcal{T})$ is a topological space and $(X, \cdot)$ is a semigroup such that the function $\cdot_x : X \to X$ defined by $\cdot_x(y) = x \cdot y$ for all $y \in X$ is continuous for all $x \in X$ we say that $(X, \cdot, \mathcal{T})$ is a semitopological semigroup.

Definition 4. A Scott quasi-uniform semigroup is a pair $(L,\mathcal{U})$ where L is a complete lattice, $\mathcal{U}$ is a quasi-uniformity on L such that $\mathcal{T}(\mathcal{U})$ is the Scott topology and $(L,\vee,\mathcal{T}(\mathcal{U}))$ is a semitopological semigroup.

Now, we prove our main result.

Theorem 1. *Let $(X,\mathcal{U})$ be a quasi-uniform space and $(L,\mathcal{V})$ a Scott quasi-uniform semigroup. The following statements are equivalent:*

i) *Every continuous function from X to L is quasi-uniformly continuous.*
ii) *The proximal topology of $(X\times L,\mathcal{U}^{-1}\times\mathcal{V})$ agrees with the topology of uniform convergence on $C(X,L)$.*
iii) *The upper Hausdorff quasi-uniform topology induced by $\mathcal{U}^{-1}\times\mathcal{V}$ agrees with the topology of uniform convergence on $C(X,L)$.*

Proof. i) $\Rightarrow$ ii) This is deduced from Proposition 3.

ii) $\Rightarrow$ iii) We only have to use Proposition 1.

iii) $\Rightarrow$ i) Suppose that there is a continuous function $f:(X,\mathcal{T}(\mathcal{U}))\to(L,\sigma(L))$ which is not quasi-uniformly continuous. Then, we can find $W\in\mathcal{V}$ and two nets $\{a_U\}_{U\in\mathcal{U}}$ and $\{b_U\}_{U\in\mathcal{U}}$ such that $(a_U,b_U)\in U$ and $(f(a_U),f(b_U))\notin W$ for all $U\in\mathcal{U}$. For each $U\in\mathcal{U}$ we define the following function $f_U:X\to L$:

$$f_U(x)=\begin{cases} f(b_U) & \text{if } x\in\overline{\{a_U\}}\\ f(x)\vee f(b_U) & \text{otherwise.}\end{cases}$$

Let us prove that f_U is continuous for each $U\in\mathcal{U}$. Let us fix $U\in\mathcal{U}$ and let $x\in X$ and $\{x_\lambda\}_{\lambda\in\Lambda}$ be a $\mathcal{T}(\mathcal{U})$-convergent net to x. We distinguish the following cases:

Case 1. $x\in\overline{\{a_U\}}$

We have that $f_U(x)=f(b_U)$, and the values of $f_U(x_\lambda)$ can be $f(b_U)$ or $f(x_\lambda)\vee f(b_U)$ for all $\lambda\in\Lambda$. It is evident that in both cases we obtain that $f_U(x)\leq f_U(x_\lambda)$, so f_U is a continuous function in x.

Case 2. $x\in X\backslash\overline{\{a_U\}}$

Since $\{x_\lambda\}_{\lambda\in\Lambda}$ is $\mathcal{T}(\mathcal{U})$-convergent to x, it is eventually in $X\backslash\overline{\{a_U\}}$. Therefore, by continuity of the sup operation, we deduce that f_U is continuous in x.

Consequently, we have shown that f_U is continuous for all $U\in\mathcal{U}$.

We now prove that $\{f_U\}_{U\in\mathcal{U}}$ converges to f with respect to $\mathcal{T}(H^+_{\mathcal{U}^{-1}\times\mathcal{V}})$.

Fix $U_0 \in \mathcal{U}$ and let $U, V \in \mathcal{U}$ such that $V^2 \subseteq U \subseteq U_0$. If $x \in \overline{\{a_V\}}$, $f_V(x) = f(b_V)$. Furthermore, $(x, a_V) \in V$, so $(x, b_V) \in V^2 \subseteq U$.

On the other hand, if $x \in X \backslash \overline{\{a_V\}}$ then $f(x) \leq f_V(x) = f(x) \vee f(b_V)$.

Hence $\{f_U\}_{U \in \mathcal{U}}$ converges to f with respect to $\mathcal{T}(H^+_{\mathcal{U}^{-1} \times \mathcal{V}})$. However, this net does not converge to f with respect to the topology of uniform convergence because $(f(a_U), f_U(a_U)) = (f(a_U), f(b_U)) \notin W$ for all $U \in \mathcal{U}$. ∎

Remark 1. As a particular case of the above theorem we can consider the continuous lattice $\mathbb{R}^*$ with the usual order where $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$. We can consider the Scott topology $\sigma(\mathbb{R}^*)$ of this continuous lattice and a quasi-uniformity $\mathcal{U}$ on $\mathbb{R}^*$ such that $\mathcal{T}(\mathcal{U}) = \sigma(\mathbb{R}^*)$. In every continuous lattice the sup operation is continuous considering the Scott topology (see [9, Chapter I, Proposition 1.11]). Therefore, we can apply the above theorem to this lattice.

In particular, we can consider the extended lower quasi-pseudo-metric on $\mathbb{R}^*$ given by

$$\ell^*(x,y) = \begin{cases} (x-y) \vee 0 & \text{if } x, y \in \mathbb{R} \\ 0 & \text{if } x = -\infty \text{ or } y = +\infty \\ +\infty & \text{if } x = +\infty \text{ or } y = -\infty \text{ and } x \neq y. \end{cases}$$

The topology generated by this extended quasi-pseudo-metric has as a base all the sets of the form $(a, +\infty] = \{b \in \mathbb{R}^* : a \ll b\}$ where $a \in \mathbb{R}^*$ and the open set $\{+\infty\}$ which coincide with the basic open sets in the Scott topology (see [9, Chapter I, Proposition 1.10]). Now, the continuous functions between X and $\mathbb{R}^*$ are the extended lower semicontinuous functions, i.e. lower semicontinuous functions with values in $\mathbb{R}^*$ (see [3, 9]). Therefore, the above result asserts that given a quasi-uniform space $(X, \mathcal{U})$ then every extended lower semicontinuous function on X is quasi-uniformly continuous if and only if the topology of uniform convergence agrees with the upper Hausdorff quasi-uniform topology induced by $\mathcal{U}^{-1} \times \mathcal{U}_{\ell^*}$ where $\mathcal{U}_{\ell^*}$ denotes the quasi-uniformity induced by ℓ^*.

Remark 2. We notice that the above theorem is also true if we consider functions from a quasi-uniform space to a lattice contained in a Scott quasi-uniform semigroup. Therefore, we not only can consider the extended real line $\mathbb{R}^*$ but the real line $\mathbb{R}$. Consequently, given a quasi-uniform space $(X, \mathcal{U})$ then $SC(X) = UC(X, \mathbb{R})$ if and only if the topology of uniform convergence coincides with the upper Hausdorff quasi-uniform topology induced by $\mathcal{U}^{-1} \times \mathcal{U}_{\ell}$.

We recall that a T_0-space Z is called *injective* if every continuous map $f : X \to Z$ extends continuously to any space Y containing X as a subspace.

Corollary 1. *Let $(X, \mathcal{U})$ be a quasi-uniform space and (Y, τ) an injective topological T_0-space. Let $\mathcal{V}$ be a quasi-uniformity compatible with τ. The following statements are equivalent:*

i) *Every continuous function from X to Y is quasi-uniformly continuous.*
ii) *The proximal topology of $(X \times Y, \mathcal{U}^{-1} \times \mathcal{V})$ agrees with the topology of uniform convergence on $C(X, Y)$.*
iii) *The upper Hausdorff quasi-uniform topology induced by $\mathcal{U}^{-1} \times \mathcal{V}$ agrees with the topology of uniform convergence on $C(X, Y)$.*

Proof. If Y is a T_0 space it is evident that the specialization order $\leq$ given by

$$x \leq y \Leftrightarrow x \in \overline{\{y\}}$$

is a partial order and it can be proved (see [9, Chapter II, Theorem 3.8]) that if Y is an injective space then Y with the specialization order is a continuous lattice. Furthermore, in a continuous lattice the sup operation is jointly continuous with respect to the Scott topology (see [9, Chapter I, Proposition 1.11]), so we can apply the theorem above, but the Scott topology (see [9, Chapter II, Theorem 3.8]) is homeomorphic to τ so we have completed the proof. ∎

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