Assessment of the maintenance cost and analysis of availability measures in a finite life cycle for a system subject to competing failures

N. C. Caballé · I. T. Castro

Abstract This paper deals with the assessment of the maintenance cost and the performance of a system under a finite planning horizon. The system is subject to two dependent causes of failure: internal degradation and sudden shocks. We assume that internal degradation follows a gamma process. When the deterioration level of the degradation process exceeds a threshold, a degradation failure occurs. Sudden shocks arrive at the system following a doubly stochastic Poisson process (DSPP). A sudden shock provokes the system failure. A condition-based maintenance (CBM) with periodic inspection times is implemented. Recursive methods combining numerical integration and Monte Carlo simulation are developed to evaluate the expected cost rate and its standard deviation. Also, recursive methods to calculate the reliability, the availability and the interval reliability of the system are given. Numerical examples are provided to illustrate the analytical results.

Keywords Availability · condition based maintenance · degradation threshold shock model · interval reliability · gamma process · doubly stochastic Poisson process

1 Introduction

A fundamental aim in the industry field is to ensure the reliability of the systems. It is well-known that some systems suffer a physical degradation process which precedes the failure. This degradation process may involve chemical and physical changes in the system complicating its maintenance. The theory of stochastic processes provides an analytical framework for modelling the impact of the uncertain and time-dependent degradation processes.

There are several approaches to model the system degradation. For example, the random coefficient model is a relatively flexible and convenient model for describing the degradation derived from physics of failures ([1], [2]). For monotonic stochastic deterioration and due to its nice mathematical properties, the gamma process is a popular degradation model in the

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literature. It is characterised by independent and non-negative gamma increments with identical scale parameters. The gamma process was first applied by Moran [3] to model water flow into a dam. From then on gamma process has been widely used in the reliability field. The survey by Van Noortwijk [4] provides many examples of the use of the gamma process in engineering.

However, some systems are not only subject to internal degradation but are also exposed to sudden shocks which can cause its failure. The class of reliability models which consider failures due to the competing causes of degradation and shocks is called Degradation-Threshold-Shock models (DTS models). In these models, the system fails when the degradation exceeds a threshold or when a fatal shock arrives whichever comes first. For example, an ammeter degrades over time and also receive shocks (such as ligthtning) that could provoke the failure of the ammeter [5]. If the fatal shock process is independent of the degradation process, the system reliability can be obtained multiplying the reliability of the two processes [6], [7]. However, there are examples where both processes are dependent [8] [9],[10]. Although the majority of existing DTS models focuses on modeling the influence of random shocks on the degradation process, however, there are systems in which the shock process is influenced by the degradation process. The intuitive idea is that the system is more sensitive to shocks when it becomes wear-out. An engineering example is found in [11]. A sliding pool is a component applied in hydraulic control systems. A sliding pool is subject to two failure processes: wear, which is modeled as a degradation process and clamping stagnation, which is modeled by random shocks. When the wear increases, more wear debris is generated. The debris will contaminate the hydraulic oil and increase the likelihood of clamping stagnation. A dependent degradation-threshold-shock model is also used to describe the reliability of tire treads [12], [13]. A tire fails when its wear exceeds a threshold. Besides the wear, tires also experience many types of shocks, e.g. road debris may puncture the tire. The probability that a spike punctures a tire depends on the current wear of the tire. The probability of a shock failure begins to increase when degradation attains a certain level and drastically increases from a threshold level.

Normally, the arrival of the shocks to the system is modelled using a point process. One of the widely used point process is the non-homogeneous Poisson process (NHPP) which assumes an intensity dependent on the time. The NHPP does not require the condition of stationary increments and the shocks are more likely to occur during specific time intervals. Assuming a NHPP model for the arrival of the shocks, the mean and the variance of the number of shocks at any given time interval are equal. However, this is not always the case since some point processes present *over-dispersion*, that is, the variance to mean ratio at any time interval is greater than one. Furthermore, enviromental conditions may lead to changes in the failure rate [14]. To overcome this problem, the doubly stochastic Poisson process is defined. A doubly stochastic process (or Cox process) is a process where the time-variable intensity of the process is in itself a stochastic process. An explanation of the use of a Cox process to model pump failures is found in [15].

Maintenance strategies regulate the different maintenance tasks which must be performed on the system. Establishing a good maintenance task, the correct functioning of the system is ensured and the maintenance cost can be optimised. Condition-based maintenance (CBM) is one of the most popular techniques used for degrading systems. CBM is a maintenance program that recommends to perform maintenance actions based on the information collected through a monitoring process using certain types of sensors or other appropriate indicators [16]. The implementation of CBM programs for DTS models is not new (see [17] and [18] among others)

Many maintenance designs are planned based on an infinite operating horizon. It means that, after any replacement, the system is renewed by a new one with the same characteristics and the same process is assumed to be repeated indefinitely. Characteristics of these systems, such as the degradation level or the age, are often selected as criteria to optimise the long-run cost rate. Due to renewal properties, this long-run cost rate is equal to the expected cost in a renewal cycle divided by the length of the renewal cycle (see e.g. [19,20,21,22]). However, most systems actually have a finite operating life cycle since the system cannot always be replaced by a new one with similar properties as the previous one an infinite number of times. For instance, in military applications, a missile launching system is only required to be functioning within the designated mission time [23]. Hence, the use of steady state formulas is questionable and the maintenance cost should be analysed under a transient approach.

Although the transient approach is more realistic than its steady-state approximation, it is less used due to the analytical and computational difficulty of treatment that it involves. However, some works can be found in the literature where authors analyse maintenance strategies under a transient approach for systems subject to competing risks. For example, Taghipour *et al.* [24] proposed a model to find the optimal interval periodic inspection interval on a finite life cycle for a system subject to different types of failure.

This paper expands the works by Cheng et al. [25] and Pandey et al. [26] considering the time as a continuous variable and by adding a new component of risk (sudden shocks), whose arrival depends on the degradation process of the system. Following the framework exposed in [27], in this paper we assume that the system is degraded following a gamma process and sudden shocks arrive at the system following a doubly stochastic Poisson process (DSPP) whose intensity depends on the degradation process. It means that the "weaker" the system, the more susceptible it is to fail due to shocks [14]. According to [12] and [13], in this paper, the probability of a shock increases from a degradation threshold. A CBM with periodical inspection times is developed using the expected cost rate in the life cycle as objective cost function. The evaluation of the maintenance cost in the life cycle of the system is performed using recursive methods. Furthermore, the results obtained using recursive methods are compared to the results obtained based on Monte Carlo simulation. Further comparisons of the maintenance cost are performed considering an infinite life cycle. The robustness of the gamma process parameters and the shock process is also analysed.

In many application fields, there is an increasing interest in evaluating the performance of maintained systems. Reliability and availability are two important performance measures in the traditional reliability field. But some situations are not covered by these indexes and new performance measures, such as the interval reliability, has been developed [28,29]. Along with the maintenance assessment, in this paper, different performance measures are also evaluated in the life cycle of the system. The evaluation of these performance measures in the life cycle of the system is performed using recursive methods. Furthermore, the results obtained using recursive methods are compared to the results obtained based on Monte Carlo simulation.

In short, the main contributions of this paper are:

1. Development of a recursive method to obtain the transient expected cost and its standard deviation in the life cycle of the system.
2. Comparing the results obtained using this recursive method to the results obtained by using Monte Carlo simulation.
3. Comparing the steady state expected cost rate and the expected cost in the life cycle of the system.
4. Analysis of the robustness of the parameters that describe the functioning of the system.
5. Assessment of the the availability, reliability, and interval reliability in the life cycle of the system using a recursive method and comparison of the results obtained by using Monte Carlo simulation.

2 Framework of the problem

A system subject to two dependent competing causes of failure, degradation and sudden shocks, is considered in this paper. The general assumptions of this model are:

1. The system starts working at time $t = 0$. This system is subject to an internal degradation process which evolves according to a homogeneous gamma process with parameters α and β, where $\alpha, \beta > 0$. Let $X(t)$ be the deterioration level of the system at time t with $X(0) = 0$. Thus, for two time instants s and t, with $s < t$, the density function of the increment deterioration level $X(t) - X(s)$ is given by
$$f_{\alpha(t-s),\beta}(x) = \frac{\beta^{\alpha(t-s)}}{\Gamma(\alpha(t-s))} x^{\alpha(t-s)-1} e^{-\beta x}, \quad x > 0, \tag{1}$$
where $\Gamma(\cdot)$ denotes the gamma function defined as
$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} \, du. \tag{2}$$
The system fails due to degradation when the deterioration level exceeds a fixed threshold L, called the breakdown threshold.
2. The system not only fails due to internal degradation, but also it is subject to sudden shocks. Sudden shocks arrive at the system according to a process $\{N_s(t), t \geq 0\}$ where $N_s(t)$ denotes the number of sudden shocks up to time t. This process shows the dependence between degradation and shocks. Following the model shown in [27], we assume that $\{N_s(t), t \geq 0\}$ is a DSPP with intensity $\lambda(t, X(t))$ given by
$$\lambda\left(t, X(t)\right) = \lambda_1(t)\mathbf{1}_{\{X(t)\leq M_s\}} + \lambda_2(t)\mathbf{1}_{\{X(t)>M_s\}}, \tag{3}$$
where λ_1 and λ_2 denote two failure rate functions which satisfy $\lambda_1(t) \leq \lambda_2(t)$, for all $t \geq 0$ and where $\mathbf{1}_{\{\cdot\}}$ denote the indicator function which equals 1 if the argument is true and 0 otherwise. The arrival of a sudden shock provokes the system failure.
3. The system degradation is hidden and the system failure is non-self-announcing. It means that the system degradation and the system failure are only revealed through an inspection. We assume that the system is inspected each T ($T > 0$) time units ($t.u.$) to check its status. If the system is down during an inspection time, a corrective maintenance (CM) is performed and the system is replaced by a new one. If the system is working during an inspection time but the degradation level exceeds M, a preventive maintenance is performed and the system is replaced by a new one. Otherwise, no maintenance action is performed during an inspection time. We assume that the time required to perform a maintenance action is negligible.
4. All maintenance actions imply a cost. A CM and a PM have associated a cost of C_c and C_p monetary units ($m.u.$), respectively, and each inspection implies a cost of C_I $m.u.$ In addition, if the system fails, the system is down until the next inspection. Each time unit that the system is down, a cost of C_d $m.u./t.u.$ is incurred. We assume $C_c > C_p > C_I$.
5. Let $(0, t_f]$ be the finite operating life cycle of the system. It means that, if the calendar time exceeds t_f, the system can no longer be replaced by a new one with the same characteristics.

Let σ_z be the random variable describing the time to reach a certain degradation level z. The distribution function of σ_z, denoted as F_{σ_z}, is given by
$$F_{\sigma_z}(t) = P[X(t) \geq z] = \int_z^\infty f_{\alpha t,\beta}(x) \, dx = \frac{\Gamma(\alpha t, z\beta)}{\Gamma(\alpha t)}, \tag{4}$$

for $t \geq 0$ where $f_{\alpha t,\beta}(x)$ and $\Gamma(\alpha t)$ are given by (1) and (2), respectively, and

$$\Gamma(\alpha, x) = \int_x^\infty u^{\alpha-1} e^{-u}\ du,$$

denotes the incomplete gamma function for $x \geq 0$ and $\alpha > 0$.

For $z_1 \leq z_2$, the survival function of $\sigma_{z_2} - \sigma_{z_1}$ is given by

$$\bar{F}_{\sigma_{z_2}-\sigma_{z_1}}(t) = P\left[\sigma_{z_2} - \sigma_{z_1} \geq t\right] = \int_{x=0}^\infty \int_{y=z_1}^\infty f_{\sigma_{z_1}, X(\sigma_{z_1})}(x,y) F_{\alpha t,\beta}(z_2 - y)\ dy\ dx, \tag{5}$$

where $F_{\alpha t,\beta}$ denotes the distribution function of $f_{\alpha t,\beta}$ and $f_{\sigma_{z_1}, X(\sigma_{z_1})}$ denotes the joint density function of $(\sigma_{z_1}, X(\sigma_{z_1}))$ provided by Bertoin [30] as

$$f_{\sigma_{z_1}, X(\sigma_{z_1})}(x,y) = \int_0^\infty \mathbf{1}_{\{z_1 \leq y < z_1 + s\}} f_{\alpha x,\beta}(y-s)\mu(ds),$$

where $\mu(ds)$ denotes the Lévy measure associated with a gamma process with parameters α and β given by

$$\mu(ds) = \alpha \frac{e^{-\beta s}}{s}, \quad s > 0.$$

From Assumption 2, the shock process follows a DSPP where the intensity of the shocks depends on time t and on the degradation level $X(t)$. It means that, given a path x of $X(t)$, the process $\{N_s(t), t \geq 0\}$ is a NHPP with intensity $\lambda(t,x)$. In absence of maintenance, the time to a system failure is defined as the minimum $W = \min(\sigma_L, Y)$ where

$$Y = \inf\left(t \geq 0, N_s(t) = 1\right),$$

with survival function

$$P(W > t) = \mathbb{E}\left[\mathbf{1}_{\{\sigma_L > t\}} exp\left(-\int_0^t \lambda(s, X(s)ds)\right)\right]$$

Let $I(v,t)$ be the survival function of Y for $t \geq v$, conditioned to $\sigma_{M_s} = v$. That is

$$I(v,t) = P\left[Y > t|_{\sigma_{M_s}=v}\right] = \bar{F}_1(v)\frac{\bar{F}_2(t)}{\bar{F}_2(v)}, \tag{6}$$

where

$$\bar{F}_j(t) = \exp\left\{-\int_0^t \lambda_j(u)du\right\}, \quad j = 1, 2, \tag{7}$$

with density function $f_j(t)$, for $j = 1, 2$.

3 Maintenance cost analysis in the life cycle of the system

A goal in industry is to find the maintenance strategy that minimises an objective cost function. One of the most used function in the literature as objective cost function is the steady-state cost rate [26] that has a simple expression if the functioning of the system is modelled as a renewal process.

Since, in this paper, the system is renewed after each preventive or corrective maintenance, let $D_1, D_2, \ldots$ be the time between successive renewals of the system and let $D_1 = D$ be the time to the system renewal. Denoting by $C^{\infty}(T, M)$ the steady-state cost and applying the *"Renewal Theorem"*, $C^{\infty}(T, M)$ is equal to the expected cost in a renewal cycle divided by the length of the renewal cycle. That is

$$C^{\infty}(T, M) = \lim_{t \to \infty} \frac{C(t)}{t} = \frac{E[C]}{E[D]},$$

where $C(t)$ denotes the maintenance cost at time t, and C and D the cost and the length of a renewal cycle, respectively. In this paper, $C^{\infty}(T, M)$ is given by

$$C^{\infty}(T, M) = \frac{\displaystyle\sum_{k=1}^{\infty} \Big[C_c P_{D,c}(kT) + C_p P_{D,p}(kT) + C_I (k-1) P_D(kT) + C_d E\left[W((k-1)T, kT)\right] P_{D,c}(kT) \Big]}{\displaystyle\sum_{k=1}^{\infty} kT \; P_D(kT)} \quad (8)$$

where $P_D(kT)$ denotes the probability of a renewal at time kT for $k = 1, 2, \ldots$ given by

$$P_D(kT) = P_{D,p}(kT) + P_{D,c}(kT), \quad (9)$$

being $P_{D,p}(kT)$ and $P_{D,c}(kT)$ the probability of a preventive and corrective maintenance action at time kT for $k = 1, 2, \ldots$, respectively, and $E\left[W((k-1)T, kT)\right]$ the expected downtime in $((k-1)T, kT]$. The analytical expressions for these quantities were provided by Huynh *et al.* [27].

Since in this paper the objective is to evaluate the maintenance cost in the life cycle of the system, the expected cost rate in this life cycle is used as objective cost function. Let R_j be the chronological time of the j-th renewal cycle, for $j = 0, 1, 2, \ldots, N(t_f) + 1$, being $N(t_f)$ the number of renewals in the life cycle of the system. Thus, R_j is given by

$$R_j = \sum_{n=1}^{j} D_n,$$

for $R_0 = 0$, where D_n denotes the length of the n-th renewal cycle, with $n = 1, 2, \ldots, N(t_f) + 1$. Hence, the length of the n-th renewal cycle D_n is given by

$$D_n = \begin{cases} R_n - R_{n-1}, & \text{if} \quad n = 1, 2, \ldots, N(t_f) \\ t_f - R_{N(t_f)}, & \text{if} \quad n = N(t_f) + 1 \end{cases}.$$

The maintenance cost in the finite life cycle is the sum of the incurred costs in the different $N(t_f)$ renewal cycles and the incurred cost in $(R_{N(t_f)}, t_f]$. That is

$$C(t_f) = \sum_{j=1}^{N(t_f)} C(R_{j-1}, R_j) + C(R_{N(t_f)}, t_f),$$

where $C(t_1, t_2)$ denotes the cost in the interval $(t_1, t_2]$, and $C(0, t_f)$ is simplified as $C(t_f)$.

Next result provides the Markov renewal equation that fulfils the expected cost at time t, $E[C(t)]$.

Theorem 1 *For $t < T$, the expected transient cost, $E[C(t)]$ is given by*

$$\begin{aligned} E[C(t)] =& C_d \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v} \left(I(u,v) \bar{F}_{\sigma_L - \sigma_{M_s}}(v-u) \right) \right] (t-v)\ dv\ du \\ &+ C_d \int_0^t f_1(u) \bar{F}_{\sigma_{M_s}}(u)(t-u) du, \end{aligned}$$

where $\bar{F}_{\sigma_{M_s}}(u)$ ($f_{\sigma_{M_s}}(u)$) denotes the survival (density) function of σ_{M_s} given by (4), $\bar{F}_{\sigma_L - \sigma_{M_s}}$ and $I(x, y)$ the survival functions given by (5) and (6), and $f_1(x)$ is the density function of the survival function given by (7).

For $t \geq T$, the expected transient cost fulfils the following recursive equation

$$E[C(t)] = \sum_{k=1}^{\lfloor t/T \rfloor} E[C(t-kT)] P_D(kT) + G(t), \tag{10}$$

where

$$\begin{aligned} G(t) =& \sum_{k=1}^{\lfloor t/T \rfloor} \Big(C_p + C_I(k-1) \Big) P_{D,p}(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} \Big(C_c + C_I(k-1) + C_d E[W((k-1)T, kT)] \Big) P_{D,c}(kT) \\ &+ \left(\lfloor t/T \rfloor C_I + C_d E[W(\lfloor t/T \rfloor T, t)] \right) \left(1 - \sum_{k=1}^{\lfloor t/T \rfloor} P_D(kT) \right), \end{aligned}$$

with initial condition $E[C(0)] = 0$ and where $\lfloor t/T \rfloor$ denotes the integer part of t/T.

Proof. It is provided in Appendix A. □

Corollary 1 *Setting $E\left[C^{(i)}(t)\right] = E[C(t)]$, for all $(i-1)T < t \leq iT$ with $i = 1, 2, \ldots, \lfloor t_f/T \rfloor$ the expected transient cost, $E\left[C^{(i)}(t)\right]$, is given by*

$$\begin{aligned} E\left[C^{(1)}(t)\right] =& C_d \left(\int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v} \left(I(u,v) \bar{F}_{\sigma_L - \sigma_{M_s}}(v-u) \right) \right] (t-v)\ dv\ du \right) \\ &+ C_d \int_0^t f_1(u) \bar{F}_{\sigma_{M_s}}(u)(t-u) du, \end{aligned}$$

and for $i \geq 1$

$$E\left[C^{(i+1)}(t)\right] = G^{(i)}(t) + \sum_{k=1}^{i} E\left[C^{(i+1-k)}(t-kT)\right] P_D(kT),$$

where

$$\begin{aligned}G^{(i)}(t) =& \sum_{k=1}^{i} \Big(C_p + C_I(k-1)\Big) P_{D,p}(kT) \\ &+ \sum_{k=1}^{i} \Big(C_c + C_I(k-1) + C_d E\left[W((k-1)T, kT)\right]\Big) P_{D,c}(kT) \\ &+ \left(iC_I + C_d E\left[W(iT,t)\right]\right) \Big(1 - \sum_{k=1}^{i} P_D(kT)\Big).\end{aligned}$$

In order to analyse the uncertainty associated with the expected transient cost, the standard deviation is calculated. Let $S(t)^2$ be the variance of expected transient cost at time t defined as

$$S^2(t) = E\left[C(t)^2\right] - \left(E\left[C(t)\right]\right)^2. \tag{11}$$

Based on Theorem 1, the following result is obtained.

Theorem 2 *For $t < T$, the expected square cost at time $t > 0$, $E\left[C(t)^2\right]$, is given by*

$$\begin{aligned}E\left[C(t)^2\right] =& C_d^2 \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v}\left(I(u,v)\bar{F}_{\sigma_L - \sigma_{M_s}}(v-u)\right)\right](t-v)^2 \, dv \, du \\ &+ C_d^2 \int_0^t f_1(u) \bar{F}_{\sigma_{M_s}}(u)(t-u)^2 du.\end{aligned}$$

For $t \geq T$, the mean square fulfils the following recursive equation

$$E\left[C(t)^2\right] = \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C(t-kT)^2\right] P_D(kT) + H(t), \tag{12}$$

where

$$\begin{aligned}H(t) =& \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_p + C_I(k-1)\right)^2 P_{D,p}(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_c + C_I(k-1) + C_d E\left[W((k-1)T, kT)\right]\right)^2 P_{D,c}(kT) \\ &+ 2\sum_{k=1}^{\lfloor t/T \rfloor} \left(C_c + C_I(k-1)\right) E\left[C(t-kT)\right] P_{D,c}(kT) \\ &+ 2\sum_{k=1}^{\lfloor t/T \rfloor} C_d E\left[W((k-1)T, kT)\right] E\left[C(t-kT)\right] P_{D,c}(kT) \\ &+ 2\sum_{k=1}^{\lfloor t/T \rfloor} \left(C_p + C_I(k-1)\right) E\left[C(t-kT)\right] P_{D,p}(kT) \\ &+ \Big(\lfloor t/T \rfloor C_I + C_d E\left[W(\lfloor t/T \rfloor T, t)\right]\Big)^2 \left(1 - \sum_{k=1}^{\lfloor t/T \rfloor} P_D(kT)\right),\end{aligned}$$

with initial condition $E\left[C(0)^2\right] = 0$.

Proof. It is given in Appendix B. □

Corollary 2 *Setting $E\left[C^{(i)}(t)^2\right] = E\left[C(t)^2\right]$, for all $(i-1)T < t \le iT$ with $i = 1, 2, \ldots, \lfloor t_f/T \rfloor$ the expected square cost, $E\left[C^{(i)}(t)^2\right]$, is given by*

$$E\left[C^{(1)}(t)^2\right] = C_d^2 \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v}\left(I(u,v)\bar{F}_{\sigma_L-\sigma_{M_s}}(v-u)\right)\right](t-v)^2 \, dv \, du$$
$$+ C_d^2 \int_0^t f_1(u)\bar{F}_{\sigma_{M_s}}(u)(t-u)^2 du,$$

and for $i \ge 1$

$$E\left[C^{(i+1)}(t)^2\right] = \sum_{k=1}^{i} E\left[C^{(i+1-k)}(t-kT)^2\right] P_D(kT) + H^{(i)}(t),$$

where

$$\begin{aligned} H^{(i)}(t) = & \sum_{k=1}^{i} \left(C_p + C_I(k-1)\right)^2 P_{D,p}(kT) \\ & + \sum_{k=1}^{i} \left(C_c + C_I(k-1) + C_d E\left[W((k-1)T, kT)\right]\right)^2 P_{D,c}(kT) \\ & + 2\sum_{k=1}^{i} \left(C_c + C_I(k-1)\right) E\left[C(t-kT)\right] P_{D,c}(kT) \\ & + 2\sum_{k=1}^{i} C_d E\left[W((k-1)T, kT)\right] E\left[C(t-kT)\right] P_{D,c}(kT) \\ & + 2\sum_{k=1}^{i} \left(C_p + C_I(k-1)\right) E\left[C(t-kT)\right] P_{D,p}(kT) \\ & + \Big(iC_I + C_d E\left[W(iT,t)\right]\Big)^2 \left(1 - \sum_{k=1}^{i} P_D(kT)\right). \end{aligned}$$

Hence, by (11) the standard deviation of the cost at time t, $S(t)$, is given by

$$S(t) = \sqrt{E\left[C(t)^2\right] - \left(E\left[C(t)\right]\right)^2}. \tag{13}$$

4 Availability measures of the system

In addition to the expected cost and its standard deviation associated, recursive expressions for the availability, the reliability and the interval reliability of the system are obtained.

Let $A(t)$ be the availability of the system at time $t > 0$, that is, the probability that the system is working at time t.

$$A(t) = \sum_{j=0}^{\infty} \mathbf{1}_{\{R_j \le t < R_{j+1}\}} P\left[X(t-R_j) < L, Y > (t-R_j)\right].$$

Next result provides the Markov renewal equation that fulfils the availability $A(t)$.

Theorem 3 *For $t < T$, $A(t)$ is given by*

$$A(t) = \bar{F}_{\sigma_{M_s}}(t)\bar{F}_1(t) + \int_0^t f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_L - \sigma_{M_s}}(t-u)I(u,t)du.$$

For $t \geq T$, $A(t)$ fulfils the following Markov renewal equation

$$A(t) = \sum_{k=1}^{\lfloor t/T \rfloor} A(t-kT)P_D(kT) + J_1(t)\mathbf{1}_{\{M \leq M_s\}} + J_2(t)\mathbf{1}_{\{M > M_s\}}, \tag{14}$$

where

$$\begin{aligned} J_1(t) =& \bar{F}_{\sigma_M}(t)\bar{F}_1(t) + \int_{\lfloor t/T \rfloor T}^t f_{\sigma_M}(u) \int_u^t f_{\sigma_{M_s} - \sigma_M}(v-u)\bar{F}_{\sigma_L - \sigma_{M_s}}(t-v)I(v,t)\ dv\ du \\ &+ \int_{\lfloor t/T \rfloor T}^t f_{\sigma_M}(u)\bar{F}_{\sigma_{M_s} - \sigma_M}(t-u)\bar{F}_1(t)\ du, \end{aligned} \tag{15}$$

and

$$\begin{aligned} J_2(t) =& \bar{F}_{\sigma_{M_s}}(t)\bar{F}_1(t) + \int_0^{\lfloor t/T \rfloor T} f_{\sigma_{M_s}}(u) \int_{\lfloor t/T \rfloor T}^t f_{\sigma_M - \sigma_{M_s}}(v-u)\bar{F}_{\sigma_L - \sigma_M}(t-v)I(u,t)\ dv\ du \\ &+ \int_0^{\lfloor t/T \rfloor T} f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_M - \sigma_{M_s}}(t-u)I(u,t)\ du + \int_{\lfloor t/T \rfloor T}^t f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_L - \sigma_{M_s}}(t-u)I(u,t)\ du, \end{aligned} \tag{16}$$

with initial condition $A(0) = 1$ and where $P_D(kT)$ is given by (9).

Proof. It is given in Appendix C. □

Often, it is also of interest the probability that the system starts working at time 0 and it continues operating for a time interval. Let $R(t)$ be the reliability of the system at time t, that is, the probability that the system is working in $(0, t]$ given by

$$R(t) = P\left[O(u) < L, \forall u \in (0,t], N_s(0,t) = 0\right],$$

where $O(t)$ denotes the deterioration level of the maintained system at time t, that is,

$$O(t) = \sum_{j=0}^{\infty} \mathbf{1}_{\{R_j \leq t < R_{j+1}\}} X(t - R_j)$$

Based on Theorem 3, the following result is obtained.

Theorem 4 *For $t < T$, $R(t)$ is given by*

$$R(t) = A(t).$$

For $t \geq T$, $R(t)$ fulfils the following Markov renewal equation

$$R(t) = \sum_{k=1}^{\lfloor t/T \rfloor} R(t-kT)P_{D,p}(kT) + J_1(t)\mathbf{1}_{\{M \leq M_s\}} + J_2(t)\mathbf{1}_{\{M > M_s\}}, \tag{17}$$

with initial condition $R(0) = 1$ where J_1 and J_2 are given by (15) and (16), respectively.

Proof. It is given in Appendix D. □

An availability measure that extends the reliability is the interval reliability, defined as the probability that the system is working at time t, and will continue working over a finite time interval of length s. The interval reliability is applied when there are periods in the lifetime cycle where a failure should be avoided with high probability. Let $IR(t, t+s)$ be the interval reliability in $(t, t+s]$. That is

$$IR(t, t+s) = P\left[O(u) < L, \forall u \in (t, t+s], N_s(t, t+s) = 0\right].$$

Availability and reliability are particular cases of the interval reliability since

$$A(t) = IR(t, t+0), \quad R(t) = IR(0, 0+t).$$

Based on Theorems 3 and 4, the following result is obtained.

Theorem 5 *For $t+s < T$, $IR(t, t+s)$ is given by*

$$IR(t, t+s) = R(t+s).$$

For $t+s \geq T$, $IR(t, t+s)$ fulfils the following Markov renewal equation

$$\begin{aligned} IR(t, t+s) = &\sum_{k=\lfloor t/T \rfloor+1}^{\lfloor (t+s)/T \rfloor} R(t+s-kT) P_{D,p}(kT) + \sum_{k=1}^{\lfloor t/T \rfloor} IR(t-kT, t+s-kT) P_D(kT) \\ &+ J_1(t+s)\mathbf{1}_{\{M \leq M_s\}} + J_2(t+s)\mathbf{1}_{\{M > M_s\}}, \end{aligned} \tag{18}$$

with initial conditions $R(0) = 1$ and $IR(0, 0) = 1$.

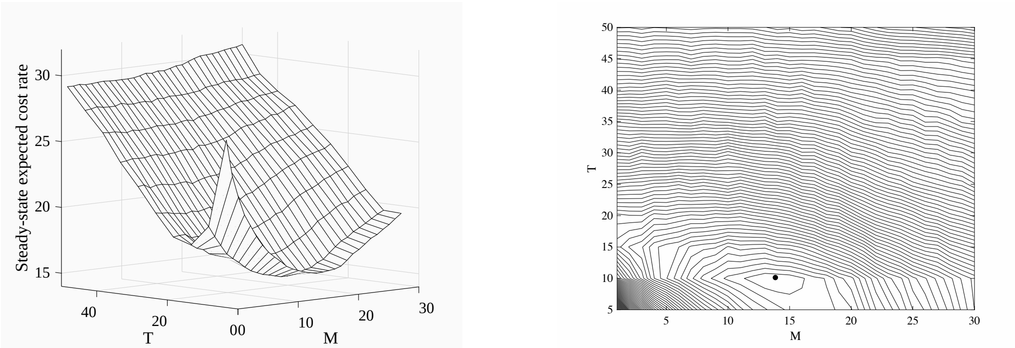
Proof. It is given in Appendix E. □

5 Numerical examples

In this section, some numerical examples are provided to illustrate the analytical results. To this end, we consider a system subject to an underlying degradation process modelled as a homogeneous gamma process with parameters $\alpha = \beta = 0.1$. We assume that the system fails when the deterioration level of the system reaches the breakdown threshold $L = 30$. The system can also fail due to a sudden shock and the sudden shock process is modelled under a DSPP with intensity

$$\lambda\left(t, X(t)\right) = 0.01 \cdot \mathbf{1}_{\{X(t) \leq M_s\}} + 0.1 \cdot \mathbf{1}_{\{X(t) > M_s\}}, \quad t \geq 0,$$

where $M_s = 20$. Under these conditions, the expected time to a degradation failure is $E\left[\sigma_L\right] = 34.0335$ $t.u.$ and the expected time due to a shock failure is $E\left[Y\right] = 28.3556$ $t.u.$ In addition, the cost sequence $C_c = 300$ $m.u.$, $C_p = 150$ $m.u.$, $C_I = 45$ $m.u.$, and $C_d = 25$ $m.u./t.u.$ is imposed. We assume that the life cycle of the system is $(0, 50]$.



(a) Mesh plot.

(b) Contour plot.

Fig. 1: Mesh and contour plot for the steady-state expected cost rate.

5.1 Steady-state expected cost rate analysis

Considering the previous dataset, the steady-state expected cost rate is first computed to establish the values of T and M which will be used subsequently in the transient approach analysis. In this way, the results obtained by using steady-state and transient formulations shall be compared.

The optimisation problem for the expected cost based on the steady-state formula given in (8) is computed as follows:

1. A grid of size 10 is obtained discretising the set $[5, 50]$ into 10 equally spaced points from 5 to 50 for T. For $i = 1, 2, \ldots, 10$, let T_i be the i-th value of the grid obtained previously.
2. A grid of size 30 is obtained by discretising the set $[1, 30]$ into 30 equally spaced points from 1 to 30 for M. For $j = 1, 2, \ldots, 30$, let M_j be the j-th value of the grid obtained previously.
3. For each fixed combination (T_i, M_j), we obtain 50000 simulations of (D, I, W_d), where D corresponds to the time to a maintenance action, I the nature of the maintenance action (corrective or preventive) and W_d the downtime in a renewal cycle. With these simulations, and applying Monte Carlo method, we obtain $\tilde{P}_{D,p}(kT_i)$, $\tilde{P}_{D,c}(kT_i)$, $\tilde{P}_D(kT_i)$, and $\tilde{E}\left[W((k-1)T_i, kT_i)\right]$ for each fixed combination (T_i, M_j) which correspond to the estimations of $P_{D,p}(kT_i))$, $P_{D,c}(kT_i))$, $P_D(kT_i))$, and $E\left[W_{T_i}((k-1)T_i, kT_i))\right]$ for $k = 1, 2, \ldots, \lfloor 50/T_i \rfloor$, respectively.
4. Quantity $C^\infty(T, M)$, which represents the steady-state expected cost rate, is evaluated by using Equation (8) replacing the corresponding probabilities by their estimations calculated in Step 3.
5. The optimisation problem is reduced to find the values T_{opt} and M_{opt} which minimise the steady-state expected cost rate $C^\infty(T, M)$. That is

$$C^\infty(T_{opt}, M_{opt}) = \min_{\substack{T \geq 0 \\ 0 \leq M \leq L}} \left\{ C^\infty(T, M) \right\}.$$

Fig. 1 shows the expected cost rate versus T and M. The values of T and M which minimise the expected cost rate are reached at $M_{opt} = 14$ $d.u.$ and $T_{opt} = 10$ $t.u.$, with an expected cost rate of 15.3819 $m.u./t.u.$ and a 95% confidence interval of (13.6658, 17.0980). Below, the expected cost in the finite life cycle will be compared to the steady-state expected cost using the values T_{opt} and M_{opt}.

5.2 Expected transient cost rate analysis for a fixed T

We consider a time between inspections $T = 10$ $t.u.$ The optimisation problem for the expected transient cost based on the recursive formula given in (10) is computed as follows.

1. A grid of size 30 is obtained by discretising the set $[1, 30]$ into 30 equally spaced points from 1 to 30 for M.
2. For fixed $T = 10$ $t.u.$ and for each value of the grid, we obtain 50000 simulations of (D, I, W_d). With these simulations, and applying Monte Carlo method, we obtain the estimations $\tilde{P}_{D,p}(10k)$, $\tilde{P}_{D,c}(10k)$, $\tilde{P}_D(10k)$, and $\tilde{E}\left[W((k-1)10, 10k)\right]$.
3. For fixed $T = 10$ $t.u.$, the expected cost in the life cycle is calculated by using the recursive formula given in (10), replacing the corresponding probabilities by their estimations and initial condition $E\left[C(0)\right] = 0$.
4. The optimisation problem is reduced to find the value M_{opt} which minimises the expected cost rate $E\left[C(50)\right]/50$.

The expected cost rate calculated using the recursive method and the expected cost rate calculated using Monte Carlo simulation are shown in Fig. 2. The expected transient cost rate based on Monte Carlo simulation was calculated for 30 equally spaced points in the interval $(0, 30]$ with 50000 simulations for each point. The expected transient cost rate calculated using the recursive method reaches its minimum value at $M = 14$ $d.u.$, with an expected transient cost rate of 14.7639 $m.u./t.u.$ and a 95% confidence interval of $(14.6927, 14.8351)$. On the other hand, the expected transient cost rate calculated by using Monte Carlo simulation reaches its minimum value at $M = 14$ $d.u.$, with an expected transient cost rate of 15.2096 $m.u./t.u.$ and a 95% confidence interval of $(15.1435, 15.2757)$.

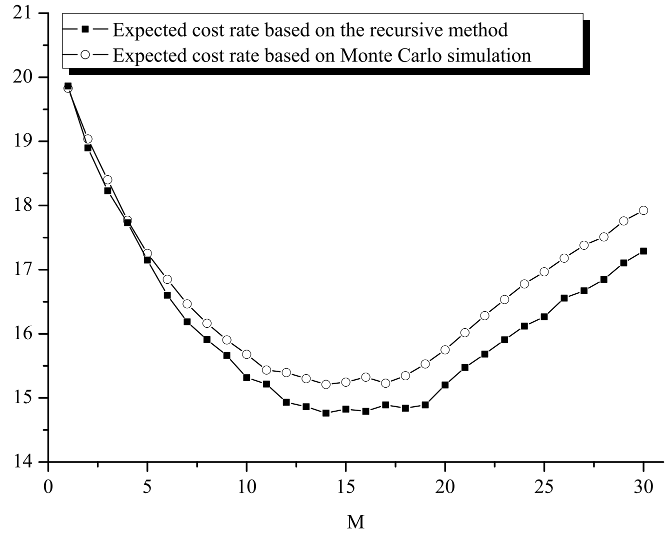


Fig. 2: Expected cost rate in the life cycle versus M.

For $T = 10$ $t.u.$, Table 1 shows the average number of completed renewal cycles in the life cycle of the system.

$\mathbf{M}$	1	2	3	4	5	6	7	8	9	10
$\mathbf{E}\left[\mathbf{N}_{10}^{\mathbf{M}}(\mathbf{50})\right]$	4.6678	4.3307	4.0400	3.7683	3.5316	3.3157	3.1261	2.9485	2.7957	2.6585
$\mathbf{M}$	11	12	13	14	15	16	17	18	19	20
$\mathbf{E}\left[\mathbf{N}_{10}^{\mathbf{M}}(\mathbf{50})\right]$	2.5253	2.4146	2.3046	2.2052	2.1092	2.0310	1.9403	1.8688	1.7999	1.7386
$\mathbf{M}$	21	22	23	24	25	26	27	28	29	30
$\mathbf{E}\left[\mathbf{N}_{10}^{\mathbf{M}}(\mathbf{50})\right]$	1.6874	1.6520	1.6247	1.5947	1.5676	1.5415	1.5234	1.4948	1.4825	1.4612

Table 1: Average number of complete renewal cycles up to $t_f = 50$ $t.u.$ versus M.

For $T = 10$ $t.u.$, Fig. 3 shows the expected transient cost rate calculated by using the recursive method and the steady-state expected cost rate calculated throughout the procedure detailed in Section 5.1 versus M. As we said previously, the value of M which minimises the expected transient cost rate is reached at $M = 14$ $d.u.$, with an expected cost rate of 14.7639 $m.u./t.u.$ and a 95% confidence interval of $(14.6927, 14.8351)$. On the other hand, the steady-state expected cost rate reaches its minimum value at $M = 14$ $d.u.$, with an expected cost rate of 15.3819 $m.u./t.u.$ and a 95% confidence interval of $(13.6658, 17.0980)$ $m.u./t.u.$

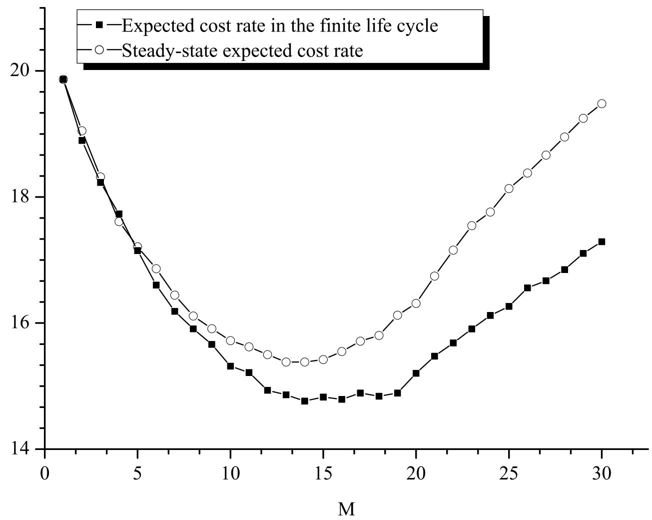


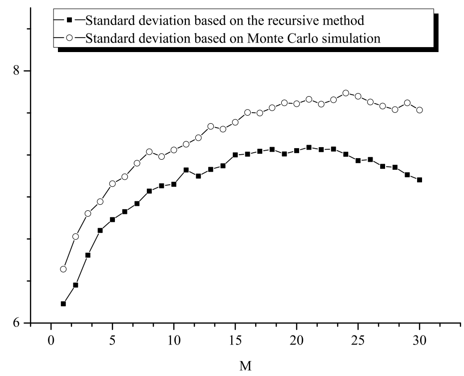
Fig. 3: Expected cost rate in the life cycle and steady-state expected cost rate versus M.

Deviation of the expected maintenance cost was calculated for 30 equally spaced points in $(0, 30]$ and shown in Fig 4a. The expected maintenance cost with its standard deviation calculated by using the recursive method and by using Monte Carlo simulation are shown in Figures 4b and 4c.

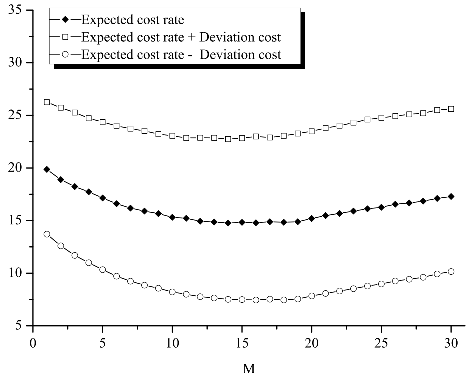
Also, we focus on the influence of the main model parameters on the expected cost in the life cycle. Firstly, a sensitivity analysis of the gamma process parameters is performed.

The values of the gamma process parameters are modified according to the following specifications:

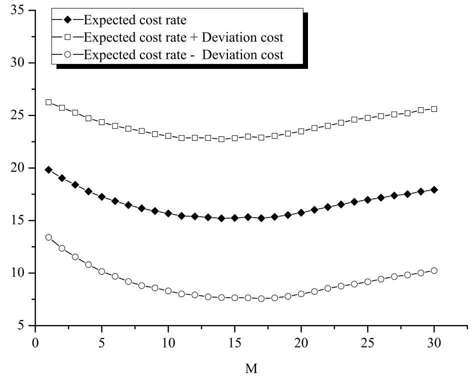
$$\alpha_{(v_i\%)} = \alpha\left[1 + \frac{v_i}{100}\right] \quad \text{and} \quad \beta_{(v_j\%)} = \beta\left[1 + \frac{v_j}{100}\right], \tag{19}$$



(a) Standard deviation versus M.



(b) Recursive method.



(c) Monte Carlo simulation.

Fig. 4: Expected cost rate and standard deviation in the life cycle versus M.

where v_i and v_j are, respectively, the i-th and j-th position of the vector $\mathbf{v} = (-10, -5, -1, 0, 1, 5, 10)$. Then, the parameter values for α and β can be simultaneous and independently modified both for increasing and decreasing changes.

Let $E\left[C_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(t_f)\right]$ be the minimal expected transient cost at time t_f obtained when the gamma process parameters (α and β) are varied according to the specifications given in (19). The expected transient cost for each combination of $\alpha_{(v_i\%)}$ and $\beta_{(v_j\%)}$ are calculated based on the recursive method following the steps detailed in 5.2. The relative measure $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ is defined as

$$\frac{\left|E\left[C(50)\right] - E\left[C_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)\right]\right|}{E\left[C(50)\right]}, \tag{20}$$

where $E\left[C(50)\right]$ is the minimal expected transient cost calculated in 5.2.

For fixed M and $T = 10$ $t.u.$, quantity $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ measures the relative difference between the minimal expected transient cost with the original parameter values and the minimal expected transient cost calculated using the modified parameter values. Values closer to zero have a lower influence on the expected transient cost rate.

Table 2 shows the relative variation percentages with a shaded grey scale. Each cell represents the quantity $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ expressed in percentage. Darker colours of cells denote a higher

	$\beta_{(-10\%)}$	$\beta_{(-5\%)}$	$\beta_{(-1\%)}$	β	$\beta_{(1\%)}$	$\beta_{(5\%)}$	$\beta_{(10\%)}$
$\alpha_{(-10\%)}$	1.1862	2.2865	5.8303	5.7383	6.4372	8.6707	11.0707
$\alpha_{(-5\%)}$	4.1067	0.9778	1.9635	2.7403	3.2077	6.0850	9.0580
$\alpha_{(-1\%)}$	7.0083	2.7336	0.6373	0.3666	1.0904	3.6113	6.1380
α	8.0187	3.8252	1.0523	0.0000	0.2624	2.8032	6.0328
$\alpha_{(1\%)}$	8.7105	4.5726	1.4507	1.0696	0.0267	2.6336	5.1357
$\alpha_{(5\%)}$	11.3000	7.0853	4.0552	3.6551	2.7247	0.2344	2.7598
$\alpha_{(10\%)}$	15.2929	10.5769	7.2532	6.6684	5.7666	3.0604	0.0958

Table 2: Relative variation percentages for the expected transient cost for the gamma process parameters for a fixed $T = 10$ $t.u.$

relative variation percentage. The results obtained show that $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ grows when α increases and β decreases and $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ decreases when α decreases and β increases. In this way, $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ reaches its minimum value when α is minimum and β is maximum and its maximum value when α is maximum and β is minimum.

By modifying $\pm 1\%$ around $\alpha = \beta = 0.1$, the relative variation percentages are small. The results also show that the relative variation percentages are lower in the diagonal of the table. In general, when α has a positive variation, $\beta_{(-5\%)}$ and $\beta_{(-10\%)}$ are higher than $\beta_{(5\%)}$ and $\beta_{(10\%)}$, respectively. Analogously, when α has a negative variation, $\beta_{(-5\%)}$ and $\beta_{(-10\%)}$ are lower than $\beta_{(5\%)}$ and $\beta_{(10\%)}$, respectively.

Similarly, the values of the parameters λ_1 and λ_2 are modified according to the following specifications:

$$\lambda_{1,(v_i\%)} = \lambda_1 \left[1 + \frac{v_i}{100}\right] \quad \text{and} \quad \lambda_{2,(v_j\%)} = \lambda_2 \left[1 + \frac{v_j}{100}\right], \tag{21}$$

Let $E^*\left[C_{\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(t_f)\right]$ be the minimal expected transient cost obtained by varying the parameters λ_1 and λ_2 simultaneously as in the scheme given in (21). Now, the relative variation $V_{\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50)$ is given by

$$\frac{\left|E^*\left[C(50)\right] - E^*\left[C_{\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50)\right]\right|}{E^*\left[C(50)\right]}. \tag{22}$$

The relative variation percentages are presented in Table 3. The results show that the parameter λ_1 has greater effects on $V_{\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50)$ than the parameter λ_2, reaching the lowest values when the variation for $\lambda_1 = 0.01$ is minimal, that is $\pm 1\%$, and the highest values when the variation for λ_1 is maximised, that is $\pm 10\%$.

5.3 Analysis of the maintenance cost in the life cycle for a fixed M

For fixed $M = 14$, the influence of parameter T is analysed. As in Section 5.2, the optimisation problem for the expected transient cost based on the recursive formula given in (10) is computed throughout the following steps.

1. A grid of size 10 for T is obtained by discretising the set $[5, 50]$ into 10 equally spaced points from 5 to 50.

	$\lambda_{2,(-10\%)}$	$\lambda_{2,(-5\%)}$	$\lambda_{2,(-1\%)}$	λ_2	$\lambda_{2,(1\%)}$	$\lambda_{2,(5\%)}$	$\lambda_{2,(10\%)}$
$\lambda_{1,(-10\%)}$	3.0794	2.4761	2.3266	1.9374	1.8849	2.0284	1.3703
$\lambda_{1,(-5\%)}$	1.6095	1.2419	1.3581	0.8946	0.9002	0.3058	0.3755
$\lambda_{1,(-1\%)}$	0.6159	0.5227	0.2235	0.0000	0.5353	0.0241	0.9365
λ_1	0.4598	0.2900	0.2235	0.0000	0.5353	0.0241	0.9365
$\lambda_{1,(1\%)}$	0.4598	0.2451	0.4946	0.0118	0.3289	1.0549	0.9833
$\lambda_{1,(5\%)}$	0.2698	0.3685	0.5846	0.7321	0.4718	0.8365	1.1554
$\lambda_{1,(10\%)}$	1.3381	2.3405	2.7045	2.2538	2.7964	2.5593	3.2779

Table 3: Relative variation percentages for the expected transient cost for parameters λ_1 and λ_2 for a fixed $T = 10$ $t.u.$

2. For fixed $M = 14$ $d.u.$ and for each fixed T_i, we obtain 50000 simulations of (D, I, W_d). With these simulations and applying Monte Carlo method, we obtain the estimations $\tilde{P}_{D,p}(kT_i)$, $\tilde{P}_{D,c}(kT_i)$, $\tilde{P}_D(kT_i)$, and $\tilde{E}\left[W((k-1)T_i, kT_i)\right]$ for $k = 1, 2, \ldots, \lfloor 50/T_i \rfloor$.
3. For each T_i and $M = 14$, the expected cost at time $t = 50$ is calculated by using the recursive formula given in (10), replacing the corresponding probabilities by their estimations and initial condition $E\left[C(0)\right] = 0$.
4. For fixed $M = 14$ $d.u.$, the optimisation problem is reduced to find the value of T which minimises the expected cost rate in the life cycle of the system $E\left[C(50)\right]/50$.

The expected cost rate evaluated using the recursive formula and using Monte Carlo simulation are shown in Fig. 5. The expected cost rate based on Monte Carlo simulation was calculated for 10 equally spaced points in $(5, 50]$ with 50000 realizations for each point. Based on Fig. 5, for

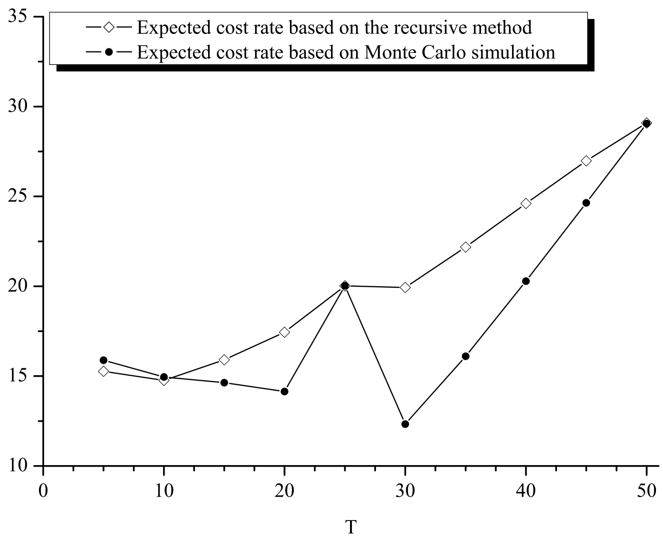


Fig. 5: Expected cost rate in the life cycle versus T.

the recursive method, the expected cost rate at time $t_f = 50$ $t.u.$ reaches its minimum value for $T = 10$ $t.u.$, with an expected cost rate of 14.7637 $m.u./t.u.$ and a 95% confidence interval of

$(14.6927, 14.8351)$. On the other hand, using Monte Carlo simulation, the expected cost rate in the life cycle reaches its minimum value at $T = 30$ $t.u.$, with an expected cost rate of 12.3259 $m.u./t.u.$ and a 95% confidence interval of $(12.2275, 12.4243)$. This difference can be explained due to the high variance in deterioration increments of the degradation process.

For $M = 14$ $d.u.$ Table 4 shows the average number of renewals of the system in its life cycle for each value of T.

$\mathbf{T}$	**5**	**10**	**15**	**20**	**25**
$\mathbf{E}\left[\mathbf{N}_{\mathbf{T}}^{\mathbf{14}}(\mathbf{50})\right]$	2.4650	2.2007	1.7772	1.4327	1.6419

$\mathbf{T}$	**30**	**35**	**40**	**45**	**50**
$\mathbf{E}\left[\mathbf{N}_{\mathbf{T}}^{\mathbf{14}}(\mathbf{50})\right]$	0.8884	0.93964	0.9682	0.9833	0.9926

Table 4: Average number of complete renewal cycles up to $t_f = 50$ versus T.

For fixed $M = 14$ $d.u.$, Fig. 6 shows the expected cost rate in the life cycle of the system calculated by using the recursive method and the steady-state expected cost rate versus T. As

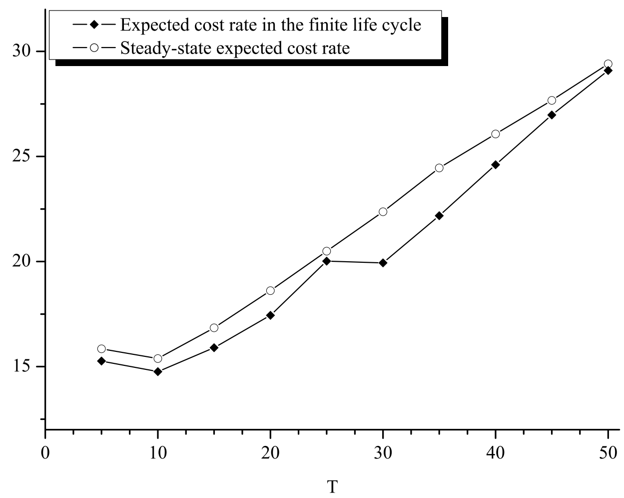
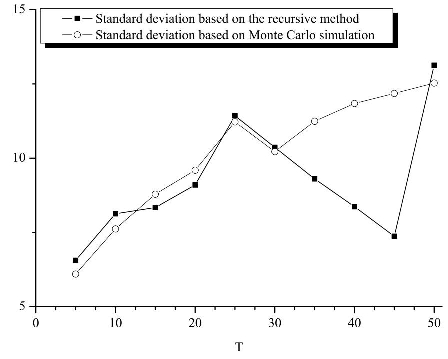


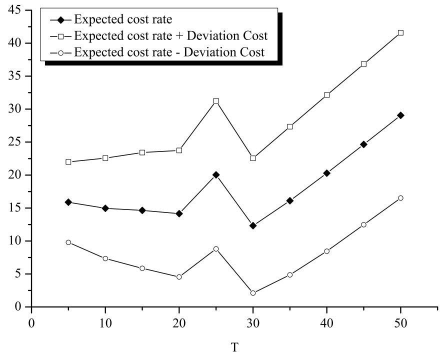
Fig. 6: Expected cost rate in the life cycle and steady-state expected cost rate versus T.

we said previously, the expected transient cost rate calculated using the recursive method at time $t_f = 50$ $t.u.$ reaches its minimum value at $T = 10$ $t.u.$, with an expected transient cost rate of 14.7637 $m.u./t.u.$ and a 95% confidence interval of $(14.6927, 14.8351)$. On the other hand, the steady-state expected cost rate reaches its minimum value at $T = 10$ $t.u.$, with a steady-state expected cost rate of 15.3819 $m.u./t.u.$ and a 95% confidence interval of $(13.6658, 17.0980)$. Steady-state expected cost rate shows a smoother behaviour compared to the expected transient cost rate.

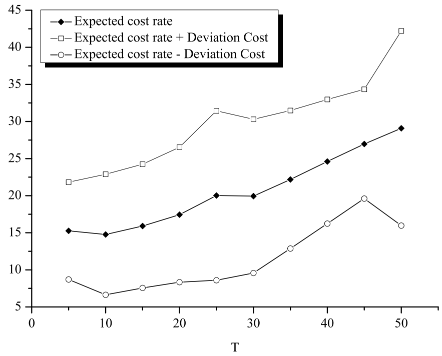
Next, the standard deviation of the expected maintenance cost is obtained. Deviation was calculated for 10 equally spaced points in $[5, 50]$ and shown in Fig 7a. Fig. 7b and Fig 7c show the expected transient cost rate calculated using the recursive method and Monte Carlo simulation with its standard deviation associated versus T.



(a) Standard deviation versus T.



(b) Recursive method.



(c) Monte Carlo method.

Fig. 7: Expected cost rate and standard deviation versus T.

Focusing now on the main model parameters influence on the solution, we analyse the gamma process parameters sensitivity.

For fixed T and $M = 14$, let $E\left[C_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)\right]$ be the minimal expected transient cost at time $t_f = 50$ $t.u.$ obtained when the gamma process parameters (α and β) are varied according to the specifications given in (19). Based on (20), $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ denotes the relative variation between the minimal expected transient cost with the original parameter values and the minimal expected transient cost calculated by using the parameter values modified according to (19).

Table 5 shows the values obtained for $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ expressed in percentage. By modifying $\pm 1\%$ around $\alpha = \beta = 0.1$, the relative variation percentages are small. The results obtained also show that $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ is lower in the diagonal of the table. That means when the parameters α and β are modified in the same direction and magnitude. Thus, as previously, when α has a positive variation, $\beta_{(-5\%)}$ and $\beta_{(-10\%)}$ are higher than $\beta_{(5\%)}$ and $\beta_{(10\%)}$, respectively. Analogously, when α has a negative variation, $\beta_{(-5\%)}$ and $\beta_{(-10\%)}$ are lower than $\beta_{(5\%)}$ and $\beta_{(10\%)}$, respectively.

On the other hand, let $E^*\left[C_{\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(t_f)\right]$ be the minimal expected transient cost obtained by varying the parameters λ_1 and λ_2 simultaneously as in the scheme given in (21). Based on (22), $V_{\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50)$ denotes the relative variation between the minimal expected transient cost with the original parameter values and the minimal expected transient cost calculated by using the parameter values modified according to (21) for variable T and $M = 14$ $d.u.$

	$\beta_{(-10\%)}$	$\beta_{(-5\%)}$	$\beta_{(-1\%)}$	β	$\beta_{(1\%)}$	$\beta_{(5\%)}$	$\beta_{(10\%)}$
$\alpha_{(-10\%)}$	1.5684	1.8862	4.7069	5.7107	6.1450	8.0609	11.2240
$\alpha_{(-5\%)}$	4.5915	0.9400	1.7235	2.5432	3.3861	5.1360	8.1300
$\alpha_{(-1\%)}$	8.5189	3.4618	0.9003	0.0450	0.6994	3.0001	6.2257
α	8.2817	4.4937	1.3744	0.0000	0.1749	2.7793	5.2119
$\alpha_{(1\%)}$	8.9568	4.7309	1.9896	1.8475	0.4936	1.4929	5.0625
$\alpha_{(5\%)}$	11.5044	7.2387	5.0048	3.5146	3.3441	0.8091	2.7099
$\alpha_{(10\%)}$	15.1620	10.7048	7.5764	6.5529	6.7841	3.6827	0.6109

Table 5: Relative variation percentages for the expected transient cost for the gamma process parameters, $M = 14$ *d.u.*

	$\lambda_{2,(-10\%)}$	$\lambda_{2,(-5\%)}$	$\lambda_{2,(-1\%)}$	λ_2	$\lambda_{2,(1\%)}$	$\lambda_{2,(5\%)}$	$\lambda_{2,(10\%)}$
$\lambda_{1,(-10\%)}$	2.1506	1.9893	1.9244	1.2564	1.7375	1.4737	1.7043
$\lambda_{1,(-5\%)}$	0.8525	0.3948	0.7135	0.2861	0.0258	0.2199	0.2733
$\lambda_{1,(-1\%)}$	0.8510	0.5039	0.4442	0.6259	0.4789	0.7896	1.3432
λ_1	0.4986	0.5000	0.4663	0.0000	0.7516	1.0475	1.1239
$\lambda_{1,(1\%)}$	0.0194	0.6036	0.9957	0.9308	0.7760	1.1128	1.9800
$\lambda_{1,(5\%)}$	1.9045	1.5869	2.8829	1.4909	2.1346	2.5697	3.0034
$\lambda_{1,(10\%)}$	2.4267	2.1395	2.8044	2.9864	2.8605	3.3628	3.0087

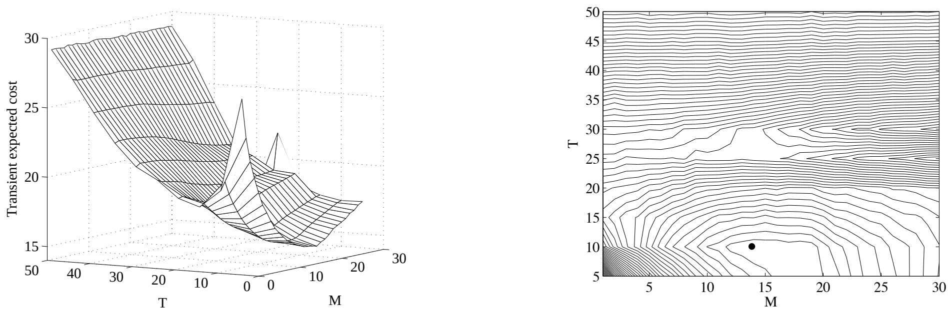
Table 6: Relative variation percentages for the expected transient cost rate for parameters λ_1 and λ_2, $M = 14$ *d.u.*

The relative variation percentages are presented in Table 6. The results show that when $\lambda_1 = 0.01$ is modified between -5% and 1%, the relative variation percentages are small. In addition, the parameter λ_1 has greater effects on $V_{\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50)$ than the parameter λ_2.

5.4 Transient two-dimensional expected cost rate analysis

The expected transient cost based on the recursive formula given in (10) versus M and T is analysed. The optimisation problem is computed as follows.

1. A grid of size 10 is obtained by discretising $[5, 50]$ into 10 equally spaced points from 5 to 50 for T. Let T_i be the i-th value of the grid obtained previously, for $i = 1, 2, \ldots, 10$.
2. A grid of size 30 is obtained by discretising $[1, 30]$ into 30 equally spaced points from 1 to 30 for M. Let M_j be the j-th value of M which corresponds to the i-th value of the grid obtained previously, for $j = 1, 2, \ldots, 30$.
3. For each fixed combination (T_i, M_j), we obtain the estimations of the probabilities involved in the model.
4. For each fixed combination T_i and M_j, the expected cost at time $t = 50$ is calculated by using the recursive formula given in (10), replacing the corresponding probabilities by their estimations and initial condition $E[C(0)] = 0$.
5. The optimisation problem is reduced to find the values T_{opt} and M_{opt} which minimise the expected cost rate $E[C(50)]/50$.



(a) Mesh plot. (b) Contour plots.

Fig. 8: Mesh and contour plots for the expected cost rate in the life cycle versus T and M.

The expected cost rate versus T and M is shown in Fig. 8. The values of T and M which minimise the expected cost rate in the life cycle of the system are reached for $M = 14$ $d.u.$ and $T = 10$ $t.u.$ with an expected cost rate of 14.7637 $m.u./t.u.$ and a 95% confidence interval of $(14.6927, 14.8351)$.

5.5 Availability measures for optimal values of T and M

For fixed $T = 10$ and $M = 14$, the availability, the reliability and the interval reliability of the system is computed.

The availability of the system based on the recursive formula given in (14) is computed throughout the following steps.

1. A grid of size 50 is obtained by discretising $[1, 50]$ into 50 equally spaced points from 1 to 50 for the instant time t. Let t_n be the n-th value of the grid obtained previously, for $n = 1, 2, \ldots, 50$.
2. The availability of the system is calculated by using the recursive formula given in (14), replacing $P_D(10k)$ by its estimation $\tilde{P}_D(10k)$ and initial condition $A(0) = 1$.

Figure 9 shows the availability of the system versus t. We can conclude that, for fixed $T = 10$ and $M = 14$, the probability that the system is working at any instant time of its life cycle is, at least, of the 82.36% with a 95% confidence interval of $(0.8203, 0.8269)$.

Next, the reliability of the system is evaluated. The reliability of the system based on the recursive formula given in (17) is computed throughout the following steps.

1. A grid of size 50 is obtained by discretising $(1, 50]$ into 50 equally spaced points from 1 to 50 for t.
2. The system reliability is calculated by using the recursive formula given in (17), replacing $P_{D,p}(10k)$ by its estimation $\tilde{P}_{D,p}(10k)$ and initial condition $R(0) = 1$.

Figure 10 shows the reliability of the system versus t. We can conclude that, for fixed $T = 10$ and $M = 14$, the probability that the system does not fail in its life cycle is of the 32.44% with a 95% confidence interval of $(0.3203, 0.3285)$.

Finally, the interval reliability of the system based on (18) is computed throughout the following steps.

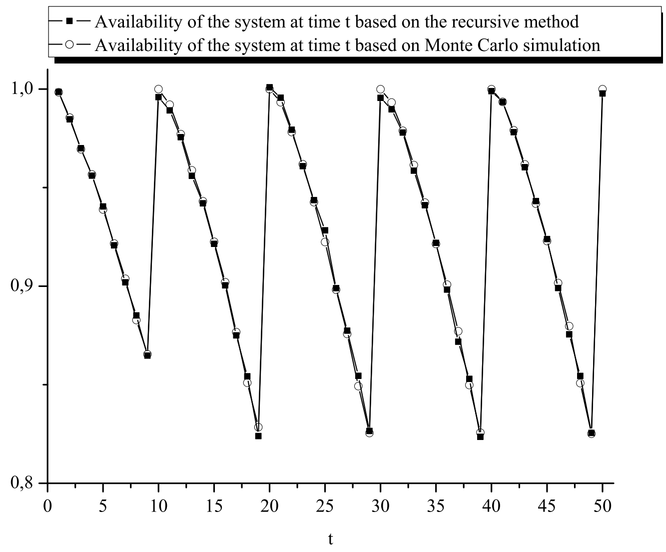


Fig. 9: Availability of the system versus t.

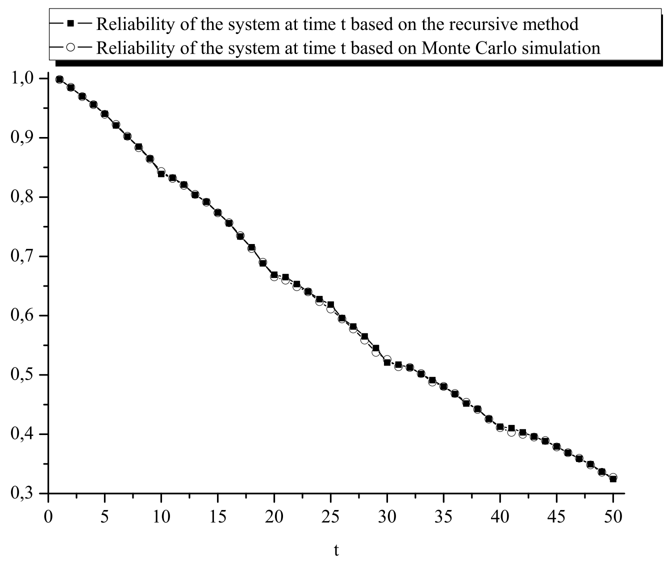


Fig. 10: Reliability of the system versus t.

1. A grid of size 10 is obtained by discretising the set $[10, 30]$ into 10 equally spaced points from 10 to 30 for t.
2. For $s = 5$, the interval reliability of the system is calculated by using the recursive formula given in (18), replacing $P_{D,p}(10k)$ and $P_D(10k)$ by their estimations $\tilde{P}_{D,p}(10k)$ and $\tilde{P}_D(10k)$, respectively, and initial conditions $IR(0, 0) = 1$ and $R(0) = 1$.

Figure 11 shows the interval reliability of the system versus t. As we can observe, the results provide for both methods are very similar. Furthermore we can conclude that, for fixed $T = 10$

and $M = 14$, the probability that the system does not fail in $[t, t+5]$ is, at least, of the 70.47% for $15 \leq t \leq 35$ with a 95% confidence interval of $(0.7008, 0.7086)$.

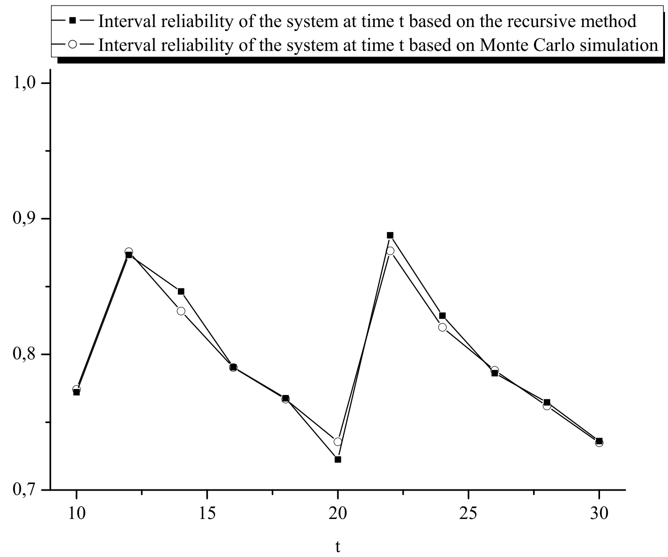


Fig. 11: Interval reliability of the system versus t.

6 Conclusions and further works

In this paper, a CBM strategy is analysed by considering a finite life cycle of the system. The system is subject to two different causes of failure, a degradation process modelled under a gamma process and a sudden shock process which follows a DSPP. We consider that both causes of failure are dependent. This dependence is reflected in that the system is more susceptible to external shocks when the deterioration level of the system reaches a certain threshold.

Under these assumptions, the expected cost rate in the life cycle is used as objective function to obtain the optimal maintenance strategy. To this end, a numerical method based on a recursive formula is provided to evaluate the expected cost rate and the standard deviation associated.

The expected transient cost rate calculated using the recursive formula is compared to both the steady-state expected cost rate and the expected transient cost rate calculated using Monte Carlo simulation. In addition, the robustness of the gamma process parameters and DSPP is analysed. Furthermore, for the comparison between transient and steady-state cost rate the results are also provided under a bivariate case, where the time between inspections and the preventive threshold vary simultaneously. In the comparison between the method based on Monte Carlo simulation and the method based on the recursive formula, for a fixed time between inspections T, similar results are observed. For the preventive threshold M, the results presented some differences, which could be explain due to the high variance in the deterioration increments of the degradation process. If the life cycle increases, the recursive method shall tend to be more costly in terms of computation than the method based on Monte Carlo simulation.

Finally, three important performance measures in the maintenance field, the availability, the reliability and the interval reliability of the system, are analysed under the bivariate optimal maintenance strategy. Recursive formulas are given to obtain these performance measures. Furthermore, the results obtained using the recursive formulation are compared to the results obtained using Monte Carlo simulation. Although the recursive equations of the paper have been derived assuming that the degradation follows a gamma process, they are adaptable for systems with different models of degradation. In this case, the structure of the recursive formulation is similar to the formulation developed in the paper although the probability expressions are different.

In this paper, we consider a system subject to a unique degradation process. However, sometimes the system is subject to multiple degradation processes. A possible further extension of this work is to consider a system subject to multiple degradation processes. With respect to the two types of failure, the analysis of this model is based on the dependence of the degradation level of the system on the intensity of shocks. An interesting extension could be to assume a bidirectional relation of dependence where the shock process also affects to the degradation process.

Appendix A

For $t < T$, $E\left[C(t)\right]$, is given by

$$\begin{aligned} E\left[C(t)\right] =& C_d E\left[(t-Y)\mathbf{1}_{\{\sigma_{M_s}<Y<t,\ Y<\sigma_L\}}\right] + C_d E\left[(t-\sigma_L)\mathbf{1}_{\{\sigma_{M_s}<\sigma_L<t,\ \sigma_L<Y\}}\right] \\ &+ C_d E\left[(t-Y)\mathbf{1}_{\{Y<t,\ Y<\sigma_{M_s}\}}\right]. \end{aligned}$$

That is

$$\begin{aligned} E\left[C(t)\right] =& C_d \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v} I(u,v)\right] \bar{F}_{\sigma_L-\sigma_{M_s}}(v-u)(t-v)\ dv\ du \\ &+ C_d \int_0^t f_{\sigma_{M_s}}(u) \int_u^t I(u,v) f_{\sigma_L-\sigma_{M_s}}(v-u)(t-v)\ dv\ du \\ &+ C_d \int_0^t f_1(u) \bar{F}_{\sigma_{M_s}}(u)(t-u) du. \end{aligned}$$

For $t \geq T$, $E\left[C(t)\right]$ is conditioned to D

$$E\left[C(t)\right] = E\left[C(t), D \leq t\right] + E\left[C(t), D > t\right].$$

Thus, if $D > t$

$$E\left[C(t), D > t\right] = \lfloor t/T \rfloor C_I \left(1 - \sum_{k=1}^{\lfloor t/T \rfloor} P_D(kT)\right) + C_d E\left[W(\lfloor t/T \rfloor T, t)\right] \left(1 - \sum_{k=1}^{\lfloor t/T \rfloor} P_D(kT)\right).$$

If $D \leq t$, $E\left[C(t)\right]$ can be split into two terms: the cost in the first renewal cycle ($C(D)$) and the cost in the remaining time horizon ($C(D,t)$). Since $C(D)$ and $C(D,t)$ are independent, we get

$$E\left[C(t), D \leq t\right] = E\left[C(D), D \leq t\right] + E\left[C(D,t), D \leq t\right].$$

Hence

$$
\begin{aligned}
E\left[C(D), D \leq t\right] &= \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C(D), D = kT\right] \\
&= \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_c + C_I(k-1) + C_d E\left[W((k-1)T, kT)\right]\right) P_{D,c}(kT) \\
&+ \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_p + C_I(k-1)\right) P_{D,p}(kT).
\end{aligned}
$$

Since $C(D,t)$ is stochastically the same as $C(t-D)$,

$$
E\left[C(D,t), D = kT\right] = E\left[C(t-kT)\right] P_D(kT).
$$

Hence, $E\left[C(t)\right]$ verifies the following recursive equation

$$
E\left[C(t)\right] = \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C(t-kT)\right] P_D(kT) + G(t),
$$

being

$$
\begin{aligned}
G(t) = &\sum_{k=1}^{\lfloor t/T \rfloor} \Big(C_p + C_I(k-1)\Big) P_{D,p}(kT) \\
&+ \sum_{k=1}^{\lfloor t/T \rfloor} \Big(C_c + C_I(k-1) + C_d E\left[W((k-1)T, kT)\right]\Big) P_{D,c}(kT) \\
&+ \left(\lfloor t/T \rfloor C_I + C_d E\left[W(\lfloor t/T \rfloor T, t)\right]\right) \Big(1 - \sum_{k=1}^{\lfloor t/T \rfloor} P_D(kT)\Big),
\end{aligned}
$$

and the result holds.

Appendix B

For $t < T$, the expected square cost, $E\left[C(t)^2\right]$, is given by

$$
\begin{aligned}
E\left[C(t)^2\right] = &C_d^2 E\left[(t-Y)^2 \mathbf{1}_{\{\sigma_{M_s} < Y < t,\ Y < \sigma_L\}}\right] + C_d^2 E\left[(t-\sigma_L)^2 \mathbf{1}_{\{\sigma_{M_s} < \sigma_L < t,\ \sigma_L < Y\}}\right] \\
&+ C_d^2 E\left[(t-Y)^2 \mathbf{1}_{\{Y < t,\ Y < \sigma_{M_s}\}}\right].
\end{aligned}
$$

That is

$$
\begin{aligned}
E\left[C(t)\right] = &C_d^2 \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v} I(u,v)\right] \bar{F}_{\sigma_L - \sigma_{M_s}}(v-u)(t-v)^2 \, dv \, du \\
&+ C_d^2 \int_0^t f_{\sigma_{M_s}}(u) \int_u^t I(u,v) f_{\sigma_L - \sigma_{M_s}}(v-u)(t-v)^2 \, dv \, du \\
&+ C_d^2 \int_0^t f_1(u) \bar{F}_{\sigma_{M_s}}(u)(t-u)^2 du.
\end{aligned}
$$

For $t \geq T$, $E\Big[C(t)^2\Big]$ is conditioned to D

$$E\left[C(t)^2\right] = E\left[C(t)^2, D \leq t\right] + E\left[C(t)^2, D > t\right].$$

Hence,

$$E\left[C(t)^2, D > t\right] = \Big(\lfloor t/T \rfloor C_I + C_d E\left[W_T^M(\lfloor t/T \rfloor T, t)\right]\Big)^2 \left(1 - \sum_{k=1}^{\lfloor t/T \rfloor} P_D(kT)\right),$$

where P_D is given by (9). On the other hand

$$E\left[C(t)^2, D \leq t\right] = E\left[\left(C(D) + C(D,t)\right)^2, D \leq t\right].$$

Developing the expression

$$E\left[\left(C(D) + C(D,t)\right)^2, D \leq t\right] = E\left[C(D)^2, D \leq t\right] + E\left[C(D,t)^2, D \leq t\right]$$
$$+E\left[2\, C(D)C(D,t), D \leq t\right].$$

Thus,

$$\begin{aligned} E\Big[C(D)^2, D \leq t\Big] &= \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C(D)^2, D = kT\right] \\ &= \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_c + C_I(k-1) + C_d E\left[W((k-1)T, kT)\right]\right)^2 P_{D,c}(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_c + C_I(k-1)\right)^2 P_{D,p}(kT). \end{aligned}$$

Following the same reasoning as in Appendix A,

$$\begin{aligned} E\left[C(D)C(D,t), D \leq t\right] &= \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_c + C_I(k-1)\right) E\left[C(t-kT)\right] P_{D,c}(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} C_d E\left[W((k-1)T, kT)\right] E\left[C(t-kT)\right] P_{D,c}(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_p + C_I(k-1)\right) E\left[C(t-kT)\right] P_{D,p}(kT). \end{aligned}$$

Hence, $E\left[C(t)^2\right]$ verifies the following recursive equation

$$E\left[C(t)^2\right] = \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C(t-kT)^2\right] P_D(kT) + H(t),$$

being

$$
\begin{aligned}
H(t) = & \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_p + C_I(k-1)\right)^2 P_{D,p}(kT) \\
& + \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_c + C_I(k-1) + C_d E\left[W((k-1)T, kT)\right]\right)^2 P_{D,c}(kT) \\
& +2 \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_c + C_I(k-1)\right) E\left[C(t-kT)\right] P_{D,c}(kT) \\
& +2 \sum_{k=1}^{\lfloor t/T \rfloor} C_d E\left[W((k-1)T, kT)\right] E\left[C(t-kT)\right] P_{D,c}(kT) \\
& +2 \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_p + C_I(k-1)\right) E\left[C(t-kT)\right] P_{D,p}(kT) \\
& +\Big(\lfloor t/T \rfloor C_I + C_d E\left[W(\lfloor t/T \rfloor T, t)\right]\Big)^2 \left(1 - \sum_{k=1}^{\lfloor t/T \rfloor} P_D(kT)\right),
\end{aligned}
$$

and the result holds.

Appendix C

For $t < T$, $A(t)$ is given by

$$
\begin{aligned}
A(t) = & P\left[t < \sigma_{M_s},\ Y > t\right] + P\left[\sigma_{M_s} < t < \sigma_L,\ Y > t\right] \\
= & \bar{F}_{\sigma_{M_s}}(t) \bar{F}_1(t) + \int_0^t f_{\sigma_{M_s}}(u) \bar{F}_{\sigma_L - \sigma_{M_s}}(t-u) I(u,t) du.
\end{aligned}
$$

For $t \geq T$, $A(t)$ is conditioned to the time to the first renewal

$$
A(t) = \sum_{j=0}^{\infty} \mathbf{1}_{\{R_j \leq t < R_{j+1}\}} \Big[P\left[O(t) < L,\ Y > (t-R_j),\ D \leq t\right] + P\left[O(t) < L,\ Y > (t-R_j),\ D > t\right]\Big].
$$

If $D > t$

$$
\begin{aligned}
A(t) = & \Big[P\left[t < \sigma_M,\ Y > t\right] \\
& + P\left[\lfloor t/T \rfloor T < \sigma_M < \sigma_{M_s} < t < \sigma_L,\ Y > t\right] \\
& + P\left[\lfloor t/T \rfloor T < \sigma_M < t < \sigma_{M_s},\ Y > t\right]\Big] \mathbf{1}_{\{M \leq M_s\}} \\
& + \Big[P\left[t < \sigma_{M_s},\ Y > t\right] \\
& + P\left[\sigma_{M_s} < \lfloor t/T \rfloor T < \sigma_M < t < \sigma_L,\ Y > t\right] \\
& + P\left[\sigma_{M_s} < \lfloor t/T \rfloor T < t < \sigma_M,\ Y > t\right] \\
& + P\left[\lfloor t/T \rfloor T < \sigma_{M_s} < t < \sigma_L,\ Y > t\right]\Big] \mathbf{1}_{\{M > M_s\}}.
\end{aligned}
$$

That is

$$\begin{aligned}
A(t) =& \Big[\bar{F}_{\sigma_M}(t)\bar{F}_1(t) \\
&+ \int_{\lfloor t/T \rfloor T}^{t} f_{\sigma_M}(u) \int_u^t f_{\sigma_{M_s}-\sigma_M}(v-u)\bar{F}_{\sigma_L-\sigma_{M_s}}(t-v) I(v,t)\ dv\ du \\
&+ \int_{\lfloor t/T \rfloor T}^{t} f_{\sigma_M}(u)\bar{F}_{\sigma_{M_s}-\sigma_M}(t-u)\bar{F}_1(t)\ du\Big]\mathbf{1}_{\{M \le M_s\}} + \Big[\bar{F}_{\sigma_{M_s}}(t)\bar{F}_1(t) \\
&+ \int_0^{\lfloor t/T \rfloor T} f_{\sigma_{M_s}}(u) \int_{\lfloor t/T \rfloor T}^{t} f_{\sigma_M-\sigma_{M_s}}(v-u)\bar{F}_{\sigma_L-\sigma_M}(t-v) I(u,t)\ dv\ du \\
&+ \int_0^{\lfloor t/T \rfloor T} f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_M-\sigma_{M_s}}(t-u) I(u,t)\ du \\
&+ \int_{\lfloor t/T \rfloor T}^{t} f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_L-\sigma_{M_s}}(t-u) I(u,t)\ du\Big]\mathbf{1}_{\{M > M_s\}} \\
&= J^M_{T,1}(t)\mathbf{1}_{\{M \le M_s\}} + J^M_{T,2}(t)\mathbf{1}_{\{M > M_s\}}.
\end{aligned}$$

If $D \le t$,

$$\begin{aligned}
&\sum_{j=0}^{\infty}\mathbf{1}_{\{R_j \le t < R_{j+1}\}} P\left[O(t) < L,\ Y > (t-R_j),\ D \le t\right] \\
=& \sum_{k=1}^{\lfloor t/T \rfloor} P_D(kT)\left[\sum_{j=0}^{\infty}\mathbf{1}_{\{R_j \le t < R_{j+1}\}} P\left[O(t-kT) < L,\ Y > (t-kT-R_j)\right]\right] \\
=& \sum_{k=1}^{\lfloor t/T \rfloor} A(t-kT)P_D(kT).
\end{aligned}$$

Then, for $t \ge T$, $A(t)$ verifies the following recursive equation

$$A(t) = \sum_{k=1}^{\lfloor t/T \rfloor} A(t-kT)P_D(kT) + J_1(t)\mathbf{1}_{\{M \le M_s\}} + J_2(t)\mathbf{1}_{\{M > M_s\}},$$

and the result holds.

Appendix D

For $t < T$, there is no maintenance action on $[0,t]$, hence $R(t)$ is equal to $A(t)$.

For $t \ge T$, $R(t)$ is conditioned to the time of the first replacement

$$\begin{aligned}
R(t) =& P\left[O(u) < L,\ \forall u \in (0,t],\ N_s(0,t) = 0,\ D \le t\right] \\
&+ P\left[O(u) < L,\ \forall u \in (0,t],\ N_s(0,t) = 0,\ D > t\right].
\end{aligned}$$

If $D > t$

$$R(t) = J_1(t)\mathbf{1}_{\{M \le M_s\}} + J_2(t)\mathbf{1}_{\{M > M_s\}}.$$

If $D \leq t$,

$$\begin{aligned} R(t) =& P\left[O(u) < L,\ \forall u \in (0,t], N_s(0,t) = 0,\ D \leq t\right] \\ =& \sum_{k=1}^{\lfloor t/T \rfloor} P_{D,p}(kT) P\left[O(u) < L, \forall u \in (0, t-kT], N_s(0, t-kT) = 0\right] \\ =& \sum_{k=1}^{\lfloor t/T \rfloor} R(t-kT) P_{D,p}(kT). \end{aligned}$$

Then, for $t \geq T$, $R(t)$ verifies the following recursive equation

$$R(t) = \sum_{k=1}^{\lfloor t/T \rfloor} R(t-kT) P_{D,p}(kT) + J_1(t)\mathbf{1}_{\{M \leq M_s\}} + J_2(t)\mathbf{1}_{\{M > M_s\}},$$

and the result holds.

Appendix E

For $(t+s) < T$, there is no maintenance action on $[0, t+s]$, hence $IR(t,t+s)$ is equal to $R(t+s)$. For $t+s \geq T$, $IR(t,t+s)$ is conditioned to the time of the first replacement

$$\begin{aligned} IR(t,t+s) =& P\left[O(u) < L, \forall u \in (t,t+s], N_s(t,t+s) = 0,\ D \leq t\right] \\ &+ P\left[O(u) < L,\ \forall u \in (t,t+s],\ N_s(t,t+s) = 0,\ t < D < t+s\right] \\ &+ P\left[O(u) < L,\ \forall u \in (t,t+s],\ N_s(t,t+s) = 0,\ D \geq t+s\right]. \end{aligned}$$

If $D \geq t+s$

$$\begin{aligned} IR(t,t+s) =& A(t+s) \\ =& J_1(t+s)\mathbf{1}_{\{M \leq M_s\}} + J_2(t+s)\mathbf{1}_{\{M > M_s\}}. \end{aligned}$$

If $t < D < t+s$

$$\begin{aligned} IR(t,t+s) =& P\left[O(u) < L,\ \forall u \in (t,t+s],\ N_s(t,t+s) = 0,\ t < D < t+s\right] \\ =& \sum_{k=\lfloor t/T \rfloor+1}^{\lfloor (t+s)/T \rfloor} P_{D,p}(kT) P\left[O(u-kT) < L,\ \forall u \in (0, t+s-kT], N_s(0, t+s-kT) = 0\right] \\ =& \sum_{k=\lfloor t/T \rfloor+1}^{\lfloor (t+s)/T \rfloor} R(t+s-kT) P_{D,p}(kT). \end{aligned}$$

If $D \leq t$

$$\begin{aligned} IR(t,t+s) =& P\left[O(u) < L, \forall u \in (t,t+s],\ N_s(t,t+s) = 0,\ D \leq t\right] \\ =& \sum_{k=1}^{\lfloor t/T \rfloor} P_D(kT) P\left[O(u-kT) < L, \forall u \in (t-kT, t+s-kT],\ N_s(t-kT, t+s-kT) = 0\right] \\ =& \sum_{k=1}^{\lfloor t/T \rfloor} IR(t-kT, t+s-kT) P_D(kT). \end{aligned}$$

Then, for $t+s \geq T$, $IR(t,t+s)$ verifies the following recursive equation

$$\begin{aligned} IR(t,t+s) = & \sum_{k=\lfloor t/T \rfloor+1}^{\lfloor (t+s)/T \rfloor} R(t+s-kT)P_{D,p}(kT) \\ & + \sum_{k=1}^{\lfloor t/T \rfloor} IR(t-kT, t+s-kT)P_D(kT) \\ & + J_1(t+s)\mathbf{1}_{\{M \leq M_s\}} + J_2(t+s)\mathbf{1}_{\{M > M_s\}}, \end{aligned}$$

and the result holds.

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