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AN INHOMOGENEOUS CONTROLLED BRANCHING PROCESS

M. González†*, C. Minuesa*†, M. Mota†*, I. del Puerto†*, A. Ramos†

† Department of Mathematics, University of Extremadura, Avda Elvas s/n, 06006 - Badajoz, Spain
*Instituto de Computación Científica Avanzada (ICCAEx), University of Extremadura, Avda Elvas s/n, 06006 - Badajoz, Spain
(e-mail: mvelasco@unex.es, cminuesaa@unex.es, mota@unex.es, idelpuerto@unex.es, aramos@unex.es)
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Abstract. A discrete time branching process is considered in which the offspring distribution is generation-dependent, and the number of reproductive individuals is controlled by a random mechanism. This model is a Markov chain, but in general the transition probabilities are non-stationary. Under not too restrictive hypotheses, this model presents the classical duality of branching processes: it either becomes extinct or grows to infinity. Sufficient conditions for the almost sure extinction and for a positive probability of indefinite growth are given. Finally, the rates of growth of the process are studied provided there is non-extinction.

Keywords: Branching process, Controlled branching process, Inhomogeneous branching process, Extinction probability, Asymptotic behaviour.

1 INTRODUCTION

Branching processes are regarded as appropriate probability models for the description of extinction / growth of populations (see [9]). The oldest and simplest discrete time branching process is the standard Bienaymé-Galton-Watson process that describes the evolution of a population in which each individual, independently of the others, gives rise to a random number of offspring (in accordance with a common reproduction law), and then dies or is not considered in the following counts. Since this standard model does not always adequately describe actual phenomena, it has many variants designed to deal with important properties of real-world populations. In particular, controlled branching processes are useful to model some situations where some kind of regulation is required. Thus, for example, the existence of predators in the environment implies that the population does not live in freedom, so that the survival of each animal (and therefore the possibility of giving new births) will be strongly affected by this factor. In such a case, therefore, a control mechanism is required at each generation that determines the number of progenitors in that generation which continue with the evolution of the population.

The development of a controlled branching process consists of two phases: a reproductive phase in which individuals give birth to their offspring according to a probability distribution called the reproduction law, and a control phase in which the number of potential progenitors of the generation is determined. In this phase some individuals can be introduced into or removed from the population according to another probability distribution called the control law.

In the literature on controlled branching processes (see [7] and [18], and references therein), the control phase is assumed to depend on the population size. In the vast majority of works, however, the reproduction law is assumed to be the same for every individual in any generation. Nevertheless, it seems reasonable to think that the reproductive abilities of the individuals of a population might vary from one generation to another, so that there have been many papers published regarding standard or multitype Bienaymé-Galton-Watson processes whose reproduction laws vary with the generation, usually to referred as varying environment models (see for example [1], [3], [4] or [5] for the standard case and [2], [10] or [12] for the multitype one).

But, until now, this possibility has not been considered in the class of controlled branching processes, at least not from a general viewpoint.

The objective of the present work was therefore to introduce and examine controlled branching processes in varying environments. The model is defined as follows:

Let $\{X_{n,i} : n = 0, 1, \ldots; i = 1, 2, \ldots\}$ and $\{\phi_n(k) : n = 0, 1, \ldots; k = 0, 1, \ldots\}$ be two independent sequences of non-negative, integer-valued random variables satisfying:

a) The variables $X_{n,i}$, $n = 0, 1, \ldots; i = 1, 2, \ldots$, are independent and, for each n, $X_{n,i}$, $i = 1, 2, \ldots$, have the same probability distribution, $\{p_{n,j}\}_{j\geq 0}$, with $p_{n,j} = P(X_{n,i} = j)$, $j \geq 0$, called the *reproduction law* of the nth generation.

b) The stochastic processes $\{\phi_n(k)\}_{k\geq 0}$, $n = 0, 1, \ldots$ are assumed to be independent and, for each k, the variables $\phi_n(k)$, $n = 0, 1, \ldots$, have the same probability distribution, called the *control law* for the population size k.

The *controlled branching process in a varying environment* (CPVE) is a sequence of random variables, $\{Z_n\}_{n\geq 0}$, defined recursively by

$$Z_0 = N, \quad Z_{n+1} = \sum_{i=1}^{\phi_n(Z_n)} X_{n,i}\,, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the empty sum is defined to be 0, and N an arbitrary non-negative integer.

Intuitively, $X_{n,i}$ represents the number of offspring produced by the ith individual in the nth generation, and Z_n represents the total number of individuals in the nth generation. Also, if $Z_n = k$ then $\phi_n(k)$ is the number of progenitors in the nth generation that will produce their offspring, according to the reproduction law $\{p_{n,j}\}_{j\geq 0}$. The offspring of these progenitors forms the $(n+1)$st generation of the population. The control is done in such a way that if $\phi_n(k) > k$ then new individuals are introduced into the population, and if $\phi_n(k) < k$ then some individuals are removed from the population.

The CPVE generalizes two classical branching models that have been extensively studied in the scientific literature on branching process theory. On the one hand, if $\phi_n(k) = k$, $n = 0, 1, \ldots, k = 0, 1, \ldots$ one has the standard Bienaymé-Galton-Watson process in varying environments. And on the other, if the reproduction law is the same for every generation, i.e., $p_{n,j}$ only depends on j, one obtains the controlled branching process with random control function (see for example [7], [17], and [19], and references therein).

It is easy to prove that the CPVE is a Markov chain, in general inhomogeneous. Our objectives in this paper are to establish the basic properties of this model and to study its long-term behaviour. For that, following this Introduction, the paper is organized as follows. In Section 2, conditions for the extinction-explosion duality to hold are stated and the extinction problem is tackled. Section 3 is devoted to studying the rate of convergence of the process on the non-extinction set. The proofs are relegated to Section 4 for the sake of readability of the paper.

2 The extinction problem

Homogeneous branching processes often show a dual long term behaviour: they either become extinct or grow to infinity. However, to obtain this behaviour in inhomogeneous processes, additional regularity conditions are required. In the following result we provide sufficient conditions, given in terms of the reproduction and control laws, for the CPVE to also present this duality.

Theorem 1. *Let $\{Z_n\}_{n\geq 0}$ be a CPVE satisfying:*

(i) $P(\phi_0(0) = 0) = 1$.

(ii) $\liminf_{n\to\infty} p_{n,0} > 0$.

Then

$$P(Z_n \to 0) + P(Z_n \to \infty) = 1. \tag{2.1}$$

Condition (i) in Theorem 1 means that 0 is an absorbing state. Condition (ii) is trivially verified for many families of offspring distributions, for example, geometric probability distributions with parameter α_n, with $0 < \liminf_{n\to\infty} \alpha_n < 1$.

At this point, it is worth mentioning that for the Galton-Watson process in varying environments $\sum_{n=0}^{\infty}(1-p_{n,1}) = \infty$ is a sufficient condition for the duality extinction-explosion to hold (see [15]). Condition (ii) of the previous theorem is stronger. Indeed, if $\liminf_{n\to\infty} p_{n,0} > 0$ then $\liminf_{n\to\infty}(1-p_{n,1}) > 0$, and therefore $\sum_{n=0}^{\infty}(1-p_{n,1}) = \infty$. The presence of the control variables makes it difficult to provide a sharper condition than (ii) or a necessary and sufficient condition for (2.1) to hold.

We are now interested in stating sufficient conditions for the almost sure extinction and for the indefinite growth of the CPVE. Such conditions will be given in terms of the first- and second-order moments of the reproduction and control laws and in terms of their respective probability generating functions. Let us now introduce the notation we shall use. For $n, k = 0, 1, \ldots,$ define:

$$m_n = E[X_{n,1}]\ , \quad \sigma_n^2 = \mathrm{Var}[X_{n,1}]\ , \quad \mathcal{E}(k) = E[\phi_0(k)]\ , \quad \tau^2(k) = \mathrm{Var}[\phi_0(k)]$$

(assumed finite) and

$$f_n(s) = E[s^{X_{n,1}}], \quad g_k(s) = E[s^{\phi_0(k)}], \quad 0 \le s \le 1.$$

From (1.1) it follows that, for $n, k = 0, 1, \ldots,$

$$E[Z_{n+1}|Z_n = k] = m_n\mathcal{E}(k), \tag{2.2}$$

$$\mathrm{Var}[Z_{n+1}|Z_n = k] = m_n^2\tau^2(k) + \sigma_n^2\mathcal{E}(k). \tag{2.3}$$

As usual, let us denote the probability of extinction by $q = P(Z_n \to 0)$. We shall assume throughout the paper that $P(\phi_0(0) = 0) = 1$. With this assumption, the extinction probability can be rewritten as $q = \lim_{n\to\infty} P(Z_n = 0)$. We shall also assume $\liminf_{n\to\infty} p_{n,0} > 0$, so that Theorem 1 applies and (2.1) holds. Consequently, we deduce that $P(Z_n \to \infty) = 1 - q = \lim_{n\to\infty} P(Z_n > 0)$. First, in the next two results, we will provide some sufficient conditions for the almost sure extinction of the process.

Theorem 2. *Let* $\{Z_n\}_{n\ge0}$ *be a CPVE such that*

$$\limsup_{k\to\infty} k^{-1}\mathcal{E}(k) < \liminf_{n\to\infty} m_n^{-1}. \tag{2.4}$$

Then $q = 1$.

Theorem 3. *Let* $\{Z_n\}_{n\ge0}$ *be a CPVE and let* $\gamma_n(s) = \inf_{k\ge1} g_k(f_n(s))$*,* $n = 0, 1, \ldots$*,* $0 \le s < 1$*. If for some* s*,* $0 \le s < 1$*,*

$$\liminf_{n\to\infty} \gamma_n(s) > 0, \tag{2.5}$$

then $q = 1$.

Notice that Theorem 2 includes as particular cases the conditions given in [5] for the almost sure extinction of the standard Bienaymé-Galton-Watson branching process in varying environments and those given in [6] for the almost sure extinction of the controlled branching processes with random control function. Moreover, Theorem 3 covers different cases not considered in Theorem 2. For example, for cases in which $\lim_{k\to\infty} \mathcal{E}(k)k^{-1} = 0$ and $m_n \to \infty$, condition (2.4) in Theorem 2 is not verified, but Theorem 3 could give an answer about the extinction of such cases, as, for instance, when $\phi_0(k) \le X$ almost surely for $k \ge 1$ and $m_n \to \infty$ (obtaining the model's almost sure extinction). From a practical outlook, this bounded control distribution could model the development of species living in environments where the number of individuals with reproductive capacity in each generation is bounded by the carrying capacity or a function of that. The carrying capacity of a biological species in an environment is the maximum population size of the species that the environment can sustain indefinitely. Another interesting case could be when one wants to control an invasive species in an environment, keeping it below some suitable limits.

In order to give sufficient conditions for a positive probability of non-extinction, we shall first provide the definition of uniformly supercritical CPVE.

Definition 1. *A CPVE is said to be* uniformly supercritical *if there exists a constant* $\eta > 1$ *such that*

$$\liminf_{n\to\infty} m_n \geq \eta\, k\mathcal{E}(k)^{-1} \quad \textit{for every } k \geq 1. \tag{2.6}$$

If $\mathcal{E}(k) = k$ for every k then (2.6) is a sufficient condition for a Galton-Watson branching process in varying environments to be uniformly supercritical according to the definition given in [4]. A uniformly supercritical CPVE has a positive probability of indefinite growth if it satisfies some conditions on the second-order moments of the reproduction and control laws. We shall establish two results. The first applies a method similar to that used in [4] for the Galton-Watson branching process in varying environments, and the second generalizes the conditions given in [6] for the controlled branching process with random control function.

Theorem 4. *Let* $\{Z_n\}_{n\geq 0}$ *be a uniformly supercritical CPVE, and let* $\eta > 1$ *satisfy (2.6). Assume also that the following conditions hold:*

(i) There exists a constant $\gamma > 0$ *such that* $\gamma/\eta^2 < 1$ *and*

$$\limsup_{n\to\infty} m_n^2 \leq \gamma\, \frac{k^2}{d^2(k)} \quad \textit{for every } k \geq 1,$$

with $d^2(k) = E[\phi_0^2(k)]$, $k = 0, 1, \ldots$

(ii)

$$\sum_{n=0}^{\infty} \frac{\sigma_n^2}{m_n^2 \eta^n} < \infty\,.$$

Then $q < 1$.

Theorem 5. *Let* $\{Z_n\}_{n\geq 0}$ *be a uniformly supercritical CPVE, and let* $\eta > 1$ *satisfy (2.6). Assume also that the sequences* $\{\mathcal{E}(k)/k\}_{k\geq 1}$ *and* $\{\tau^2(k)/k\}_{k\geq 1}$ *are bounded, and that there exists* $\delta > 0$ *such that*

$$\sum_{n=0}^{\infty} \frac{m_n^2 + \sigma_n^2}{(\eta - \delta)^n} < \infty.$$

Then $q < 1$.

From Theorems 2, 4, and 5, one deduces that the behaviour of the sequence of the expected growth rates per individual when, in a certain generation, there are k individuals, that is, $E[Z_{n+1}Z_n^{-1} \mid Z_n = k] = m_n k^{-1}\mathcal{E}(k)$, $k = 1, 2, \ldots$, seems quite important in order to determine the extinction probability. Indeed, this is quite usual in most branching models. Again, it is not surprising that its behaviour relative to the value unity establish in some form the threshold for the process's extinction or non-extinction. Indeed, conditions (2.4) and (2.6) can be rewritten as $m_n k^{-1}\mathcal{E}(k) < 1$ for all $k \geq k_0$ and $n \geq n_0$, $k_0, n_0 > 0$, and $m_n k^{-1}\mathcal{E}(k) > 1$ for all $n \geq N_0$, for all $k \geq 1$, $N_0 > 0$, respectively. It is a matter for further research to study the behaviour of the process when this double indexed sequence $\{m_n k^{-1}\mathcal{E}(k)\}_{n,k\geq 0}$ approaches unity.

3 Asymptotic behaviour

If $q < 1$, we are interested in the rate of growth of $\{Z_n\}_{n\geq 0}$ on the non-extinction set. In particular, do there exist sequences of positive constants $\{r_n\}_{n\geq 0}$ such that $\lim_{n\to\infty} Z_n/r_n$ exists almost surely and $P(0 < \lim_{n\to\infty} Z_n/r_n < \infty) > 0$? To this end, let us assume asymptotic linear growth of the mathematical expectations of the control means, i.e., we shall assume that $\tau = \lim_{k\to\infty} k^{-1}\mathcal{E}(k)$ exists and is finite. Let us consider the sequences

$$r_n = \tau^n \prod_{i=0}^{n-1} m_i \quad \text{and} \quad W_n = r_n^{-1} Z_n.$$

By the supermartingale convergence theorem, if $\{k^{-1}\mathcal{E}(k)\}_{k\geq 1}$ is a monotonic increasing sequence, then $\{W_n\}_{n\geq 0}$ converges almost surely to a non-negative and finite random variable, W, as $n \to \infty$. So, from now on we shall assume that the sequence $\{k^{-1}\mathcal{E}(k)\}_{k\geq 1}$ is increasing, and write $\delta_k = \tau - k^{-1}\mathcal{E}(k)$. Consequently $\{\delta_k\}_{k\geq 0}$ is a non-increasing sequence with limit equal to zero. Moreover we assume that the process $\{Z_n\}_{n\geq 0}$ is uniformly supercritical so that $\tau \liminf_{n\to\infty} m_n \geq \eta$, for some constant $\eta > 1$, i.e., there exists n_0 such that for all $n \geq n_0$, $\tau m_n \geq \eta$. For simplicity, we shall assume without loss of generality that $n_0 = 0$.

The following result establishes a condition for the existence of the limit of $\{E[W_n]\}_{n\geq 0}$ as $n \to \infty$. Such a limit will be positive and finite if the process starts with a large enough number of individuals.

Proposition 1. *Let $\{Z_n\}_{n\geq 0}$ be a uniformly supercritical CPVE, and let $\eta > 1$ satisfy (2.6). Assume that the sequence $\{\delta_k\}_{k\geq 1}$ is non-increasing and that $\sum_{k=1}^{\infty} k^{-1}\delta_k < \infty$. Then there exists N_0 such that*

$$\lim_{n\to\infty} E[W_n] > 0, \text{ if } Z_0 = N > N_0 \text{ with } q < 1.$$

Finally we provide the following result in which we prove the almost sure convergence in L^1 and in L^2 of $\{W_n\}_{n\geq 0}$ to a non-degenerate random variable by assuming that $q < 1$, and therefore we establish the geometric growth of the CPVE in uniformly supercritical cases.

Theorem 6. *Let $\{Z_n\}_{n\geq 0}$ be a uniformly supercritical CPVE, and let $\eta > 1$ satisfy (2.6). Assume that:*

(i) The sequence $\{\delta_k\}_{k\geq 1}$ is non–increasing and $\sum_{k=1}^{\infty} k^{-1}\delta_k < \infty$.

(ii) The sequence $\{k^{-2}\tau^2(k)\}_{k\geq 1}$ is non–increasing and $\sum_{k=1}^{\infty} k^{-3}\tau^2(k) < \infty$.

(iii) $\sum_{n=0}^{\infty} \frac{\sigma_n^2}{m_n^2 \eta^n} < \infty$.

Then $\{W_n\}_{n\geq 0}$ is an L^2-bounded supermartingale, and converges almost surely in L^1 and in L^2 to the random variable W that is finite almost surely and non-degenerate at zero.

Concluding remark

We have introduced a new branching model which we have termed the CPVE presenting the novelty of conjoining the possibility of the reproduction laws varying with the generation and the incorporation of a random mechanism that determines the number of progenitors in each generation. We adapted the methodological approaches developed independently for controlled branching processes and for Galton-Watson processes in varying environments so as to be able to study the extinction problem and the rates of growth for the CPVE. It is interesting to mention that a CPVE could also be non-trivially thought of as a general branching process with size- and time-dependent reproduction laws. In particular, using the notation of (1.1)

$$Z_{n+1} = \sum_{i=1}^{Z_n} Y_{n,i}(Z_n), n = 0, 1, \ldots, \quad \text{with}$$

$$Y_{n,i}(Z_n) = X_{n,i} + Z_n^{-1} \sum_{j=1}^{\phi_n(Z_n)-Z_n} X_{n,j} I_{\{\phi_n(Z_n)>Z_n\}} - Z_n^{-1} \sum_{j=1}^{Z_n-\phi_n(Z_n)} X_{n,j} I_{\{\phi_n(Z_n)<Z_n\}}.$$

In this case, the expected value $\mu_{n,k} = E[Y_{n,i}(Z_n) \mid Z_n = k] = k^{-1} m_n \mathcal{E}(k)$, $n, k = 1, 2, \ldots$, depends on the generation and the population size. Therefore, one could expect that by adapting techniques of size-dependent branching processes it would be possible to obtain some results about this process (indeed Theorem 6 follows these ideas). While this is an interesting open topic for further research, this approach involves a certain dilution of the mathematical modeling of the control of the population sizes achieved at each generation. In this sense, as has been done in the present paper, it is worth conjoining the ideas of control on the population and a generation-dependent reproduction law in such a way that the two features both appear explicitly in the definition of the model. This allows one to study potential regularity conditions for the control and reproduction laws that lead to the extinction or survival of the process. It also provides a clearer viewpoint from which to consider the problem of estimating the model's parameters, and facilitates the task of developing potential relevant applications of the process.

4 Proofs

Proof of Theorem 1 From condition (i), one has that 0 is an absorbing state. The result is obtained as a consequence of the very general Theorem 2 in [11] if one proves that for any x there is a $\delta > 0$ such that $P(\exists n : Z_n = 0 \mid Z_1 = z_1, \ldots, Z_k = z_k) \geq \delta$ if only $z_k \leq x$, with $z_1, \ldots, z_k$ being non-negative integers. Indeed, since (ii) holds, there exist $n_1 > 0$ and $0 < a < 1$ such that $p_{n,0} \geq a > 0$ for every $n \geq n_1$. Moreover, $g_k(a) > 0$ for every $k = 0, 1, \ldots,$ Therefore, for $n \geq n_1$,

$$\begin{aligned} P(Z_{n+1} = 0 \mid Z_1 = z_1, \ldots, Z_n = z_n) &= P\left(\sum_{l=1}^{\phi_n(z_n)} X_{n,l} = 0\right) = E[p_{n,0}^{\phi_n(z_n)}] \\ &= g_{z_n}(p_{n,0}) \geq g_{z_n}(a) \geq \min_{k \leq x} g_k(a) > 0 \end{aligned}$$

if only $z_n \leq x$.

Proof of Theorem 2 One needs the following auxiliary lemma, for which we shall just sketch the proof. The details can be found in [16], p. 41:

Lemma 1. *Let $\{X_n\}_{n\geq 0}$ be a sequence of non-negative random variables and $\{\mathcal{F}_n\}_{n\geq 0}$ a sequence of σ-algebras such that X_n is $\mathcal{F}_n$-measurable for all n. If there exists a constant $A > 0$ such that, for every n, $E[X_{n+1}|\mathcal{F}_n] \leq X_n$ almost surely on $\{X_n \geq A\}$, then $P(X_n \to \infty) = 0$.*

Proof of the Lemma:

Let $A > 0$ satisfy the hypothesis of the lemma. It is enough to prove that, for every $N > 0$, $P(\inf_{n\geq N} X_n \geq A, X_n \to \infty) = 0$. Fixed $N > 0$, define the *stopping time* $T(A)$ by $\inf\{n \geq N : X_n < A\}$ if $\inf_{n\geq N} X_n < A$ and by ∞ otherwise. Define also the sequence of random variables $\{Y_n\}_{n\geq 0}$, with Y_n for $n \geq 0$ as follows

$$Y_n = \begin{cases} X_{N+n} & \text{if } N + n \leq T(A), \\ X_{T(A)} & \text{if } N + n > T(A). \end{cases}$$

Since $E[X_{n+1}|\mathcal{F}_n] \leq X_n$ almost surely on $\{X_n \geq A\}$, $\{Y_n\}_{n\geq 0}$ is a non-negative supermartingale. Applying the martingale convergence theorem, one obtains the almost sure convergence of the sequence $\{Y_n\}_{n\geq 0}$ to a non-negative and finite limit, and therefore the proof of the Lemma ends.

Now let us prove the theorem. By hypothesis there exist $A > 0$ and $n_0 > 0$ such that

$$k^{-1}\mathcal{E}(k) < m_n^{-1}, \qquad \text{for all } k \geq A \text{ and } n \geq n_0. \tag{4.1}$$

Assume without loss of generality that $n_0 = 0$. Otherwise one would proceed with the sequence $\{Z_n\}_{n\geq n_0}$.

Let us write $\mathcal{F}_n = \sigma(Z_0, \ldots, Z_n)$, $n = 0, 1, \ldots$, i.e., the σ-algebra generated by the random variables $\{Z_0, \ldots, Z_n\}$. Since $\{Z_n\}_{n\geq 0}$ is a Markov chain, and using (2.2) and (4.1), one deduces that, for all n,

$$E[Z_{n+1}|\mathcal{F}_n] = E[Z_{n+1}|Z_n] = m_n\mathcal{E}(Z_n) \leq Z_n \quad \text{ on } \{Z_n \geq A\}.$$

Now, since we are assuming that $P(Z_n \to \infty) = 1 - q$, applying Lemma 1 the proof is finished.

Proof of Theorem 3 Let us denote the probability generating function of Z_n by $F_n(s)$, $0 \leq s \leq 1$. One has, for $n = 0, 1, \ldots$, and $0 \leq s < 1$,

$$F_{n+1}(s) = E[g_{Z_n}(f_n(s))] = P(Z_n = 0) + E[g_{Z_n}(f_n(s))I_{\{Z_n>0\}}].$$

Using (2.1), $\lim_{n\to\infty} s^{Z_n} = I_{\{Z_n \to 0\}}$ almost surely and therefore $\lim_{n\to\infty} F_{n+1}(s) = q$.

Hence, for all $0 \leq s < 1$,

$$\lim_{n\to\infty} E[g_{Z_n}(f_n(s))I_{\{Z_n>0\}}] = 0,$$

and by Fatou's lemma,

$$E[\liminf_{n\to\infty} g_{Z_n}(f_n(s))I_{\{Z_n>0\}}] = 0.$$

Now on $\{Z_n \to \infty\}$,

$$\liminf_{n\to\infty} g_{Z_n}(f_n(s))I_{\{Z_n>0\}} \geq \liminf_{n\to\infty} \gamma_n(s),$$

and therefore $0 = (1-q)\liminf_{n\to\infty} \gamma_n(s)$ for all $0 \leq s < 1$, so that, using (2.5), one deduces that $q = 1$.

Proof of Theorem 4 Since the CPVE is uniformly supercritical and hypothesis (i) holds, one can take n_0 such that for all $n \geq n_0$

$$\mathcal{E}(k) \geq m_n^{-1}\eta k \quad \text{ for every } k \geq 1 \tag{4.2}$$

and

$$d^2(k) \leq m_n^{-2}\gamma k^2 \quad \text{ for every } k \geq 1. \tag{4.3}$$

Assume without loss of generality that $n_0 = 0$. Otherwise one would proceed with the sequence $\{Z_n\}_{n\geq n_0}$.

We shall make use of the fact that, for every non-negative random variable Y, the following inequality holds:

$$P(Y > 0) \geq \left(\frac{\text{Var}[Y]}{E[Y]^2} + 1\right)^{-1}. \tag{4.4}$$

Let us write $T_n = \phi_n(Z_n)$. Using (2.2), (2.3) and

$$\text{Var}[Z_{n+1}] = \text{Var}[E[Z_{n+1}|Z_n]] + E[\text{Var}[Z_{n+1}|Z_n]],$$

one obtains

$$\frac{\text{Var}[Z_{n+1}]}{E[Z_{n+1}]^2} = \frac{\text{Var}[T_n]}{E[T_n]^2} + \frac{\sigma_n^2}{m_n^2 E[T_n]} = \frac{E[T_n^2]}{E[T_n]^2} - 1 + \frac{\sigma_n^2}{m_n^2 E[T_n]}\,.$$

Using (4.2) recursively, one has

$$E[T_n] = E[\mathcal{E}(Z_n)] \geq m_n^{-1}\eta E[Z_n] = \eta E[T_{n-1}] \geq E[T_0]\eta^n,$$

and, using (4.3),

$$E[T_n^2] = E[d^2(Z_n)] \leq m_n^{-2}\gamma E[Z_n^2].$$

Since $\gamma/\eta^2 < 1$, one deduces from the previous equations that

$$\frac{\text{Var}[Z_{n+1}]}{E[Z_{n+1}]^2} \leq \frac{E[Z_n^2]\gamma}{E[Z_n]^2\eta^2} - 1 + \frac{\sigma_n^2}{m_n^2 E[T_0]\eta^n} \leq \frac{\text{Var}[Z_n]}{E[Z_n]^2} + \frac{\sigma_n^2}{m_n^2 E[T_0]\eta^n}.$$

By iteration, one obtains that, for every $n \geq 0$,

$$\frac{\text{Var}[Z_{n+1}]}{E[Z_{n+1}]^2} \leq \frac{1}{E[T_0]}\sum_{j=0}^{\infty}\frac{\sigma_j^2}{m_j^2\eta^j},$$

and, by hypothesis (ii), this series is convergent. So, applying (4.4), for every $n \geq 0$ one obtains

$$P(Z_{n+1} > 0) \geq \left(\frac{1}{E[T_0]}\sum_{j=0}^{\infty}\frac{\sigma_j^2}{m_j^2\eta^j} + 1\right)^{-1}.$$

The right hand side of this inequality is positive and does not depend on n. Therefore

$$P(Z_n \to \infty) = \lim_{n\to\infty} P(Z_n > 0) \geq \left(\frac{1}{E[T_0]}\sum_{j=0}^{\infty}\frac{\sigma_j^2}{m_j^2\eta^j} + 1\right)^{-1} > 0,$$

which finishes the proof.

Proof of Theorem 5 We shall prove that $P(Z_n \to \infty) = 1 - q > 0$. Since the process is uniformly supercritical, there exists n_0 such that for all $n \geq n_0$ and for every $k \geq 1$

$$m_n\frac{\mathcal{E}(k)}{k} \geq \eta. \tag{4.5}$$

Assume without loss of generality that $n_0 = 0$. Otherwise one would proceed with the sequence $\{Z_n\}_{n\geq n_0}$ by showing that $P(Z_n \to \infty | Z_{n_0} = N') > 0$ for some $N' > 0$.

Take $\delta' > 0$ such that $\delta' < \min\{\eta - 1, \delta\}$ and write $A_n = \{Z_{n+1} > (\eta - \delta')Z_n\}$. Since $\eta - \delta' > 1$, it is immediate that $\cap_{n=0}^{\infty} A_n \subseteq \{Z_n \to \infty\}$, so that it is enough to prove that $P(\cap_{n=0}^{\infty} A_n) > 0$.

By the Markov property and using that $Z_0 = N$, one has

$$\begin{aligned} P\left(\bigcap_{n=0}^{\infty} A_n\right) &= \lim_{l\to\infty} P\left(\bigcap_{n=0}^{l} A_n\right) = P(A_0)\lim_{l\to\infty}\prod_{n=1}^{l} P\left(A_n \mid \bigcap_{j=0}^{n-1} A_j\right) \\ &\geq P(A_0)\prod_{n=1}^{\infty}\inf_{k>(\eta-\delta')^n N} P(A_n | Z_n = k). \end{aligned} \tag{4.6}$$

Take a and b to be bounds for the sequences $\{\mathcal{E}(k)/k\}_{k\geq 1}$ and $\{\tau^2(k)/k\}_{k\geq 1}$, respectively. Applying (2.3), (4.5) and Chebyshev's inequality, one obtains

$$\begin{aligned} P(A_n^c|Z_n = k) &\leq P(Z_{n+1} \leq m_n\mathcal{E}(k) - k\delta'|Z_n = k) \\ &\leq P(|Z_{n+1} - m_n\mathcal{E}(k)| \geq k\delta' \mid Z_n = k) \\ &\leq \frac{\mathrm{Var}[Z_{n+1}|Z_n = k]}{k^2\delta'^2} = \frac{m_n^2\tau^2(k) + \sigma_n^2\mathcal{E}(k)}{k^2\delta'^2} \leq \frac{m_n^2 a + \sigma_n^2 b}{k\delta'^2}. \end{aligned}$$

Therefore, from (4.6),

$$\begin{aligned} P\left(\bigcap_{n=0}^{\infty} A_n\right) &\geq P(A_0)\prod_{n=1}^{\infty}\inf_{k>(\eta-\delta')^n N}\left(1 - \frac{m_n^2 a + \sigma_n^2 b}{k\delta'^2}\right) \\ &= P(A_0)\prod_{n=1}^{\infty}\left(1 - \frac{m_n^2 a + \sigma_n^2 b}{(\eta-\delta')^n N\delta'^2}\right). \end{aligned}$$

Since $\delta' < \delta$, by hypothesis the series $\sum_{n=1}^{\infty} m_n^2/(\eta-\delta')^n$ and $\sum_{n=1}^{\infty}\sigma_n^2/(\eta-\delta')^n$ are convergent, and consequently

$$P\left(\bigcap_{n=0}^{\infty} A_n\right) \geq P(A_0)\prod_{n=1}^{\infty}\left(1 - \frac{m_n^2 a + \sigma_n^2 b}{(\eta-\delta')^n N\delta'^2}\right) > 0,$$

which finishes the proof.

Proof of Proposition 1 In [14], it was proved that, under the hypotheses satisfied by the sequence $\{\delta_k\}_{k\geq 0}$, there exists a positive and non-increasing function $\delta(x)$ such that $\delta_k \leq \delta(k)$ for all k, $x\delta(x)$ is concave, and $\sum_{k=1}^{\infty} k^{-1}\delta(k) < \infty$. Thus, by Jensen's inequality, one can check that

$$0 \leq E[W_n] - E[W_{n+1}] \leq \tau^{-1}E[W_n]\delta(\eta^n E[W_n]),$$

.

Now, one can use Lemma 2 of [13], by assuming $a_n = E[W_n]$, $f = \tau^{-1}\delta$, and $m = \eta$, to conclude the result.

Proof of Theorem 6 Under the hypotheses of the theorem, one has that $\{W_n\}_{n\geq 0}$ is a supermartingale. It will be enough to check that it is L^2-bounded to obtain its L^1-convergence to W. Moreover, the limit W is non-degenerate at 0 because, under L^1-convergence, $\lim_{n\to\infty} E[W_n] = E[W]$, and using Proposition 1, this limit is greater than 0 if N is large enough.

Let us prove that $\{E[W_n^2]\}_{n\geq 0}$ is a bounded sequence. The proof is an appropriate adaptation of Theorem 3 in [8], so that we shall only show the main steps. Some calculations lead us to

$$E[W_{n+1}^2] = E[W_n^2] + \frac{1}{\tau^2}E\left[W_n^2\left(\frac{\tau^2(Z_n)}{Z_n^2} + \delta_{Z_n}^2 - 2\tau\delta_{Z_n}\right)\right] + \sigma_n^2\frac{E[\mathcal{E}(Z_n)]}{r_{n+1}^2}.$$

By the properties of the sequences $\{\delta_k\}_{k\geq 0}$ and $\{\tau^2(k)\}_{k\geq 0}$ and results in [8], there exist positive and non-increasing functions $\delta(x)$ and $h(x)$ such that:

a) $\delta_k \leq \delta(k)$, for all $k \geq 0$, $\sum_{k=1}^{\infty} k^{-1}\delta(k) < \infty$ and the functions $x\delta(x)$, $x\delta(x^{1/2})$, and $x\delta^2(x^{1/2})$ are concave.

b) $k^{-2}\tau^2(k) \leq h(k)$, for all $k \geq 0$, $\sum_{k=1}^{\infty} k^{-1}h(k) < \infty$, and the function $xh(x^{1/2})$ is concave.

Therefore,

$$E[W_{n+1}^2] \le E[W_n^2]\left(1+\frac{1}{\tau^2}(h(E[Z_n])+\delta^2(E[Z_n])+2\tau\delta(E[Z_n]))\right)+\sigma_n^2\frac{E[\mathcal{E}(Z_n)]}{r_{n+1}^2}.$$

By Proposition 1, there exists $c>0$ such that $E[W_n]>c$ for all n, and since the process is uniformly supercritical, one deduces that $E[Z_n]\ge c\eta^n$. Hence, using that $h(x)$, $\delta(x)$, and $\delta^2(x)$ are non-increasing functions, one obtains that

$$\begin{aligned} E[W_{n+1}^2] &\le Z_0^2\prod_{i=0}^{n}\left(1+\frac{1}{\tau^2}(h(c\eta^i)+\delta^2(c\eta^i)+2\tau\delta(c\eta^i))\right) \\ &+ \sum_{i=0}^{n}\sigma_i^2\frac{E[\mathcal{E}(Z_i)]}{r_{i+1}^2}\prod_{j=i+1}^{n}\left(1+\frac{1}{\tau^2}(h(c\eta^j)+\delta^2(c\eta^j)+2\tau\delta(c\eta^j))\right). \end{aligned}$$

To conclude that $\{E[W_n^2]\}_{n\ge0}$ is a bounded sequence, it is enough to check that

$$\sum_{n=0}^{\infty}\sigma_n^2\frac{E[\mathcal{E}(Z_n)]}{r_{n+1}^2}<\infty \tag{4.7}$$

and

$$\prod_{n=0}^{\infty}\left(1+\frac{1}{\tau^2}(h(c\eta^n)+\delta^2(c\eta^n)+2\tau\delta(c\eta^n))\right)<\infty. \tag{4.8}$$

In respect of (4.7), using that $\{k^{-1}\mathcal{E}(k)\}_{k\ge1}$ converges to τ in a non-increasing way, that $\{W_n\}_{n\ge0}$ is a supermartingale, that the process is uniformly supercritical, and condition (iii), one has that

$$\sum_{n=0}^{\infty}\sigma_n^2\frac{E[\mathcal{E}(Z_n)]}{r_{n+1}^2}<\sum_{n=0}^{\infty}\tau\sigma_n^2\frac{E[Z_n]}{(\tau m_n^2)r_n}\le\frac{N}{\tau}\sum_{n=0}^{\infty}\frac{\sigma_n^2}{m_n^2 r_n}<\infty.$$

The convergence in (4.8) follows from $\sum_{k=1}^{\infty}k^{-1}\delta(k)<\infty$ and $\sum_{k=1}^{\infty}k^{-1}h(k)<\infty$ assumed in a) and b).

Finally, the L^2-convergence of $\{W_n\}_{n\ge0}$ is proved using Doob's decomposition and following similar ideas to those used in Theorem 3 in [8].

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