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The fundamental theorem of affine geometry

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Abstract: We deal with a natural generalization of the classical Fundamental Theorem of Affine Geometry to the case of non bijective maps. This extension geometrically characterizes semiaffine morphisms. It was obtained by W. Zick in 1981, although it is almost unknown. Our aim is to present and discuss a simplified proof of this result.

Key words: Fundamental Theorem, semiaffine morphisms, parallel morphisms.

MSC (2020): 51A05, 51A15.

Introduction

A map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ between real affine spaces is an *affine morphism* if it has equations of the form

$$\begin{cases} y_1 \; = \; a_{11}x_1 + \cdots + a_{1n}x_n + b_1 \\ \quad \vdots \\ y_m \; = \; a_{m1}x_1 + \cdots + a_{mn}x_n + b_m \end{cases}$$

with $a_{ij}, b_i \in \mathbb{R}$. Its equations are polynomials of degree ≤ 1 hence, in some sense, affine morphisms are the simplest maps, apart of constant maps.

If moreover φ is bijective, that is, $n = m$ and $\det(a_{ij}) \neq 0$, then φ is an *affinity*. The Fundamental Theorem geometrically characterizes affinities: For $n \geq 2$, collineations $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ (bijections transforming lines into lines) are just affinities.

The Fundamental Theorem holds more generally for affine spaces over arbitrary fields of scalars:

Let $\mathbb{A}$, $\mathbb{A}'$ be affine spaces of dimensions ≥ 2 over division rings K, K', respectively (of orders $\neq 2$). The classical Fundamental Theorem states that collineations $\mathbb{A} \to \mathbb{A}'$ are just semiaffinities.

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This theorem was first proved by E. Kamke [10] and is collected in many textbooks (see [2, 3, 4, 15]). More information about its history can be found in [11, pp. 51 – 52].

The classical theorem is restricted to bijective maps; it leaves open the question of a geometrical characterization of non-bijective semiaffine morphisms. In [9, Part I, Chapter V, Theorem 1], Frenkel characterized *injective* semiaffine maps, with an associated *bijective* ring morphism $K \to K'$. In 1981, W. Zick obtains a general result without any injective or surjective condition. To improve Frenkel's result, he introduces a notion of morphism preserving parallelism, valid for non-injective maps, and he removes the traditional (and artificial) condition that the ring map $K \to K'$, associated with a semilinear map, be bijective.

In our opinion, Zick's result is the ultimate version of the Fundamental Theorem of Affine Geometry. Unfortunately, his work hasn't been published and it seems to be almost totally unknown (we learned about its existence from the paper [13]). Our purpose is to give a simplified proof of this result and to explain its interest for the foundations of affine geometry.

The geometric characterization of affine maps is only one aspect of the Fundamental Theorem. It has also a role in the foundations of affine geometry. There essentially exist two ways to define affine space. On the one hand, a synthetic definition using axioms based on the intuitive properties of points, lines and parallelism. On the other hand, an algebraic definition using algebraic structures such as fields and vector spaces. Both definitions are equivalent, but they apparently suggest very different notions of morphism between affine spaces. Its equivalence is the substance of the Fundamental Theorem.

This article is divided into three sections. In the first one, we recall the synthetic and algebraic definitions of affine space; their equivalence is not trivial and for its proof the reader is addressed to the literature. In the second section, we explain that both definitions of affine space induce different notions of morphism: Parallel morphisms in the synthetic case and semiaffine morphisms in the algebraic case. The last section contains the proof of the general version of the Fundamental Theorem, which states the equivalence between parallel and semiaffine morphisms. The classical version for bijective maps is obtained as a consequence.

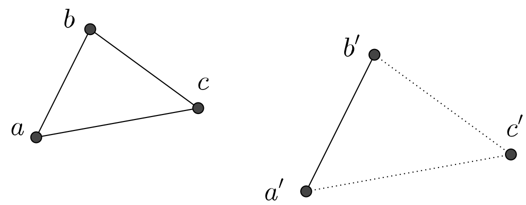
1. The definition of affine space

The synthetic point of view

A synthetic definition of affine space is given by means of terms and axioms that are evident to our geometric intuition, without using of coordinates or algebraic structures. In the literature there are several of these definitions. The definition that we state below is due to O. Tamaschke [16]. We prefer this definition because it emphasizes parallelism as a primitive element in the concept of affine space (for a definition where parallelism is not involved, using only incidence axioms, see [17]).

Definition 1.1. An *affine space* is a set $\mathbb{A} \neq \emptyset$ (whose elements are named *points*), with a family $\mathcal{L}$ of subsets (named *lines*) endowed with an equivalence relation $\parallel$ (named *parallelism*), satisfying the following axioms:

- A1. any two different points lie in a unique line;
- A2. any line has at least two points;
- A3. (*parallel axiom*) given a line L and a point p there is a unique parallel line to L passing through p;
- A4. (*similar triangles axiom*) let a, b, c be three non-collinear points and let a', b' be two different points such that $ab \parallel a'b'$. The line parallel to ac through a' and the line parallel to bc through b' intersect at a point c'.



Definition 1.2. A subset $S \subseteq \mathbb{A}$ is said to be a *subspace* when it fulfills the following conditions:

- (a) the line joining any two different points of S is contained in S;
- (b) for any line $L \subseteq S$ and any point $p \in S$, the parallel line to L passing through p also is contained in S.

Condition (b) is superfluous when the lines of $\mathbb{A}$ have at least three points.

Definition 1.3. The *dimension* of a non empty subspace $S \subseteq \mathbb{A}$ is the supremum of the naturals n such that there exists a strictly increasing sequence of subspaces $\emptyset \neq S_0 \subset \cdots \subset S_n = S$.

Points and lines are just subspaces of dimension 0 and 1, respectively. Subspaces of dimension 2 are named *planes*.

Definition 1.4. Two non-empty subspaces S and S' are said to be *parallel* (we put $S \parallel S'$) when for any line $L \subseteq S$ there is a parallel line $L' \subseteq S'$ and, conversely, for any line $L' \subseteq S'$ there is a parallel line $L \subseteq S$.

The algebraic point of view

In undergraduate courses it is usual to define affine spaces in terms of certain algebraic structures (fields or, more generally, division rings, vector spaces or group actions).

Definition 1.5. An *affine space* is a set $\mathbb{A} \neq \emptyset$ (whose elements are named *points*) together with a vector space V over a division ring K and a map $+ \colon \mathbb{A} \times V \to \mathbb{A}$, $(p, v) \mapsto p + v$, such that the following axioms are satisfied:

(1) $(p + v_1) + v_2 = p + (v_1 + v_2)$ for all $p \in \mathbb{A}$, $v_1, v_2 \in V$;

(2) $p + v = p \iff v = 0$ for all $p \in \mathbb{A}$, $v \in V$;

(3) given two points $p, \bar{p} \in \mathbb{A}$ there is a vector $v \in V$ (necessarily unique) such that $\bar{p} = p + v$.

An affine space $(\mathbb{A}, V, +)$ will be simply denoted $\mathbb{A}$.

Note that each vector $v \in V$ defines a bijective map $\tau_v \colon \mathbb{A} \to \mathbb{A}$, $p \mapsto p + v$, named *translation* with respect to v. Definition 1.5 captures the idea that an affine space is a set with a distinguished group of transformations (the group of translations) isomorphic to the additive group $(V, +)$ of a vector space.

Alternatively, in the language of group actions, one may define an affine space as a set $\mathbb{A}$ endowed with a free and transitive action $+ \colon \mathbb{A} \times V \to \mathbb{A}$ of the additive group of a vector space V.

The *dimension* of an affine space $\mathbb{A}$ is defined to be the dimension of the vector space V (possibly infinite).

1.6. Coordinates. Let $\mathbb{A}_n$ be an affine space of finite dimension n. An *affine reference* is a sequence $\{p_0, v_1, \ldots, v_n\}$ where $p_0 \in \mathbb{A}$ is a point (named *origin* of the reference) and $\{v_1, \ldots, v_n\}$ is a basis of V.

Now, given a point $p \in \mathbb{A}_n$ we have $p = p_0 + v$ for a unique vector $v \in V$. Writing $v = x_1 v_1 + \cdots + x_n v_n$, we obtain

$$p \;=\; p_0 + x_1 v_1 + \cdots + x_n v_n$$

for a unique sequence of scalars $x_1, \ldots, x_n \in K$, named *affine coordinates* of p. Assigning its coordinates to each point, we obtain a bijection

$$\mathbb{A}_n \xrightarrow{\;\sim\;} K \times \overset{n}{\cdots} \times K.$$

Definitions 1.7. A non-empty subset $S \subseteq \mathbb{A}$ is a *affine subspace* when it is

$$S \;=\; p + W := \{p + w : w \in W\}$$

where $p \in \mathbb{A}$ is a point and $W \subseteq V$ is a vector subspace. Then W is said to be the *direction* of S.

We agree that the empty subset also is a subspace.

Remark that a non-empty subspace S of direction W is an affine space $(S, W, +)$. So the *dimension* of S is the dimension of its direction W as a vector space.

Points are just subspaces of dimension 0. Subspaces of dimension 1 are named *lines* and subspaces of dimension 2, *planes*.

Definition 1.8. Two non-empty subspaces $S = p + W$ and $S' = p' + W'$ are said to be *parallel* when both have the same direction: $S \parallel S' \iff W = W'$.

Note that parallel subspaces have the same dimension.

Of course, two distinct lines are parallel if and only if they are coplanar and do not intersect.

Equivalence of definitions

The algebraic definition 1.5 of affine space is deep and very convenient for an efficient development of Affine Geometry. Although, in a certain sense it is not a primary definition, since it requires some motivation or explanation. There is a great gap between ordinary spatial intuition, with its informal ideas of point, line, parallelism, and an abstract definition in terms of algebraic structures such as a field of scalars or a vector space. The emergence of these structures is a beautiful surprise, formulated as Theorem 1.9 below.

It is an easy exercise to check that any affine space, in the sense of the algebraic definition 1.5, fulfills the synthetic definition 1.1.

The converse is not so easy. An essential role is played by Desargues's Theorem, which holds in any algebraic affine space and also in any synthetic affine space of dimension ≥ 3. However, there exist synthetic affine planes where Desargues's Theorem fails to hold (an easy example is the Moulton plane [2]). With the exception of non-Desarguesian planes, the algebraic and synthetic notions of affine space are equivalent:

THEOREM 1.9. *Let $\mathbb{A}$ be a synthetic affine space of dimension ≥ 3 (or of dimension 2 and Desarguesian). There exist, canonically associated to $\mathbb{A}$, a division ring K, a K-vector space V and a map $\mathbb{A} \times V \xrightarrow{+} \mathbb{A}$ such that:*

(i) $(\mathbb{A}, V, +)$ *is an algebraic affine space;*

(ii) *subspaces of the algebraic affine space $(\mathbb{A}, V, +)$ are just subspaces of the synthetic affine space $\mathbb{A}$.*

Variations of this theorem can be found in [2, 4, 16].

2. WHICH ARE THE *morphisms* BETWEEN AFFINE SPACES?

The algebraic and synthetic definitions of affine space, although equivalent by Theorem 1.9, suggest different definitions of "morphism" between affine spaces. We will show that the Fundamental Theorem states the equivalence of both notions of morphism.

MORPHISMS IN THE ALGEBRAIC CASE

In the case of algebraic structures, such as group, ring or vector space, the (homo)morphisms are defined to be maps *preserving the structure*. Typically, an algebraic structure consists of some sets (and their direct products) with certain maps between them (named operations) satisfying certain identities (named axioms). A map between two structures of the same kind is said to preserve the structure when it is compatible with the operations in an obvious sense.

For example, a group is a set G with operations $G \times G \xrightarrow{\cdot} G$, $G \xrightarrow{\text{inv}} G$, $* \xrightarrow{1} G$, satisfying the usual axioms. A morphism between groups $\varphi\colon G \to G'$

is defined to be a map preserving the structure, in the sense that the following diagrams are commutative,

$$\begin{array}{ccc} G\times G & \xrightarrow{\varphi\times\varphi} & G'\times G' \\ \cdot\big\downarrow & & \big\downarrow\cdot \\ G & \xrightarrow{\varphi} & G' \end{array}\ , \qquad \begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ \mathrm{inv}\big\downarrow & & \big\downarrow\mathrm{inv} \\ G & \xrightarrow{\varphi} & G' \end{array}\ , \qquad \begin{array}{ccc} * & = = = & * \\ 1\big\downarrow & & \big\downarrow 1 \\ G & \xrightarrow{\varphi} & G' \end{array}\ .$$

The commutativity of the first diagram states that $\varphi(g_1 \cdot g_2) = \varphi(g_1)\cdot\varphi(g_2)$ for all $g_1, g_2 \in G$. The other two diagrams state that $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$ and $\varphi(1) = 1$ (in fact both follow from the former condition, due to the axioms of group). Hence a map $\varphi\colon G \to G'$ preserves the structure when it fulfills the condition $\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$ for all $g_1, g_2 \in G$, which is the standard definition of group morphism.

Now let us consider the case of vector spaces. A vector space is a list $(V, K, \cdot)$ where V is an abelian group, K is a division ring and $K \times V \xrightarrow{\cdot} V$, $(\lambda, v) \mapsto \lambda \cdot v$, is a map satisfying certain axioms. Therefore, a *morphism* between vector spaces $(V, K, \cdot)$ and $(V', K', \cdot)$ should be defined by two maps $\phi\colon V \to V'$, $\sigma\colon K \to K'$, where ϕ is a morphism of groups, σ is a morphism of rings and the following diagram is commutative

$$\begin{array}{ccc} K\times V & \xrightarrow{(\sigma,\phi)} & K'\times V' \\ \cdot\big\downarrow & & \big\downarrow\cdot \\ V & \xrightarrow{\phi} & V' \end{array}$$

that is to say, $\phi(\lambda \cdot v) = \sigma(\lambda) \cdot \phi(v)$ for all $\lambda \in K$, $v \in V$. Assuming that ϕ is not null then σ is uniquely determined by ϕ. So we arrive to the following definition.

Definition 2.1. Let V, V' be vector spaces over division rings K, K', respectively. A map $\varphi\colon V \to V'$ is said to be *semilinear* when:

(a) it is additive: $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$ for all $v_1, v_2 \in V$;

(b) there is a ring morphism $\sigma\colon K \to K'$ such that $\varphi(\lambda v) = \sigma(\lambda)\varphi(v)$ for all $\lambda \in K$, $v \in V$. We do not require that $\sigma\colon K \to K'$ be surjective.

Analogously, a *morphism* between algebraic affine spaces $(\mathbb{A}, V, +)$ and $(\mathbb{A}', V', +)$ should be defined by two maps $\varphi\colon \mathbb{A} \to \mathbb{A}'$, $\vec{\varphi}\colon V \to V'$, where $\vec{\varphi}$ is semilinear, satisfying the commutative diagram

$$\begin{array}{ccc} \mathbb{A} \times V & \xrightarrow{(\varphi,\vec{\varphi})} & \mathbb{A}' \times V' \\ {\scriptstyle +}\downarrow & & \downarrow{\scriptstyle +} \\ \mathbb{A} & \xrightarrow{\varphi} & \mathbb{A}' \end{array}$$

that is to say, $\varphi(p+v) = \varphi(p) + \vec{\varphi}(v)$. This equality implies that $\vec{\varphi}$ is determined by φ. So we arrive to the following definition.

DEFINITION 2.2. Let $(\mathbb{A}, V, +)$ and $(\mathbb{A}', V', +)$ be affine spaces over division rings K and K', respectively. A map $\varphi\colon \mathbb{A} \to \mathbb{A}'$ is a *semiaffine morphism* when there is a semilinear map $\vec{\varphi}\colon V \to V'$ such that

$$\varphi(p+v) \;=\; \varphi(p) + \vec{\varphi}(v) \qquad\qquad \forall\, p \in \mathbb{A}\,,\; v \in \mathbb{V} \quad .$$

The semilinear map $\vec{\varphi}$ is unique and it is named *differential* of φ.

A semiaffine morphism $\varphi\colon \mathbb{A} \to \mathbb{A}'$ is a *semiaffine isomorphism* or a *semiaffinity* when both φ and the ring morphism $\sigma\colon K \to K'$ (associated to $\vec{\varphi}$) are bijective. In such case the inverse map $\varphi^{-1}\colon \mathbb{A}' \to \mathbb{A}$ also is a semiaffine isomorphism.

The prefix *semi* in the terms *semilinear, semiaffine, semiaffinity* is deleted when $K = K'$ and the associated ring morphism $\sigma\colon K \to K$ is the identity.

2.3. Any vector space V has an underlying structure of affine space ($\mathbb{A} = V$, $V, +$), where the map $+\colon \mathbb{A} \times V \to \mathbb{A}$ is just the addition of vectors,

$$\mathbb{A} \times V = V \times V \xrightarrow{\;+\;} V = \mathbb{A}\,.$$

Conversely, given an affine space $(\mathbb{A}, V, +)$ and a fixed point $p_0 \in \mathbb{A}$ we have an affine isomorphism

$$V \quad \xrightarrow{\;\sim\;} \quad \mathbb{A}\,, \qquad v \longmapsto p_0 + v\,.$$

Observe that $0 \mapsto p_0$. This isomorphism supports the colloquial statement that an affine space is a vector space where we have forgotten the origin; once we fix a point $p_0 \in \mathbb{A}$ as the *origin* we have an identification $\mathbb{A} = V$.

2.4. Let V and V' be vector spaces, hence also affine spaces, over division rings K and K', respectively. A map $\varphi\colon V \to V'$ is a semiaffine morphism if and only if it is

$$V \xrightarrow{\;\varphi\;} V'\,, \qquad \varphi(v) = \vec{\varphi}(v) + \mathbf{b}\,,$$

where $\vec{\varphi}\colon V \to V'$ is a semilinear map and $\mathbf{b} := \varphi(0)$.

2.5. Equations of a semiaffine morphism. Let $\varphi\colon \mathbb{A}_n \to \mathbb{A}'_m$ be a semiaffine morphism, between affine spaces of finite dimension, with associated ring morphism $K \to K'$, $x \mapsto x'$. Given affine coordinates $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ of $\mathbb{A}_n$ and $\mathbb{A}'_m$, respectively, the equations of φ are

$$\begin{cases} y_1 = x'_1 a_{11} + \cdots + x'_n a_{1n} + b_1 \\ \quad\vdots \\ y_m = x'_1 a_{m1} + \cdots + x'_n a_{mn} + b_m \end{cases}$$

where (a_{ij}) is the matrix of the semilinear map $\vec{\varphi}\colon V \to V'$, and $(b_1, \dots, b_m)$ are the coordinates of $\mathbf{b} = \varphi(p_0)$.

Morphisms in the synthetic case

Now, the synthetic definition of affine space is not algebraic as the previous structures, so that it is not evident what does it mean to say that a map $\varphi\colon \mathbb{A} \to \mathbb{A}'$, between synthetic affine spaces, *preserves the structure*. Let us consider the proposal of W. Zick. First, we introduce the following

2.6. Notation. Given points $p_0, p_1 \in \mathbb{A}$, let $p_0 \vee p_1$ be the smallest affine subspace containing p_0, p_1. If p_0, p_1 are distinct points, then $p_0 \vee p_1$ is a line. If $p_0 = p_1$ then $p_0 \vee p_1 = p_0 = p_1$.

Recall that any two parallel subspaces have equal dimension. Therefore the expression $(a \vee b) \parallel (c \vee d)$ means that both $a \vee b$ and $c \vee d$ are parallel lines or both are points ($a = b$ and $c = d$).

Definition 2.7. (Zick) A map $\varphi\colon \mathbb{A} \to \mathbb{A}'$, between affine spaces, is a *parallel morphism* when

$$(a \vee b) \parallel (c \vee d) \quad \Rightarrow \quad \big(\varphi(a) \vee \varphi(b)\big) \parallel \big(\varphi(c) \vee \varphi(d)\big)$$

for all $a, b, c, d \in \mathbb{A}$.

Since the synthetic notion 1.1 of affine space is based on the relations of collinearity and parallelism, it seems reasonable at first sight to say that morphism $\varphi\colon \mathbb{A} \to \mathbb{A}'$ preserving the structure are parallel morphisms. This intuition is confirmed by the Fundamental Theorem 3.9, stating the equivalence between the semiaffine and the parallel morphisms.

In conclusion, Theorem 1.9 and the Fundamental Theorem 3.9 are the mathematical formulation of the equivalence between the algebraic and synthetic points of views on Affine Geometry.

3. Fundamental Theorem

Semiaffine morphisms are parallel morphisms

Lemma 3.1. *Any parallel morphism $\varphi\colon \mathbb{A} \to \mathbb{A}'$ satisfies the property*

$$x_0 \in x_1 \vee x_2 \quad \Rightarrow \quad \varphi(x_0) \in \varphi(x_1) \vee \varphi(x_2)$$

for any $x_0, x_1, x_2 \in \mathbb{A}$.

Proof. If $x_1 = x_2$ then $x_0 = x_1 = x_2$ and it is clear. Otherwise x_0 is not x_1 or x_2, let us assume that $x_0 \neq x_2$. We have $x_1 \vee x_2 = x_0 \vee x_2$, hence $x_1 \vee x_2 \parallel x_0 \vee x_2$, so that $\varphi(x_1) \vee \varphi(x_2) \parallel \varphi(x_0) \vee \varphi(x_2)$, and then $\varphi(x_1) \vee \varphi(x_2) = \varphi(x_0) \vee \varphi(x_2) \ni \varphi(x_0)$. ∎

As a consequence, the restriction of a parallel morphism $\varphi\colon \mathbb{A} \to \mathbb{A}'$ to a line is constant or it is an injection into a line of $\mathbb{A}'$. Now the next statement directly follows from the definition.

3.2. *A map $\varphi\colon \mathbb{A} \to \mathbb{A}'$, between affine spaces, is a parallel morphism if and only if it satisfies the following condition:*
For any two parallel lines $L_1, L_2 \subseteq \mathbb{A}$ the restrictions $\varphi_{|L_1}$ and $\varphi_{|L_2}$ are both constant or both injective, and in such case $\varphi(L_1) \subseteq L_1'$, $\varphi(L_2) \subseteq L_2'$, where L_1', L_2' are two parallel lines in $\mathbb{A}'$.

Proposition 3.3. *Any semiaffine morphism $\varphi\colon \mathbb{A} \to \mathbb{A}'$ is a parallel morphism.*

Proof. Let $L_1 = p_1 + \langle v \rangle$, $L_2 = p_2 + \langle v \rangle$ be two parallel lines of $\mathbb{A}$. If $\vec{\varphi}(v) = 0$ then $\varphi(L_1) = \varphi(p_1)$ and $\varphi(L_2) = \varphi(p_2)$. If $\vec{\varphi}(v) \neq 0$ then φ embeds the lines $L_i = p_i + \langle v \rangle$ into the lines $L_i' = \varphi(p_i) + \langle \vec{\varphi}(v) \rangle$, $(i = 1, 2)$, which are parallel. By 3.2, φ is a parallel morphism. ∎

Parallel morphisms are semiaffine

In this subsection, V, V' are vector spaces over division rings K, K', respectively.

Lemma 3.4. ([6]) *Let $\varphi, \phi\colon V \to V'$ be additive maps. If for any $x \in V$ we have $\phi(x) \in K' \cdot \varphi(x)$ and the image of φ contain two linearly independent vectors, then there is a scalar $\lambda \in K'$ such that $\phi = \lambda \cdot \varphi$.*

Proof. For any $x \in V \setminus \ker\varphi$ we have $\phi(x) = \lambda_x \cdot \varphi(x)$ for a unique scalar $\lambda_x \in K'$. We have to show that λ_x does not depend on x. Let $x, y \in V \setminus \ker\varphi$; we distinguish two cases.

1. $\varphi(x)$ and $\varphi(y)$ are linearly independent. Then $x, y, x+y \in V \setminus \ker\varphi$ and the equality $\phi(x+y) = \phi(x)+\phi(y)$ shows that $\lambda_{x+y}\varphi(x+y) = \lambda_x\varphi(x)+\lambda_y\varphi(y)$, that is to say, $\lambda_{x+y}\varphi(x)+\lambda_{x+y}\varphi(y) = \lambda_x\varphi(x)+\lambda_y\varphi(y)$, hence $\lambda_x = \lambda_{x+y} = \lambda_y$.

2. $\varphi(x)$ and $\varphi(y)$ are linearly dependent. Take $z \in V$ such that $\varphi(z)$ is linearly independent of both vectors. According to the former case, we have $\lambda_x = \lambda_z = \lambda_y$. ∎

Proposition 3.5. (Zick) *Let $\varphi\colon V \to V'$ be an additive map, such that $\varphi(Kx) \subseteq K'\varphi(x)$ for all $x \in V$, and such that the image contains two linearly independent vectors. Then $\varphi\colon V \to V'$ is semilinear.*

Proof. ([6]) Given $\lambda \in K$ we define the additive map $\phi_\lambda(x) := \varphi(\lambda x)$. By Lemma 3.4, there is a scalar $\sigma(\lambda) \in K'$ such that $\phi_\lambda = \sigma(\lambda)\varphi$, that is to say, $\varphi(\lambda x) = \sigma(\lambda)\varphi(x)$. We have to check that $\sigma\colon K \to K'$ is a ring morphism.

Taking $x \in V \setminus \ker\varphi$ we have

$$\varphi((\lambda_1 + \lambda_2)x) \;=\; \sigma(\lambda_1 + \lambda_2)\varphi(x)$$

and moreover

$$\begin{aligned}\varphi((\lambda_1 + \lambda_2)x) \;&=\; \varphi(\lambda_1 x + \lambda_2 x)\\ &=\; \sigma(\lambda_1)\varphi(x) + \sigma(\lambda_2)\varphi(x) \;=\; (\sigma(\lambda_1) + \sigma(\lambda_2))\varphi(x)\,,\end{aligned}$$

so that $\sigma(\lambda_1 + \lambda_2) = \sigma(\lambda_1) + \sigma(\lambda_2)$. Analogously we prove that $\sigma(\lambda_1\lambda_2) = \sigma(\lambda_1)\sigma(\lambda_2)$. ∎

Recall (see 2.3) that a vector space V also is an affine space.

Lemma 3.6. *Let $\varphi\colon V \to V'$, $x \mapsto x'$, be a parallel morphism. If $\varphi(0) = 0$ and $\dim\langle\varphi(V)\rangle \geq 2$, then $\varphi\colon V \to V'$ is additive.*

Proof. Since φ transforms the parallelogram (eventually degenerated) with vertices $0, x, y, x+y$ into a parallelogram $0, x', y', (x+y)'$, we have

$$(x+y)' \;=\; \lambda x' + y' \;=\; x' + \mu y' \tag{1}$$

for certain $\lambda, \mu \in K'$.

When $y' \notin \langle x' \rangle$ then $(x+y)' = x' + y'$, because either $x' = 0$, so that we put $\lambda x' = x'$ en (1), or x' and y' are linearly independent (so that $\lambda = \mu = 1$). The case $x' \notin \langle y' \rangle$ is similar.

Otherwise we have $\langle x' \rangle = \langle y' \rangle$. By hypothesis, there exists $z \in V$ such that $z' \notin \langle x' \rangle = \langle y' \rangle$ and by (1) we also have $z' \notin \langle (x+y)' \rangle$. By the former case, we have $(y+z)' = y' + z' \notin \langle x' \rangle$ and

$$x' + y' + z' \,=\, x' + (y+z)' \,=\, (x+y+z)' \,=\, (x+y)' + z' \,,$$

hence $x' + y' = (x+y)'$. ∎

Lemma 3.7. *Let $\varphi \colon V \to V'$ be a parallel morphism. If $\varphi(0) = 0$ then we have $\varphi(Kx) \subseteq K' \varphi(x)$ for all $x \in V$.*

Proof. By Lemma 3.1 we have $\varphi(x_1 \vee x_2) \subseteq \varphi(x_1) \vee \varphi(x_2)$, so that

$$\varphi(Kx) \,=\, \varphi(0 \vee x) \,\subseteq \varphi(0) \vee \varphi(x) \,=\, 0 \vee \varphi(x) \,=\, K'\varphi(x) \,.$$

∎

Proposition 3.8. *Let $\varphi \colon V \to V' \neq 0$ be a parallel morphism. If the image of φ is not contained in an affine line, then φ is a semiaffine morphism, that is to say, we have*

$$\varphi(x) \,=\, \vec{\varphi}(x) + \mathbf{b} \,,$$

where $\vec{\varphi} \colon V \to V'$ is a semilinear map and $\mathbf{b} = \varphi(0)$.

Proof. Composing φ with a translation we may assume that $\varphi(0) = 0$. The above two lemmas show that $\varphi \colon V \to V'$ fulfills the hypotheses of Proposition 3.5, hence $\varphi \colon V \to V'$ is semilinear. ∎

According to 2.3 any affine space is isomorphic (an affinity) to its direction: $\mathbb{A} \simeq V$. Combining 3.3 and 3.8 we finally obtain

3.9. Fundamental Theorem (Zick [18]) *Let $\varphi \colon \mathbb{A} \to \mathbb{A}' \neq *$ be a map such that the image is not contained in a line. Then φ is a semiaffine morphism if and only if it is a parallel morphism.*

3.10. *Case $K = K' = \mathbb{R}$.* It is elementary that the only ring morphism $\mathbb{R} \to \mathbb{R}$ is the identity (see [15, p. 86]). So, in the case of real affine spaces, we may drop the prefix semi in the above theorem. A stronger result may be obtained:

Let $\varphi\colon \mathbb{A} \to \mathbb{A}'$ be a map, between real affine spaces, such that the image is not contained in a line. Then φ is an affine morphism if and only if for any $p_0, p_1, p_2 \in \mathbb{A}$ the following condition holds:

$$p_0 \in p_1 \vee p_2 \qquad \Rightarrow \qquad \varphi(p_0) \in \varphi(p_1) \vee \varphi(p_2)\,. \tag{2}$$

This statement is an easy consequence of the following more general result (Lenz [14, Hilfssatz 3]): *Let $\mathbb{P}$ and $\mathbb{P}'$ be real projective spaces, with* $\dim \mathbb{P} < \infty$*, and let $U \subseteq \mathbb{P}$ be an open set. Let $\varphi\colon U \to \mathbb{P}'$ be a map satisfying* (2) *such that the image is not contained in a line. Then φ is a linear map in homogeneous coordinates.*

Lenz's result was generalized by Frank [8] for projective spaces, endowed with a linear topology, over division rings.

3.11. Over arbitrary division rings, maps $\mathbb{A} \to \mathbb{A}'$ satisfying condition (2) are algebraically characterized as *fractional semiaffine morphisms* (see [19]).

A map $\varphi\colon \mathbb{A} \to \mathbb{A}'$ is called a *lineation* if the image by φ of any three collinear points are collinear. It is a weaker condition than (2). See [5] for a version of the fundamental theorem for surjective lineations.

In [1], several generalizations of the fundamental theorem are obtained, where collinearity preservation is assumed only for a finite number of directions of lines.

3.12. In the case of projective spaces, Faure-Frölicher [7] and Havlicek [12] extended the classical Fundamental Theorem of Projective Geometry to non necessarily bijective maps. See also [6].

The classical Fundamental Theorem

Now we examine the case of bijective maps to obtain the classical version of the Fundamental Theorem.

Lemma 3.13. *Let V be a vector space over a division ring K with $|K| \neq 2$. Let $W \subseteq V$ be a subset such that:*

(a) $0 \in W$*;*

(b) *if $w_1, w_2 \in W$, then $(1-t)w_1 + tw_2 \in W$ for all $t \in K$ (that is to say, $w_1, w_2 \in W \;\Rightarrow\; w_1 \vee w_2 \subseteq W$).*

Then W is a vector subspace.

Proof. It is enough to show that $\langle w_1, w_2\rangle \subseteq W$ whenever $w_1, w_2 \in W$.

Remark that if $w \in W$ then $\langle w\rangle \subseteq W$: For any $t \in K$ we have $tw = (1-t)0 + tw \in W$.

Now, given $w_1, w_2 \in W$, for all $x, y \in K$ we have $xw_1, yw_2 \in W$, hence

$$W \ni (1-t)xw_1 + tyw_2 = \bar{x}w_1 + \bar{y}w_2 , \qquad \text{where } \bar{x} := (1-t)x ,\ \bar{y} := ty .$$

Taking $t \neq 0, 1$ (since $|K| \neq 2$), the values of $\bar{x}, \bar{y}$ are arbitrary, so that the vector $\bar{x}w_1 + \bar{y}w_2$ is any vector of $\langle w_1, w_2\rangle$. ∎

Any affine subspace $S \subseteq \mathbb{A}$ satisfies

$$x_1, x_2 \in S \quad \Rightarrow \quad x_1 \vee x_2 \subseteq S .$$

That is to say, any affine subspace $S \subseteq \mathbb{A}$ contains the line joining any two different points of S. Reciprocally,

Proposition 3.14. *Let $\mathbb{A}$ be an affine space over a division ring K, with $|K| \neq 2$. If a subset $S \subseteq \mathbb{A}$ contains the line joining any two different points of S, then S is an affine subspace.*

Proof. Fix $p_0 \in S$ and consider the affine isomorphism $V \simeq \mathbb{A}$, $v \mapsto p_0 + v$. Via this isomorphism, the subset S corresponds to a subset $W \subseteq V$ fulfilling the conditions of the lemma, so that W is a vector (hence affine) subspace of V and, therefore, S is a subspace of $\mathbb{A}$. ∎

Definition 3.15. A bijective map $\varphi\colon \mathbb{A} \to \mathbb{A}'$ is a *collineation* if the image of each line L of $\mathbb{A}$ is a line $\varphi(L)$ of $\mathbb{A}'$.

Note that the inverse of a collineation is also a collineation.

Theorem 3.16. *Let $\mathbb{A}$ and $\mathbb{A}'$ be affine spaces of dimensions ≥ 2 over division rings K and K', respectively, with $|K|, |K'| \neq 2$. A bijective map $\varphi\colon \mathbb{A} \to \mathbb{A}'$ is a semiaffinity if and only if it is a collineation.*

Proof. $(\Rightarrow)$: Any line $L = p + \langle v\rangle$ goes to a line $\varphi(L) = \varphi(p) + \langle\vec{\varphi}(v)\rangle$.

$(\Leftarrow)$: By Proposition 3.14, any collineation $\varphi\colon \mathbb{A} \to \mathbb{A}'$ transforms subspaces into subspaces. In particular, φ transforms planes into planes and then preserves parallelism, that is to say, φ is a parallel morphism. By the Fundamental Theorem 3.9, φ is a semiaffine morphism. Analogously, the inverse collineation φ^{-1} is a semiaffine morphism, hence φ is a semiaffinity. ∎

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