Math.Comput.Sci. (2023) 17:11
https://doi.org/10.1007/s11786-023-00564-9
Mathematics in Computer Science

Some Reflections About the Success and Bibliographic Impact of the Dynamic Geometry System *GeoGebra*

Eugenio Roanes-Lozano · Carmen Solano-Macías

Received: 27 October 2022 / Revised: 18 December 2022 / Accepted: 23 January 2023 / Published online: 29 May 2023

Abstract The authors were surprised by the number of articles that used or cited the computer algebra system *DERIVE* more than 10 years after it was discontinued and developed a small bibliographic study about it, published in 2019. Now they address in a similar way the very successful dynamic geometry system *GeoGebra* that, although created 20 years ago, later than the other great dynamic geometry systems (*Cabri Geometry II*, *The Geometer's Sketchpad* and *Cinderella*), has now dozens of millions of users around the world. Not surprisingly, the cites to *GeoGebra* in the well known bibliographic databases *Scopus*, *Web of Science* and *Google Scholar* show an impressive growth.

Keywords Dynamic geometry system · Computer algebra system · GeoGebra · Educational software · Symbolic computation · Technology in mathematics education

Mathematics Subject Classification 68W30 · 97P40 · 97P70 · 68N15 · 68T35

1 Introduction

1.1 Previous Related Work

In 2019 the authors, surprised by the remanence of the Computer Algebra System (CAS) *DERIVE*[1] more than 10 years after it was discontinued, published an article [19] about how it was still used for teaching and how it was still mentioned in several papers. It included a bibliographic study about the papers (most devoted to educational issues) that still referred to *DERIVE*.

[1] All product names, trademarks and registered trademarks are property of their respective owners.

E. Roanes-Lozano
Departamento de Didáctica de las Ciencias Experimentales, Instituto de Matemática Interdisciplinar (IMI), Sociales y Matemáticas, Facultad de Educación, Universidad Complutense de Madrid, c/ Rector Royo Villanova 1, 28040 Madrid, Spain
e-mail: eroanes@ucm.es

C. Solano-Macías (✉)
Departamento de Información y Comunicación, Facultad de Ciencias de la Documentación y Comunicación, Universidad de Extremadura, Grupo de Investigación ARDOPA, Plaza de Ibn Marwan s/n, 06001 Badajoz, Spain
e-mail: csolano@unex.es

1.2 First Notes About the Dynamic Geometry System *GeoGebra* and this Study

Nowadays the Dynamic Geometry System (DGS) *GeoGebra* has become a very successful piece of software, claiming over 100 million users (!!!) all around the world (many more than any other DGS or CAS). It now addresses not only dynamic geometry but also includes algebraic capabilities.

GeoGebra is a free piece of software that welcomes the contributions and ideas from its users and is spread through a network of the so called *GeoGebra Institutes*.

We shall give afterwards a brief overview of DGS in general, as well as a summary of the main characteristics and capabilities of *GeoGebra*. Finally, a bibliographic study of the evolution of the papers mentioning *GeoGebra* will be presented.

1.3 About the Authors

The first author has taught computational mathematics to students from the School of Education at the Universidad Complutense de Madrid for 36 years within the frame of different subjects about the use of information and communication technologies (ICT) in mathematics teaching. In these subjects he has used different hardware and languages (in the past, mainly, *Logo*, *Derive* and *The Geometer's Sketchpad* and now, mainly, *Scratch*, *Maple* and *GeoGebra*). He hs also taught computational mathematics to postgraduates at the School of Mathematics along these years. He was beta tester of the DGS *The Geometer's Sketchpad*.

The second author has been a teacher of the School of Librarians of the Universidad de Extremadura for 27 years. She is specialized in quantitative studies in Social Sciences and Humanities.

2 About *DGS*

2.1 The Pioneers

Cabri Géomètre (later renamed *Cabri Geometry II* [28] and now *Cabri II Plus*) and *The Geometer's Sketchpad* [29,34], were available in the early '90 s. They included the main features of DGS: dynamism and a mouse-based data introduction, allowing to comfortably experiment with plane geometry (Fig. 1).

2.2 Some Ulterior DGS

Many other DGS were developed afterwards. We could underline:

- *Cinderella* [5,13], that performs internal computations in $\mathbb{C}$ (in order to avoid discontinuity problems in animations).
- *Calques 3D* [3,31], devoted to 3D geometry.
- *Geometry Expressions* [9,30], that includes a small internal CAS that allows to directly perform symbolic computations derived from the geometric constructions.

2.3 DGS and Symbolic Computation: Possibilities

Providing the DGS with the possibility to perform symbolic computations opens new and exciting fields like:

- Automatic Theorem Proving in geometry (ATP) [22]: In a naïve way: the geometric conditions are translated into algebraic conditions, and it is checked whether the (algebraic) thesis condition follows from assuming the (algebraic) hypotheses conditions. Let's see a trivial example.

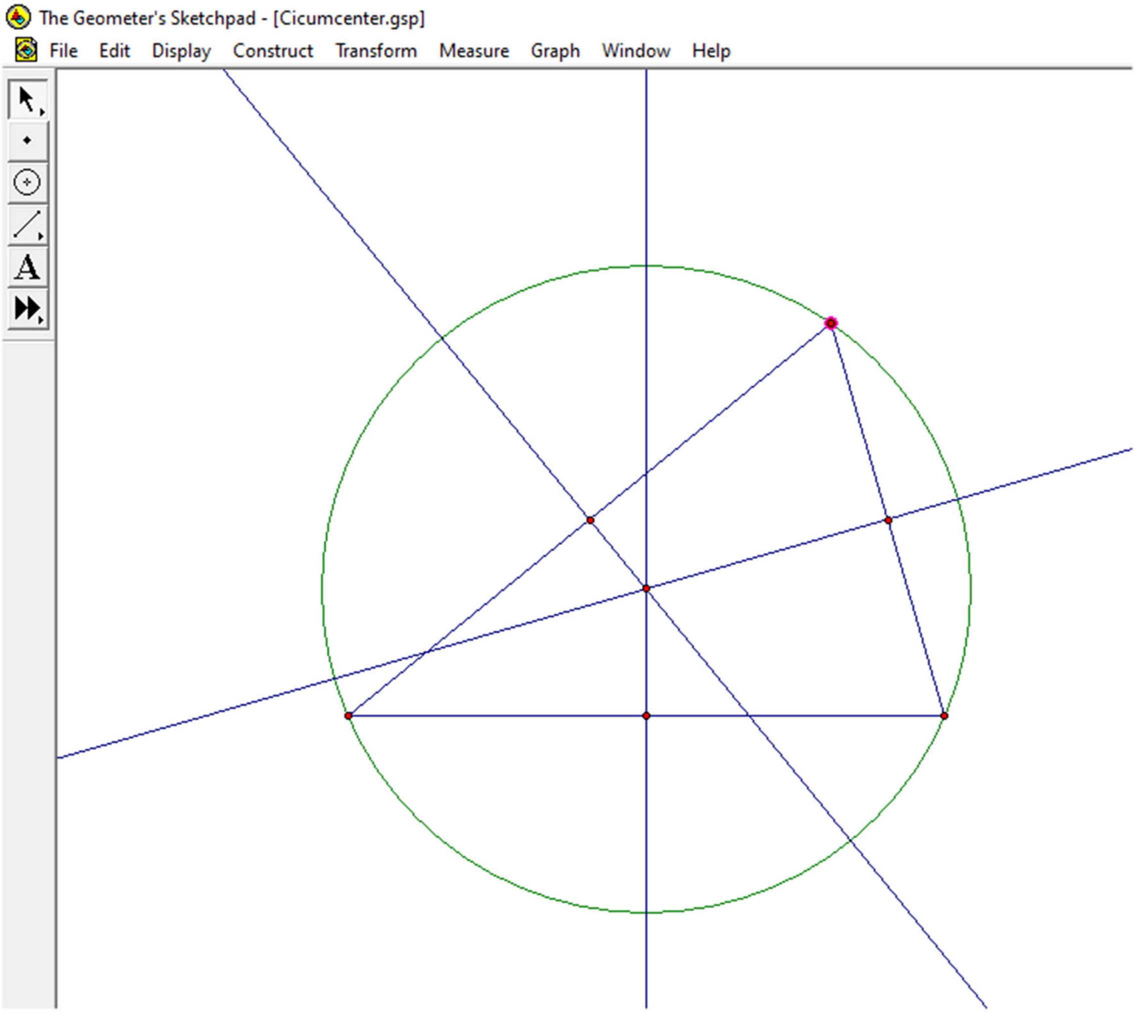

Fig. 1 Circumcircle of a triangle constructed with *The Geometer's Sketchpad v. 4.01*. The initial elements of the construction (in this case the vertices of the triangle) can be dragged and dropped with the mouse, consequently altering the whole construction. It is therefore trivial to check that the circumcentre can lie on the outside, the inside or the border of the triangle

Theorem: The segment bisectors of the sides of a triangle are concurrent (existence of circumcentre).
Proof: Assign general coordinates to the vertices of the triangle, for instance A=(0,0), B=(b1,0), C=(c1,c2) (the reference system is properly chosen). The linear system consisting in the equations of these three lines is compatible, so they are collinear (Fig. 2). The equations systems are not always linear (if circumferences or distances are involved, second degree equations arise and the corresponding equations system are algebraic but not linear). The best known solving methods in such case are Wu's pseudoremainder method [4,32,33] and Gröbner bases method [2,16] (both even allowing to prove new theorems [23–25]).

- Automatic discovery of theorems in geometry (derived from the previous one, oriented to hypotheses completion) [17].
- Exact geometric loci finding [1,23].
- Applications in physics, for instance to linkages [14].

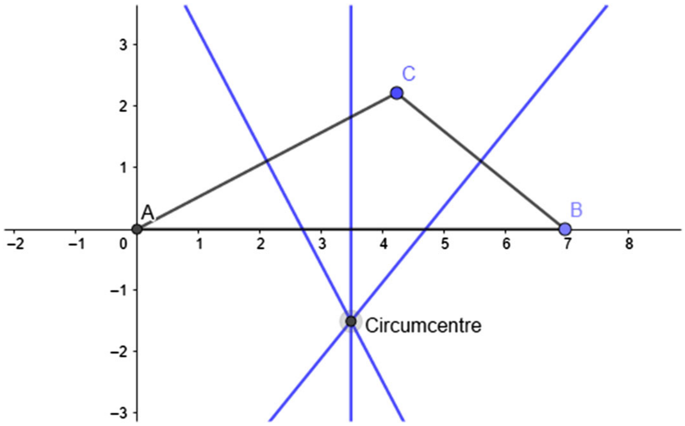

Fig. 2 The perpendicular bisectors of a triangle are collinear (constructed with *GeoGebra 6*)

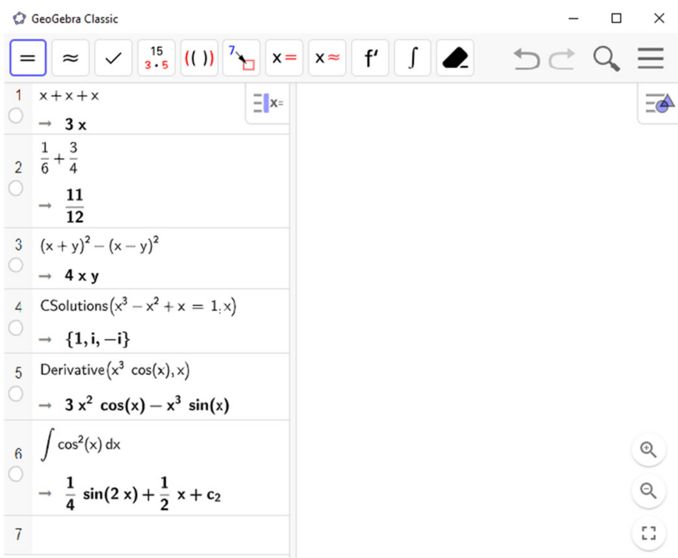

Fig. 3 *GeoGebra 6* "CAS View" showing an exact arithmetic calculation and some symbolic computations

2.4 Incorporating Symbolic Capabilities to DGS

That a DGS can perform symbolic computations can be achieved by different ways:

- The designers and developers of the DGS can incorporate a CAS to the DGS. That is the case of *Geometry Expressions* and *GeoGebra* (Fig. 3).
- the designers and developers of the DGS can facilitate the communication between the DGS and external CAS (another possibility of *Geometry Expressions* and *GeoGebra*).
- external designers and developers develop a connection between a DGS and a CAS using the output file of the DGS (2D case [18,20], 3D case [21]).

2.5 About *GeoGebra*

According to the epigraph "Short History of GeoGebra" of [10]:

> "GeoGebra was created by Markus Hohenwarter in 2001/2002 as part of his master's thesis [11] in mathematics education and computer science at the University of Salzburg in Austria. Supported by a DOC scholarship from the Austrian Academy of Sciences he was able to continue the development of the software as part of his PhD project in mathematics education [12]. During that time, GeoGebra won several international awards, including the European and German educational software awards, and was translated by math instructors and teachers all over the world to more than 25 languages."

GeoGebra has always been freely available, initially thanks to the support of the Austrian Ministry of Education and later thanks to the American NSF project "Standard Mapped Graduate Education and Mentoring".

The graphic interface of *GeoGebra* is similar to those of other DGS (see Figs. 1 and 2). However there is a difference with respect to other DGS like, for instance, *The Geometer's Sketchpad*: in *The Geometer's Sketchpad* the constructible geometric objects (the selectable "tools") depend on the already selected geometric objects, while, in *GeoGebra* the "tools" are firstly chosen and the input geometric objects are selected a posteriori.

2.6 Main Milestones in the Development of *GeoGebra*

GeoGebra "basic" windows (the *Graphical View* and the *Algebraic View*) have a bidirectional connection (Fig. 4) [10]:

- Changes introduced with the mouse in the *Graphical View* induce the corresponding changes in the *Algebraic View*, and,
- Conversely, the changes introduced through the keyboard in the *Algebraic View* induce the corresponding changes in the *Graphical View*.

Since version 3.2 (2009) [7] it includes a *Spreadsheet View*, that allows to easily check geometric theorems (Fig. 5).

Since version 5.04 (beta v. 2012) [7] it includes 3D capabilities, opening a whole new world of possibilities (Fig. 6).

The mathematical software *GeoGebra* has reached an unprecedented success, claiming, as said above, over 100 million users.

We'll analyse afterwards through a bibliographic study its impact in academic papers.

3 Bibliographic Data from *Scopus* (as on April 29th 2022)

3.1 General *Scopus* Data

The search for *GeoGebra* in the database *Scopus* [27] in "Title–Abstract–Keywords" ("T–A–K") finds 832 references. The search for *GeoGebra* in "All Fields" finds 2264 references. They are distributed as shown in Table 1 and Fig. 7.

The values in "T–A–K" are close to those of a monotonically increasing function.

The values in "All Fields" correspond to a monotonically increasing function if we exclude the 2008 value.

3.2 *Scopus* Data by Author

According to *Scopus*, the top authors citing *GeoGebra* in "T–A–K" are:

1) Kovács, Z. (39)
2) Recio, T. (25)
3) Botana, F. (15)

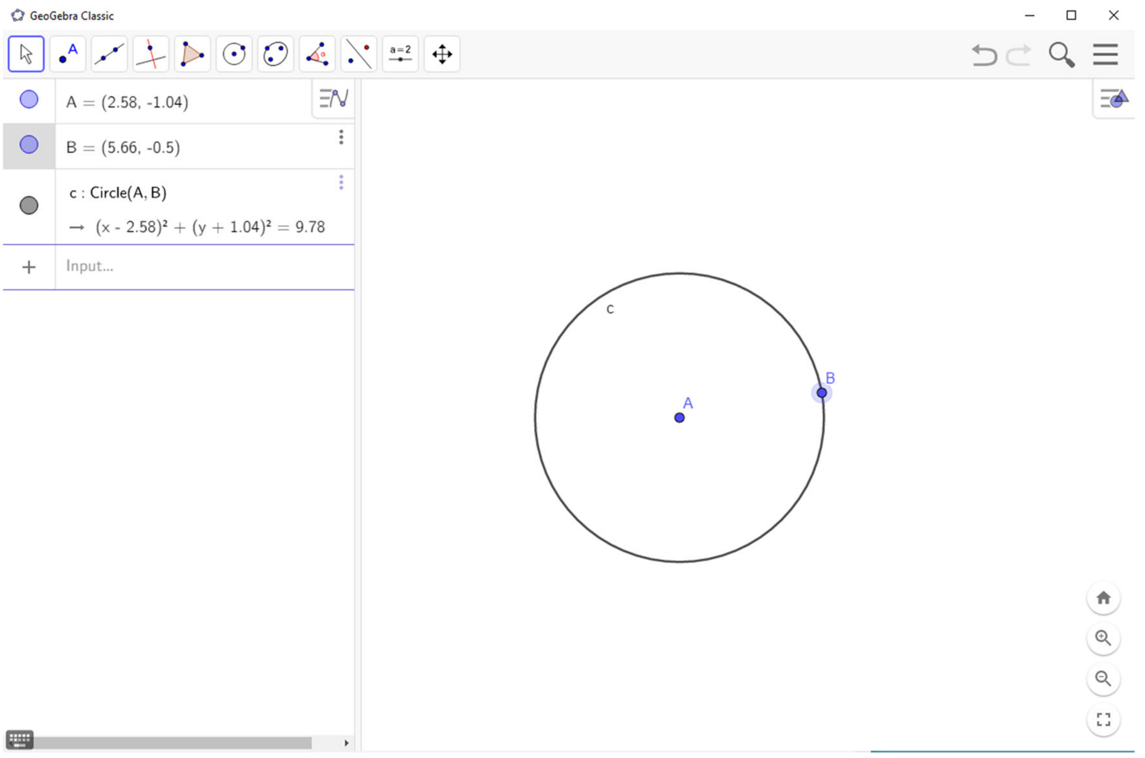

Fig. 4 *GeoGebra 6 Algebraic View* and *Graphical View* windows

4) Hohenwarter, M. (11)

Meanwhile, the top authors citing *GeoGebra* in "All Fields" are:

1) Kovács, Z. (41)
2) Recio, T. (27)
3) Botana, F. (22)
...
10) Hohenwarter, M. (12)

It has to be noticed that the first three authors in the previous lists are working in a remarkable "official" extension of *GeoGebra* (*GeoGebra Discovery*), that is able to find and formally proof theorems directly from geometric constructions (using algebraic ATP techniques) [15].

3.3 *Scopus* Data by Subject Area

The top subject areas where *GeoGebra* is cited in "T–A–K" in *Scopus* database are (Fig. 8):

1) Social Sciences (404)
2) Computer Science (299)
3) Mathematics (296)
4) Physics and Astronomy (171)
5) Engineering (85)

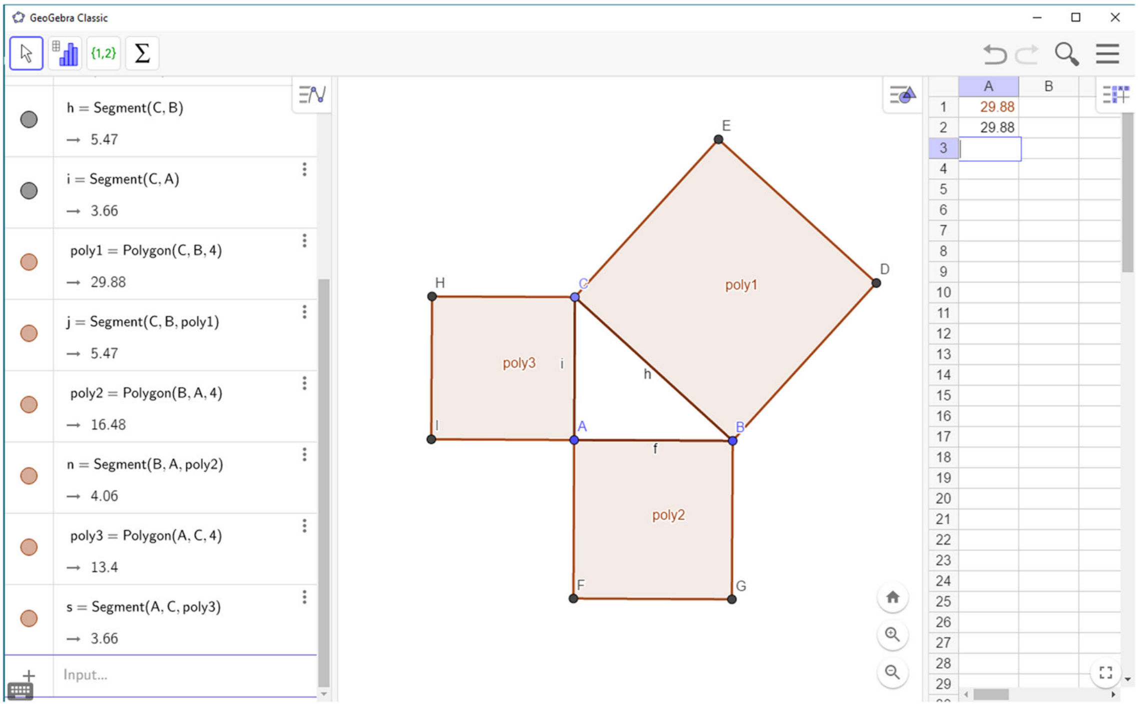

Fig. 5 Checking (not proving) Pythagoras Theorem with *GeoGebra 6* and its *Spreadsheet View* (the area of *poly1* is compared with the sum of areas of *poly2* and *poly3*)

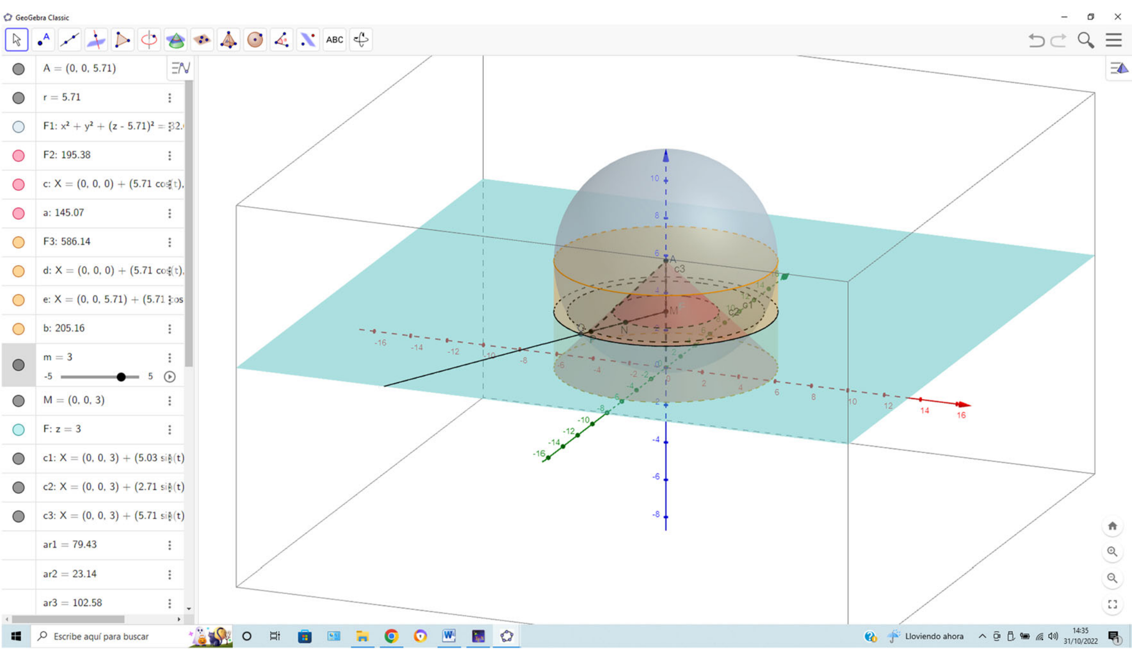

Fig. 6 Arquimedes calculation of the volume of a sphere with *GeoGebra* (the details can be found in [26])

Table 1 Evolution of the cites to *GeoGebra* in the database *Scopus*

	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
TAK	0	0	0	1	3	0	5	10	22	20	46	31	46	50	76	93	124	119	159
All	1	0	1	1	8	3	14	18	46	47	75	85	93	116	186	224	335	416	495

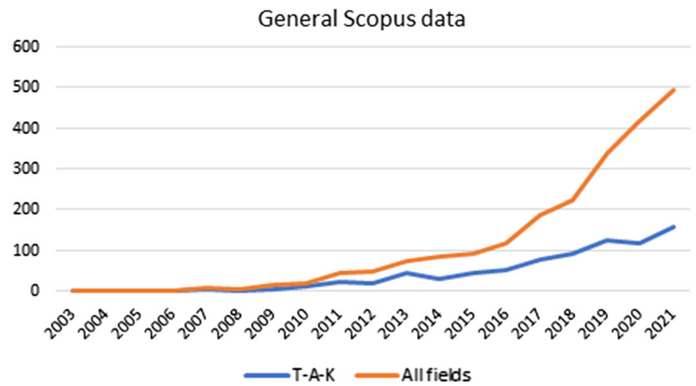

Fig. 7 Evolution of the cites to *GeoGebra* in the database *Scopus*

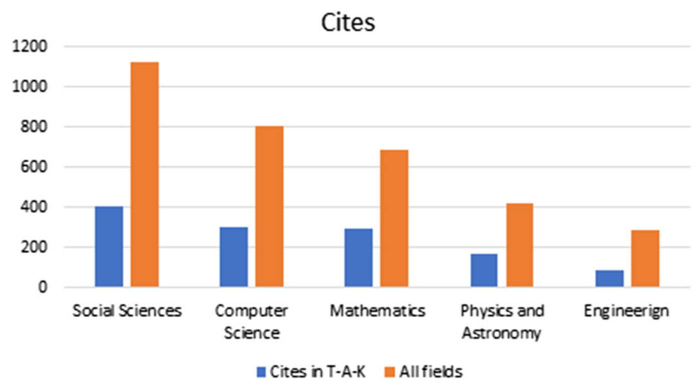

Fig. 8 Top subject areas where *GeoGebra* is cited (*Scopus* database)

Meanwhile, the top subject areas where *GeoGebra* is cited in "All Fields" in *Scopus* database are (Fig. 8):

1) Social Sciences (1119)
2) Computer Science (808)
3) Mathematics (687)
4) Physics and Astronomy (424)
5) Engineering (286)

Due to the characteristics and purpose of *GeoGebra*, we believe that most "Social Sciences" papers will correspond to educational papers This is confirmed if we check the journals where the papers in this area have been published. For instance, the most recent publications in this area indexed in *Scopus* are published in the journals:

- International Journal of STEM Education
- Thinking Skills and Creativity
- Journal of Mathematical Behavior
- Computer Applications in Engineering Education
- i-com (Human-Computer Interaction)

- Creativity Studies
- International Journal of Instruction
- Nurse Education Today (mentioned in a reference, nothing to do)
- International Journal of Science and Mathematics Education
- Education Sciences

3.4 *Scopus* Data by Country

The top 10 countries when looking for *GeoGebra* cites in “T–A–K” (*Scopus* database) are (Fig. 9):

1) Indonesia (127)
2) Spain (78)
3) Turkey (71)
4) Austria (60)
5) Czech Republic (50)
6) Brazil (43)
7) United States (42)
8) Slovakia (34)
9) Italy (28)
10) Malaysia (26)

It is surprising to us that the US occupies the 7th place, the UK the 18th, Germany the 24th and China the 26th. Meanwhile, the top 14 countries when looking in “All Fields” instead are (Fig. 9):

1) Indonesia (395)
2) Spain (175)
3) United States (171) (7th in “T–A–K”)
4) Turkey (151)
5) Brazil (98)
6) China (96) (26th in “T–A–K”)
7) Czech Republic (81)
8) Austria (78)
9) Malaysia (78)
10) Italy (69)
11) Slovakia (67)
12) Germany (62) (24th in “T–A–K”)
13) Israel (59)
14) United Kingdom (58) (18th in “T–A–K”)

The position changes of the US, China and Germany are remarkable and worth a deeper study. Can they be related, for instance, to cultural issues?

The distribution by countries can also be visualized in the maps of Figs. 10 and 11.

4 Bibliographic Data from *Web of Science* (as on April 29th 2022)

The search for *GeoGebra* in the database *Web of Science* [6] in “Title” finds 330 references. The search for *GeoGebra* in “Topic” finds 800 references. They are distributed as shown in Table 2 and Fig. 12. Both lists of values show an increasing general tendency, although with more oscillations than when using *Scopus* as data source.

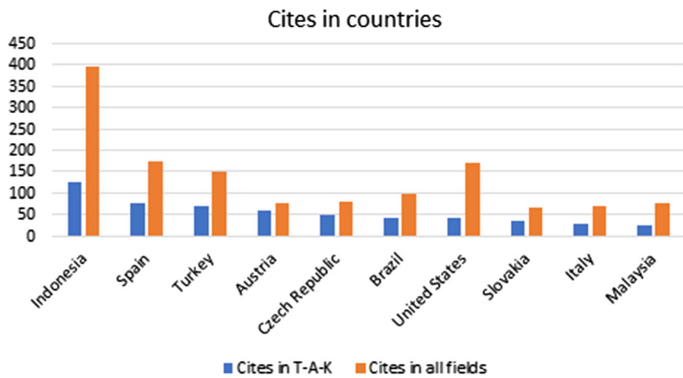

Fig. 9 Top countries when looking for *GeoGebra* cites in "T–A–K" and in "All Fields" in *Scopus* database

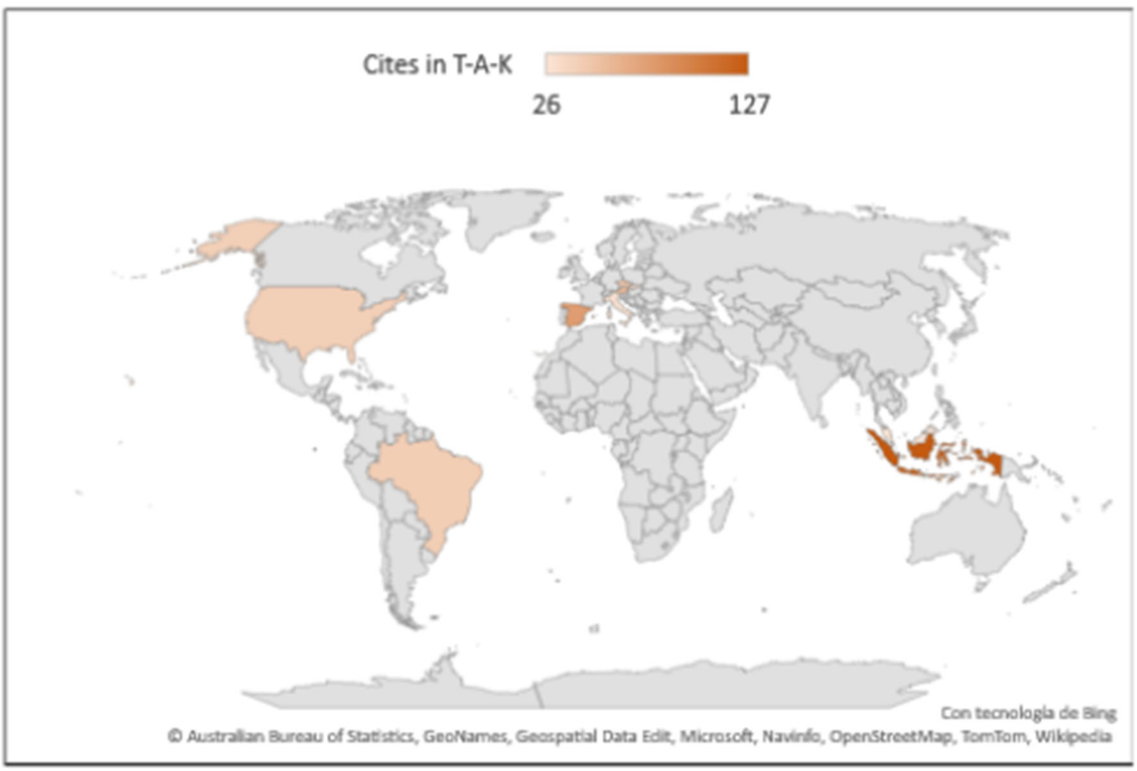

Fig. 10 Top countries when looking for *GeoGebra* cites in "T–A–K" in *Scopus* database

5 Bibliographic Data from *Google Scholar* (as on April 29th 2022)

The general search for *GeoGebra* in the database *Google Scholar* [8] results in an impressive ∼73,800 references. The advanced search in "Title" finds 9820 references. They are distributed as shown in Table 3 and Fig. 13.

The values for the general case correspond to a monotonically increasing function from 2006 onwards. The values for the search in "Title" also show an increasing tendency, with a slight maximum in 2018, and are more or less stabilized since 2017.

The dates in this source are less accurate than those in *Scopus* and *WoS* (e.g., many documents are dated before *GeoGebra* existed due to dating errors!).

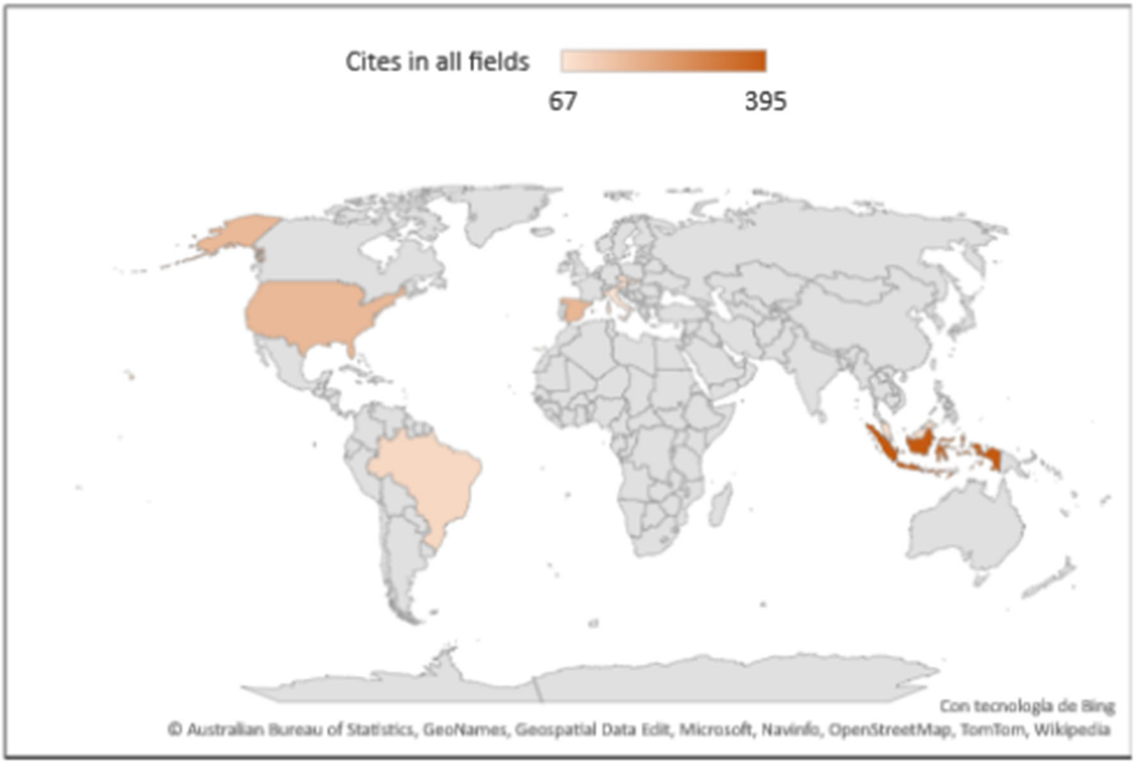

Fig. 11 Top countries when looking for *GeoGebra* cites in "All Fields" in *Scopus* database

Table 2 Evolution of the cites to *GeoGebra* in the database *Web of Science*

	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
Title	2	10	9	4	17	13	29	25	52	38	43	43	38
Topic	2	14	25	14	27	34	69	65	109	110	121	90	104

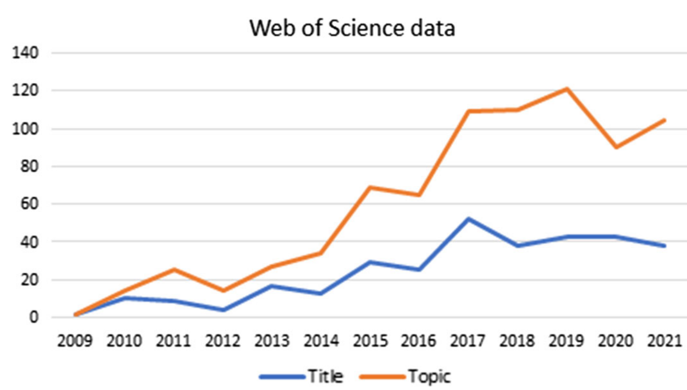

Fig. 12 Evolution of the cites to *GeoGebra* in the database *Web of Science*

Table 3 Evolution of the cites to *GeoGebra* in the database *Scopus*

	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
Title	4	11	8	16	38	78	149	257	352	467	657	586	717	762	994	999	993	982	928
All	39	49	37	69	208	288	460	782	1520	1730	2430	2870	3640	4510	5280	5700	6390	6850	8120

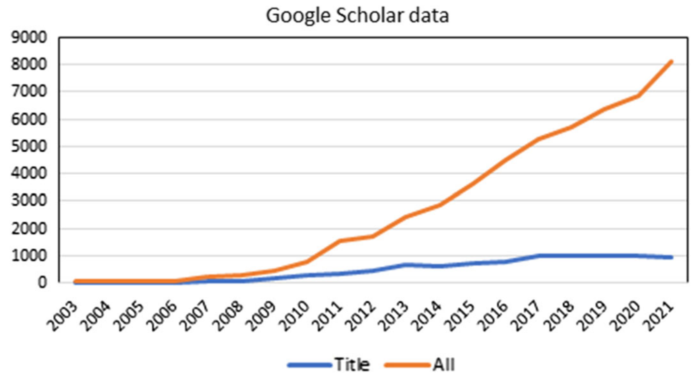

Fig. 13 Evolution of the cites to *GeoGebra* in the database *Google Scholar*

6 Conclusions

The available DGS are great tools for exploring geometry. The huge number of papers using *GeoGebra* and their constant growth confirms this fact and, moreover, the success of this particular piece of software.

The three bibliographic sources used (*Scopus*, *Web of Science* and *Google Scholar*) provide data with similar tendencies (constant growth). In the three sources consulted, we perceive a slight decrease in the number of citations in 2020, coinciding with the pandemic.

It is noticeable that very many papers are published in educational journals.

We guess that the success is due to the good policy of this software:

- It is free,
- Training has been provided but the *GeoGebra Institutes*,
- It is multilingual,
- It development has been opened to the contribution and suggestions of the users community.

There are open questions:

- Which are the reasons for the changes in the positions of the US, the UK, Germany and China if ordering countries by publications indexed in *Scopus* mentioning *GeoGebra* in “T–A–K” or in “All Fields”?
- Which are the reasons for the growth of references in *Google Scholar* in the general search and the stabilization of references in the search in the title?

Acknowledgements This paper is dedicated to the memory of Eugenio Roanes-Lozano. This work was partially supported by the research projects PGC2018-096509-B-I00 and PID2021-122905NB-C21 (Government of Spain).

Funding Information Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

References

1. Botana, F., Valcarce, J.L.: A software tool for the investigation of plane loci. Mat. Comput. Simul. **61**(2), 141–154 (2003). https://doi.org/10.1016/S0378-4754(02)00173-8
2. Buchberger, B.: Bruno Buchberger's PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. J. Symb. Comput. **41**(3–4), 475–511 (2006). https://doi.org/10.1016/j.jsc.2005.09.007
3. Calques3D. (n.a.). http://www.calques3d.org/
4. Chou, S.C.: Mechanical Geometry Theorem Proving. Reidel, Dordrecht (1988)
5. Cinderella.: (2019). https://cinderella.de/tiki-index.php
6. Clarivate. : Web of Science. (n.a.). https://www.webofscience.com/wos/alldb/basic-search
7. GeoGebra. (n.a.). In Wikipedia. https://es.wikipedia.org/wiki/GeoGebra#:~:text=Es
8. Google Académico. (n.a.). https://scholar.google.es/
9. GXWeb / Geometry Expressions. (n.a.). https://geometryexpressions.com/
10. Hohenwarter, M., Preiner, J.: Dynamic mathematics with GeoGebra. J. Online Math. Appl. (JOMA) **7**, 1448 (2007)
11. Hohenwarter, M.: GeoGebra - ein Softwaresystem für dynamische Geometrie und Algebra der Ebene (Master's thesis). University of Salzburg, (2002)
12. Hohenwarter, M.: GeoGebra - didaktische Materialien und Anwendungen für den Mathematikunterricht (PhD thesis). University of Salzburg, (2006)
13. Kortenkamp, U.: Foundations of dynamic geometry (PhD. Thesis). Swiss Fed. Inst. Tech. Zurich, (1999)
14. Kovács, Z., Recio, T., Vélez, P.: Reasoning about linkages with dynamic geometry. J. Symb. Comput. **97**, 16–30 (2020). https://doi.org/10.1016/j.jsc.2018.12.003
15. Kovács, Z., Recio, T., Vélez, M.P.: Automated reasoning tools in GeoGebra Discovery. ACM Commun. Comput. Algebra **55**(2), 39–43 (2021). https://doi.org/10.1145/3493492.3493495
16. Kutzler, B., Stifter, S.: On the application of Buchberger's algorithm to automated geometry theorem proving. J. Symb. Comput. **2**(4), 389–397 (1986)
17. Recio, T., Vélez, M.P.: Automatic discovery of theorems in elementary geometry. J. Autom. Reason. **23**, 63–82 (1999). https://doi.org/10.1023/A:1006135322108
18. Roanes-Lozano, E.: Boosting the geometrical possibilities of dynamic geometry systems and computer algebra systems through cooperation. In: M. Borovcnik, H. Kautschitsch (Eds.), Technology in Mathematics Teaching. Proceedings of ICTMT-5, öbv & hpt, Schrifrenreihe Didaktik der Mathematik 25, Viena, pp. 335–348 (2002)
19. Roanes-Lozano, E., Galín-García, J.L., Solano-Macías, C.: Some reflections about the success and impact of the computer algebra system DERIVE with a 10-year time perspective. Math. Comput. Sci. **13**, 417–431 (2019). https://doi.org/10.1007/s11786-019-00404-9
20. Roanes-Lozano, E., Roanes-Macías, E., Villar-Mena, M.: A bridge between dynamic geometry and computer algebra. Math. Comp. Mod. **37**(9–10), 1005–1028 (2003). https://doi.org/10.1016/S0895-7177(03)00115-8
21. Roanes-Lozano, E., van Labeke, N., Roanes-Macías, E.: Connecting the 3D DGS Calques3D with the CAS Maple. Math. Comput. Simul. **80**(6), 1153–1176 (2010). https://doi.org/10.1016/j.matcom.2009.09.008
22. Roanes-Macías, E., Roanes-Lozano, E.: Nuevas Tecnologías en Geometría. Editorial Complutense, Madrid (1994)
23. Roanes-Macías, E., Roanes-Lozano, E.: Automatic determination of geometric Loci. 3D-extension of Simson-Steiner theorem. In: Campbell, J.A., Roanes-Lozano, E. (eds) Artificial Intelligence and Symbolic Computation. AISC 2000. Lecture Notes in Computer Science, vol 1930. Springer, Berlin, Heidelberg, 2001, pp. 157–173 (2001). https://doi.org/10.1007/3-540-44990-6_12
24. Roanes-Macías, E., Roanes-Lozano, E.: 3D-extension of Steiner chains problem. Math. Comput. Model. **45**, 137–148 (2007). https://doi.org/10.1016/j.mcm.2006.04.012
25. Roanes-Macías, E., Roanes-Lozano, E., Fernández-Biarge, J.: Obtaining a 3D extension of Pascal theorem for non-degenerated quadrics and its complete configuration with the aid of a computer algebra system. RACSAM Rev. R. Acad. A **103**(1), 93–109 (2009). https://doi.org/10.1007/BF03191837
26. Roanes Macías, E., Roanes Lozano, E.: Recreando con GeoGebra el cálculo del volumen de la esfera ideado por Arquímedes. Bol. Soc. "Puig Adam" Prof. Mat. **102**, 62–72 (2016)
27. Scopus. (n.a.). https://www.scopus.com/search/form.uri?display=basic#basic
28. Instruments, Texas: Cabri Geometry II. Windows y MS-DOS. Texas Instruments Incorporated, Manual para Macintosh (1999)
29. The Geometer's Sketchpad Version 5.06. (n.a.). https://sketchpad.keycurriculum.com/
30. Todd, P.: Geometry Expressions: a constraint based interactive symbolic geometry system. In: F. Botana, T. Recio (Eds.), Automated Deduction in Geometry, 6th International Workshop, ADG 2006, Springer-Verlag Lecture Notes in Artificial Intelligence 4689, Berlin, Heidelberg, New York, pp. 189–202 , (2007). https://doi.org/10.1007/978-3-540-77356-6_12
31. van Labeke, N.: Calques 3D: a microworld for spatial geometry learning. ITS'98 - System Demonstrations, San Antonio (Texas), August 16–19. (1998) . Available from: http://www.calques3d.org/docs/its98-demo.pdf

32. Wu, W.-T.: On the decision problem and the mechanization of theorem-proving in elementary geometry. In: W. W. Bledsoe and D. W. Loveland (Eds.) Automated Theorem Proving. After 25 Years. Contemporary Mathematics 29, AMS, Providence, Rhode Island, pp. 213–234 (1984)
33. Wu, W.-T.: Some recent advances in mechanical theorem-proving of geometries. In: W. W. Bledsoe and D. W. Loveland (Eds.) Automated Theorem Proving. After 25 Years. Contemporary Mathematics 29, AMS, Providence, Rhode Island, pp. 235–242 (1984)
34. Wu, W.-T.: The Geometer's Sketchpad 4 User Guide and Reference Manual. Key Curriculum, Emeryville, CA, (2001)