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COMPLETION OF (LF)-SPACES

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Once the existence of metrizable (LF)-spaces was discovered by A. Grothendieck [1], the problem whether the completion of an (LF)-space is or is not an (LF)-space is answered in the negative, because no (LF)-space can be a Fréchet space. However, some (non-metrizable) (LF)-spaces are complete, e.g. the classical Köthe's strict (LF)-spaces. In this paper we will carry out a thorough study of the completeness of (LF)-spaces stressing upon the rather stable completion properties of (LB)-spaces. A basic tool for handling this problem is an Open Mapping Theorem for completions of (LF)-spaces, proved as well in the paper.

We recall that a Hausdorff locally convex space (E, τ) is an (LF)-space if there exists a strictly increasing sequence (E_n, τ_n) of Fréchet spaces, called defining sequence for E, such that $E = \bigcup_n E_n$, $\tau_{n+1}|_{E_n} \leq \tau_n$ for every $n \in \mathbb{N}$ and τ is the finest Hausdorff locally

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convex topology on E such that $\tau|_{E_n} \leq \tau_n$ for all $n \in \mathbb{N}$. We write $(E, \tau) = \lim (E_n, \tau_n)$. If each space E_n is a Banach space, the (LF)-space E is called an (LB)-space. If $\tau_{n+1}|_{E_n} = \tau_n$ for every $n \in \mathbb{N}$, the (LF)- or (LB)-space is said to be strict.

According to S. Saxon and P. P. Narayanaswami [4], an (LF)-space E is said to be of type i or an $(LF)_i$-space ($i = 1,2,3$), if it satisfies the following condition (i) :

(1) E has a defining sequence $\{E_n\}$ such that no E_n is dense in E.

(2) E is not metrizable and it has a defining sequence $\{E_n\}$ such that some E_n is dense in E.

(3) E is metrizable

While it is true that the completion of an $(LF)_3$-space is never an (LF)-space (because it is a Fréchet space), the completion of an $(LF)_1$- or an $(LF)_2$-space can be an (LF)-space. As a matter of fact, any strict (LF)-space is a complete $(LF)_1$-space and the $(LB)_2$-spaces $\ell_{p^-} = \lim \ell_{p_n}$ ($1 < p_n < p_{n+1} < p$, $\lim_{n\to\infty} p_n = p$) defined by S. Saxon and P. P. Narayanaswami in [3] are complete as well. On the other hand we supply in the paper examples of $(LF)_i$-spaces, $i = 1,2$, whose completion is not an (LF)-space.

For the subclass of (LB)-spaces we have instead, the following main result:

<u>Theorem.</u> a) The completion of an $(LB)_1$-space is an $(LB)_1$-space. b) The completion of an $(LB)_2$-space is either an $(LB)_2$-space or a Banach space.

This Theorem has nontrivial applications because non-complete $(LB)_1$- and $(LB)_2$-spaces do exist. Note, however that (LB)-spaces of type 3 do not exist by the Amemiya-Komura Theorem.

The proof of the main Theorem relies heavily on the following Open Mapping Theorem for completions of (LF)-spaces:

Theorem. Let Ψ be an (LF) topology on the completion $(\tilde{E}, \tilde{\tau})$ of the (LF)-space (E, τ), such that the identity map $I : (\tilde{E}, \Psi) \longrightarrow (\tilde{E}, \tilde{\tau})$ is continuous. Then I is a topological isomorphism.

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