ON THE THREE SPACE PROBLEM FOR COUNTABLE BARRELLEDNESS

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ABSTRACT:

Let F be a closed subspace of a Hausdorff locally convex space E such that F and the Hausdorff quotient E/F enjoy a property (M). Does the whole space E enjoy (M)?.

This problem is called "the three-space-problem" and it is a common problem in the framework of twisted exact sequences (see |3|). It is been already considered by several authors, e.g. |PB|, |4|.

In the present paper we shall provide a negative answer to the problem for $\mathbb{P}$-quasi-barrelledness, $\mathbb{P} \in \{\aleph_0, l_\infty, c, c_0\}$ and for df spaces. We shall also supply a thoroughly positive answer for l_∞-barrelledness and a partial affirmative answer for c_0-barrelledness.

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Our first two examples are modifications of constructions in |4| (see also |PB, 8.3.47|).

Let F be an infinite dimensional Frechet-Montel space admitting continuous norm. We set E:=F' and we consider $|E, \mu(E,F)|$ (Note that $|E, \mu(E,F)|$ is a non-normable bornological DF-space). By |4| there exists a fundamental sequence of bounded sets $\{B_n\}_{n \in \mathbb{N}}$ in E and a countably-codimensional dense subspace $L \subset E$ such that

a) $[B_n]$ is dense in E and $\dim|[B_{n+1}]\setminus[B_n]| \geq n$ for all $n \in \mathbb{N}$.

b) $\dim |(L+[B_n])/L| \leq n$ for every $n \in \mathbb{N}$.

Let τ_0 be the projective topology on E, with respect to the linear maps

$$I: E \longrightarrow |E, \mu(E,F)|$$

$$q: E \longrightarrow |E/L, \sigma|$$

where γ is any normed topology in E/L. Then, by |2|, $\tau_0|L = \mu(E,F)|L$ and $\tau_0/L = \gamma$.

Proposition.

With all the previous notations.

i) $(L, \tau_0|L)$ is a barrelled space.

ii) $(E/L, \tau_0/L)$ is a normed space.

iii) (E, τ_0) is not c_0-quasi-barrelled.

Proof.

i) and ii) are consequences of the preceding considerations iii). The proof will be complete if we show that (E, τ_0) has a fundamental sequence of bounded sets and yet it is not a df space. By b), there must exist finite dimensional linear subspace $M_n \subset L$ such that $L+[B_n]=L\oplus M_n$, and we may assume $M_n \subset M_{n+1}$, $\forall n \in \mathbb{N}$.

Let C_n be a bounded zero-neighbourhood in M_n, $n \in \mathbb{N}$ We choose them so that $C_n \subset C_{n+1}$, $\forall n \in \mathbb{N}$. We claim that $(H_n := (B_n \cap L)+(n.C_n))_{n \in \mathbb{N}}$ constitutes a fundamental sequence of bounded sets in (E, τ_0). Let B be any bounded set in (E, τ_0) (hence it is $\mu(E,F)$-bounded) then there exists

$$n_0 \in \mathbb{N} \ : \ B \subset B_n \subset L+[B_n]=L \oplus M_n, \ \forall n \in \mathbb{N}, \ n \geq n_0$$

Thus, there is $m \in \mathbb{N}.\ m \geq n_0 : B \subset B_m \cap L + m \cdot C_m$.

Suppose $|E, \tau_0|$ is a df space then for every bounded zero neighbourhood B in $|E/L, \tau_0/L|$, by |(J), 12.4.8|, there exists $n \in \mathbb{N}$ such that $B \subset \overline{q(H_n)}$. Whence $[q(H_n)]$ is dense in $|E/L, \tau_0/L|$ and consequently $[B_n]+L$ is dense in $|E, \tau_0|$ which contradicts that L is closed in $|E, \tau_0|$ and $\dim|(L+[B_n])/L|$ is finite.

As a corollary we establish the main results of the paper.

Theorem.

The three-space problem fails for $\mathbb{P}$-quasi-barrelledness where $\mathbb{P} \in \{\chi_0, l_\infty, c, c_0\}$.

Since $(E, \mathcal{T}_0)$, $(L, \mathcal{T}_0|L)$, and $(E/L, \mathcal{T}_0/L)$ possess fundamental sequences of bounded sets, we also infer, from the above counterexample the following result:

Proposition.

The three-space problem does not hold for df-spaces.

Next we provide another counterexample for the classess of $\mathbb{P}$-quasi-barrelled spaces, $\mathbb{P} \in \{l_\infty, c\}$.

Let $(E, \mathcal{T})$ be a barrelled Hausdorff locally convex space containing a dense linear subspace L of countably infinite codimension such that every bounded set in $(E, \mathcal{T})$ has finite-dimensional linear span. (see [1]). Take the Hausdorff locally convex topology $\mathcal{T}_0$ on E such that $\mathcal{T}_0|L = \mathcal{T}|L$, $\mathcal{T}_0/L = \gamma$ being γ any prefixed normed topology on E/L.

Proposition.

Under the above assumptions,

i) $(L, \mathcal{T}_0|L)$ is barrelled.

ii) $(E/L, \mathcal{T}_0/L)$ is normable.

iii) $(E, \mathcal{T}_0)$ is non-$\mathbb{P}$-quasi-barrelled. $\mathbb{P} \in \{l_\infty, c\}$.

Proof.

iii)

It suffices to show $(E, \mathcal{T}_0)$ is non-$\mathbb{P}$-barrelled. Note that every $\mathbb{P}$-quasi barrelled space whose bounded sets have finite-dimensional linear spans is a $\mathbb{P}$-barrelled space, $\mathbb{P} \in \{l_\infty, c\}$. Assume $(E, \mathcal{T}_0)$ is $\mathbb{P}$-barrelled, $\mathbb{P} \in \{l_\infty, c\}$, then $(E/L, \mathcal{T}_0/L)$ would also be $\mathbb{P}$-barrelled $\mathbb{P} \in \{l_\infty, c\}$ and applying theorem 1.1 in |6| and |PB.8.2.8|, it would carry the finest locally convex topology, contradicting $\mathcal{T}_0/L = \gamma$.

We finally supply a thoroughly, positive answer for l_∞-barrelledness and a partial one for c_0-barrelledness.

Theorem.

Let E be a Hausdorff locally convex space. Let L be a closed linear l_∞-barrelled subspace of E such that E/L is l_∞-barrelled. Then E is l_∞-barrelled.

Proof.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a $\sigma(E',E)$-bounded sequence in E'. Then $\{f_n|L\}_{n\in\mathbb{N}}$ is $\sigma(L',L)$-bounded and hence L-equicontinuous in L'. By Hahn-Banach, we construct for every $n\in\mathbb{N}$, an extension $g_n\in E'$ of the mapping $f_n|L\in L'$, so that $\{g_n\}_{n\in\mathbb{N}} \subset E'$ is E-equicontinuous. Thus $(g_n-f_n)|L=0$, $\forall n\in\mathbb{N}$ and we can define $h_n\in(E/L)':h_n(Q(x))=(g_n-f_n)(x)$ $\forall x\in E$ (Q being the quotient mapping $Q:E \longrightarrow E/L$). So $\{h_n\}_{n\in\mathbb{N}}$ is a $\sigma(L^\circ,E/L)$-bounded (hence E/L-equicontinuous) set. Since the transposed mapping of Q, $Q^t: L^\circ := (E/L)' \longrightarrow E'$ maps equicontinuous sets into equicontinuous sets and Q^T is inyective, we conclude $\{g_n-f_n\}_{n\in\mathbb{N}}$ (and consequently $\{f_n\}_{n\in\mathbb{N}}$ is E-equicontinuous.

An identical proof holds for the following partial positive answer for c_0-barrelledness.

Proposition.

Let E be a Hausdorff locally convex space. Let L be a closed linear c_0-barrelled subspace of E such that E/L is l_∞-barrelled. Then E is c_0-barrelled.

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