On the Completion of (LF)-Spaces

By

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Abstract. Once the existence of metrizable (LF)-spaces was discovered, the problem whether the completion of an (LF)-space is or is not an (LF)-space was answered in the negative, because no (LF)-space can be a Fréchet space. However, some (non-metrizable) (LF)-spaces are complete, e.g. the classical Köthe's strict (LF)-spaces. In this paper we will carry out a thorough study of the completeness of (LF)-spaces stressing upon the stable completion properties of (LB)-spaces. A basic tool for handling this problem is an Open Mapping Theorem for completions of (LF)-spaces, which is also proved in the present paper.

1. Introduction and Preliminary Results. In [8], Saxon and Narayanaswami partition the class of all the (LF)-spaces into three mutually disjoint, non-empty classes, denoted (LF)₁-spaces or (LF)-spaces of type i (i = 1, 2, 3) (see definitions below). (LF)₁-spaces include the well known class of strict (LF)-spaces, while (LF)₃-spaces are those (LF)-spaces which are metrizable (the first example of a metrizable (LF)-space is due to A. Grothendieck [2]; these spaces were later on studied in [5], [7] and [8]).

Certain completeness properties of (LF)-spaces depend upon the type of the space. For example, no (LF)₃-space is complete, because no (LF)-space can be Fréchet, while many (LF)₁-spaces are complete, for example all strict (LF)-spaces. We will prove that (LB)-spaces exhibit a fair behaviour with respect to the completion, while the remaining (LF)-spaces are, in general, not stable for complete hulls.

For the general theory of (LF)-spaces, we follow throughout [3], §19. We recall that a Hausdorff locally convex space \((E, \tau)\) is an

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(LF)-space if there exists a strictly increasing sequence \(\{(E_n, \tau_n)\}\) of Fréchet spaces, called a defining sequence for \(E\), such that \(E = \bigcup E_n\) and \(\tau_{n+1} \leq \tau_n\) for every \(n \in \mathbb{N}\) and \(\tau\) is the finest Hausdorff locally convex topology on \(E\) such that \(\tau|_{E_n} \leq \tau_n\) for all \(n \in \mathbb{N}\). We write \((E, \tau) = \lim (E_n, \tau_n)\). If each space \(E_n\) is a Banach space, the (LF)-space \(E\) is called an (LB)-space. If \(\tau_{n+1}|_{E_n} = \tau_n\) for every \(n \in \mathbb{N}\), the (LF)- or (LB)-space is said to be strict.

1.1. Definition. An (LF)-space \(E\) is said to be of type \(i\) or an (LF)
space \((i = 1, 2, 3)\), if it satisfies the following condition (i):

(1) \(E\) has a defining sequence \(\{E_n\}\) such that no \(E_n\) is dense in \(E\).

(2) \(E\) is not metrizable and it has a defining sequence \(\{E_n\}\) such that some \(E_n\) is dense in \(E\).

(3) \(E\) is metrizable.

The disjointness of the three classes is readily seen: Since any two defining sequences for an (LF)-space are equivalent we have \((\text{LF})_1 \cap (\text{LF})_2 = \emptyset\); \((\text{LF})_2 \cap (\text{LF})_3 = \emptyset\) is obvious; \((\text{LF})_1 \cap (\text{LF})_3 = \emptyset\) follows from the fact that every metrizable, barrelled space is Baire like (cf. [6] Corollary 2.5.).

In the sequel, \(\varphi\) will denote a countable-dimensional linear space endowed with the finest locally convex topology.

2. Open Mapping Theorem for (LF)-Spaces. In this Section we will make use of the classical Pták’s Open Mapping Theorem to deduce a similar theorem for completions of (LF)-spaces. The following lemma is obvious.

2.1. Lemma. If \((E_1, \Gamma_1)\) and \((E_2, \Gamma_2)\) are Fréchet spaces continuously included in a Hausdorff topological space, then the locally convex topology \(\Gamma\) on \(E_1 \cap E_2\) with neighborhood basis \(\{U \cap V; U \in \Gamma_1, V \in \Gamma_2\}\) is Fréchet.

2.2. Theorem. Let \(\Psi\) be an (LF) topology on the completion \((\tilde{E}, \tilde{\tau})\) of the (LF)-space \((E, \tau)\), such that the identity map \(I: (\tilde{E}, \Psi) \to (\tilde{E}, \tilde{\tau})\) is continuous. Then \(I\) is a topological isomorphism.

Proof. Let us assume that \((E, \tau) = \lim (E_n, \tau_n)\) and \((\tilde{E}, \Psi) = \lim (\tilde{E}_j, \Psi_j)\) for Fréchet spaces \(E_n, \tilde{E}_j (n, j \in \mathbb{N})\). Fix \(n \in \mathbb{N}\). Since \(E_n \subset \tilde{E}_j, E_n \bigcup (E_n \cap \tilde{E}_j)\) and, therefore, there exists \(j \in \mathbb{N}\) such that \(H = \bigcup \).
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\( E_n \cap F_j \) is a dense and barrelled subspace of \( (E_n, \tau_n) \) (let us note that \( E_n \), being Fréchet, is a \((db)\)-space, see [4]). By hypothesis, \( E_n \) and \( F_j \) are continuously included in the (Hausdorff) space \( (\tilde{E}, \tilde{\tau}) \). By means of the Lemma 2.1., \( H \) is endowed with a Fréchet topology \( H' \). Since \( \Gamma \geq \tau_n|_H \), Pták’s Open Mapping Theorem applied to the identity map \( (H, \Gamma) \to (H, \tau_n|_H) \) yields \( \Gamma = \tau_n|_H \). We deduce that \( H \) is closed in \( (E_n, \tau_n) \), that is \( H = E_n \subset F_j \). We also deduce that the inclusion map \( (E_n, \tau_n) \to (F_j, \Psi_j) \) is continuous and, thus, \( (E_n, \tau_n) \leftrightarrow (\tilde{E}, \Psi) \) is continuous as well. Since \( n \in \mathbb{N} \) was arbitrarily fixed, we conclude that the canonical immersion \( (E, \tilde{\tau}) \leftrightarrow (\tilde{E}, \Psi) \) is continuous. Extending by continuity to the completions, we prove the continuity of the identity \( (\tilde{E}, \tilde{\tau}) \to (\tilde{E}, \Psi) \).

3. Completion of (LF)-Spaces. While it is true that the completion of an \((LF)_2\)-space is never an \((LF)_1\)-space (because it is a Fréchet space), the completion of an \((LF)_1\)-space or an \((LF)_2\)-space can be an \((LF)_1\)-space. As a matter of fact, any strict \((LF)_1\)-space is a complete \((LF)_1\)-space ([3], 19.5.3.) and the \((LB)_2\)-spaces \( \lim_{n} \) of type 2 defined by SAXON and NATAYANASWAMI in [7] are complete as well. However, we will show in the next examples that the completion of an \((LF)_i\)-space, \( i = 1, 2 \), is not, in general, an \((LF)_i\)-space.

3.1. Example. Let \( E = \lim E_n \) be a complete \((LF)_2\)-space. For each \( n \in \mathbb{N} \) we define \( F_n = E_n \times E_n \times \ldots \) endowed with the (Fréchet) product topology denoted by \( \Psi_n \). The \((LF)_1\)-space \( (E, \Psi) = \lim (F_n, \Psi_n) \) is a dense, proper subspace of \( E \times E \times \ldots \) because of its “diagonal” construction. Using the fact that \( E \) is of type 2 it is easy to prove that \( \Psi \) is the relative topology induced in \( F \) by the product topology of \( E \times E \times \ldots \). We first observe that \( (E, \Psi) \) is not metrizable, because \( E \) is not metrizable. Since \( E \) is of type 2, some \( F_n \) is dense in \( E \times E \times \ldots \) and hence also in \( F \). So \( (F, \Psi) \) is a \((LF)_2\)-space and its completion is \( E \times E \times \ldots \) which is not an \((LF)_2\)-space because no infinite product of \((LF)_1\)-spaces is an \((LF)_2\)-space.

3.2. Example (P. Pérez-Carreras) Let \( F \) be the non-complete \((LF)_2\)-space described in Example 3.1. Then obviously \( F \times \varphi \) is an \((LF)_1\)-space. The completion \( \tilde{F} \times \varphi \) of \( F \times \varphi \) is not an \((LF)_2\)-space because it has a quotient \( \tilde{F} \) which is neither an \((LF)_2\)-space nor a Fréchet space.
We will now study separately the completion of \((LB)_1\)-spaces and \((LB)_2\)-spaces (note that \((LB)\)-spaces of type 3 do not exist because no \((LB)\)-space is metrizable). The following lemma will be needed:

### 3.3. Lemma

Let \(E\) be a normed space with closed unit ball \(B\) and let \(F\) be a Hausdorff complete locally convex space such that \(E\) is continuously included in \(F\). If \(\tilde{B}\) is the closure of \(B\) in \(F\), the linear span \(F_0 = \text{sp}(\tilde{B})\) is a Banach space under the norm topology for which \(\tilde{B}\) is the closed unit ball.

**Proof.** Since \(F\) is complete and the canonical injection \(E \hookrightarrow F\) is continuous, \(\tilde{B}\) is a complete bounded disk in \(F\). Therefore the norm gauge of \(\tilde{B}\) defines on \(F_0\) a Banach topology ([3] 18.4.4.) with closed unit ball equal to \(\tilde{B}\).

If \(\tilde{E}\) is the completion of \(E\) and \(H \subset E\), we will denote in the sequel by \(\tilde{H}\) and \(\tilde{H}\) the closures of \(H\) in \(E\) and in \(\tilde{E}\) respectively.

### 3.4. Theorem

The completion of an \((LB)_1\)-space is an \((LB)_1\)-space.

**Proof.** For each \(n \in \mathbb{N}\), let \((E_n, \tau_n)\) be a Banach space with closed unit ball \(B_n\), such that \((E, \tau) = \lim_n (E_n, \tau_n)\) is an \((LB)_1\)-space. Let \((\tilde{E}, \tilde{\tau})\) be the completion of \((E, \tau)\). We can assume, without loss of generality, that \(B_n \subset B_{n+1}\) for every \(n \in \mathbb{N}\) (change, if necessary, \(B_n\) by a suitable multiple of \(B_n\)). By Lemma 3.3., \(F_n = \text{sp}(\tilde{B}_n)\) becomes a Banach space for a topology \(\Psi_n\), for which \(\tilde{B}_n\) is the closed unit ball. Furthermore, for each \(n \in \mathbb{N}\)

\[
\tilde{\tau}|_{F_n} \leq \Psi_n. \tag{1}
\]

Indeed, if \(V\) is a closed 0-neighborhood in \(\tilde{\tau}\), \(V \cap E_n \in \tau_n\) and, consequently, \(\lambda B_n \subset V \cap E_n\) for some \(\lambda > 0\). Since \(F_n \subset \tilde{E}_n\) we deduce \(\lambda \tilde{B}_n \subset V \cap F_n\) and (1) is proved.

Let \(F = \bigcup F_n\). If for some \(m \in \mathbb{N}\) \(F_m = F\), then \(\tilde{E}_m \supset F_m \supset E\) and \(\tilde{E}_m = E \cap \tilde{E}_m = E\) contradicting that \(E\) is of type 1. Therefore, a strictly increasing subsequence \(\{F_{n_j}\}_{j \in \mathbb{N}} \subset \{F_n\}_{n \in \mathbb{N}}\) can be chosen such that \(F = \bigcup F_{n_j}\). We rename this subsequence as \(\{F_n\}\) again and define the \((LB)\)-space \((F, \Psi) = \lim_n (F_n, \Psi_n)\) which is (algebraically) a linear subspace of \(\tilde{E}\). The absolutely convex hull \(U = \Gamma B_n = \bigcup B_n\) is a neighborhood of 0 in \((E, \tau)\), and, in particular, is absorbing in \(E\). By [1], Theorem 2, for every \(\varepsilon > 0\)
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\[ \tilde{U} \subset (1 + \epsilon) \bigcup_{n \in \mathbb{N}} \tilde{B}_n \subset (1 + \epsilon) \bigcup_{n \in \mathbb{N}} F_n = F. \]  \hspace{1cm} (2)

Since \( \tilde{U} \) is a \( \tilde{r} \)-neighborhood of 0 in \( \tilde{E} \), we deduce that \( \tilde{E} = \text{sp}(\tilde{U}) \subset F \) and, hence, \( \tilde{E} = F \).

By (1) the inclusion map \((F_n, \Psi'_n) \hookrightarrow (\tilde{E}, \tilde{r})\) is continuous for every \( n \in \mathbb{N} \). Thus, the identity map \((F, \Psi') \rightarrow (\tilde{E}, \tilde{r})\) is continuous and by Theorem 2.2. it is a topological isomorphism. We have, hence, proved that \((\tilde{E}, \tilde{r})\) is an (LB)-space. If for some \( n \in \mathbb{N}, \tilde{F}_n = \tilde{E} \), one easily gets \( \tilde{E}_n = E \) which is impossible because \( E \) is of type 1. So \((\tilde{E}, \tilde{r})\) is of type 1.

### 3.5. Theorem.

**The completion of an (LB)

### 3.5. Theorem. The completion of an (LB)\(_2\)-space is either an (LB)\(_2\)-space or a Banach space.

**Proof:** We keep all the notations of the Theorem 3.4. but that \((E, r)\) is now of type 2. If for some \( m \in \mathbb{N}, \tilde{F}_m = F \), then the relation (2) yields now \( \tilde{U} \subset F_m \). \( \tilde{U} \) being a \( \tilde{r} \)-neighborhood of 0 in \( \tilde{E} \), we deduce that \( \tilde{E} = F_m \) and the Open Mapping Theorem applies to conclude \( \tilde{r} = \Psi'_m \), that is \((\tilde{E}, \tilde{r})\) is a Banach space. Otherwise, there exists a strictly increasing subsequence \( \{F_n\}_{j \in \mathbb{N}} \subset \{F_n\}_{n \in \mathbb{N}} \) such that \( F = \bigcup_j F_n \).

We then proceed as in the proof of the Theorem 3.4. to conclude that \((\tilde{E}, \tilde{r})\) is an (LB)-space. Since some \( E_n \) is dense in \( E \), some \( F_n \) is dense in \( \tilde{E} \) and \((\tilde{E}, \tilde{r})\) is of type 2.

### 3.6. Remark.

The Theorems 3.4. and 3.5. have nontrivial applications because non-complete (LB)$_1$- and (LB)$_2$-spaces do exist. Indeed, the (LB)-space \( E \) of [3] 31.6. is an example of a non-complete (LB)$_2$-space. The topological product of this space \( E \) and \( \varphi \) is an (LB)$_1$-space that is not complete, because the closed subspace \( E \) of \( E \times \varphi \) is not complete. Since every strict (LF)-space is complete, \( E \times \varphi \) supplies as well an example of a non-strict (LB)$_1$-space.

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