UNIVERSAL NUCLEAR G_∞-SPACES

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1. Introduction. The class of nuclear smooth sequence spaces $\Lambda(P)$ of infinite type (also called nuclear G_∞-spaces) has been introduced and studied by T. Terzioglu [14] and it is a intermediate categorie between power series spaces $\Lambda(\alpha)$ of infinite type and general Köthe spaces $\Lambda(P)$.

In [11] , M. S. Ramanujan introduces the notion of $\Lambda(\alpha)$-nuclearity that is a natural generalization of s-nuclearity of V. S. Brudovskii [1] and A. Martineau [8] , and he gives a complete account of the structure of spaces $\Lambda(\alpha)$ and of $\Lambda(\alpha)$-nuclear locally convex spaces. The class of nuclear G_∞-spaces is strictly larger than the class of nuclear power series spaces of infinite type (cf. [3]) . Furthermore, G_∞-spaces enjoy additional properties described mainly in [3] , [9] and [16] .

This course is a survey of the theory of G_∞-spaces $\Lambda(P)$ and the associated notion of $\Lambda(P)$-nuclearity, including such topics as uniform $\Lambda(P)$-nucleariy, and the classic Embedding Theorem of $\Lambda(P)$-nuclear spaces into products of duals $\Lambda(P)'$

For the notations, terminology and the not proved results of the general theory, we will refer throughout to [6] and [10] .

2. <u>Preliminaries</u>. We recall that a set P of scalar sequences $a = (a_n)$ is called a Köthe set, if it satisfies the following conditions:

(K1) For each $a \in P$, $a_n \geq 0$ for all $n \in N$

(K2) For each $a, b \in P$, there exists $c \in P$ such that $\sup (a_n b_n) \leq c_n$ for all $n \in N$

(K3) For each $n \in N$, there exists $a \in P$ such that $a_n > 0$

The corresponding Köthe space associated to P is the vector space of scalar sequences

$$\lambda(P) = \{ \xi = (\xi_n) : \sum a_n |\xi_n| < +\infty \quad \text{for all} \quad (a_n) \in P \}$$

Under the natural topology $\lambda(P)$ becames a complete locally convex space and $\lambda(P)$ is nuclear if and only if the following Grothendieck-Pietsch criterion is satisfied:

(K4) For each $a \in P$, there exists $b \in P$ and $c \in l^1$ such that $a_n \leq b_n c_n$ for all $n \in N$

Among the most interesting Köthe spaces we find the power series spaces $\Lambda(\alpha)$ of infinite type, wich are the Köthe spaces generated by the Köthe set $P = \{ (k^{\alpha_n})_n : k \in N\}$ where $\alpha = (\alpha_n)$ is assumed to be a sequence of real numbers such that $0 \leq \alpha_1 \leq \alpha_2 \leq \dots \uparrow \infty$. The (K4) criterion is equivalent, for $\Lambda(\alpha)$ spaces, to the following growth condition

(K4') For some $R > 1$, $\sum R^{-\alpha_n} < +\infty$

If (K4') is satisfied, α is called exponent sequence and then

$$\Lambda(\alpha) = \{ \xi = (\xi_n) : |\xi_n|^{1/\alpha_n} \longrightarrow 0 \} \tag{1}$$

(see [11]).

A Köthe set P is called a power set if P satisfies the following conditions

(G1) P is countable (We shall denote by $(a_{kn})_n$ the k-th sequence in P)

(G2) $1 \leq a_{kn} \leq a_{k+1,n}$ for all $k, n \in N$

(G3) $a_{kn} \leq a_{k,n+1}$ for all $k, n \in N$

(G4) For every $k \in N$ there is $j \in N$ such that $\sup_n a_{kn}^2 a_{jn}^{-1} < +\infty$

The power set P is called stable if it satisfies adittionally the so-called "stability condition"

(G5) For every $k \in N$ there is $j \in N$ such that $\sup_n a_{k,2n} a_{jn}^{-1} < +\infty$

The corresponding Köthe space $\lambda(P)$ associated to a power set P is a Fréchet space called smooth sequence space of infinite type or G_∞-space and it will be denoted by $\Lambda(P)$. If P is stable, $\Lambda(P)$ is called also stable. G_∞-spaces were introduced and studied by T. Terzioglu [14] and they are a intermediate class between Köthe spaces and power series spaces because, as shown by E. Dubinsky and M. S. Ramanujan [3], there exist G_∞-spaces wich are not power series spaces, while every power series space is, obviously a G_∞-space.

Conditions (G4) and (K4) together are equivalent to the following

(G4N) For every $k \in N$ there is $j \in N$ such that $\sum a_{kn}^2 a_{jn}^{-1} < +\infty$

wich is, therefore, a necessary and sufficient condition for a G_∞-space $\Lambda(P)$ to be nuclear. We shall call P nuclear power set in case the stronger condition (G4N) is satisfaid instead condition (G4).

As usual we shall denote the Köthe dual of $\Lambda(P)$ by

$$\Lambda^{\times}(P) = \{ \eta = (\eta_n) : \sum |\eta_n| \cdot |\xi_n| < +\infty \quad \text{for all} \quad \xi \in \Lambda(P) \}$$

and the positive elements of $\Lambda(P)$ by

$$\Lambda^{+}(P) = \{ \eta \in \Lambda(P) : \eta_n \geq 0 \quad \text{for all} \quad n \in N \}$$

3. $\Lambda(P)$-Nuclearity. In this section we shall deal with the notion of λ-nuclearity introduced and studied by E. Dubinsky and M. S. Ramanujan [3] and whose methods are entirely valid into the framework of $\Lambda(P)$-nuclearity. We start with the following

Definition 3.1. (Dubinsky, Romonujan) Let λ be a sequence space. A linear map $T : E \longrightarrow F$ between Banach spaces is said to be λ-nuclear (resp. pseudo-λ-nuclear) if there exist a sequence $\xi \in \lambda$, a bounded sequence $(a_n) \subset E'$ and a sequence $(y_n) \subset F$ wich satisfies $(< y_n \; b >) \in \lambda^\times$ for all $b \in F'$ (resp. wich is bounded in F) , such that the following condition holds

$$Tx = \sum \xi_n < x \; a_n > y_n \qquad x \in E$$

If P is a nuclear power set both definitions of $\Lambda(P)$-nuclearity are equivalent , as we see in the following

Proposition 3.2. Let P be a nuclear power set. A linear map $T : E \longrightarrow F$ is $\Lambda(P)$-nuclear if and only if it is pseudo-$\Lambda(P)$-nuclear.

Proof. Since $a_{kn} \geq 1$ for all $(a_{kn})_n \in P$, we have $\Lambda(P) \subset l^1$ and thus $l^\infty \subset \Lambda(P)^\times$. We deduce that if the map T is pseudo-$\Lambda(P)$-nuclear , then T is $\Lambda(P)$-nuclear. Conversely, let $T : E \longrightarrow F$ be a $\Lambda(P)$-nuclear map and ξ , (a_n) and (y_n) as in definition 3.1. The sequence τ defined by $\tau_n = \xi_n^{1/2}$ belongs to $\Lambda(P)$. Indeed, by nuclearity of $\Lambda(P)$, given $k \in N$ we can choose $j \in N$ such that condition (G4N) is satisfied. We then have

$$\sum_n a_{kn} |\tau_n| = \sum_n a_{kn} a_{jn}^{-1/2} a_{jn}^{1/2} |\xi_n|^{1/2} \leq$$

$$\leq (\sum_n a_{kn}^2 a_{jn}^{-1})^{1/2} (\sum_n a_{jn} |\xi_n|)^{1/2} < +\infty$$

Finally, if we define $z_n = \tau_n y_n$, then

$$Tx = \sum \tau_n < x \; a_n > z_n \qquad x \in E$$

, where the sequence $\langle z_n\ b\rangle = \tau_n \langle y_n\ b\rangle$ is bounded for each $b \in F'$ (it is even summable) and, in view of the uniform boundedness theorem, the sequence (z_n) is bounded in F . Thus, we conclude that T is pseudo-$\Lambda(P)$-nuclear.

For each continuous linear map T between Banach spaces E and F , and for each $n \in N$, we define the n-th approximation number of T by

$$\alpha_n(T) = \inf\{\, \| T - A_n \| \,\}$$

where A_n ranges over the set of all continuous linear maps from E into F such that $A_n(E)$ has at most dimension n . If the (decreasing) sequence $(\alpha_n(T))$ belongs to a given sequence space λ , we shall say that T is of type λ . If for some $p > 0$, $(\alpha_n(T)^p) \in \lambda$, T is called of type λ^p .

Proposition 3.3. Let P be a nuclear power set. Each pseudo-$\Lambda(P)$-nuclear map is of type $\Lambda(P)$.

Proof. Let $Tx = \sum \xi_n \langle x\ a_n\rangle\, y_n$ be a representation of the pseudo-$\Lambda(P)$-nuclear map T , as in definition 3.1. We have

$$\alpha_n(T) \leq \sum_{i=n}^{\infty} |\xi_i| \cdot \|a_i\| \cdot \|y_i\| \leq c \sum_{i=n}^{\infty} |\xi_i| \qquad (1)$$

where $c > 0$ is a number which does not depend on n . Since $a_{1i} \geq 1$ for every $i \in N$, condition (K4) of nuclearity gives in particular an index $r \in N$ such that, for every $n \in N$

$$\frac{(n+1)}{a_{rn}} \leq \sum_{i=0}^{n} \frac{1}{a_{ri}} \leq \sum_{i=0}^{\infty} \frac{a_{1i}}{a_{ri}} = M < +\infty$$

Let $k \geq r$ be and by condition (G4) let us take $j \in N$ and $M' > 0$ such that $a_{kn}^2 \leq M' a_{jn}$ for every $n \in N$. We then have

$$c^{-1} \sum_{n=0}^{\infty} a_{kn}\alpha_n(T) \leq \sum_{n=0}^{\infty} \Big(\sum_{i=n}^{\infty} |\xi_i| \Big) a_{kn} = \sum_{n=0}^{\infty} |\xi_n| \sum_{i=0}^{n} a_{ki} \leq$$

$$\sum_{n=0}^{\infty} |\xi_n| (n+1) a_{kn} = \sum_{n=0}^{\infty} |\xi_n| \frac{n+1}{a_{rn}} a_{rn} a_{kn} \leq M \sum_{n=0}^{\infty} |\xi_n| a_{kn}^2 \leq MM' \sum_{n=0}^{\infty} |\xi_n| a_{jn} < +\infty$$

Since this relation is also true for $k < r$ (see condition (G2)!), it follows that $(\alpha_n(T)) \in \Lambda(P)$ and T is of type $\Lambda(P)$

The following theorem is a generalization of a result of G. Köthe [5], about s-nuclearity of diagonal maps in l^1 and supplies plenty of $\Lambda(P)$-nuclear maps in the Banach space l^1. We recall that if ξ is an element of the space ϕ of finitely non-zero sequences, there exists a unique sequence $\bar{\xi}$ with the property $0 \leq \bar{\xi}_{n+1} \leq \bar{\xi}_n$ for all $n \in N$ and there exists a bijection $\pi : N \longrightarrow N$ such that $|\xi_{\pi(n)}| = \bar{\xi}_n$ for all $n \in N$. If $\xi \in c_0 - \phi$ there exists a unique sequence $\bar{\xi}$ such that $0 < \bar{\xi}_{n+1} \leq \bar{\xi}_n$ for all $n \in N$ and there exists an injection $\pi : N \longrightarrow N$ such that $\pi(N) = \{ n \in N : \xi_n \neq 0 \}$ and $|\xi_{\pi(n)}| = \bar{\xi}_n$ for all $n \in N$. In either case we shall call $\bar{\xi}$ the decreasing rearrangement of ξ. We then have

Proposition 3.4. Let $\xi = (\xi_n)$ be a sequence of non-negative numbers converging to 0, and let $D_\xi : l^1 \longrightarrow l^1$ be the diagonal transformation $D_\xi(x_n) = (\xi_n x_n)$. Then the following assertions are equivalent

a) D_ξ is $\Lambda(P)$-nuclear

b) $\bar{\xi} \in \Lambda(P)$

Proof. We exclude the trivial case $\xi \in \phi$. Then let $\pi : N \longrightarrow \pi(N)$ be the bijection defining the decreasing rearrangement of ξ and let $\psi_\pi : l^1 \longrightarrow l^1[\pi(N)]$ be the isomorphism $\psi_\pi(x_n) = (x_{\pi(n)})$ with inverse $\psi_{\pi^{-1}}$.

b)$\Longrightarrow$ a). If $\bar{\xi} \in \Lambda(P)$, it is straightforward to check that the diagonal map $D_{\bar{\xi}} : l^1[\pi(N)] \longrightarrow l^1[\pi(N)]$ is $\Lambda(P)$-nuclear and, thus, so is $D_\xi = \psi_{\pi^{-1}} \circ D_{\bar{\xi}} \circ \psi_\pi$.

a)$\Longrightarrow$ b). If D_ξ is $\Lambda(P)$-nuclear, it is of type $\Lambda(P)$ (Proposition 3.3.) Therefore, $D_{\bar{\xi}}$ is also of type $\Lambda(P)$. But, since $\bar{\xi}$ is decreasing, we have $\alpha_n(D_{\bar{\xi}}) = \bar{\xi}_n$ (see [10], 8.1.5.). Hence, $\bar{\xi} \in \Lambda(P)$.

Let E be a locally convex space and U a closed absolutely convex neighbourhood of 0 in E with corresponding gauge p_U. As usual we shall denote by E_U the completion of the normed space $E/p_U^{-1}(0)$ endowed with the norm p_U. Let P be a nuclear power set. Following [3], a locally convex space E is said to be $\Lambda(P)$-nuclear if a basis $\mathfrak{U}$ (equivalent, any basis) of closed absolutely convex zero neighbourhoods in E has the following property: For every $U \in \mathfrak{U}$ there exists $V \in \mathfrak{U}$ such that $V \subset U$ and the canonical map $T_{VU} : E_V \longrightarrow E_U$ is $\Lambda(P)$-nuclear (in view of Proposition 3.2. is equivalent to say that T_{VU} is pseudo-$\Lambda(P)$-nuclear).

It is worth noting that, in view of Proposition 3.2., the above definition is equivalent to Ramanujan's definition of $\Lambda(\alpha)$-nuclearity (see [11]) and, as we shall see later, to Moscatelli's definition of λ-nuclearity (see [9]). In order to prove this assertion, we shall need some properties relatives to nuclear G_∞-spaces wich are collected in the following proposition

Proposition 3.5. Let P be a power set. Then

a) $\Lambda(P) \subset l^1$

b) $\Lambda(P)$ is normal, that is if $\xi \in \Lambda(P)$ and $|\eta_n| \leq |\xi_n|$ for all $n \in N$, then $\eta \in \Lambda(P)$

c) If P is stable, $\Lambda(P)$ is additive, that is for every $\xi, \eta \in \Lambda(P)$, there exists a bijection $\pi : N \longrightarrow N$ such that $(\tau_{\pi(n)}) \in \Lambda(P)$ where $\tau = \xi \cup \eta$ is defined by $\tau_{2n-1} = \xi_n$, $\tau_{2n} = \eta_n$

d) If P is nuclear, $\Lambda(P)$ is decreasing rearrangement invariant, that is $\xi \in \Lambda(P)$ implies $\bar{\xi} \in \Lambda(P)$

Proof. Parts a) and b) are easy consequences of definitions. For c) , let $\xi , \eta \in \Lambda(P)$ be, and let us prove that the sequence $\tau = \xi \cup \eta$ belongs to $\Lambda(P)$. If $k \in N$, let $j \in N$ be the number given in axiom (G5). Then, there exists a number $c > 0$ such that

$$\sum_n |\tau_n| a_{kn} = \sum_n |\eta_n| a_{k\,2n} + \sum_n |\xi_n| a_{k\,2n-1} \leq$$

$$\leq c \sum_n |\eta_n| a_{jn} + c \sum_n |\xi_n| a_{jn} < +\infty$$

Finally d) is proved noting that for every $\xi \in \Lambda(P)$, the diagonal map $D_\xi : l^1 \longrightarrow l^1$ is $\Lambda(P)$-nuclear and applying Proposition 3.4.

In the remainder of this paragraph, we shall assume that P is a stable nuclear power set, that is P satisfies conditions (G1) , (G2) , (G3) , (G4N) and (G5) .

Proposition 3.6. Let $T : E \longrightarrow F$, $S : F \longrightarrow G$ be continuous linear maps of type $\Lambda^p(P)$ $(p > 0)$. Then $S \circ T : E \longrightarrow G$ is of type $\Lambda^{p/2}(P)$.

Proof. Bt hypothesis $(\alpha_n(T)^2) \in \Lambda^{p/2}(P)$ and $(\alpha_n(S)^2) \in \Lambda^{p/2}(P)$. From the inequality

$$\alpha_{2n}(S \circ T) \leq \alpha_n(T)\ \alpha_n(S) \leq 1/2\ [\alpha_n(T)^2 + \alpha_n(S)^2]$$

and from normality of $\Lambda(P)$ we get $\xi = (\alpha_{2n}(S \circ T)) \in \Lambda^{p/2}(P)$. Since $\alpha_{2n+1} \leq \alpha_{2n}$, again by normality we have $\eta = (\alpha_{2n+1}(S \circ T)) \in \Lambda^{p/2}(P)$. Finally, by Proposition 3.5. c) and d) , we conclude

$$(\alpha_n(S \circ T)) = \overline{\xi \cup \eta} \in \Lambda^{p/2}(P)$$

These previous results lead to the following characterization theorem for $\Lambda(P)$-nuclear spaces

Theorem 3.7. Let E be a locally convex space with basis of closed absolutely convex zero-neighbourhoods $\mathcal{U}$. The following assertions are equivalents

1) E is $\Lambda(P)$-nuclear

2) For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \subset U$ and the canonical map T_{VU} is pseudo-$\Lambda(P)$-nuclear

3) For some, resp. for each, number $p > 0$, the following condition is satisfied : For each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V \subset U$ and the canonical map T_{VU} is of type $\Lambda^p(P)$

Proof. By [9] , I.2. Theorem 1 , we have 3)$\Longrightarrow$ 2) . Equivalence 1)$\Longleftrightarrow$ 2) is the Proposition 3.2. Finally we shall prove 2)$\Longrightarrow$ 3) . Let $p > 0$ be and choose $n \in N$ such that $2^{-n} < p$. By Proposition 3.3., given $U = U_0 \in \mathcal{U}$ there exist neighbourhoods $U_k \in \mathcal{U}$, $k = 1 , 2 , \ldots ,$ n , such that

$U_{k+1} \subset U_k$ and the canonical map $T_k : E_{U_{k+1}} \longrightarrow E_{U_k}$ is of type $\Lambda(P)$. If $V = U_n$, the canonical map $T_{VU} = T_0 \circ T_1 \circ \ldots \circ T_{n-1}$ is of type $\Lambda^{2^{-n}}(P)$ because of Proposition 3.6. Consequently T_{VU} is of type $\Lambda^p(P)$ and the theorem is entirely proved.

Next we investigate $\Lambda(P)$-nuclearity of sequence spaces. We shall be mainly concerned with Köthe spaces $\lambda(Q)$, where Q is a Köthe set as defined in paragraph 1 . We have the following Köthe-Pietsch-Grothendieck criterion wich comes from a corrected version due to Köthe [5] of a theorem of Brudovskii about s-nuclearity of Köthe spaces $\lambda(Q)$.

Theorem 3.8. Let Q be a Köthe set. $\lambda(Q)$ is $\Lambda(P)$ nuclear if and only if it is satisfied some of the two following equivalent conditions

a) For each $a \in Q$, there exists $b \in Q$ such that $a_n \leq b_n$ for all $n \in N$, and $\overline{ab^{-1}} \in \Lambda(P)$ (here ab^{-1} stands for the sequence $(a_n b_n^{-1})$ where ratios $0/0$ are assumed to be 0)

b) For each $a \in Q$, there exists $b \in Q$ such that $a_n \leq b_n$ for all $n \in N$ and the sequence $(a_n b_n^{-1})$, $a_n \neq 0$, can be rearranged into a sequence in $\Lambda(P)$

Proof. The equivalence of a) and b) is a strightforward consequence of Proposition 3.5. d) , because if η is the rearrangement of $(a_n b_n^{-1})$, $a_n \neq 0$, belonging to $\Lambda(P)$, we have $\bar{\eta} \in \Lambda(P)$. But $\bar{\eta} = \overline{ab^{-1}}$ and the equivalence is proved.

Let $a \in Q - \phi$ be and let $\sigma_a : N \longrightarrow N$ a strictly increasing map such that $a_{\sigma_a(n)} \neq 0$ for all $n \in N$ and let $U_a = \{ \xi : \sum a_n |\xi_n| \leq 1 \}$ be a neighbourhood of 0 in $\lambda(Q)$. Noting that $[\lambda(Q)]_{U_a}$ is norm-isomorphic to $l^1[\sigma_a(N)]$ (the map $(\xi + U_a) \longmapsto (\xi_{\sigma_a(n)} a_{\sigma_a(n)})$ being such an isomorphism), then the canonical map $T_{U_b U_a}$, $b \in Q$, $b_n \geq a_n$ for all $n \in N$, from $[\lambda(Q)]_{U_b}$ into $[\lambda(Q)]_{U_a}$ can be identified with the diagonal transformation $D_\tau : l^1[\sigma_b(N)] \longrightarrow l^1[\sigma_a(N)]$ where $\tau_n = a_n b_n^{-1}$, $n \in \sigma_a(N)$. Now, by Proposition 3.4. D_τ (equivalent $T_{U_b U_a}$) is $\Lambda(P)$-nuclear if and only if $\bar{\tau} = \overline{ab^{-1}}$ belongs to $\Lambda(P)$, and our theorem easily follows.

V. B. Moscatelli has shown that for a general sequence space λ satisfying the properties collected in Proposition 3.5., the class of λ-nuclear spaces is closed under the formation of countable direct sums (see [9] , II,1, Lemma 1 and I,4, Theorem 5) . In the same way, permanence of λ-nuclearity under the formation of product is closely related to the property c) of Proposition 3.5. (see [3] Lemma 2.7. and Theorem 2.9.). Because $\Lambda(P)$ satisfies the properties of the above mentioned theorems, we state without proof the following

Theorem 3.9. If P is a stable nuclear power set, every product and every countable direct sum of $\Lambda(P)$-nuclear spaces is $\Lambda(P)$-nuclear.

<u>4. Uniform $\Lambda(P)$-nuclearity</u>. In the condition of Theorem 3.8. about $\Lambda(P)$-nuclearity of Köthe spaces we claim for a rearrangement of the sequence ab^{-1} that depends on the element a . In some applications (see [15]) and specially in the study of dual spaces, it is of great interest the existence of a "universal" permutation π valid for every element a . This leads G. Köthe to define a sequence space $\lambda(Q)$ to be uniformly $\Lambda(P)$-nuclear if there exists a bijection $\pi : N \longrightarrow N$ such that for each $a \in Q$, there exists $b \in Q$ such that $a_n \leq b_n$ for all $n \in N$ and

$$a_{\pi(n)} = b_{\pi(n)} c_n \qquad n \in N$$

for some sequence $(c_n) \in \Lambda(P)$.

We then have

Proposition 4.1. If $\lambda(Q)$ is uniformly $\Lambda(P)$-nuclear, $\lambda(Q)$ is $\Lambda(P)$-nuclear.

Proof. Let π be the "universal" bijection associated to $\lambda(Q)$ and let $a \in Q$ be. (We exclude, by trivial, the case $a \in \phi$) . Then let $\sigma : N \longrightarrow N$ be a strictly increasing sequence such that $a_{\pi(\sigma(n))} \neq 0$ for all $n \in N$. We choose the element $b \in Q$ given by uniform $\Lambda(P)$-nuclearity, wich satisfies "a fortiori" $b_{\pi(\sigma(n))} \neq 0$ for all $n \in N$ and we finally show that

$$\frac{a_{\pi(\sigma(n))}}{b_{\pi(\sigma(n))}} \qquad n \in N$$

is a rearrangement of non-zero entries of the sequence ab^{-1} that belongs to $\Lambda(P)$. Indeed, let $k \in N$ and , being σ a increasing map we have

$$\sum_n \frac{a_{\pi(\sigma(n))}}{b_{\pi(\sigma(n))}} a_{kn} \leq \sum_n \frac{a_{\pi(\sigma(n))}}{b_{\pi(\sigma(n))}} a_{k\sigma(n)} =$$

$$= \sum_n c_{\sigma(n)} a_{k\sigma(n)} \leq \sum_n c_n a_{kn} < +\infty$$

From Theorem 3.8. we now deduce that $\lambda(Q)$ is $\Lambda(P)$-nuclear.

As shown by E. Dubinsky and M. S. Ramanujan, [3] Theorem 2.7. , not every $\Lambda(P)$-nuclear space is uniformly $\Lambda(P)$-nuclear and in [5] , G. Köthe shows that if Q is countable, then $\lambda(Q)$ is s-nuclear if and only if $\lambda(Q)$ is uniformly s-nuclear. We give now a generalization of this result valid for $\Lambda(P)$-nuclearity that contains, as a particular case, Dubinsky and Ramanujan's theorem ([3] , Theorem 2.6.) about $\Lambda(\alpha)$-nuclearity . The stability of the sequence exponent α in the above mentioned theorem (that is $\sup \alpha_{2n}/\alpha_n < +\infty$) and the stability of P (condition (G5)) in our following theorem, are essential assumptions for the equivalence.

In order to prove this theorem we shall need the two following lemmas, the first of wich is due to Dubinsky and Ramanujan [3] , and thus, its proof is omitted

Lemma 4.2. (E. Dubinsky, M. S. Ramanujan) Let $\beta : N \longrightarrow N \times N$ be a bijection and $\pi_k : N \longrightarrow N$, $k \in N$, be a sequence of injections such that for each $n \in N$, there exist k , $m \in N$ with $\pi_k(m) = n$. Then there exist an injection $\gamma : N \longrightarrow N \times N$ ($\gamma(n) = (\gamma_1(n) , \gamma_2(n))$ for all $n \in N$) and a bijection $\pi : N \longrightarrow N$ satisfying the following conditions

a) $n \leq \beta^{-1} \circ \gamma(n)$ for all $n \in N$

b) If $\pi_{\gamma_1(n)}(\gamma_2(n)) = \pi_k(m)$, then $\beta^{-1} \circ \gamma(n) \leq \beta^{-1}(k\ m)$

c) $\pi(n) = \pi_{\gamma_1(n)}(\gamma_2(n))$ for all $n \in N$

Lemma 4.3. If P is a stable power set, there exists a bijection $\beta : N \longrightarrow N \times N$ with the following property: For each $r \in N$ and $k \in N$, there exists $j \in N$ such that

$$\sup_m \frac{a_{r,\beta^{-1}(k\ m)}}{a_{jm}} < +\infty$$

Proof. Let β be the bijection defined by $\beta^{-1}(k\ m) = 2^{k-1}(2m-1)$. Given $r = r_o \in N$ and $k \in N$ we take integers $r_i \in N$ ($1 \leq i \leq k$) supplied by condition (G5) and numbers γ_i ($1 \leq i \leq k$) such that

$$a_{r_{i-1},2n} \leq \gamma_i\, a_{r_i,n} \qquad \text{for all } n \in N\ ,\ 1 \leq i \leq k$$

We then have for $j = r_k \in N$

$$\sup_m \frac{a_{r,\beta^{-1}(k\ m)}}{a_{jm}} = \sup_m \frac{a_{r,2^{k-1}(2m-1)}}{a_{jm}} \leq$$

$$\leq \sup_m \frac{a_{r,2^k m}}{a_{jm}} \leq \sup_m \prod_{i=1}^{k} \frac{a_{r_{i-1},2^{k-i+1}m}}{a_{r_i,2^{k-i}m}} \leq \prod_{i=1}^{k} \gamma_i$$

Theorem 4.4. Let Q be a countable Köthe set and P a stable power set. If $\lambda(Q)$ is $\Lambda(P)$-nuclear, then $\lambda(Q)$ is uniformly $\Lambda(P)$-nuclear.

Proof. Let us denote $Q = \{ c^k : k \in N \}$. By the Grothendieck-Pietsch-Köthe criterion (Theorem 3.8.b)) , for each $k \in N$ we can find an integer $j(k) \in N$ and an injection $\pi_k : N \longrightarrow N$ such that $c_n^k \leq c_n^{j(k)}$ for all $n \in N$, $\pi_k(N) = \{ n \in N : c_n^k \neq 0 \}$ and the sequence

$$b^k_{\pi_k(n)} = \frac{c^k_{\pi_k(n)}}{c^{j(k)}_{\pi_k(n)}} \qquad n \in N$$

belongs to $\Lambda(P)$.

If $n \in N$, by condition (K3) there exists $k \in N$ such that $c^k_n \neq 0$, that is $n \in \pi_k(N)$ and , thus, there is $m \in N$ such that $n = \pi_k(m)$. On the other hand, by stability of P , we can have the bijection β given in Lemma 4.3. and then we can apply Lemma 4.2. to get the injection γ and the bijection π . We finally show that π is the desired bijection in order to check uniform $\Lambda(P)$-nuclearity of $\lambda(Q)$, that is, we shall prove that for each $k \in N$, $(b^k_{\pi(n)})_n \in \Lambda(P)$.

If $k \in N$, choose $n \in N$ such that $b^k_{\pi(n)} \neq 0$ and take now the (unique) integer $m \in N$ such that $\pi(n) = \pi_k(m)$. By Lemma 4.2.c) we have

$$\pi_k(m) = \pi_{\gamma_1(n)}(\gamma_2(n))$$

and by Lemma 4.2. a) and b) we get

$$n \leq \beta^{-1}\ \gamma(n) \leq \beta^{-1}(k\ m) \qquad (1)$$

Then, let $(a_{rn})_n \in P$ be arbitrary, and let $j \in N$ be the number associated to k and r given in Lemma 4.3., satisfying

$$a_{r,\beta^{-1}(k\ m)} \leq C\ a_{jm} \qquad \text{for all } m \in N$$

(here C is a number that does not depend on $m \in N$) . Consequently, by applying (1) and axiom (G3) we have, for each $n \in N$ with $b^k_{\pi(n)} \neq 0$

$$b^k_{\pi(n)} a_{rn} = b^k_{\pi_k(m)} a_{rn} \leq b^k_{\pi_k(m)} a_{r,\beta^{-1}(k\ m)} \leq$$

$$\leq C\ b^k_{\pi_k(m)} a_{jm}$$

But each such $n \in N$ has associated a unique $m \in N$ with $\pi(n) = \pi_k(m)$ because π_k is injective. Consequently

$$\sum_n b^k_{\pi(n)} a_{rn} \leq C \sum_m b^k_{\pi_k(m)} a_{jm}$$

Since $(b^k_{\pi_k(m)})_m \in \Lambda(P)$, the last series is finite and, thus, $(b^k_{\pi(n)})_n \in \Lambda(P)$. The theorem is now completely proved.

5. Dual Spaces. It is known, by a result of A. Martineau [8] and V. S. Brudovskii [1], that the strong topological dual of every metrizable nuclear locally convex space is s-nuclear. In particular, the dual space s' of the space s of rapidly decreasing sequences is s-nuclear and the question arises whether an analogous theorem holds for a nuclear G_∞-space $\Lambda(P)$.

Theorem 5.1. Let P be a stable nuclear power set. The strong topological dual $\Lambda(P)'$ of the nuclear G_∞-space $\Lambda(P)$ is $\Lambda(P)$-nuclear.

Proof. We recall (see [5]) that the strong topological dual of $\Lambda(P)$ is topologically isomorphic to $\lambda(\Lambda^+(P))$ endowed with its natural topology of Köthe space. Let $c \in \Lambda^+(P)$ be and let b be the sequence defined by $b_n = +c_n^{1/2}$. If we show $b \in \Lambda^+(P)$, then $cb^{-1} = b \in \Lambda(P)$ and by Proposition 3.5.d) we have $\overline{cb^{-1}} \in \Lambda(P)$. Applying now Theorem 3.8., $\Lambda(P)$-nuclearity of $\Lambda(P)'$ is proved. Clearly $b_n \geq 0$ and, if $(a_{kn})_n \in P$, we choose $j \in N$ given in condition (G4N). We then have

$$\sum_n a_{kn} b_n = \sum_n a_{kn} a_{jn}^{-1/2} a_{jn}^{1/2} b_n \leq$$

$$\leq (\sum_n a_{kn}^2 a_{jn}^{-1})^{1/2} (\sum_n a_{jn} c_n)^{1/2} < +\infty$$

Consequently $b \in \Lambda(P)$ and our assertion is proved.

Under the stronger assumption of uniform $\Lambda(P)$-nuclearity we have the following sufficient condition for a Köthe space to have a strong topological dual $\Lambda(P')$-nuclear.

Theorem 5.2. Let P and P' be nuclear power sets and let us assume that there is a sequence $b \in \Lambda(P')$ such that $b^{-1} = (b_n^{-1}) \in P$. Then the strong topological dual of each Köthe space uniformly $\Lambda(P)$-nuclear is $\Lambda(P')$-nuclear.

Proof. Let $\lambda(Q)$ be a Köthe space uniformly $\Lambda(P)$-nuclear and let $\pi : N \longrightarrow N$ be the "universal" bijection associated. If $\xi \in \lambda^+(Q)$, we can construct a strictly increasing map $\sigma : N \longrightarrow N$ such that $\xi_n \neq 0$ for each $n \in \sigma(N)$. Let τ the sequence defined by $\tau_n = \xi_n (b_{\pi^{-1}(n)})^{-1}$, $n \in N$. If we show that $\tau \in \lambda^+(Q)$, then, the non-zero entries of the sequence (ξ_n/τ_n) can be rearranged by means of the injection $\sigma \circ \pi$, into the sequence $(b_{\sigma(n)})$ wich belongs to $\Lambda(P')$ because, by hypothesis

$$\sum_n b_{\sigma(n)} a_{kn} \leq \sum_n b_{\sigma(n)} a_{k\sigma(n)} \leq \sum_n b_n a_{kn} < +\infty \qquad \text{for all } (a_{kn})_n \in P'$$

From Theorem 3.8. we conclude that $\lambda(\lambda^+(Q))$, and hence the strong dual $\lambda(Q)'$, is $\Lambda(P')$-nuclear.

In order to prove that $\tau \in \lambda^+(Q)$, let $\eta \in Q$ be. By hypothesis there is $\beta \in Q$ and $c \in \Lambda(P)$ such that

$$\eta_{\pi(n)} = \beta_{\pi(n)} c_n \qquad \text{for all } n \in N$$

Consequently we have

$$\sum \tau_n \eta_n = \sum \xi_n (b_{\pi^{-1}(n)})^{-1} \eta_n = \sum \xi_{\pi(n)} b_n^{-1} \eta_{\pi(n)} =$$

$$= \sum \xi_{\pi(n)} b_n^{-1} \beta_{\pi(n)} c_n$$

But $(b_n^{-1} c_n)$ is a bounded sequence (it is even summable because, by hypothesis,

$b^{-1} \in P$) . Therefore, we can find a number $M > 0$ such that the inequality

$$\sum \tau_n \eta_n \leq M \sum \xi_{\pi(n)} \beta_{\pi(n)} < +\infty$$

holds. Since, obviously, $\tau_n \geq 0$, we finally get $\tau \in \lambda^+(Q)$.

Corollary 1. If α and β are exponent sequences such that (α_n/β_n) converges to ∞ , then the strong topological dual of each Köthe space uniformly $\Lambda(\alpha)$-nuclear is $\Lambda(\beta)$-nuclear.

Proof. Let $P = \{ (k^{\alpha_n}) : k \in N \}$ and $P' = \{ (k^{\beta_n}) : k \in N \}$ be the nuclear power set associated to α and β respectively. The sequence (2^{α_n}) belongs to P and by hypothesis $(2^{-\alpha_n/\beta_n}) \longrightarrow 0$ and, thus, $(2^{-\alpha_n}) \in \Lambda(\beta) = \Lambda(P')$ (see the statement (1) in paragraph 2) . We now apply the theorem.

Corollary 2. If $P = \{ (e^{n^k})_n : k \in N \}$, the strong topological dual of each Köthe space uniformly $\Lambda(P)$-nuclear is s-nuclear.

Proof. We know that $s = \Lambda(P')$ where $P' = \{ ((n+1)^k)_n : k \in N \}$. The corollary easily follows from the theorem noting that $(e^n) \in P$ and $(e^{-n}) \in \Lambda(P')$ because for all $k \in N$

$$\sum_n e^{-n}(n+1)^k < +\infty$$

Corollary 1 is also obtained by M. S. Ramanujan (see [11] Proposition 8). Since $\Lambda(P)$ in Corollary 2 is not a power series space (see [3] Theorem 2.25.), our Theorem 5.2. improves Ramanujan's result of Corollary 1, supplying a stronger tool to investigate s-nuclearity of dual spaces.

6. Universal $\Lambda(P)$-nuclear Locally Convex Spaces. By the celebrated theorem of T. Kōmura and Y. Kōmura [4] , it is known that the Fréchet space s of rapidly decreasing sequences is a universal generator for the variety

of nuclear locally convex spaces, in the sense that every nuclear space can be embedded in an I-fold topological product s^I . A further universality theorem is due to A. Martineau ([8]) valid for the variety of s-nuclear locally convex spaces and in this paragraph we have the following embedding theorem for the variety of $\Lambda(P)$-nuclear locally convex spaces

Theorem 6.1. Let P be a stable nuclear power set and let $\Lambda(P)'$ the strong topological dual of $\Lambda(P)$. A locally convex space E is $\Lambda(P)$-nuclear if and only if it is topologically isomorphic to a subspace of a suitable I-fold topological product $[\Lambda(P)']^I$

Proof. Necessity. Let us assume that E is $\Lambda(P)$-nuclear. By the equivalence $1 \Leftrightarrow 2$ of the Theorem 3.7., for each closed absolutely convex neighbourhood U of O in E , there is a closed absolutely convex neighbourhood V of O such that $V \subset U$ and the canonical map $T_{VU} : E_V \longrightarrow E_U$ is pseudo-$\Lambda(P)$-nuclear. Hence, there are sequences $(\lambda_n) \in \Lambda(P)$, $(b_n) \subset (E_V)' \approx E'_{V^o}$ with $\|b_n\| \leq 1$ and $(y_n) \subset E_U$ with $p_U(y_n) \leq 1$, such that

$$x(U) = T_{VU}(x(V)) = \sum \lambda_n < x(V)\ b_n > y_n \qquad x \in E \tag{1}$$

From (1) we get at once

$$< x\ b_n > = 0 \quad \text{for all} \quad n \in N \implies p_U(x) = p_U(x(U)) = 0 \tag{2}$$

We next prove that for each $x \in E$

$$(< x\ b_n >) \in \Lambda(P)' \tag{3}$$

Indeed, let $(\xi_n) \in \Lambda^+(P)$ be. Because $(b_n) \subset V^o$ we have, by virtue of Theorem 3.5.a),

$$\sum |< x\ b_n>| \xi_n \leq p_V(x) \sum \xi_n \quad < +\infty \tag{4}$$

Consequently, the map $\phi_U : E \longrightarrow \Lambda(P)'$ given by

$$\phi_U(x) = (< x\ b_n >)_n \qquad x \in E \tag{5}$$

is well defined and , obviously is a linear map. It is also continuous because of (4).

Let $\mathcal{U} = \{U_i : i \in I\}$ a basis of closed absolutely convex neighbourhoods of 0 in E . By the above argument, for each $i \in I$ we are able to define a continuous linear map ϕ_{U_i} as in (5) and then we define $\phi : E \longrightarrow [\Lambda(P)']^I$ by $\phi(x) = (\phi_{U_i}(x))_{i \in I}$, $x \in E$, wich is also a continuous linear map. Moreover ϕ is one-to-one because if $\phi_{U_i}(x) = 0$ for all $i \in I$, the statement (2) gives $p_{U_i}(x) = 0$ for all $i \in I$ and, consequently, $x = 0$. Finally we show that ϕ is a open map. Let $U_j \in \mathcal{U}$ be and $W = \prod_{i \in I} W_i$ a zero neighbourhood in $[\Lambda(P)']^I$, where $W_i = \Lambda(P)'$ if $i \neq j$ and

$$W_j = \{ \xi \in \Lambda(P)' : \sum |\lambda_n| \, |\xi_n| \leq 1 \}$$

(here (λ_n) is the sequence in $\Lambda(P)$ associated to $U = U_j$ and given in representation (1)). All we have to show is

$$\phi(E) \cap W \subset \phi(U)$$

Let $\phi(x) \in W$ be. We have $(< x\ b_n >) = \phi_U(x) = \phi_{U_j}(x) \in W_j$ and, from (1) we finally get

$$p_U(x) = p_U(x(U)) \leq \sum |\lambda_n| \, |< x\ b_n >| \; p_U(y_n) \leq 1$$

We have thus proved $x \in U$ and the necessity.

Sufficiency. From Theorems 5.1. and 3.9. we can assure that $[\Lambda(P)']^I$ is a $\Lambda(P)$-nuclear locally convex space and, obviously, so is every subspace of $[\Lambda(P)']^I$. The theorem is now completely proved.

In the proof of sufficiency in the last theorem , the stability of P plays a prominent role as it has been pointed out by P. Spuhler [13] and by E. Dubinsky and M. S. Ramanujan [3] , because the product of even two $\Lambda(P)$-nuclear spaces need not be $\Lambda(P)$-nuclear if P fails to be stable (see [13]).

From the above theorem we thus deduce that the variety $\mathcal{N}_{\Lambda(P)}$ of $\Lambda(P)$-nuclear locally convex spaces has the $\Lambda(P)$-nuclear space $\Lambda(P)'$ as universal generator in the sense that every $\Lambda(P)$-nuclear locally convex space is topologically isomorphic to a subspace of a suitable I-fold topological product of spaces $\Lambda(P)'$.

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