# WEIGHTED INDUCTIVE LIMITS OF CONTINUOUS FUNCTION SPACES WITH NON–LOCALLY COMPACT DOMAIN

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Dedicated to Professor J. J. Gutiérrez Suárez
on the occasion of his 65th birthday

## INTRODUCTION

In this paper we construct a projective description of inductive limits of weighted spaces of continuous functions defined in a paracompact, not necessarily locally compact, topological space. The locally compact setting of this question has been extensively studied in [2] , [3] , [4] , [5] , [6] and [7] and in the present work we develop new techniques to address the weakened hypothesis situation. The paracompact setting pays off by permitting the extension of the theory to the class of spaces of continuous functions defined on metric spaces (without any other hypothesis of compactness).

This kind of problems was first studied in [4] for the special case of echelon and

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coechelon Köthe sequence spaces which is nothing but the study of weighted spaces of continuous functions defined in a discrete (and hence locally compact) topological space. However the original impetus comes back to the description of inductive limits of weighted spaces of continuous and holomorphic maps (see [2] and [11] ) defined in topological spaces always assumed, at least, locally compact. More recently, [8] , has been obtained a projective description of weighted inductive limits of spaces of null sequences with values in a Fréchet space.

The notations and results on Hausdorff locally convex spaces (l. c. s.) will be taken from [13] and [14] . Sometimes a l. c. s. E is assumed to have the countable neighborhood property (c. n. p.), i. e. for every sequence $\{p_n\}_{n\in\mathbb{N}} \subset cs(E)$ ( = the set of all continuous seminorms in E ), there exist scalars $a(n) > 0$ , $n \in \mathbb{N}$ , and $p \in cs(E)$ such that $p_n(x) \leq a(n)p(x)$ for all $x \in E$ and $n \in \mathbb{N}$ . The scalar field, every normed space and even every (DF) space have the c. n. p. (see, for instance, [9] and [15] ).

If X is any completely regular topological space we denote by $C(X,E)$ the linear space of all the continuous functions from X into E and by $C_c(X,E)$ the subspace of those functions of $C(X,E)$ which have compact support.

A "weight" on the completely regular topological space X is a non–negative real valued upper–semicontinuous (u. s. c.) function defined on X . A system of weights in X is a directed upward set V of weights in X such that for all $x \in X$ there exists $v \in V$ such that $v(x) > 0$ . The system of weights V on X generates the "weighted" space

$$CV(X,E) := \{ f \in C(X,E) \,;\, \text{for all } p \in cs(E) \text{ and } v \in V ,\, q_{vp}(f) := \sup_{t\in X} v(t)p(f(t)) < +\infty \}$$

and its subspace

$$CV_0(X,E) := \{ f \in C(X,E) \,;\, \text{for all } p \in cs(E) \text{ and } v \in V ,\, v(p_\circ f) \text{ vanishes at infinity} \}$$

The space $CV(X,E)$ is topologized by the set of seminorms $\{q_{vp}\}_{p\in cs(E),v\in V}$ and then $CV_0(X,E)$ is a closed subspace which is topologized with the relative topology. If v is a strictly positive weight and $V := \{ \lambda v \, ; \, \lambda > 0 \}$ , the weighted spaces $CV(X,E)$ and $CV_0(X,E)$ will be denoted respectively $C(v)(X,E)$ and $C(v)_0(X,E)$ . Let $\mathscr{V} = \{v_n\}_{n \in \mathbb{N}}$ be a decreasing sequence of strictly positive weights $v_n$ on $X$ . We consider the weighted inductive limits

$$\mathscr{V}C(X,E) := \operatorname*{ind}_{n \to} C(v_n)(X,E)$$

$$\mathscr{V}_0C(X,E) := \operatorname*{ind}_{n \to} C(v_n)_0(X,E)$$

endowed with the usual Hausdorff inductive locally convex topology.

Associated to the above sequence of weights $\mathscr{V} = \{v_n\}$ we construct its maximal Nachbin family $\overline{V} = \overline{V}(\mathscr{V})$ of all the u. s. c. functions $\overline{v} : X \longrightarrow [0 \, , \infty)$ such that for some sequence $r_m > 0$ , $m \in \mathbb{N}$ , $\overline{v} \leq \inf_{m \in \mathbb{N}} r_m v_m$ . $\overline{V}$ is a system of weights in $X$ and by the very definition of $\overline{V}$ we have the obvious (continuous) inclusions

$$\mathscr{V}C(X,E) \longrightarrow C\overline{V}(X,E) \qquad \qquad \mathscr{V}_0C(X,E) \longrightarrow C\overline{V}_0(X,E)$$

Beside the maximal Nachbin family $\overline{V} = \overline{V}(\mathscr{V})$ associated to a decreasing sequence of strictly positive continuous weights $\mathscr{V} = \{v_n\}$ , some subfamilies occasionally arise in the theory. We define $\hat{V} := \{ v \in \overline{V} \, ; \, v \text{ is continuous} \}$ and $\tilde{V} := \{ v \in \hat{V} \, ; \, v > 0 \}$ . If $\hat{V}$ is a system of weights, the natural inclusion $C\overline{V}_0(X,E) \longrightarrow C\hat{V}_0(X,E)$ is continuous because $\hat{V} \subset \overline{V}$ . But in the presence of local compactness, the systems $\overline{V}$ and $\hat{V}$ are equivalent, i. e., for every $\overline{v} \in \overline{V}$ there exists $w \in \hat{V}$ such that $\overline{v} \leq w$ (which obviously ammounts to the algebraic and topological identity $C\overline{V}_0(X,E) = C\hat{V}_0(X,E)$ ). Indeed, given $\overline{v} \in \overline{V}$ we take $r_m > 0$ for all $m \in \mathbb{N}$ such that $\overline{v} \leq \inf_{m \in \mathbb{N}} r_m v_m$ . The function

$w := \inf_{m \in \mathbb{N}} r_m v_m$ majorizes $\bar{v}$ and it is pointwise limit of decreasing continuous functions on a compact neighborhood of each point of X (namely, $w = \lim_{n\to\infty} ( \inf_{1\leq m\leq n} r_m v_m ))$. The Dini's Theorem forces this convergence to be uniform on such neighborhood and, hence, w is continuous (cf. [10], 3.2.18. ).

It is not such a trivial matter the question whether $\bar{V}$ is equivalent to $\tilde{V}$ (which results in $C\bar{V}_o(X,E) = C\tilde{V}_o(X,E)$ ). In order to answer this question let us introduce the following condition on $\mathcal{V} = \{v_n\}$ (see [5], 0.4.):

(V) For all $n \in \mathbb{N}$ there exists $m > n$ such that $v_m/v_n$ vanishes at infinity.

If a completely regular topological space X possesses a decreasing sequence $\mathcal{V} = \{v_n\}$ of strictly positive continuous weights satisfying the condition (V) then X is $\sigma$–compact: as a matter of fact the compact sets $A_k := \{ t \in X ; \frac{v_m(t)}{v_n(t)} \geq 1/k \}$, $k \in \mathbb{N}$, satisfy $A_k \subset \mathring{A}_{k+1}$ for all $k \in \mathbb{N}$, and $X = \bigcup_{k=1}^{\infty} A_k$ (here n is any integer and m is as in the condition (V) ). If X is even a paracompact topological space then, by [4] Lemma 1.8. and following, for every $\bar{v} \in \bar{V}$ there exists $\tilde{v} \in \tilde{V}$ such that $\bar{v} \leq \tilde{v}$.

## WEIGHTED SPACES WITH PARACOMPACT DOMAIN

The following result about general countable inductive limits is well known (see [12] Proposición 2.5. and [15] Observation 8.4.7.(b) ) and we will use it in the sequel

PROPOSITION 1. Let $E = \operatorname{ind}_{n\to} E_n$ be a countable inductive limit of l. c. s. such that for every $n \in \mathbb{N}$, $E_n$ is dense in $E_{n+1}$. Then $E_1$ (and "a fortiori" every $E_n$) is dense in E.

PROOF. If $U$ is an open neighborhood in $E$ of some $x \in E$ and $n \in \mathbb{N} \cup \{0\}$ is such that $x \in E_{n+1}$, then $U \cap E_{n+1}$ is an open neighborhood of $x$ in $E_{n+1}$. By hypothesis $(U \cap E_{n+1}) \cap E_n = U \cap E_n$ is a non–void (open) set of $E_n$. Proceed step by step until get $U \cap E_1 \neq \phi$.

The following is an example of the above situation in the framework of the spaces of weighted continuous functions

PROPOSITION 2. Let $\mathcal{V} = \{v_n\}$ be a decreasing sequence of strictly positive continuous weights on the completely regular topological space $X$ and let $E$ be any l. c. s. Then for every $n \in \mathbb{N}$, $C(v_n)_o(X,E)$ is a dense subspace of $C(v_{n+1})_o(X,E)$. Consequently $C(v_1)_o(X,E)$ is a dense subspace of $\mathcal{V}_o C(X,E)$.

PROOF. Fix $f \in C(v_{n+1})_o(X,E)$, $p \in cs(E)$ and $\varepsilon > 0$. We take a compact set $K \subset X$ such that $\sup_{t \in X \setminus K} v_{n+1}(t)p(f(t)) \leq \varepsilon$. Consider the function $g := \frac{v_{n+1}}{v_n} \varphi f \in C(X,E)$ where $\varphi := \inf\left( \sup_{y \in K} \frac{v_n(y)}{v_{n+1}(y)}, \frac{v_n}{v_{n+1}} \right)$. It is clear that $g \in C(v_n)_o(X,E)$. Indeed if $r \in cs(E)$ and $\varepsilon_1 > 0$ are fixed, we take a compact set $K_1 \subset X$ such that if $t \in X \setminus K_1$ then $v_{n+1}(t)r(f(t)) \leq \dfrac{\varepsilon_1}{\sup_{y \in K} \frac{v_n(y)}{v_{n+1}(y)}}$. Then we have

$$\sup_{t \in X \setminus K_1} v_n(t)r(g(t)) = \sup_{t \in X \setminus K_1} v_n(t) \frac{v_{n+1}(t)}{v_n(t)} \varphi(t)r(f(t)) \leq$$

$$\leq \left[ \sup_{y \in K} \frac{v_n(y)}{v_{n+1}(y)} \right] \sup_{t \in X \setminus K_1} v_{n+1}(t)r(f(t)) \leq \varepsilon_1 .$$

We will finally prove that $q_{v_{n+1}p}(g-f) \leq \varepsilon$. For $t \in K$ one has $\varphi(t) = \dfrac{v_n(t)}{v_{n+1}(t)}$ and thus $v_{n+1}(t)p(\frac{v_{n+1}(t)}{v_n(t)}\varphi(t)f(t) - f(t)) = v_{n+1}(t)\, p(f(t)-f(t)) = 0$ and for $t \in X\backslash K$, taking in account that $0 < \varphi(t) \le \frac{v_n(t)}{v_{n+1}(t)}$, one has

$v_{n+1}(t)p(\frac{v_{n+1}(t)}{v_n(t)}\varphi(t)f(t) - f(t)) = v_{n+1}(t)\,(1 - \frac{v_{n+1}(t)}{v_n(t)}\varphi(t))\, p(f(t)) \le \varepsilon$. This proves the first assertion and the second one is now a consequence of the Proposition 1.

It should be noted that, under the hypothesis of local compacity on $X$, the "smaller" subspace $C_c(X,E)$ turns out to be also dense in $\mathcal{V}_oC(X,E)$ as it can be easily checked.

The following is the main technical result

THEOREM 3. Let $\mathcal{V} = (v_n)$ be a decreasing sequence of strictly positive continuous weights on the paracompact topological space $X$ and let $E$ be a l. c. s. with the c. n. p. Then the inductive limit space $\mathcal{V}_oC(X,E)$ and the weighted space $C\overline{V}_o(X,E)$ induce the same topology on their common subspace $C(v_1)_o(X,E)$.

PROOF. Let $U := \overline{\Gamma(\bigcup_{n=1}^{\infty} \rho_n B_{p_n})}$ be a basic 0–neighborhood in $\mathcal{V}_oC(X,E)$ (here $\rho_n > 0$, $p_n \in cs(E)$ and $B_{p_n} := \{ f \in C(v_n)_o(X,E) ; q_{v_n p_n}(f) \le 1 \}$ is a typical 0–neighborhood in $C(v_n)_o(X,E)$; $\Gamma$ denotes, as usual, the absolutely convex hull and the closure is understood, of course, in the topology of $\mathcal{V}_oC(X,E)$). Since $E$ possesses the c. n. p., there exist scalars $a(n) > 0$, $n \in \mathbb{N}$, and $p \in cs(E)$ such that $p_n \le a(n)p$ for all $n \in \mathbb{N}$. We then define $\overline{v} := \inf_{n\in\mathbb{N}} a(n)2^n\rho_n^{-1}v_n \in \overline{V}$ and we construct the neighborhood in $C\overline{V}_o(X,E)$

$$W := \{ \varphi \in C\overline{V}_o(X,E) ; q_{\overline{v}p}(\varphi) = \sup_{t\in X} \overline{v}(t)p(\varphi(t)) < 1 \} .$$

Take $\varphi \in W \cap C(v_1)_o(X,E)$ and consider for each $n \in \mathbb{N}$ the open set

$$U_n := \{ t \in X ; v_n(t)2^n \rho_n^{-1} p_n(\varphi(t)) < 1 \}$$

Since $q_{\overline{v}p}(\varphi) < 1$ , $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of the paracompact topological space $X$, and we take a continuous locally finite partition of the unity $\{f_n\}_{n\in\mathbb{N}}$ of $X$ subordinated to this cover. Since $\varphi \in C(v_1)_o(X,E)$ and $\{v_n\}_{n\in\mathbb{N}}$ is decreasing, it follows that $f_n\varphi \in C(v_n)_o(X,E)$ for all $n \in \mathbb{N}$ . Furthermore for each $n \in \mathbb{N}$ , $\mathrm{supp}(f_n) \subset U_n$ and $0 \leq f_n \leq 1$ . Therefore $q_{v_n p_n}(2^n f_n \varphi) = \sup_{t\in U_n} v_n(t)2^n f_n(t)p_n(\varphi(t)) \leq \rho_n$ , i. e.,

$$2^n f_n \varphi \in \rho_n B_{p_n} \tag{1}$$

We will next prove that

$$\varphi = \lim_{m\to\infty} \sum_{n=1}^{m} f_n\varphi \ \left( = \lim_{m\to\infty} \sum_{n=1}^{m} 2^{-n}(2^n f_n \varphi) \right) \tag{2}$$

where the convergence is assumed in the topology of $\mathcal{V}_o C(X,E)$ . Since the partition of the unity $\{f_n\}_{n\in\mathbb{N}}$ is locally finite, the expansion $\varphi(t) = \sum_{n=1}^{\infty} f_n(t)\varphi(t)$ converges uniformly on a neighborhood of each point of $X$ , and, consequently, converges uniformly on each compact set of $X$ . Let then $\varepsilon > 0$ and $r \in cs(E)$ be fixed. Take a compact set $K \subset X$ such that $v_1(t)r(\varphi(t)) \leq \varepsilon$ for all $t \in X\backslash K$ and then choose an integer $m_o \in \mathbb{N}$ such that for $m \geq m_o$ , $\sup_{t\in K} r(\varphi(t) - \sum_{n=1}^{m} f_n(t)\varphi(t)) \leq \dfrac{\varepsilon}{\sup_{y\in K} v_1(y)}$ . It follows that for $m \geq m_o$

$$\sup_{t\in K} v_1(t)r(\varphi(t) - \sum_{n=1}^{m} f_n(t)\varphi(t)) \leq \varepsilon \tag{3}$$

On the other hand, in view of $\sum_{n=1}^{\infty} f_n = 1$ , one has for every $m \in \mathbb{N}$

$$\sup_{t\in X\backslash K} v_1(t)r(\varphi(t) - \sum_{n=1}^{m} f_n(t)\varphi(t)) \leq \sup_{t\in X\backslash K} v_1(t)(1 - \sum_{n=1}^{m} f_n(t))r(\varphi(t)) \leq \varepsilon \tag{4}$$

From (3) and (4) we conclude that $q_{v_1 r}(\varphi - \sum_{n=1}^{m} f_n \varphi) \le \varepsilon$ for all $m \ge m_0$, i. e., $\varphi = \lim_{m\to\infty} \sum_{n=1}^{m} f_n \varphi$, the limit in the topology of $C(v_1)_0(X,E)$ and, "a fortioiri", in the topology of $\mathcal{V}_0 C(X,E)$. The relations (1) and (2) yield $\varphi \in U$ and now the fact $W \cap C(v_1)_0(X,E) \subset U$ together with the continuity of the embeddings $C(v_1)_0(X,E) \longrightarrow \mathcal{V}_0 C(X,E) \longrightarrow C\overline{V}_0(X,E)$ prove our result.

REMARK 1. The above theorem applies, in particular, to topological spaces X which are both locally compact and $\sigma$–compact. A characterization of these spaces in terms of the weights of $\mathcal{V}$ can be seen in [1].

REMARK 2. From this Theorem also results that on the subspace $C_c(X,E)$ of $C(v_1)_0(X,E)$, the relative topologies of $C\overline{V}_0(X,E)$ and $\mathcal{V}_0 C(X,E)$ coincide. This important fact was first proved in [5] for locally compact spaces X, and the above Theorem 3 extends [5], Lemma 1.1. to paracompact topological spaces.

COROLLARY. Let X be a paracompact topological space and let E be any l. c. s. with the c. n. p. Then for every decreasing sequence of strictly positive continuous weights $\mathcal{V} = \{v_n\}$ on X, the continuous injection $\psi : \mathcal{V}_0 C(X,E) \longrightarrow C\overline{V}_0(X,E)$ is a topological isomorphism into.

PROOF. By the Theorem 3 the restriction $\psi|_{C(v_1)_0(X,E)}$ is a topological isomorphism into. Since $C(v_1)_0(X,E)$ is a dense subspace of $\mathcal{V}_0 C(X,E)$ (Proposition 2), $\psi$ itself is a topological isomorphism into.

The techniques developed in the Proposition 2 extend, with minor modification, to the following

PROPOSITION 4. Let $\mathscr{V} = \{v_n\}$ be a decreasing sequence of strictly positive continuous weights on the completely regular topological space X and let $\tilde{V}$ be the family of strictly positive and continuous elements of the maximal Nachbin family $\overline{V}$ associated to $\mathscr{V}$. Then if E is any l. c. s., for every $n \in \mathbb{N}$, $C(v_n)_o(X,E)$ is a dense subspace of $C\tilde{V}_o(X,E)$. In particular $\mathscr{V}_oC(X,E)$ is a dense subspace of $C\tilde{V}_o(X,E)$.

PROOF. Let us fix the step $C(v_n)_o(X,E)$ and take $f \in C\tilde{V}_o(X,E)$, $v \in \tilde{V}$, $p \in cs(E)$ and $\varepsilon > 0$. Choose a compact set $K \subset X$ such that $\sup_{t \in X \setminus K} v(t)p(f(t)) \leq \varepsilon$. Since v is strictly positive, we are allowed to consider the (continuous) function $\varphi := \inf (\sup_{y \in K} \frac{v_n(y)}{v(y)}, \frac{v_n}{v})$. Now it is routine to check (just as in the proof of Proposition 2) that the continuous function $g := \frac{v}{v_n} \varphi f$ is an element of $C(v_n)_o(X,E)$ such that $q_{vp}(f-g) \leq \varepsilon$. This proves the first assertion and the second one follows obviously from it.

## MAIN RESULTS

If $\mathscr{V}$ is a decreasing sequence of strictly positive continuous weights satisfying the condition (V) and $\overline{V} = \overline{V}(\mathscr{V})$ is its maximal Nachbin family, then $C\overline{V}_o(X,E) = C\tilde{V}_o(X,E)$ whenever X is a paracompact topological space and E any l. c. s. (see Introduction). Thus the Corollary of Theorem 3 and the Proposition 4 yield the following fundamental theorem of representation of inductive limits of weighted spaces of continuous functions with paracompact domain.

THEOREM 5. Let $\mathscr{V} = \{v_n\}$ be a decreasing sequence of strictly positive continuous weights satisfying the condition (V) on the paracompact topological space X and let E be a l. c. s. with the c. n. p. Then $\mathscr{V}_oC(X,E)$ is a topological dense subspace of $C\overline{V}_o(X,E)$.

This theorem also invites to find conditions of completeness of $C\overline{V}_0(X,E)$ in order to study the completeness of the inductive limit $\mathcal{V}_0C(X,E)$ (a classical problem to deal with). It is known that if E is complete and X is a $k_{\mathbb{R}}$–space, then $C\tilde{V}_0(X,E)$ is a complete l. c. s. (because $\inf_{t\in K} v(t) > 0$ for all $v \in \tilde{V}$ and every compact set $K \subset X$ ). Therefore under these additional assumptions, the Theorem 5 allows us to conclude that $C\overline{V}_0(X,E) = C\tilde{V}_0(X,E)$ is the completion of $\mathcal{V}_0C(X,E)$. But if the l. c. s. E is even a Banach space then each step $C(v_n)(X,E)$ is a Banach space and much more can be said by entering the theory of (DF) spaces and using the fact that quasi–complete (DF) spaces are complete

THEOREM 6. Let X be a paracompact $k_{\mathbb{R}}$–space, E a Banach space and $\mathcal{V} = \{v_n\}$ a decreasing sequence of strictly positive continuous weights on X satisfying the condition (V) . Then $C\overline{V}_0(X,E) = \mathcal{V}_0C(X,E)$ (algebraic and topologically) and this common space is complete.

PROOF. In view of the representation Theorem 5, all that remains to be proved is that $\mathcal{V}_0C(X,E)$ is itself complete. This is accomplished by using a standard regularity argument. Indeed, under the condition (V), $\mathcal{V}C(X,E) = \mathcal{V}_0C(X,E)$ (algebraic and topologically) and also the condition (V) forces the (LB)–space $\mathcal{V}C(X,E) = \operatorname{ind}_{n\to} C(v_n)(X,E)$ to be boundedly retractive ([3] , pg. 96) and hence (quasi–)complete.

FINAL REMARK. The representation theorems 5 and 6 have its chief application to topological spaces X which are metrizable (metric spaces are both paracompact and $k_{\mathbb{R}}$ ). As a matter of fact these theorems provide a projective description of inductive limits of weighted spaces of continuous functions defined in non–locally compact topological spaces, covering such an important case as the above mentioned metric setting.

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