UNIVERSIDAD DE EXTREMADURA

TESIS DOCTORAL / PHD THESIS

Análisis de Fiabilidad y del Mantenimiento de Sistemas en Deterioro

Reliability and Maintenance Analysis for Deteriorating Systems

Nuria Caballé Cervigón

Departamento de Matemáticas

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Conformidad de los directores / Advisors' conformity:

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“The enchanting charms of this sublime science, the Mathematics, reveal themselves in all their beauty only to those who have the courage to go deeply into it.”

Carl Friedrich Gauss.

Abstract

A system is understood as a set of components organised according to a given design with the objective of achieving the performance of some determined functions. However, some systems are affected by a physical phenomenon named degradation which reduces its capability for fulfilling these functions. Even, many systems present multiple sources of degradation. A classical example of this multiple degradation is the pitting corrosion consisting in the appearance of different pits or holes which evolve simultaneously in the system. For degrading systems, it is assumed that the system fails when the magnitude of the degradation exceeds a predetermined threshold.

Furthermore, some systems are not only subject to internal degradation, but also they can fail due to other causes. Factors such as the difference in pressure, temperature, or humidity could affect the failure mechanism of a system. Shocks are important mechanisms of system failures. Some of these shocks can be viewed as the accumulation of some random stress on a system and others result in an immediate system failure. These failures are called traumatic failure. A tire thread is an example of system subject to competing failures. A tire fails when its wear exceeds a threshold determined by some industrial standards. Besides the wear, tires also experience shocks such as the road debris, which may puncture the protector zone. Thereby, the failure probability of a tire tread depends both on the natural degradation and sudden shocks.

In order to a system works properly, it needs a set of actions such as inspections, repairs or replacements. This set of actions is defined as maintenance. The main aim of maintenance is to prevent the system failure and, if it is not possible to avoid its appearance, to reduce or mitigate its consequences. So that, maintenance plays an important role within the organisation and functioning of the industry. In maintenance, a widely used technique is the condition-based maintenance. This type of maintenance uses monitoring techniques for checking the state of the system and, throughout the obtained information, to determine the maintenance tasks required. The set of activities performed is denoted as a maintenance strategy.

Traditionally, the search for the optimal maintenance strategy has been based on an asymptotic approach, that is, considering an infinite life cycle. In this approach, the general assumption is that the system can be replaced an infinite number of times by a new and identical system. However, this situation seldom holds in practice since the technology does not remain constant and the system models are renovated using the advance of the technology. Hence, the steady state assumption and, consequently the use of the asymptotic approach seems to be questionable. This fact has motivated the development of maintenance strategies based on a transient approach, that is, considering a finite life cycle.

In this thesis, we extend the current survey about condition-based maintenance for systems subject to two causes of failure, internal degradation and sudden shocks, proposing, developing, and analysing new condition-based maintenance strategies. To this end, we propose two maintenance strategies under an asymptotic approach for systems subject to multiple degradation processes and sudden shocks where these two causes of failure were considered both independent and dependent. Furthermore, we propose two maintenance strategies under a transient approach for systems subject to a unique degradation process and sudden shocks where these two causes of failure were considered both independent and dependent. These new maintenance strategies are very useful due to their wide applicability in the optimisation of resources in the general industry.

Additionally, there is an increasing interest in evaluating the performance assessment of maintained systems. In this thesis, several availability measures proposed in the current scientific literature for describing the maintained system behaviour are evaluated.

Resumen

Un sistema es un conjunto de componentes dispuestos de acuerdo a un diseño establecido con el objetivo de lograr el cumplimiento de unas determinadas funciones. Sin embargo, algunos sistemas están sometidos a un proceso de deterioro. Se denomina deterioro a un proceso de degradación física que sufren los sistemas que reduce su capacidad para realizar estas funciones. Incluso, muchos sistemas se deterioran en más de una dirección. Un ejemplo clásico de esta degradación múltiple es la corrosión por picado, consistente en la aparición de diferentes fisuras que evolucionan simultáneamente en el sistema. Suponemos que un sistema sometido a deterioro falla cuando la magnitud de la degradación excede un umbral predeterminado.

Además, algunos sistemas no sólo están sometidos a degradación interna, si no que también pueden fallar debido a otras causas. Factores como pueden ser la diferencia de tensión, presión, temperatura o humedad podrían influir en el mecanismo de fallo de un sistema. Los choques son importantes mecanismos de fallo de los sistemas. Algunos de estos choques se pueden reflejar en la acumulación aleatoria de deterioro y otros derivar en un fallo inmediato del sistema. Estos últimos se denominan fallos traumáticos. Los neumáticos son un ejemplo de un sistema sometido a estas dos causas competitivas de fallo. Un neumático falla cuando su desgaste excede un umbral determinado por algún estándar industrial. Además del desgaste, los neumáticos también están expuestos a los restos existentes en la calzada, que pueden perforar su zona protectora. De este modo, la probabilidad de fallo del neumático depende tanto de la degradación natural como de choques repentinos.

Para que un sistema funcione adecuadamente, precisa de una serie de acciones como inspecciones, reparaciones o reemplazamientos. El conjunto de estas acciones se denomina mantenimiento. El principal objetivo del mantenimiento es el de prevenir dichos fallos o bien, si no es posible eliminar su aparición, disminuir o atenuar sus consecuencias. Debido a esto, el mantenimiento juega un papel cada vez más importante dentro de la organización y funcionamiento de los sistemas industriales. Dentro del mantenimiento, una técnica clave que ha sido ampliamente desarrollada en los últimos tiempos es el mantenimiento basado en la condición del sistema. Este tipo de mantenimiento utiliza técnicas de monitorización para analizar el estado del sistema y, a través de la información obtenida, programar las actividades de mantenimiento requeridas. El conjunto de actividades a realizar se denomina estrategia de mantenimiento.

Tradicionalmente, la búsqueda de estas estrategias de mantenimiento ha estado basado en un enfoque asintótico, esto es, considerando un horizonte de operatividad infinito. En este enfoque, la hipótesis general es que el sistema puede ser reemplazado un número ilimitado de veces por uno nuevo e idéntico. Sin embargo,

esta situación rara vez sucede en la práctica, ya que la tecnología mejora y los modelos de los sistemas evolucionan. Por tanto, la hipótesis del enfoque asintótico en la modelización de sistemas es cuestionable. De esta manera, se desarrollan estrategias de mantenimiento basadas en ciclos de vida finitos donde el sistema sólo puede ser reemplazado un número limitado de veces.

En esta tesis, se ampliará el estudio existente sobre estrategias de mantenimiento basadas en la condición de sistemas sometidos a ambas causas de fallo, degradación interna y choques repentinos, planteando, desarrollando y analizando nuevas estrategias de mantenimiento basadas en la condición del sistema. Para ello, se propondrán dos estrategias de mantenimiento basadas en un enfoque asintótico, donde el sistema esta sometido a múltiples procesos de degradación y choques repentinos, donde estas dos causas de fallo serán consideradas tanto independientes como dependientes. Por otro lado, también se propondrán dos estrategias de mantenimiento basadas en un enfoque transitorio, donde el sistema está sometido a un único proceso de deterioro y choques repentinos, donde estas dos causas de fallo serán consideradas tanto independientes como dependientes. Estas nuevas estrategias de mantenimiento son muy útiles debido a la amplia aplicabilidad que tienen dentro de la optimización de recursos en la industria en general.

Además, en las últimas décadas, se ha generado un especial interés en evaluar el rendimiento de los sistemas mantenidos. En esta tesis, se evaluarán varias medidas de rendimiento propuestas en la literatura actual que describen el comportamiento de los sistemas mantenidos.

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First of all, I would like to thank my advisors, Dr. Inmaculada Torres Castro and Dr. Carlos Javier Pérez Sánchez. We know that these years have been hard and intense, with lots of ups and downs. However, step by step, we were getting goals, seeing everything started to fit and acquire sense and form. I thank you all your good words and support in hard times, that you believed in me and all your help and knowledge that you have given me.

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When I started this adventure which is the PhD Thesis, my advisor Inma said to me that, up to this moment, all the achievements were sprints, the goal was got objectives in as short a time as possible. When you initiate a thesis, everything changes, and sprints become a long-distance race, where those ones which arrive are who endure, resist, and fight. About this statement, I have two very clear conclusions. The first one is that she could not be more accurate in this simile. The second one is that I would not have arrived at the goal without each and every one of you.

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Cuando empecé toda esta aventura que es una tesis doctoral, mi directora Inma me dijo que, hasta ese momento, todos los logros eran sprints, carreras de corta duración, la meta era conseguir objetivos en el menor tiempo posible. Cuando comienzas una tesis, todo cambia, y los sprints se transforman en una carrera de fondo, en la que aquél que llega es el que aguanta, resiste y lucha. Sobre esa afirmación tengo dos conclusiones muy claras después de todos estos años. La primera, es que ella no podía haber estado más acertada en el símil. La segunda, es que yo no hubiera alcanzado la meta si no hubiera sido por todos y cada uno de vosotros.

Gracias,

Nuria Caballé

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Scientific diffusion

Throughout this thesis the following original publications in international journals has been produced:

- I. T. Castro, N. C. Caballé, C. J. Pérez. A condition-based maintenance for a system subject to multiple degradation processes and external shocks. *International Journal of Systems Sciences*, 46:1692-1704, 2015. JCR impact factor of 1.947 (Q_1, T_1). Citations from SCI: 10. Citations from Google Scholar: 25.
- N. C. Caballé, I. T. Castro, C. J. Pérez, J. M. Lanza-Gutiérrez. A condition-based maintenance of a dependent degradation-threshold-shock model in a system with multiple degradation processes. *Reliability Engineering and System Safety*, 134:98-109, 2015. JCR impact factor of 2.498 (Q_1, T_1). Citations from SCI: 22. Citations from Google Scholar: 35.
- N. C. Caballé, I. T. Castro. A degradation-threshold-shock model for a system. The case of dependent causes of failure in finite-time. *Journal of Polish Safety and Reliability Association Summer Safety and Reliability.* 7:13-20, 2016. Citations from SCI: 0. Citations from Google Scholar: 0.

The following publications are currently under review:

- N. C. Caballé, I. T. Castro. Performance measures in a degradation-threshold-shock model under a condition-based maintenance. *Applied Mathematical Modelling.* Impact factor of 2.291 (Q_1, T_1).
- N. C. Caballé, I. T. Castro. Maintenance cost and some performance measures analysis for a system subject to different types of failures under a finite life cycle. *OR Spectrum.* Impact factor of 1.395 (Q_2, T_2).

Furthermore, some results directly related to this thesis have been published and presented in the following international conferences:

- N. C. Caballé, I. T. Castro, C. J. Pérez. Condition-based maintenance for systems subject to multiple deterioration processes and external shocks. *APARM 2012, 5th Asia- Pacific International Symposium*, 2012.
- N. C. Caballé, I. T. Castro, C. J. Pérez. A model for a system subject to multiple degradation processes and external shocks: the case of dependent causes of failure. *ALT 2014, 5th International Conference on Accelerated Life Testing and Degradation Models*, 2014.

- N. C. Caballé, I. T. Castro. Optimizing maintenance cost for a degradation-threshold-shock model in finite time. *ASMDA 2015, 16th Conference of the Applied Stochastic Models and Data Analysis International Society*, 2015.

- N. C. Caballé, I. T. Castro. Assessment of a condition-based-maintenance strategy in finite time for a system subject to internal degradation and sudden shocks. *II International Workshop on Statistical Models for Business, Engineering and Sciences*, (Awarded as the Best Poster) 2015.

- N. C. Caballé, I. T. Castro. Comparison of condition-based maintenance strategies under different approaches in degradation-threshold-shock models. *CMSTATISTICS 2016, 9th International Conference on the ERCIM WG on Computational and Methodology Statistics* (Invited Presentation), 2016.

- N. C. Caballé, I. T. Castro. Performance assessment measures for deteriorating systems. *JMEEM 2017, II Joint Meeting Évora-Extremadura on Mathematics*, 2017.

- N. C. Caballé, I. T. Castro. Performance measures for a system subject to degradation and sudden shocks. *ESREL 2017, European Safety and Reliability Conference*, 2017.

Acronyms

CBM	Condition-Based Maintenance.
CM	Corrective Maintenance.
DSPP	Doubly Stochastic Poisson Process.
DTS	Degradation-Threshold-Shock.
d.u.	degradation unit.
HPP	Homogeneous Poisson Process.
i.i.d	independent and identically distributed.
ISO	International Organization for Standards.
m.u.	monetary unit.
NHPP	Non-Homogeneous Poisson Process.
PM	Preventive Maintenance.
t.u.	time unit.

Notation

a, b	Shape and scale parameters of the power-law.
$A_T^M(t)$	Point availability of the system at time t with time between inspections T and preventive threshold M.
$\tilde{A}_T^M(t)$	Estimation of $A_T^M(t)$.
$a_{(v_i\%)}$, $b_{(v_j\%)}$	Power-law parameters modified under the i-th and j-th positions for the vector $\mathbf{v}$.
α, β	Shape and scale parameters of the gamma process.
$\alpha_{(v_i\%)}, \beta_{(v_j\%)}$	Gamma process parameters modified under the i-th and j-th positions for the vector $\mathbf{v}$.
c, d	Sudden shock process parameters with independent causes of failure.
$c_{(v_i\%)}, d_{(v_j\%)}$	Sudden shock process parameters modified under the i-th and j-th positions for the vector $\mathbf{v}$.
C_c	Cost associated with a CM.
C_d	Downtime cost.
C_I	Cost associated with each inspection.
C_p	Cost associated with a PM.
$C(t)$	Cost in the interval $(0, t]$.
$C(t_1, t_2)$	Cost in the interval $(t_1, t_2]$.
$C^\infty(T, M)$	Asymptotic expected cost rate for a time between inspections T and preventive threshold M.
$\tilde{C}^\infty(T, M)$	Estimation of the asymptotic expected cost rate for a time between inspections T and preventive threshold M.
$\tilde{C}^{M,\infty}_{T,a_{(v_i\%)},b_{(v_j\%)}}$	Minimal expected cost rate obtained by varying the power-law parameters simultaneously.
$\tilde{C}^{M,\infty}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}$	Minimal expected cost rate obtained by varying the gamma process parameters simultaneously.
D_j	Length of the j-th renewal cycle.

$E\left[C_T^M(t)\right]$	Expected cost in the life cycle of the system at time t with time between inspections T and preventive threshold M.
$\tilde{E}\left[C_T^M(t)\right]$	Estimation of $E\left[C_T^M(t)\right]$.
$E\left[C_T^M(t)^2\right]$	Expected square cost at time t with time between inspections T and preventive threshold M.
$\tilde{E}^*\left[C_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^M(t_f)\right]$	Minimal expected cost in the life cycle of the system varying gamma process parameters.
$\tilde{E}^*\left[C_{T,c_{(v_i\%)},d_{(v_j\%)}}^M(t_f)\right]$	Minimal expected cost in the life cycle of the system varying sudden shock process parameters.
$\tilde{E}^*\left[C_{T,\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}^M(t_f)\right]$	Minimal expected cost in the life cycle of the system varying sudden shock process parameters.
$E_{opt,T}^M(t_{f_r})$	Minimal expected cost rate in the life cycle of the system varying at time t_{f_n} with time between inspections T and preventive threshold M.
$\tilde{E}_{opt,T}^M(t_{f_r})$	Estimation of the $E_{opt,T}^M(t_{f_r})$.
$E\left[W_T^M(t_1,t_2)\right]$	Expected downtime in $(t_1,t_2]$ for a time between inspections T and preventive threshold M.
$\tilde{E}\left[W_T^M(t_1,t_2)\right]$	Estimation of $E\left[W_T^M(t_1,t_2)\right]$.
$f_{\alpha t,\beta}$	Density function of the gamma process.
$\bar{F}_{S_{[1]}}$	Survival function of $S_{[1]}$.
F_{σ_z}	Distribution function of σ_z.
$\bar{F}_{\sigma_{z_2}-\sigma_{z_1}}$	Survival function of $\sigma_{z_2}-\sigma_{z_1}$.
$\bar{F}_{V_{[1]}}$	Survival function of $V_{[1]}$.
$\bar{F}_{W_{[1]}}$	Survival function of $W_{[1]}$.
$\bar{F}_Y$	Survival function of Y.
$\Gamma(t)$	Gamma function at time t.
$\Gamma(\alpha,t)$	Incomplete gamma function.
$I(v,t)$	Survival function of Y conditioned to $\sigma_{M_s}=v$.
$IR_T^M(t,t+h)$	Reliability interval in $(t,t+h]$ with time between inspections T and preventive threshold M.
$\tilde{IR}_T^M(t,t+h)$	Estimation of $IR_T^M(t,t+h)$.
$JA_T^M(t_1,t_2,\ldots,t_n)$	Joint availability of the system in $t_1,t_2,\ldots,t_n$ with time between inspections T and preventive threshold M.
$\tilde{JA}_T^M(t_1,t_2,\ldots,t_n)$	Estimation of $JA_T^M(t_1,t_2,\ldots,t_n)$.

$JIR_T^M(t_1, t_1+h_1, \ldots, t_n, t_n+h_n)$	Joint interval reliability of the system in $(t_1, t_1+h_1], \ldots, (t_n, t_n+h_n]$ with time between inspections T and preventive threshold M.
$J\tilde{I}R_T^M(t_1, t_1+h_1, \ldots, t_n, t_n+h_n)$	Estimation of $JIR_T^M(t_1, t_1+h_1, \ldots, t_n, t_n+h_n)$.
L	Breakdown threshold.
$\lambda(t)$	Intensity of the sudden shock process with independent causes of failure.
$\lambda(t, X(t))$	Intensity of the sudden shock process with dependent causes of failure.
$\Lambda(t)$	Cumulative intensity of the sudden shock process.
$\lambda_1(t, X(t))$, $\lambda_2(t, X(t))$	Intensity parameters of the sudden shock process with dependent causes of failure.
$\lambda_{1,(v_i\%)}, \lambda_{2,(v_j\%)}$	Sudden shock process parameters modified under the i-th and j-th positions of the vector $\mathbf{v}$.
M	Preventive threshold.
$m(t)$	Arrival intensity of the degradation processes.
$M(t)$	Cumulative arrival intensity of the degradation processes.
$m_M(t)$	Intensity of the counting process $N_M(t)$ at time t.
$m_{M_s}(t)$	Intensity of the counting process $N_{M_s}(t)$ at time t.
$m_L(t)$	Failure rate function of $W_{[1]}$ at time t.
M_s	Threshold from which the system is more susceptible to sudden shocks.
$N_d(t)$	Number of degradation processes in the system at time t.
$N_M(t)$	Number of degradation processes whose deterioration level exceeds M at time t.
$N_{M_s}(t)$	Number of degradation processes whose deterioration level exceeds M_s at time t.
$N_T^M(tf)$	Total number of complete renewal cycles up to t_f.
$N_s(t)$	Number of sudden shocks at time t.
$N_s(t_1, t_2)$	Number of sudden shocks arrived at the system in $(t_1, t_2]$.
$O(t)$	Deterioration level of the maintained system at time t.
$P_{R_1}^M(kT)$	Probability of the first preventive maintenance action at k-th inspection for a time between inspections T and preventive threshold M.
$\tilde{P}_{R_1}^M(kT)$	Estimation of $P_{R_1}^M(kT)$

$P^M_{R_{1,c}}(kT)$	Probability of the first corrective maintenance action is at k-th inspection for a time between inspections T and preventive threshold M.
$\tilde{P}^M_{R_{1,c}}(kT)$	Estimation of $P^M_{R_{1,c}}(kT)$.
$P^M_{R_{1,p}}(kT)$	Probability of the first preventive maintenance action is at k-th inspection for a time between inspections T and preventive threshold M.
$\tilde{P}^M_{R_{1,p}}(kT)$	Estimation of $P^M_{R_{1,p}}(kT)$.
R_j	Time instant of the j-th renewal cycle.
$r(t)$	Failure rate function of the system at time t.
$R^M_T(t)$	Reliability of the system at time t with time between inspections T and preventive threshold M.
$\tilde{R}^M_T(t)$	Estimation of $R^M_T(t)$.
S_i	Time from $t=0$ of the i-th degradation process to reach the deterioration level M.
$S_{[1]}$	Time instant at which, for the first time, the deterioration level of a given degradation process exceeds the preventive threshold M.
$S^M_T(t)$	Standard deviation associated with the expected cost in the life cycle of the system at time t with time between inspections T and preventive threshold M.
σ_z	Time to reach the deterioration level z.
T	Time between inspections.
t_f	Time at which the life cycle ends.
t_{f_r}	Random finite life cycle.
$\mathbf{v}$	Variation vector for model parameters.
V_i	Time from $t=0$ of the i-th degradation process to reach the deterioration level M_s.
$V_{[1]}$	Time instant at which, for the first time, the deterioration level of a given degradation process exceeds the preventive threshold M_s.
$V^M_{T,a_{(v_i\%)},b_{(v_j\%)}}(t_f)$	Relative variation percentage for the power-law parameters.
$V^M_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(t_f)$	Relative variation percentage for the gamma process parameters.
$V^M_{T,c_{(v_i\%)},d_{(v_j\%)}}(t_f)$	Relative variation percentage for sudden shock process parameters.
$V^M_{T,\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(t_f)$	Relative variation percentage for the sudden shock process parameters.

$VP_T^M(t_f)$	Relative variation percentage for the expected maintenance cost at time $t_F > 0$ with time between inspections T and preventive threshold M.
W_i	Time from $t = 0$ of the i-th degradation process to reach the deterioration level L.
$W_{[1]}$	Time instant at which, for the first time, the deterioration level of a given degradation process exceeds the preventive threshold L.
$X(t)$	Deterioration level of the system at time t.
$X_i(t)$	Deterioration level of the i-th degradation process t time units after its initiation.
$X_i^*(t)$	Deterioration level of the i-th degradation process in t.
Y	Time to the first sudden shock in absence of maintenance.
Z	Time to the system failure in absence of maintenance.

CHAPTER

1

Introduction

In the field of engineering systems and structures, a fundamental aim is to ensure the reliability of the systems. It is well-known that some systems suffer a physical degradation process which precedes the failure. This degradation is due to the irreversible accumulation of damage through life and may involve corrosion, material fatigue, wearing out, and fracturing [1]. Consequently, this degradation process complicates the maintenance since it is uncertain and depends on the time. The theory of stochastic processes provides an analytical framework for modelling the impact of the uncertain and time-dependent degradation process.

The gamma process is a stochastic cumulative process considered as one of the most appropriated processes for modelling the damage involved by the cumulative deterioration of systems and structures [2]. It is characterised by independent and non-negative gamma increments with identical scale parameter. The gamma process was first applied by Moran [3–6] to model water flow into a dam. Later, Abdel-Hammed [7] proposed the gamma process as a specific model for deterioration occurring randomly in time. From then on, the gamma process has been widely used in the maintenance field (see e.g. [8–14]). From a practical point of view, the gamma process has been applied to model different real situations such as to determine optimal dike heightenings [15], optimal sand nourishment sizes [16], optimal maintenance decisions for steel coatings [17], and optimal inspection intervals for high-speed railway tracks [18], bern breakwaters [19], steel pressure vessels [20], automobile brake pads [21], or the block mats and rock dumping [22, 23].

In the current literature about degradation models, often, only one degradation source is considered (see, e.g., [24–29]). However, it does not embrace all cases. In practice, many systems are subject to multiple degradation processes due to they can be degraded in more than one way [30]. A multiple degradation process is a combination of two different processes: the initiation and the growth. Several authors studied systems subject to multiple degradation processes proposing different stochastic models, both for the degradation initiation and for its growth (see e.g. [31–35]). Based on [36] and [37], in this thesis, we provide a new approach combining the non-homogeneous Poisson process (NHPP) and the gamma process for the initiation and growth process, respectively.

However, systems are not only subject to internal degradation, but they are also exposed to sudden shocks which can cause its failure. To the best of our knowledge, Lemoine and Wenocur [38] were the first to combine both causes of failure, proposing the degradation-threshold-shock (DTS) models with the end of developing analytical formulas which model the time to the failure of this kind of systems. In the literature, DTS models has been analysed from both a theoretical point of view ([39–41]) and applied to real situations such as the accelerated degradation data from plastic substrate active matrix light-emitting diodes [42], a real dataset of tire treads [43], or to a micro-electro-mechanical system to evaluate the efficiency of the developed reliability [44]. All they assumed that both causes of failure (degradation and sudden shocks) were independent. However, it seems reasonable to think that the system degradation could reduce the ability to resist shocks and make it more prone to sudden shocks, *i.e.*, the system strength could decrease by the degradation. This fact could be modelled considering as dependent both causes of failure. There are some works related to this in the scientific literature (see e.g. [45–50]).

From a practical point of view, the deterioration and failures of the systems incur high costs. That is why, nowadays, maintenance plays an important role in most companies, which try to provide high quality products, but minimising the production cost [51]. Maintenance refers to the set of necessary operations applied to a system so that it can work properly. Maintenance activities performed on an industrial system can increase not only its safety, but also ensure its availability and correct functioning. Traditionally, these maintenance tasks were performed based on requirements in legislation, company standards or in-house maintenance experience. However, in the early 1960s, authors as Barlow and Hunter, Radner and Jorgenson or McCall [52–54] started developing mathematical models, which aim to quantify costs and to find the optimum balance between the maintenance cost on one side, and the associated cost (benefit) on the other. Maintenance strategies regulate the different maintenance tasks which will be performed on systems. Thus, establishing a good maintenance strategy is absolutely necessary to ensure the availability, the correct functioning of the system, and optimising its maintenance cost. It exists different classifications for the maintenance strategies. Because of its properties, the condition-based maintenance (CBM) is a kind of maintenance strategy which is usually programmed ([55–57]) and in DTS models in particular ([58–60]) for controlling the system reliability. In this line, this thesis provides a new approach proposing a CBM in a DTS model for a system subject to multiple degradation processes and sudden shocks where both causes of failure were considered as independent [61] or dependent [62].

The majority of the current scientific literature assumes that maintenance tasks are planned based on an infinite operating horizon. In most of these papers, the criterion for optimising the CBM strategies is the asymptotic cost rate [63–68]. In the asymptotic criterion, the general assumption is that the system can be replaced an infinite number of times by a new and identical system. However, this situation seldom holds in practice because the life cycle of the systems is often finite. For instance, in military applications, a missile launching system is only required to be functioning within the designated mission time whose length is uncertain. The residual life cycle for such system measured from the present time to the end of the mission is typically finite and decreases over time. When the mission is close to end, replacement of a functioning system becomes less necessary and traditional maintenance strategies turn out to be very costly to the stakeholders [69]. In these cases, the use of the asymptotic approach seems to be questionable since the policies obtained are only limiting approximations to reality. Although maintenance strategies which consider a finite time horizon are more realistic than those considering an asymptotic approach, the first ones are less used due to the analytical and computational treatment difficulties that they involve. Therefore, maintenance policies under a finite life cycle have not been much studied in the literature about degradation systems (see e.g. [70–73]). In this thesis, we progress in the study of the transient approach proposing CBM strategies for a system subject to a degradation process and sudden shocks, where both causes of failure are considered as independent [74] or dependent [75].

Additionally, in the last decades, there is an increasing interest in evaluating the performance of maintained systems in many application fields. Availability measures are used in various stages of the system life cycle to measure and predict the system behaviour [76]. An availability measure is defined as a quantifiable indicator which reports information related to the performance of a system or component. The interest developed by these measures has triggered their use in the literature (see e.g. [77–81]). Along with the maintenance assessment, in this thesis, some availability measures are also evaluated in the system life cycles.

1.1 Motivation and objectives

The economic losses caused by a system failure or a production which does not fulfil its reliability in accordance with certain standards could become very important. Special mention deserves the life-threatening systems. The nuclear power plant accidents or the air crashes reveal the serious consequences of a deficient maintenance. Therefore, the maintenance strategy implementations have an extraordinary importance as actions aimed to combat the age effect and the system failures.

In the last decades, the economic effort invested in system maintenance has increased considerably. Hence, the optimisation from the point of view of these maintenance strategy costs acquires a singular interest. The use of scientific techniques such as the optimisation mathematical models and the analysis of availability measures has increased in the last few years to measure and predict the system behaviour and, thus, reduce the maintenance cost. However, an important characteristic of the system maintenance and the cost optimisation is that the decisions must be taken under a state of uncertainty such as the physical system degradation.

Furthermore, the systems are not only subject to internal degradation. They can also fail due to other reasons such as the difference in stress, pressure, temperature or humidity. For systems operating under these conditions, it is necessary to program activities which minimise the operative cost, and maximise the reliability.

The principal reason to consider this research line as relevant is by proposing new failure models, more complex, which are better adapted to some situations found in practice such as the deterioration in train railways, hydraulic dams or in biofuel pipes, and does not yet solved in the scientific literature. The concrete objectives of this thesis are listed below.

1. Proposing, developing, and analysing new deteriorating stochastic models. These models are addressed considering that the systems are subject to a degradation represented by both one and multiple processes. Furthermore, the system is also subject to sudden shocks considered as dependent or independent on the degradation process of the system.

2. Analysing a CBM strategy for a system, assuming the possibility that the system can either be replaced by a new one with the same characteristics as the previous one an infinite number of times (asymptotic approach), or up to a certain time instant the system cannot be replaced by a new one with the same characteristics at the failure time (transient approach). Some advanced programming techniques are applied to this end.

3. Comparing the results obtained from both approaches, asymptotic and transient.

4. Analysing some availability measures of the maintenance models.

The fact that these models are analysed under a transient approach complicates considerably the computational treatment. Thus, new programming knowledge was developed in different software such as R and MATLAB for using the high-performance computation.

1.2 Research planning

Designing a good roadmap is fundamental to ensure the achievement of the goal proposed at the beginning of the project. This section analyses the research planning followed to develop this thesis. The planning consists of six main phases with different activities each one enunciated as follows:

A. Acquiring general knowledge.

B. Analysing an independent DTS model with multiple degradation processes.

C. Analysing a dependent DTS model with multiple degradation processes.

D. Analysing an independent DTS model with a degradation process.

E. Analysing a dependent DTS model with a degradation process.

F. Disseminating the work performed.

The concrete activities and aims are detailed in Table 1.1.

Phase	Activity	Aim of the activity
A	Acquiring knowledge about the scientific method	Studying the methodologies for getting a good research work.
	Acquiring knowledge about the maintenance theory	Studying the background of the maintenance theory of reliability.
	Acquiring knowledge about the programming software	Developing programming capabilities in R and MATLAB software.
	Reviewing the state-of-the-art	Performing the current state-of-the-art review about maintenance theory.
B	Defining the model	Defining the DTS model for a system subject to multiple degradation processes and sudden shocks with independent competing causes of failure.
	Preliminary analysis	Analysing the assumptions underlying the model.
	Problem formulation	Modelling the system behaviour, identifying the different stochastic processes and the associated operative measures.
	Maintenance strategy description	Describing a CBM strategy for a system operating under an asymptotic approach.
	Computational hedge achievement	Obtaining a computational hedge of the analytical results.
C	Defining the model	Defining the DTS model for a system subject to multiple degradation processes and sudden shocks with dependent competing causes of failure.
	Preliminary analysis	Analysing the assumptions underlying the model.
	Problem formulation	Modelling the system behaviour, identifying the different stochastic processes and the associated operative measures.
	Maintenance strategy description	Describing a CBM strategy for a system operating under an asymptotic approach.
	Computational hedge achievement	Obtaining a computational hedge of the analytical results.

Phase	Activity	Aim of the activity
D	Defining the model	Defining the DTS model for a system subject to a degradation process and sudden shocks with independent competing causes of failure.
	Preliminary analysis	Analysing the assumptions underlying the model.
	Problem formulation	Modelling the system behaviour, identifying the different stochastic processes.
	Obtaining availability measures	Obtaining measures of interest which provide information about the system performance, such as the point availability, the joint availability, the reliability, the interval reliability, and the joint interval reliability.
	Maintenance strategy description	Describing a CBM strategy for a system operating under a transient approach.
	Computational hedge achievement	Obtaining a computational hedge of the analytical results.
	Comparing the results obtained	Comparing the numerical results obtained considering both transient and asymptotic approach.
E	Defining the model	Defining the DTS model for a system subject to a degradation process and sudden shocks with dependent competing causes of failure.
	Preliminary analysis	Analysing the assumptions underlying the model.
	Problem formulation	Modelling the system behaviour, identifying the different stochastic processes.
	Obtaining availability measures	Obtaining measures of interest which provide information about the system performance, such as the point availability, the reliability, and the interval reliability.
	Maintenance strategy description	Describing a CBM strategy for a system operating under a transient approach.
	Computational hedge achievement	Obtaining a computational hedge of the analytical results.
	Comparing the results obtained	Comparing the numerical results obtained considering both transient and asymptotic approach.
F	Disseminating the results obtained	Writing papers for relevant conferences and ISI-SCI journals with the aim of disseminating this research.
	Writing the Ph. D. thesis	Writing a full document, including all the details of this research for getting the degree of *Philosophæ Doctor* in Mathematics.

Table 1.1: Activities and aims developed in this Ph. D. thesis.

Additionally, Figure 1.1 shows the relationship between the different phases of this planning research.

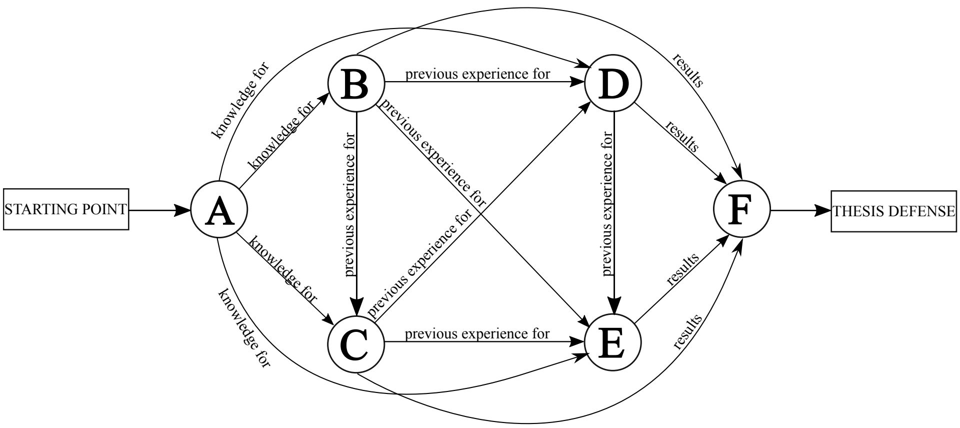


Figure 1.1: Relationship between Ph. D. thesis phases.

1.3 Thesis structure

This document is divided into seven chapters. Chapter 1 introduces the issues tackled in this thesis, detailing the motivation for considering this research line and the concrete objectives reached, exposing the planning followed, and relating the different phases of the research.

Chapter 2 defines the most important concepts related to the mathematical maintenance theory in reliability, resuming its principal aspects.

Chapter 3 analyses a CBM for a system subject to multiple degradation processes and sudden shocks, where both causes of failure are independent, optimising the asymptotic expected cost rate in a bivariate case.

Chapter 4 analyses a CBM for a system subject to multiple degradation processes and sudden shocks, where both causes of failure are dependent. Numerical examples are provided to optimise the asymptotic expected cost rate in a bivariate case and study the robustness of some parameters of the maintenance model.

Chapter 5 analyses a CBM for a system subject to a degradation process and sudden shocks, where both causes of failure are independent. The expected cost rate in the life cycle of the system and its standard deviation are analysed developing a recursive method which combines numerical integration and Monte Carlo simulation. Numerical examples are provided to optimise the expected cost rate in the life cycle. The results obtained for the expected cost rate in the life cycle from the recursive method are compared to the results obtained by the method based on strictly Monte Carlo simulation and to the asymptotic expected cost rate. Additionally, some availability measures of the system and the robustness of some parameters of the model are also analysed.

Chapter 6 analyses a CBM for a system subject to a degradation process and sudden shocks, where both causes of failure are dependent. The expected cost rate in the life cycle of the system and its standard deviation are analysed throughout a recursive method. Numerical examples are provided to optimise the expected cost rate in the life cycle in an univariate and a bivariate case. The results obtained for the expected cost rate in the life cycle from the recursive method are compared to the results obtained by the method based on strictly Monte Carlo simulation and to the asymptotic expected cost rate. Furthermore, some availability measures of the system and the robustness of some parameters of the model are also analysed.

Chapter 7 provides the main conclusions derived from this research and some possible future research lines. Finally, the references which have been cited throughout this thesis are listed.

CHAPTER

2

Background

In this chapter, the background necessary to develop this thesis is detailed. Some types of stochastic processes are shown in Section 2.1. The counting processes are described in Section 2.2. Multiple degradation process and DTS models are studied in Sections 2.3 and 2.4, respectively. Section 2.5 deepens in the different kinds of maintenance for a system. Finally, Section 2.6 studies different availability measures of the system.

2.1 Stochastic deteriorating processes

The natural degradation of a system is dependent on the time, *i.e.*, it depends on the own age of the system. This degradation is not deterministic but it has a random component, which makes that it must be analysed from a stochastic point of view. We suppose that degradation can be represented by a non-decreasing (the greater the age, the greater the deterioration), non-negative (the system cannot suffer an auto-repair), and measurable quantity (for each time instant, it must exist a value representing the deterioration level of the system at this time). Generally, due to the random component of the degradation process, the most appropriated tool is a monotonic stochastic process. Van Noortwijk [2] proposed different models to describe the degradation processes such as the failure rate function, the Markov processes, the Brownian motion, and the gamma process.

2.1.1 Failure rate function

A lifetime distribution represents the uncertainty in the time to the system failure. Let X be the random variable representing the lifetime of a system with distribution function $F(t)$ and density function $f(t)$. The failure rate function $r(t)$ of X is defined as

$$r(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{\bar{F}(t)}, \quad t \geq 0, \tag{2.1}$$

where $\bar{F}(t) = 1 - F(t)$ denotes the survival function defined as

$$\bar{F}(t) = 1 - F(t).$$

The expression $r(t)dt$ represents the probability of a component of age t fails in the interval time $(t, t + dt)$ if it has survived to t. The failure survival function at time t defined as the probability of the system survives at time instant t is obtained from (2.1) as

$$\bar{F}(t) = \exp\left\{-\int_0^t r(\tau)\, d\tau\right\}, \quad t \geq 0.$$

This approach considers that the system must be observed as a unique component and its internal structure cannot be analysed. This approach is named *"black box"* [82]. The failure rate function and life distribution approach are usually used in the mechanic and electric engineering fields, where it is very common to assume that an item can be found into two different states: functioning or failure. However, a system can be found in a range of states depending on its degrading conditions. To analyse these more general situations, the Markov processes were introduced.

2.1.2 Markov processes

A Markov process is a special kind of stochastic process which satisfies the Markovian property. Let $X(t)$ be the state of the process at the time instant t, the values of $X(\tau)$, with $\tau > t$, are independent of the values of $X(u)$, with $u < t$. That means that, when the current state of the system at time t, $X(t)$, is known, the future state of the system $X(\tau)$ is independent of anything happened in the past $X(u)$. Thus, the new state is simply a function of the last state and an auxiliary random variable [83]. The set of all possible states of the system is named state space Ω and it can be a finite set or a countably infinite set. In the case of a finite set, $\Omega = \{0, 1, 2, \ldots, n\}$ with $n + 1$ possible states, the process is named Markov chain or discrete time Markov process.

We consider a system with three components where the failure times of each components are modelled under an exponential distribution. Table 2.1 shows the possible states of such system.

State	Component 1	Component 2	Component 3
7	Functioning	Functioning	Functioning
6	Functioning	Functioning	Failure
5	Functioning	Failure	Functioning
4	Failure	Functioning	Functioning
3	Functioning	Failure	Failure
2	Failure	Functioning	Failure
1	Failure	Failure	Functioning
0	Failure	Failure	Failure

Table 2.1: Possible states of a system with three components.

Despite each component only presents two states, functioning or failure, the system can be found in eight different states. Therefore, this situation cannot be modelled under a failure rate function and we must use a Markov chain.

In the case that the state space Ω is continuous, the process is named continuous time Markov process. Some examples of this kind of processes are the Brownian motion and the gamma process.

2.1.3 The Brownian motion

The Brownian motion is a stochastic process $\{X(t),\ t \geq 0\}$ in continuous time with parameters μ and σ^2 verifying the following properties [84]:

1. $X(t)$ is modelled under a Gaussian distribution with mean μt and variance $\sigma^2 t$, $\forall t \geq 0$.
2. $X(t)$ has independent and stationary increments.
3. $P[X(0) = 0] = 1$.
4. $X(t)$ is continuous in $[0, \infty)$ with probability 1.

This stochastic process is often used in fields such as economy, queuing theory, engineering, robotics, chemistry, biology, and physics (see, e.g., [85–88]). However, the Brownian motion presents a great handicap to model the deterioration of a system. Although the process is positive, the increment may be negative. That means that, the state of the system could suffer an auto-repair. However, one of the main assumptions of the natural degradation of a system is that this one is non-negative. Hence, to justify the use of the Brownian motion as a deteriorating stochastic model, it is said that the model can either suffer measure mistakes or some external maintenance actions which improve the system condition.

2.1.4 The gamma process

In response to the problem raised by the Brownian motion, Abdel-Hameed [7] proposed the gamma process as a model to express mathematically the system deterioration. In general, a gamma process is a stochastic process with non-negative and independent increments following a gamma distribution with identical scale parameter. It is a very adequate process and it has been widely applied to model the monotonic wear accumulated over time in a sequence of small increments such as wear, stress, corrosion, erosion... (see, e.g., [89–91]).

Let $X(t)$ be the system deterioration at time $t \geq 0$. The process $\{X(t), t \geq 0\}$ is a gamma process with parameters $\alpha(t) > 0$ and $\beta > 0$ if it verifies the following properties [2]:

1. The increment $X(t+s) - X(t)$ follows a gamma probability distribution $Ga(\alpha(t+s) - \alpha(t), \beta)$ with density function

$$f_{\alpha(t),\beta}(x) = \frac{1}{\Gamma(\alpha(t))}\beta^{\alpha(t)}x^{\alpha(t)-1}e^{-\beta x}, \quad x \geq 0,$$

for $t \geq 0$ and $s \geq 0$, and where $\Gamma(\cdot)$ denotes the gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx. \tag{2.2}$$

2. $X(t)$ has independent increments, that is, if $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$, then $X(t_1) - X(s_1)$ and $X(t_2) - X(s_2)$ are independent random variables.
3. $P[X(0) = 0] = 1$.

The mean and variance of the gamma process are

$$E[X(t)] = \frac{\alpha(t)}{\beta}, \quad Var(X(t)) = \frac{\alpha(t)}{\beta^2}.$$

Since the ratio of variance and mean is equal to $1/\beta$, it does not depends on the time. Additionally, the coefficient of variation is

$$CV(X(t)) = \frac{\sqrt{Var(X(t))}}{E[X(t)]} = \frac{1}{\sqrt{\alpha(t)}}.$$

If $\alpha(t)$ is an increasing function, the coefficient of variation decreases when t increases.

When $\alpha(t) = \alpha t$, then the gamma process is called a homogeneous gamma process. In this particular case, the increments are independent and also stationary. Under this assumption, the wear difference between the instants $t+s$ and t is a random variable with distribution $Ga(\alpha s, \beta)$ and density function

$$f_{\alpha s,\beta}(x) = \frac{1}{\Gamma(\alpha s)}\beta^{\alpha s}x^{\alpha s-1}e^{-\beta x}, \quad x \geq 0. \tag{2.3}$$

Let σ_z be the random variable describing the time to reach a certain deterioration level z. Modelling the degradation as a gamma process, the distribution function of σ_z is

$$F_{\sigma_z}(t) = P[X(t) \geq z] = \int_z^\infty f_{\alpha t,\beta}(x)dx = \frac{\Gamma(\alpha t, z\beta)}{\Gamma(\alpha t)}, \quad t \geq 0, \tag{2.4}$$

where $\Gamma(\alpha t)$ and $f_{\alpha t,\beta}$ are given in (2.2) and (2.3), respectively, and

$$\Gamma(\alpha, x) = \int_x^\infty u^{\alpha-1}e^{-u}du,$$

denotes the incomplete gamma function for $x \geq 0$ and $\alpha > 0$. The distribution function F_{σ_z} given in (2.4) is known as the first hitting time distribution to the level z with density

$$f_{\sigma_z}(t) = \frac{\alpha}{\Gamma(\alpha t)}\int_{z\beta}^\infty \left\{\log(u) - \psi(\alpha t)\right\} u^{\alpha t-1}e^{-u}du, \quad t \geq 0,$$

where ψ is the digamma function corresponding to the derivative of the logarithm of the gamma function. That is

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{\partial \log\Gamma(x)}{\partial x}.$$

The probability distribution of the random variable $\sigma_{z_2} - \sigma_{z_1}$ is used in the subsequent mathematical development of this thesis. This distribution is difficult to derive due to the "overshoot behaviour" of the gamma process. Overshoot behaviour means that if $X(t)$ is the deterioration level of a degradation process t, then it does not exactly reach the level z_1 but attains it with a non-degenerative random overshoot (see, e.g., [82] and [92]). This fact occurs due to the gamma process is a jump process. Thus, $X(\sigma_{z_1}) \geq z_1$ and hence $\sigma_{z_2} - \sigma_{z_1}$ does not have the same probability distribution as $\sigma_{z_2-z_1}$. By Bertoin [93], the joint probability density of $(\sigma_{z_1}, X(\sigma_{z_1}))$ is

$$f_{\sigma_{z_1},X(\sigma_{z_1})}(x, y) = \int_0^\infty \mathbf{1}_{\{z_1 \leq y < z_1+s\}} f_{\alpha x,\beta}(y-s)\mu(ds), \tag{2.5}$$

for $i = 1, 2, \ldots$, where $\mu(ds)$ denotes the Lévy measure of the gamma process with parameters α and β defined as

$$\mu(ds) = \alpha\frac{e^{-\beta s}}{s}, \quad s > 0.$$

The survival function of $\sigma_{z_2} - \sigma_{z_1}$ is obtained as

$$\begin{aligned}\bar{F}_{\sigma_{z_2}-\sigma_{z_1}}(t) &= P\left[\sigma_{z_2} - \sigma_{z_1} \geq t\right] \\ &= \int_{x=0}^{\infty}\int_{y=z_1}^{\infty} f_{\sigma_{z_1},X(\sigma_{z_1})}(x,y) F_{\alpha t,\beta}(z_2 - y)\, dy\, dx,\end{aligned} \tag{2.6}$$

where $F_{\alpha t,\beta}$ denotes the distribution function of (2.3) and $f_{\sigma_{z_1},X(\sigma_{z_1})}(x,y)$ is given in (2.5).

In the last years, there has been a considerable activity in reliability theory in the development of models that describe events observed over time. Next, some results about counting processes are shown.

2.2 Counting processes

Let $N(t)$ be the number of events in the time interval $(0,t]$. A counting process $\{N(t), t \geq 0\}$ is a stochastic process if it fulfils the following properties [94]:

1. $N(0) = 0$.
2. $P[0 < N(t) < \infty] = 1$.
3. The sample paths of $N(t)$ are right continuous and piecewise constant with jump size $+1$.

Some events which can be modelled using a counting process are the number of repairs performed on a system in an time interval or the number of system failures in a time period. One of the most widely used example of counting process is the Poisson process [94].

2.2.1 Poisson processes

It is usually used in scenarios where event occurrences happened randomly at a certain rate are counted.

The homogeneous Poisson process

Ross [95] defined the HPP with failure rate constant λ as a counting process $\{N(t), t \geq 0\}$ with the following properties:

1. $N(0) = 0$.
2. $P[N(\triangle t) = 1] = \lambda \triangle t + o(\triangle t)$, where $o(\triangle t)$ denotes the infinitesimal of order t, that is, $\lim\limits_{\triangle t \to 0} \dfrac{o(\triangle t)}{\triangle t} = 0$.
3. The number of events in any interval of length t is distributed as a Poisson distribution with parameter λt.

If the arrival of failures at the system is modelled under an HPP with rate λ, the probability of a system failure in $(t, t+\triangle t]$ is $\lambda\triangle t$. Under a HPP, all times between failures are independent and exponentially distributed with the same parameter λ.

HPP is appropriate to model the arrival of failures when the system is not affected by the age. However, in some real situations, system failure is affected by the system age. In these cases, it requires that the failure intensity of the systems is not constant. A practical extension of the HPP is the NHPP, which allows that the system failure intensity changes with the age [96]. Then, in the NHPP the interoccurrence times are neither independent nor identically distributed.

The non-homogeneous Poisson process

The process $\{N(t), t \geq 0\}$ is an NHPP with intensity $\lambda(t)$ if it verifies the following properties [97]:

1. $N(0) = 0$.
2. $P[N(\triangle t) = 1] = \lambda(t)\triangle t + o(\triangle t)$.
3. $P[N(\triangle t) \geq 2] = o(\triangle t)$.

The cumulative intensity function $\Lambda(t)$ characterises the NHPP. It represents the cumulative number of events in the interval $(t, t+h]$. It is given by

$$\Lambda(t, t+h) = \int_t^{t+h} \lambda(\tau) d\tau,$$

where $\lambda(\tau)$ is the failure rate of such counting process. It can then be shown that, for $s < t$, the increment $N(t) - N(s)$ follows a Poisson distribution with intensity

$$\Lambda(s, t) = \int_s^t \lambda(\tau) d\tau.$$

An important issue is the function used to model the intensity $\lambda(t)$. The Weibull distribution is probably the most widely distribution to model the intensity of the NHPP [98]. The Weibull process has been referred to by many different terms such as the power-law process, Weibull restoration process, NHPP with Weibull intensity function, Weibull process, and more recently, as the power-law NHPP [99]. To the best of our knowledge, this model was first analysed by Crow [96], who defined the intensity function as

$$\lambda(t) = \alpha\beta t^{\beta-1}, \quad t \geq 0. \tag{2.7}$$

If the arrival of failures is modelled using an NHPP with Weibull intensity given in (2.7), the system improves over time if $0 < \beta < 1$ and is deteriorated if $\beta > 1$. If $\beta = 1$, the model is reduced to an NHPP. This model is one of the most used in the scientific literature to generate an NHPP [100, 101], and therefore it will be used in the numerical examples of this thesis.

The doubly stochastic Poisson process

Sometimes, the arrival process intensity of the events is not deterministic, but stochastic. In 1955, Cox [102] introduced the notion of doubly stochastic Poisson processes (DSPP) that are more commonly referred to as Cox process nowadays. A Cox process $N(\cdot)$ is a conditional Poisson process where, for almost every given paths of the process $X(t)$, $N(\cdot)$ is a Poisson process with stochastic intensity $\lambda(t, X(t))$. That is, the intensity of the process is modulated by an "outside" process $(X(t))$ influencing the evolution of the counting process $(N(\cdot))$. By choosing $X(\cdot)$, we may get a large class of conditional Poisson processes which are used in the system reliability theory describing failures of complex systems [103]. The process $\{N(t), t \geq 0\}$, given $X(t)$, is an DSPP with intensity $\lambda(t, X(t))$ if it verifies the following conditions:

1. $N(0) = 0$.
2. $P[N(\triangle t) = 1] = \lambda(t, X(t))\triangle t + o(\triangle t)$.

3. The number of events in any interval of length t is distributed as a Poisson distribution with parameter $\Lambda(t, X(t))$.

If the Cox process $N(t)$ describes the number of system failures in $(0, t]$, the process $X(t)$ may include other relevant variables for predicting the likelihood of failure.

In essence, a Poisson process is a continuous-time Markov process on positive integers (usually starting at 0) which has independent and identically distributed ($i.i.d.$) holding times. That means that, in a Poisson process each integer i has a holding time before advancing to the next integer $i+1$ distributed under an exponential $\forall i = 1, 2, \ldots$ with probability 1. Then, the holding times between two successive integers are $i.i.d.$ random variables following an exponential distribution. In the same informal spirit, we may define a renewal process to be the same thing, except that the holding times follows a more general distribution than in a Poisson process.

2.2.2 Renewal processes

Renewal theory has its origin in the study of strategies for replacement of technical components, but later it was developed as a general theory within the stochastic processes. As the name of the processes indicate, they are used to model renewals or replacements of systems.

Let $N(t)$ be the number of renewals in $(0, t]$. A renewal process is a counting process $\{N(t), t \geq 0\}$ with interoccurrence times $D_1, D_2, \ldots$ $i.i.d.$ with distribution

$$F_D(t) = P[D_j \leq t], \quad \text{for } t \geq 0,\ j = 1, 2, \ldots$$

The events observed at times $D_1, D_2, \ldots$ are called renewals, $F_D(t)$ is called the underlying distribution of the renewal process[1].

Let R_j be the time up to the j-th renewal, that is, R_j is the time instant of the j-th renewal cycle given by

$$R_j = \sum_{n=1}^{j} D_n, \tag{2.8}$$

with $R_0 = 0$. Then,

$$N(t) = \max\{j : R_j \leq t\}.$$

An important role in renewal theory is played by the renewal function $Q(t)$ defined by

$$Q(t) = E[N(t)], \quad t \geq 0,$$

where $Q(t)$ is the mean number of renewals in the time interval $(0, t]$.

Let $F^{(j)}(t)$ be the distribution function of the random variable measuring the time up to the j-th renewal R_j. For $j = 1, 2, \ldots$, $F^{(j)}(t)$ is[2]

$$F^{(j)}(t) = P[R_j \leq t], \quad t \geq 0.$$

To find the exact value of $F^{(j)}(t)$ is often complicated. We will outline an approach that may be used, at least, in some cases [104]. Since based on (2.8), R_j can be expressed as

[1] Note that the HPP is a renewal process where the underlying distribution is exponential with parameter λ. Thus, a renewal process is a generalisation of the HPP.

[2] Note that $F^{(1)}(t) = F_D(t)$.

$R_j = R_{j-1} + D_j$, and R_{j-1} and D_j are independent, $F^{(j)}(t)$ is the convolution function of R_{j-1} and D_j. Then

$$F^{(j)}(t) = F_D * F^{(j-1)}(t) = \int_0^t F_D(x) F^{(j-1)}(t-x)\, dx. \tag{2.9}$$

When $F_D(t)$ is absolutely continuous with density function $f_D(t)$, the density function of R_j is

$$f^{(j)}(t) = \int_0^t f_D(x) f^{(j-1)}(t-x)\, dx, \quad \text{for } j = 1, 2, \ldots$$

Many stochastic processes fulfil the property of regenerating themselves at certain points in time, so that the behaviour of the process after the regeneration epoch is a probabilistic replica of the behaviour starting point at time $t = 0$ and is independent of the behaviour before the regeneration epoch.

Regenerative processes

A stochastic process $\{J(t), t \in \Phi\}$ with time-index set Φ is regenerative if there exists a random epoch R_1 such that [105]:

1. $\{J(t + R_1), t \in \Phi\}$ is independent of $\{J(t), 0 \leq t < R_1\}$.
2. $\{J(t + R_1), t \in \Phi\}$ has the same distribution as $\{J(t), t \in \Phi\}$.
3. The existence of the regeneration epoch R_1 implies the existence of further regeneration epoch $R_2, R_3, \ldots$, having the same property as R_1.

We assume that the set Φ is either the interval $\Phi = [0, \infty)$ in the case that we have a continuous-time regenerative process or the countable set $\Phi = \{0, 1, 2, \ldots\}$ in the case of a discrete-time regenerative process.

Previously, we defined D_j as the j-th renewal corresponding to the length of the j-th renewal cycle. Based on (2.8), $D_j = R_j - R_{j-1}$ for $j = 1, 2, \ldots$ with $R_0 = 0$. The random variables $D_1, D_2, \ldots$ are $i.i.d.$ In fact, the sequence $\{D_1, D_2, \ldots\}$ underlies a renewal process where the events are the occurrence of the regeneration epochs. In the following it is assumed that[1]

$$0 < E[D_1] < \infty.$$

Intuitively speaking, a regenerative process can be split into $i.i.d.$ renewal cycles. A cycle is defined as the time interval between two consecutive regeneration epochs. On the other hand, in many practical situations, a reward structure is imposed on the regenerative process $\{J(t), t \in \Phi\}$. The renewal-reward processes are a simple appealing model dealing with a so-called regenerative process on which a cost or reward structure is imposed. They are a powerful tool in the analysis of numerous applied probability models and are very useful for theoretical purposes [105].

[1] Note that the cycle length D_j assumes values from the index set Φ.

Renewal-reward processes

Let RW_j be the total reward earned in the j-th renewal cycle, for $j = 1, 2, \ldots$ We assume that $RW_1, RW_2, \ldots$ are $i.i.d.$ random variables. In applications, RW_j typically depends on the length of the j-th renewal D_j. Let $RW(t)$ be the cumulative reward earned up to time t. Then, the process $\{RW(t), t \geq 0\}$ is a renewal-reward process. With these assumptions about $\{RW(t), t \geq 0\}$, the renewal-reward theorem is enunciated as follows

Theorem 2.1. *For almost any realisation of the process $\{RW(t), t \geq 0\}$, the long-run average reward per time unit is equal to the expected reward earned during one cycle divided by the expected length of one cycle. That is*

$$P\left[\lim_{t\to\infty} \frac{RW(t)}{t} = \frac{E[RW_1]}{E[D_1]}\right] = 1.$$

Proof. It is provided by Tijms [105] on Chapter 2, pages 41-42.

□

With a proper choice of the "rewards", many interesting renewal processes can be turned into renewal reward processes. For instance, the renewal-reward processes will be applied in Section 2.5 by considering the reward as the maintenance cost of the system.

Next, alternating renewal processes and Markov renewal processes will be shown.

Alternating renewal processes

Suppose a system is alternately up and down. Denote by $U_1, U_2, \ldots$ the length of successive up-periods. Let us assume that the times to failure are $i.i.d.$ with distribution function

$$F_U(x) = P[U_i \leq x], \quad \text{for } x \geq 0,\ i = 1, 2, \ldots$$

Likewise, we assume the corresponding down-periods $V_1, V_2, \ldots$ to be $i.i.d.$ with distribution function

$$F_V(y) = P[V_i \leq y], \quad \text{for } y \geq 0,\ i = 1, 2, \ldots$$

The sequences $\{U_i\}$ and $\{V_i\}$ are not required to be independent of each other.

Let $\{I(t), t \geq 0\}$ be the continuous-time stochastic process where

$$I(t) = \begin{cases} 1 & \text{if the system is up at time } t \\ 0 & \text{otherwise} \end{cases}. \tag{2.10}$$

The process $\{I(t), t \geq 0\}$ is a regenerative process where the epoch at which an up-period starts can be taken as regeneration epoch. We assume that an up-period starts at epoch 0.

If we define the complete up/down-period cycle to be renewals, we obtain an ordinary renewal process with renewal interoccurrence times

$$D_i = U_i + V_i, \quad \text{for } x \geq 0,\ i = 1, 2, \ldots$$

This resulting process is called an alternating renewal process. The underlying distribution function, $F_D(t)$, is the convolution of the distribution functions $F_U(t)$ and $F_V(t)$. That is

$$\begin{aligned} F_D(t) &= P[D_i \leq t] = P[U_i + V_i \leq t] = F_U * F_V(t) \\ &= \int_0^t F_U(t-x) F_V(x)\, dx = \int_0^t F_U(t-x) dF_V(x). \end{aligned}$$

This process will be applied for calculating different availability measures in Chapters 5 and 6 by considering the reward as the availability of the system. However, sometimes the system presents more than two states (up or down). Markov renewal processes analyse this kind of situations.

Markov renewal processes

In Subsection 2.1.2, Markov processes are defined as a special kind of stochastic processes satisfying the Markovian property. That is, if the current state of the system is known, the future state is independent of anything happened in the past. Let $\Omega = \{1, 2, \ldots, n\}$ be the state space of the Markov process.

A Markov renewal process is a process which records at each time t, the number of times that a system visits each one of the possible states up to time t [106]. That means, if $N_i(t)$ denotes the number of times in $[0, t]$ that the process is in the i-th state, the vector

$$[N_0(t), N_1(t), \ldots, N_n(t)], \quad t \geq 0,$$

is called a Markov renewal process.

Let $D_{1i}, D_{2i}, \ldots$ denote the interoccurrence times between successive transitions into the i-th state $\forall i \in \Omega$. The visits to the i-th state will be now a renewal process and we may use the theory of renewal processes described previously[1]. Then, $D_{1i}, D_{2i}, \ldots \forall i \in \Omega$ are $i.i.d.$ with distribution function

$$F_{D_i}(t) = P[D_{ji} \leq t], \quad \text{for } t \geq 0,\ j = 1, 2, \ldots, \text{ and } \forall i \in \Omega.$$

The events observed at times $D_{1i}, D_{2i}, \ldots \forall i \in \Omega$ are now called renewals and $F_{D_i}(t)$ is the underlying distribution of the renewal process.

Let R_{ji} be the time up to the j-th renewal of the i-th state. That is, R_{ji} is the time instant of the j-th renewal cycle of the i-th state given by

$$R_{ji} = \sum_{m=1}^{j} D_{mi},$$

with $R_{0i} = 0, \forall i \in \Omega$.

Let $F_i^{(j)}(t)$ be the distribution function of the random variable measuring the time up to the j-th renewal in the i-th state. If we consider each possible value of the state space $\Omega = \{1, 2, \ldots, n\}$ of the j-th renewal, by (2.9), the joint distribution function for all possible states of the system is

$$F^{(j)}(t) = \sum_{i=1}^{n} \int_0^t F_{D_i}(x) F_i^{(j-1)}(t - x)\, dx.$$

This kind of functions are called Markov renewal equations and will be used in Chapters 5 and 6 to obtain analytical results for different measures of the system.

[1] A renewal process is equivalent to the special case of a Markov renewal process with one state.

2.3 Multiple degradation processes

In the current scientific literature about degradation models, often, only one degradation source is considered (see, e.g., [24–29]). However, it does not embrace all cases. In practice, many systems are subject to multiple degradation processes due to which they can be degraded in more than one way. It is unlikely that all degradation processes appear at the same time. They rather initiate at random times and then grow depending on the environment and conditions of the system [36]. In most of works about systems subject to multiple degradation processes, the authors used the stochastic processes detailed in Section 2.1 for modelling the growth process and the counting processes detailed in Section 2.2 for modelling the initiation process.

A characteristic example of this type of multiple degradation processes is the pitting corrosion process consisting in the appearance of small pits on the surface of some metals and alloys. The more conventional explanation for pitting corrosion is that it is an autocatalytic process. That means that, when a pit starts to grow, the conditions generated encourage its own growth. This kind of corrosion is extremely insidious, as it causes little loss of material with the small effect on its surface, while it damages the deep structures of the metal. It is well established the pitting corrosion process has a stochastic nature [107, 108]. Each pit is considered as a degradation process. Pits are deteriorated in a way that, when they start growing, the environment stimulates their own growth [109]. Therefore, a pitting corrosion model is a combination of two different processes: the initiation process and the growth process.

Let $(T_1, T_2, \ldots, T_n)$ be the random vector representing the first n arrival times from the NHPP with intensity function $m(t)$ and cumulative intensity function $M(t)$. Given $N(t) = n$, the conditional joint density function of the vector $(T_1, T_2, \ldots, T_n)$ is

$$f_{T_1,T_2,\ldots,T_n|N(t)}(u_1, u_2, \ldots, u_n|n) = \frac{n! \prod_{i=1}^{n} m(u_i)}{[M(t)]^n},$$

for $0 < u_1 < u_2 < \ldots < u_n < t$. From the fact that each arrival time of the NHPP depends on the number of arrivals, the vector $(T_1, T_2, \ldots, T_n)$ can be treated as an order statistic from a sequence of $i.i.d.$ random variables. Based on [110] and conditioning to $N(t) = n$, the vector $(T_1, T_2, \ldots, T_n)$ has the same distribution that the order statistics of the random vector of size n $(V_1, V_2, \ldots, V_n)$. This distribution is

$$P[V \leq u] = \frac{M(u)}{M(t)}, \tag{2.11}$$

for $0 \leq u \leq t$. Then, we can write the density function of the k-th arrival time T_k for $1 \leq k \leq n$ conditioned to n arrivals up to a time t as

$$f_{T_k|N(t)=n}(u) = \frac{n!}{(k-1)!(n-k)!} \left(\frac{M(u)}{M(t)}\right)^{k-1} \left(1 - \frac{M(u)}{M(t)}\right)^{n-k} \frac{m(u)}{m(t)}. \tag{2.12}$$

When a degradation process arrives, the growth process is activated. Let $\{X(t), t \geq 0\}$ be the underlying degradation process and let $X_k(t)$ be the deterioration level of the k-th degradation process at time t. That is

$$X_k(t) = X(t - T_k).$$

By [36],

$$P[X_k(t) \leq x | N(t) = n] = \int_0^t P[X(t-s) \leq x] f_{T_k|N(t)=n}(s) ds, \tag{2.13}$$

where $f_{T_k|N(t)=n}(\cdot)$ denotes the density function given in (2.12).

Let $X_r(t)$ be the deterioration level of a randomly selected degradation process at time t. Conditioning to $N(t)=n$, $X_r(t)$ is

$$X_r(t)|N(t)=n=\begin{cases} X(t-T_{\xi_n}) & \text{if } n\geq 1 \\ 0 & \text{otherwise} \end{cases},$$

where ξ_n is a uniform random variable taking values $1,2,\ldots,n$ with probability $1/n$. The cumulative density function of $X_r(t)$ is

$$\begin{aligned} P[X_r(t)\leq x] &= \sum_{n=0}^{\infty} P[X_r(t)\leq x|N(t)=n]P[N(t)=n] \\ &= P[N(t)=0]+\sum_{n=1}^{\infty} P[X_r(t)\leq x|N(t)=n]P[N(t)=n]. \end{aligned}$$

Based on (2.13), the conditional probability $P[X_r(t)\leq x|N(t)=n]$ for $n\geq 1$ is calculated as

$$\begin{aligned} P[X_r(t)\leq x|N(t)=n] &= \frac{1}{n}\sum_{k=1}^{\infty} P[X_k(t)\leq x|N(t)=n] \\ &= \int_0^t P[X(t-s)\leq x]\frac{1}{n}\sum_{k=1}^{\infty} f_{T_k|N(t)=n}(s)ds \\ &= \int_0^t P[X(t-s)\leq x]\frac{m(s)}{M(t)}ds \\ &= P[X(t-V)\leq x], \end{aligned} \tag{2.14}$$

where V is given in (2.11). Note that Equation (2.14) does not depends on n. Hence,

$$\begin{aligned} P[X_r(t)\leq x|N(t)>0] &= \frac{\sum_{i=1}^{\infty} P[X_r(t)\leq x|N(t)=n]P[N(t)=n]}{P[N(t)>0]} \\ &= P[X(t-V)\leq x]. \end{aligned}$$

Finally, the distribution function of the deterioration level at time t for a random degradation process is

$$P[X_r(t)\leq x]=P[N(t)=0]+P[X_r(t)\leq x|N(t)>0]P[N(t)>0].$$

2.4 Degradation-threshold-shock models

Additionally, some systems are subject to failures provoked by different causes, both internal and external. The internal failures are provoked by the own system structure such as the material ageing. Nevertheless, it is possible that the system does not fail due to its internal structure. For example, the systems often fail due to the environmental conditions where they are working. Gorjian *et al.* [111] and Yang [112] classified the failures into two different kinds:

1. *Gradual failure:* This kind of failure can be predicted by monitoring one or several state indicators of the system.

2. *Sudden failure:* This system failure cannot be predicted and the system stop working without any previous indication.

To the best of our knowledge, Lemoine and Wenocur [38] were the first to combine both causes of failure, internal degradation and sudden shocks. This kind of models are called *DTS models*.

We suppose that the internal degradation level of a system is described by a stochastic process $\{X(t), t \geq 0\}$. This system is regarded as failed when the degradation exceeds a critical threshold L independent of $\{X(t), t \geq 0\}$. Additionally, the system can also fail when a shock occurs although the degradation process has not yet reached the threshold L. The time to a shock can be modelled as the first point of a DSPP $\{\psi(t), t \geq 0\}$ with stochastic intensity $\lambda(t, X(t))$ that depends on the time t and on the degradation level $X(t)$. That means that, given a realisation $x(t)$ of $X(t)$, $\{\psi(t), t \geq 0\}$ is an NHPP with intensity $\lambda(t, x(t))$. Hence, the lifetime of the system is defined as the minimum

$$Z = \min\{\sigma_L, Y\},$$

of the degradation failure time $\sigma_L = \inf\{t : X(t) \geq L\}$ and the sudden shock time $Y = \inf\{t : \psi(t) = 1\}$. Let X_t denote the path of $\{X(t), t \geq 0\}$, that is $X_t = \{X(s) : 0 \leq s \leq t\}$. Given the degradation path X_t, the conditional survival function of Y is

$$P[Y > t|X_t] = \exp\left\{-\int_0^t \lambda(s, X_s)\, ds\right\}.$$

Thus, the survival function of Z is[1]

$$P[Z > t] = E\left[\mathbf{1}_{\{\sigma_L > t\}} \exp\left\{-\int_0^t \lambda(s, X_s)\, ds\right\}\right].$$

An expression of the survival function Z is provided by the following result.

Theorem 2.2. *Let the sudden shock time Y has the stochastic failure rate $\lambda(t, X_t)$ with $E[\int_0^t \lambda(s, X_s) ds] < \infty$ for all $t \geq 0$ and assume that, given L independent of X_t, the degradation failure time σ_L has the conditional failure rate $\omega(t, L)$ with $E[\int_0^t \omega(s, L) ds] < \infty$ for all $t \geq 0$. Then*

$$P[Y > t] = \exp\left\{-\int_0^t \check{\lambda}(s) ds\right\},$$

and

$$P[\sigma_L > t] = \exp\left\{-\int_0^t \bar{\omega}(s) ds\right\},$$

where the failure rate $\check{\lambda}(\cdot)$ and $\bar{\omega}(\cdot)$ are given by $\check{\lambda}(t) = E[\lambda(t, X_t)|Y > t]$ and $\bar{\omega}(t) = E[\omega(t, L)|\sigma_L > t]$, respectively. The survival function of Z can be expressed as

$$P[Z > t] = \exp\left\{-\int_0^t (\bar{\lambda}(s) + \bar{\omega}(s)) ds\right\},$$

where $\bar{\lambda}(t) = E[\lambda(t, X_t)|Z > t]$ is the failure rate of a sudden shock if a degradation failure is not occurred.

[1] Symbol $\mathbf{1}_{\{\}}$ denotes the indicator function which equals 1 if the argument is true and 0 otherwise.

Proof. It is provided by Lehman [113], page 290.

□

If σ_L has a failure rate function $\omega(t)$ which is independent of L and if Y has a deterministic failure rate $\lambda(t)$, then the survival function of Z is simplified as

$$P[Z > t] = \exp\left\{ -\int_0^t (\lambda(s) + \omega(s))ds \right\}.$$

2.5 Maintenance policies

Based on the International Organization for Standards (ISO) $8402 : 1994$ [114], the maintenance of an item is defined as *"the operation and care set necessary for an item continues to work adequately"*. This operations are named *"maintenance tasks"*. Here, the term item refers to *"any entity, which may be a component, system (a set of components) or a subsystem (a part of a system)"*. A required function refers to *"any function that is necessary to be performed by the entity and can be a single function or a combination of multiple functions"*. The maintenance tasks are either performed to prevent future failures or to reset the system to an operating condition when the system failure has already happened.

Rausand and Høyland [104] classified the different maintenance tasks into two types:

1. *Corrective maintenance (CM):* It is performed after a system is no longer functioning. The aim of the CM is to return to the system to an operative condition, either by repairing or replacing by a new one.

2. *Preventive maintenance (PM):* It is a planned maintenance performed when the system is working properly, but with the end of preventing future failures. This kind of maintenance can involve different tasks such as inspections, settings, lubrication, or system component replacement by wear. These PM tasks are in turn classified into the following categories:

   - *Age-based maintenance:* The maintenance tasks are performed when the system exceeds a certain age, measured either in operative time system or in other concepts such as the number of kilometres in a car or the number of landing and take-offs in air-crafts.
   - *Calendar based maintenance:* The maintenance tasks are performed based on calendar instants fixed. A calendar-based maintenance is, generally, easier to administrate than an age-based maintenance since the maintenance tasks are programmed at predefined time instants and they are not based on the system internal structure.
   - *Condition-based maintenance (CBM):* The maintenance tasks are based on one or several state variable measures. Generally, the system maintenance is performed when a state variable measure exceeds a certain threshold. Some state variable examples are temperature, the number of particles in the lubrication oil, or degradation. This kind of maintenance is also named *"predictive"*. Compared to the previous ones, the CBM tends to reduce the costs and provides a better system operability since it uses the real time information of the system condition to identify the most appropriated instant to perform the maintenance tasks.

Generally, the state of the system is not monitored continuously since it supposes a high cost, but it is evaluated at predetermined instants named inspection times. In these instants, the information about the system condition is provided to analyse if the system must be replaced by a new one in a preventive or corrective way, or if, by contrast, it is not necessary to perform any maintenance task. In addition, each maintenance action implies both a cost and a profit, and a key objective in maintenance is finding a balance between both of them. To this end, it is fundamental to find an adequate maintenance strategy for each system. By maintenance strategy (or policy), we mean a decision rule establishing the sequel of maintenance actions (inspections, maintenance tasks preventive or corrective...) [115]. The maintenance strategy design is to build a mathematical model describing adequately the system mechanism functioning.

Most maintenance models assume that, when a system fails, it is replaced by another new and identical system. The resulting process is an identical and independent probabilistic replica of the previous one and these maintenance policies can be analytically treated as stochastic renewal-reward processes. An accurate evaluation of mean, variance, and other high order moments of the reward is an analytically challenging task. Commonly in maintenance policies, the reward is the maintenance cost of the system. Based on the *"Renewal Theory"*, asymptotic cost rate is often analysed in the scientific literature.

Let C^∞ be the asymptotic cost rate. Following the background about renewal-reward processes detailed in Subsection 2.2.2, we can obtain an expression for C^∞. Based on Theorem 2.1, the asymptotic cost rate is

$$C^\infty = \lim_{t\to\infty} \frac{C(t)}{t} = \frac{E[C_1]}{E[D_1]}, \tag{2.15}$$

where $C(t)$ denotes the total maintenance cost of the system up to t, and $E[C_1]$ and $E[D_1]$ denote the expected cost and the expected length of the first renewal cycle, respectively. This function is also called long-run average cost and it will be applied in Chapters 3 and 4 for analysing the asymptotic expected cost rate.

In maintenance policies which consider the asymptotic approach, optimal maintenance strategies are often based on the minimisation of (2.15) under the assumption that systems are used over infinite life cycle [116]. That means that, when a system fails, it is replaced by a new one and with the same characteristics an infinite number of times. However, this situation seldom holds in practice since most of the systems are designed to experience a finite but uncertain life cycle [117, 118]. For instance, in military applications, a missile launching system is only required to be functioning within the designated mission time whose length is uncertain. In this kind of systems, the use of the asymptotic approach seems to be questionable [69]. Considering an asymptotic approach, though bringing technical convenience, is not realistic under these circumstances [119]. For this reason, maintenance optimisation models for systems with finite life cycles are required in the engineering field.

Let $N(t)$ be the total number of complete system replacements up to t. Considering the renewal process $\{N(t), t \geq 0\}$, we are interested in analysing the expected cost for a system with a finite life cycle. Let $C(t)$ be the total maintenance cost of the system up to t given by

$$C(t) = \sum_{j=1}^{N(t)} C(R_{j-1}, R_j) + C(R_{N(t)}, t),$$

where $C(t_1, t_2)$ denotes the cost of the interval time $(t_1, t_2]$, $C(0, t)$ is expressed in a compact way as $C(t)$, and R_j denotes the time instant of the j-th replacement, for $j = 0, 1, 2, \ldots, N(t)+$

1 defined as

$$R_j = \sum_{n=1}^{j} D_n,$$

for $R_0 = 0$, where D_n denotes the length of the n-th renewal cycle, with $n = 1, 2, \ldots, N(t)+1$. Thus, the length of the n-th renewal cycle is

$$D_n = \begin{cases} R_n - R_{n-1}, & \text{if} \quad n = 1, 2, \ldots, N(t) \\ t - R_{N(t)}, & \text{if} \quad n = N(t) + 1 \end{cases}.$$

Although maintenance policies based on finite life cycles are more realistic than based on infinite life cycles, they are less used in the scientific literature since the analytical and computational treatment is more complex for mean, variance, and other higher moments. Cheng [120] obtained expressions for the first and second moments and generalised them to the m-th order moment of $C(t)$. Next, we analyse these expressions.

Let $U_1(t)$ be the first order moment of $C(t)$, that is, the expected cost at time t.

$$U_1(t) = E[C(t)].$$

Conditioning to the first replacement time R_1, and using the law of total expectation, the expected cost at time t $U_1(t)$ is written as

$$U_1(t) = \sum_{0<\tau\leq t} E[C(t)|R_1 = \tau] f_{R_1}(\tau) + E[C(t)|R_1 > t]\bar{F}_{R_1}(t), \tag{2.16}$$

where $\bar{F}_{R_1}(t) = P[R_1 > t]$ is the survival function of R_1. In the above equation, $U_1(t)$ is partitioned into two terms: $R_1 \leq t$ and $R_1 > t$. When $R_1 = \tau < t$, $C(t)$ is split into two terms; the cost in the first renewal interval $C(0, \tau) = C(\tau)$, and the cost of the remaining time horizon $C(\tau, t)$, such that

$$E[C(t)|R_1 = \tau] = E[C(\tau)|R_1 = \tau] + E[C(\tau, t)|R_1 = \tau] = E[C(\tau)|R_1 = \tau] + U_1(t - \tau). \tag{2.17}$$

In the above equation, the renewal argument is used, that is

$$E[C(\tau, t)|R_1 = \tau] = U_1(t - \tau).$$

That is because, given the first renewal point $R_1 = \tau$, $C(\tau, t)$ is stochastically the same as $C(0, \tau) = C(\tau)$. Substituting (2.17) into (2.16), we get

$$U_1(t) = (U_1 * f_{R_1})(t) + G_1(t),$$

where

$$G_1(t) = \sum_{0<\tau\leq t} E[C(\tau)|R_1 = \tau] f_{R_1}(\tau) + E[C(t)|R_1 > t]\bar{F}_{R_1}(t). \tag{2.18}$$

Let $U_2(t)$ be the second order moment of $C(t)$, that is, the expected square cost at time t

$$U_2(t) = E[C(t)^2].$$

Similar to (2.16), $E[C(t)^2]$ can be written as

$$U_2(t) = \sum_{0<\tau\leq t} E[C(t)^2|R_1 = \tau] f_{R_1}(\tau) + E[C(t)^2|R_1 > t]\bar{F}_{R_1}(t), \tag{2.19}$$

when $R_1 = \tau < t$, we split $C(t)$ into $C(\tau) + C(\tau, t)$, allowing to write the first expectation term in the right hand side of (2.19) as

$$E[C(t)^2|R_1 = \tau] = E[C(\tau)^2|R_1 = \tau] + 2E[C(\tau)C(\tau,t)|R_1 = \tau] + E[C(\tau,t)^2|R_1 = \tau]. \tag{2.20}$$

Based on the renewal argument, the last two terms in (2.20) can be simplified as

$$E[C(\tau)C(\tau,t)|R_1 = \tau] = E[C(\tau)|R_1 = \tau]U_1(t-\tau), \tag{2.21}$$

and

$$E[C(\tau,t)^2|R_1 = \tau] = E[C(t-\tau)^2|R_1 = \tau] = U_2(t-\tau). \tag{2.22}$$

Substituting (2.20), (2.21), and (2.22) into (2.19), the following renewal equation is obtained

$$U_2(t) = (U_2 * f_{R_1})(t) + G_2(t),$$

where

$$\begin{aligned} G_2(t) = &\sum_{0<\tau\leq t} \Big(E[C(\tau)^2|R_1 = \tau]f_{R_1}(\tau) + 2E[C(\tau)|R_1 = \tau]f_{R_1}(\tau)U_1(t-\tau) \Big) \\ &+ E[C(t)^2|R_1 > t]\bar{F}_{R_1}(t). \end{aligned} \tag{2.23}$$

With the first and the second order moment of $C(t)$, we can get the variance $S(t)^2$ and the standard deviation $S(t)$ at time t as

$$S(t)^2 = U_2(t) - U_1(t)^2,$$

and

$$S(t) = \sqrt{S(t)^2} = \sqrt{U_2(t) - U_1(t)^2}.$$

We now generalise to the expression of the m-th order moment of $C(t)$. Let $U_m(t)$ be the m-th order moment of $C(t)$

$$U_m(t) = E[C(t)^m],$$

with initial condition $U_m(0) = 0$. For simplicity, we define

$$h_m(\tau) = E[C(\tau)^m|R_1 = \tau]f_{R_1}(\tau),$$

and

$$\bar{H}_m(t) = E[C(t)^m|R_1 > t]\bar{F}_{R_1}(t).$$

Then, $h_m(\tau)$ is the partition of $E[C(\tau)^m]$ over the set $\{R_1 = \tau\}$ and $\bar{H}_m(t)$ is that of $E[C(t)^m]$ over the set $\{R_1 > t\}$. Thus, we have

$$\sum_{\tau>0} h_m(\tau) = \sum_{\tau>0} E[C(\tau)^m|R_1 = \tau]f_{R_1}(\tau) = E[C(\tau)^m],$$

and

$$\lim_{t\to\infty} \bar{H}_m(t) = 0.$$

Using the above definitions, $G_1(t)$ given in (2.18) and $G_2(t)$ given in (2.23) can be simplified as

$$G_1(t) = \sum_{0<\tau\leq t} h_1(\tau) + \bar{H}_1(t),$$

and

$$G_2(t) = \sum_{0<\tau\leq t} h_2(\tau) + 2(h_1 * U_1)(t) + \bar{H}_2(t).$$

Similar to the derivation of $U_2(t)$, the renewal equation for the m-th moment of $C(t)$ is obtained as

$$U_m(t) = (U_m * f_{R_1})(t) + G_m(t),$$

where

$$G_m(t) = \sum_{0<\tau\leq t} h_m(\tau) + \sum_{j=1}^{m-1} \left(\begin{array}{c} m \\ j \end{array} \right) (h_j * U_{m-j})(t) + \bar{H}_m(t),$$

being

$$\left(\begin{array}{c} m \\ j \end{array} \right) = \frac{m!}{j!(m-j)!},$$

the binomial coefficient.

First and second order moments of $C(t)$ will be applied in Chapters 5 and 6 for obtaining the expressions of the expected cost in the finite life cycle and its associated standard deviation, and the results obtained shall be compared to the provided by the asymptotic expected cost rate.

2.6 Availability measures of the system

In many application fields, there is an increasing interest in analysing the performance assessment of a maintained system. For that, various availability measures are proposed to describe the behaviour of a maintained system and its properties [121]. An availability measure is defined as a quantifiable indicator which reports information related to the performance of a system or component.

Let $I(t)$ be a binary random variable given in (2.10). Hence, we can define probabilities of the form

$$P[I(t) = 1, \forall t \in \Phi], \tag{2.24}$$

where Φ is a set comprising a finite series of points or intervals of time.

We consider a system with two states (operating and repair) where the breakdown/reactivation of the system can be modelled using an alternating renewal process [122]. Under this assumption, we can derive a variety of useful formulae for predicting the availability of the system.

Let $U_1, U_2, \ldots$ and $V_1, V_2, \ldots$ be the failure and repair times, respectively, and let F and G be the length of the up and down period distributions, respectively. Based on alternating renewal process theory, let $J(t)$ be the convolution of F and G at time t. That is

$$J(t) = F * G(t) = \int_0^t F(t-u)dG(u). \tag{2.25}$$

The renewal counting function of the number of failures in $(0, t]$ is denoted by $N(0, t)$. Assuming that the system is up at time $t = 0$, let $H(t)$ be the number of the expected number of failures in $(0, t]$. Then

$$H(t) = E[N(0,t)] = \sum_{n=1}^{\infty} nP[N(0,t) = n],$$

where
$$P[N(0,t)=n]=J^{(n)}(t)-J^{(n+1)}(t), \quad \text{for } n\geq 0,$$
being $J^{(n)}(t)$ the n-th recursive convolution of $J(t)$ for $n \geq 0$ with $J^{(0)}(t)=1$.

If $\Phi = t_1$, by (2.24), we can define the point availability of the system at time t_1, that is, the probability that the system is up at time t_1 [123] as
$$A(t_1)=P[I(t_1)=1].$$
Assuming that the system is up at time $t=0$, $A(t_1)$ fulfils the following recursive equation
$$\begin{aligned}
A(t_1) &= P[U_1>t_1]+\sum_{n=1}^{\infty}P[V_n<t_1, N(0,t_1)=n]\\
&= \bar{F}(t_1)+\int_0^{t_1}\bar{F}(t_1-u)\sum_{n=1}^{\infty}P[N(0,t_1-u)=n]\,du\\
&= \bar{F}(t_1)+\int_0^{t_1}\sum_{n=1}^{\infty}P[I(t_1-u)=1, N(0,t_1-u)=n, N(t_1-u,t_1)=0]\,du\\
&= \bar{F}(t_1)+\int_0^{t_1}P[I(t_1-u)=1]dH(u)\,du\\
&= \bar{F}(t_1)+\int_0^{t_1}A(t_1-u)h(u)\,du,
\end{aligned}$$
where $N(t_1-u,t_1)$ is the number of failures in $(t_1-u,t_1]$ and $h(u)=dH(u)/du$.

Often, also of interest is the probability that the system is operating at a specified time and it continues operating for a specified time interval [124]. This is the case of the reliability of the system and the interval reliability of the system.

If $\Phi=(0,t_1]$, the reliability of the system in t_1, $R(t_1)$, is defined as the probability that the system is up in $(0,t_1]$ [125]. That is
$$R(t_1)=P\left[I(u)=1, u\in(0,t_1]\right].$$
Assuming that the system is up at time $t=0$, $R(t_1)$ fulfils the following equation
$$R(t_1)=P[U_1>t_1]=\bar{F}(t_1).$$

If $\Phi=(t_1,t_1+s]$, the interval reliability of the system in $(t_1,t_1+s]$, $IR(t_1,t_1+s)$, is defined as the probability that the system is up in a time interval $(t_1,t_1+s]$ [126]. That is
$$IR(t_1,t_1+s)=P\left[I(u)=1, u\in(t_1,t_1+s]\right].$$
Assuming that the system is up at time $t=t_1$, $IR(t_1,t_1+s)$ fulfils the following recursive equation
$$\begin{aligned}
IR(t_1,t_1+s) &= P[U_1>t_1+s]+\sum_{n=1}^{\infty}P[I(u)=1, u\in(t_1,t_1+s], N(0,t_1)=n]\\
=&\bar{F}(t_1+s)+\int_0^{t_1}\sum_{n=1}^{\infty}\Big(P[I(u-v)=1, u\in(t_1-v,t_1+s-v], N(0,t_1-v)=n,\\
&N(t_1+s-v,t_1+s)=0]\,dv\Big)\\
=&\bar{F}(t_1+s)+\int_0^{t_1}IR(t_1-v,t_1+s-v)h(v)\,dv.
\end{aligned}$$

Point availability and reliability of the system are particular cases of the *interval reliability* since

$$IR(t_1, t_1 + 0) = A(t_1),$$

and

$$IR(0, 0 + s) = R(s).$$

Two more complex availability measures named *"compound availability measures"* extend the previous ones. They are the joint availability and the joint interval reliability, respectively. The joint availability of the system in $\Phi = \{t_1, t_2, \ldots, t_n\}$, $JA(t_1, t_2, \ldots, t_n)$ is defined as the probability that the system is up at Φ, with $t_1, t_2, \ldots, t_n > 0$ [127]. The joint interval reliability of the system, $JIR(t_1, t_1 + h_1, t_2, t_2 + h_2, \ldots, t_n, t_n + h_n)$, is defined as the probability that the system is up in n disjoint time intervals [128]. This joint interval reliability is applied when there are periods in the life cycle where failure should be avoided with high probability.

Availability measures are of use in assessing the likelihood that the system is available at specified times. If the probability falls bellow a certain threshold, arrangements for securing an adequate back-up can be made. So that, reliability, interval reliability, and joint interval reliability of the system are specially important for systems which must be working when an emergency situation arises [129]. Thus, they are applied when there are periods in the life cycle where failure should be avoided with high probability.

In Chapters 5 and 6, some availability measures are analysed for DTS models subject to a CBM. Due to the complexity of the models, there is not hope to find an explicit solution for the availability measures but they are obtained in a recursive way using renewal equations.

CHAPTER

3

Independent DTS model with multiple degradation processes

This chapter[1] analyses a CBM for a system subject to different competing causes of failure: internal degradation and sudden shocks. Internal degradation is the result of the arrival at the system of multiple degradation processes that grow according to a gamma process. Sudden shocks arrive at the system at random times and provoke the failure of the system. A CBM model with periodic inspection times is developed for this competing failure model. Under this maintenance strategy, a preventive replacement is performed when the degradation level of a given degradation process in an inspection time exceeds a predetermined threshold. Under these assumptions, the analytical expression of the asymptotic expected cost rate is obtained. Numerical examples are provided to illustrate this complex maintenance model.

In short, the main aspects covered in this chapter are:

1. Analysing the asymptotic expected cost rate by implementing a CBM strategy in a DTS model.

2. Considering two causes of failure, internal degradation and sudden shocks.

3. Considering multiple degradation processes.

4. Optimising the asymptotic expected cost rate in a bivariate case.

This chapter is structured as follows. In Section 3.1 the general framework of the problem is described. Section 3.2 details the description of the maintenance strategy. Section 3.3 analyses the asymptotic expected cost rate. Section 3.4 shows numerical examples which illustrate this maintenance model and Section 3.5 presents the conclusions and shows further possible extensions.

[1]This chapter is based on the works by Castro *et al.* [61] and Caballé *et al.* [130].

3.1 Framework of the model

This section deals with the general framework of the model and its mathematical formulation.

3.1.1 General assumptions

The assumptions of the model are the following:

1. The system is subject to different internal degradation processes initiated at random times, according to an NHPP $\{N_d(t), t \geq 0\}$ with intensity $m(t)$ and cumulative intensity

$$M(t) = \int_0^t m(u)du, \quad t \geq 0,$$

   where t is the age of the system and $m(t)$ is a non-decreasing function in t. The sequential initiation points of the degradation processes are denoted by $0 \leq T_1 \leq T_2 \leq \ldots$, where T_i is the initiation point of the i-th degradation process.

2. Degradation processes grow according to gamma processes. Let $X_i(t)$ be the deterioration level of the i-th degradation process t time units after its initiation, for $i = 1, 2, \ldots$ We assume $\{X_i(t),\, t \geq 0\}$ follows a homogeneous gamma process with parameters αt and β ($\alpha, \beta > 0$) and density function given in (2.3). The system is said to fail due to degradation when the deterioration level of a degradation process exceeds a threshold L.

3. The system not only fails due to internal degradation. Sudden shocks arrive at the system according to an NHPP $\{N_s(t), t \geq 0\}$ with intensity $\lambda(t)$ and cumulative intensity

$$\Lambda(t) = \int_0^t \lambda(u)du, \quad t \geq 0, \tag{3.1}$$

   where t is the age of the system and $\lambda(t)$ is a non-decreasing function in t. Sudden shocks provoke the system failure.

4. We assume both causes of failure are independent.

5. The system is inspected each T time units ($t.u.$). In these instants, it is checked if the system is working or is down. If the system is down at the inspection time, a CM is performed and the system is replaced by a new one and with the same characteristics of the previous one. A CM event simulation is shown in Figure 3.1. On the other hand, if the system is working at the inspection time, the deterioration level of each degradation process is checked. Let M be the deterioration level from which the system is considered as too worn ($M < L$). If the deterioration level of a given degradation process exceeds the preventive threshold M, a PM is performed and the system is replaced by a new one and with the same characteristics of the previous one. A PM event simulation is shown in Figure 3.2. Otherwise, no maintenance task is performed. We suppose that the time necessary to perform a maintenance action is negligible.

6. All maintenance actions imply a cost. The cost of a CM is C_c monetary units ($m.u.$). The cost of a PM is C_p $m.u.$ Each inspection performed implies a cost C_I $m.u.$ In addition, if the system fails, the system is down until the next inspection. Each time unit that the system is down, a cost C_d $m.u./t.u.$ is incurred. We assume that $C_c > C_p > C_I$.

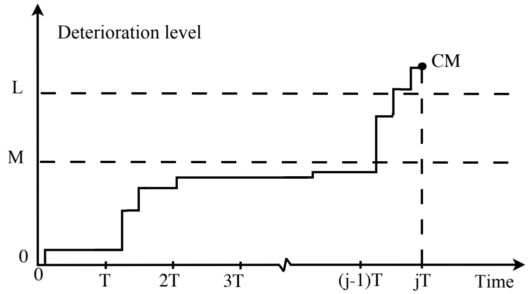


Figure 3.1: A corrective maintenance event.

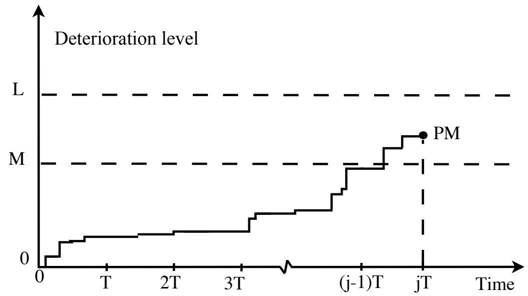


Figure 3.2: A preventive maintenance event.

3.1.2 Distribution of counting processes

Since we assume the degradation processes arrive at the system following a random counting process, a stochastic model is developed to find the distribution of the time to attain a deterioration level M from $t = 0$ resulting from the combination of the initiation process and growth process. Let S_i be the random variable denoting the time from the origin where the deterioration level of the i-th degradation process exceeds the threshold M. That is

$$S_i = T_i + \sigma_M, \quad i = 1, 2, \ldots$$

where σ_M is the random variable describing the time up to a degradation process exceeds the preventive threshold M.

Let W_i be the random variable denoting the time from the origin where the deterioration level of the i-th degradation process exceeds the failure threshold L. That is

$$W_i = T_i + \sigma_L, \quad i = 1, 2, \ldots$$

where σ_L is the random variable describing the time up to a degradation process exceeds the breakdown threshold L. Figure 3.3 shows a degradation process realisation.

Next, we are interested in the number of degradation processes whose deterioration levels exceed M at time t. Let $N_M(t)$ be the number of degradation processes exceeding the preventive threshold M in the time interval $(0, t]$, that is,

$$N_M(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{S_i \leq t\}}. \tag{3.2}$$

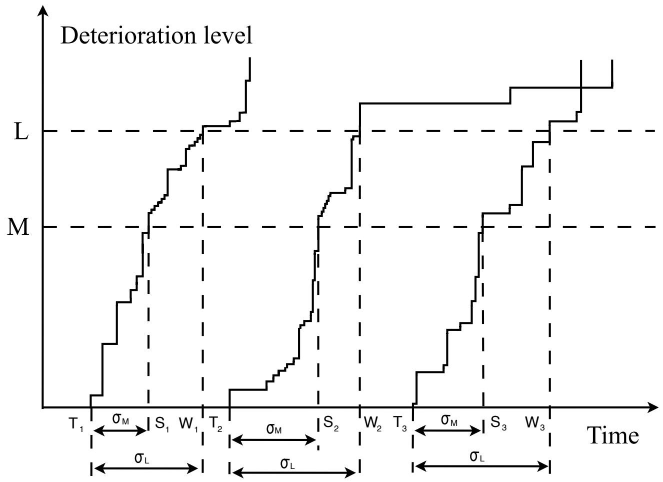


Figure 3.3: Realization of the degradation processes.

Considering a result of [36] and [131], we can obtain the distribution of the counting process from the combination of the initiation process and growth process and, consequently, the distribution of the counting process given in (3.2).

Lemma 3.1. *Let $\{T_n,\, n \geq 1\}$ be an NHPP with intensity function $\lambda(t)$ and let $\{U_n,\, n \geq 1\}$ be a sequence of independent and identical distributed random variables with density function f and let T'_n be the random variable defined as*

$$T'_n = T_n + U_n, \quad n = 1, 2, \dots$$

Let $\{N'(t),\, t \geq 0\}$ be the counting process associated with the random variables T'_n, that is

$$N'(t) = \sum_{n=1}^{\infty} \mathbf{1}_{\{T'_n \leq t\}}, \quad t \geq 0.$$

Then $\{N'(t),\, t \geq 0\}$ is an NHPP with intensity $\lambda * f$.

Proof. It is provided by [36] (Section 4.1). □

Considering Lemma 3.1, we shall obtain the distribution of the counting process $\{N_M(t),\, t \geq 0\}$ given in (3.2).

Corollary 3.1. *Under the assumptions of the model, $\{N_M(t), t \geq 0\}$ is an NHPP with intensity*

$$m_M(t) = (m * f_{\sigma_M})(t) = \int_0^t m(v) F_{\sigma_M}(t-v)dv, \tag{3.3}$$

where $m(t)$ denotes the intensity of the counting process $\{N_d(t), t \geq 0\}$ and f_{σ_M} the density of σ_M.

The setting of this problem can be considered as a particular case of the framework analysed by Cha and Filkenstein in [132], where a system subject to different events that arrive according to an NHPP with rate $\nu(t)$ was considered. Each event triggers an effective event after random time D_i for $i = 1, 2, \ldots$ with distribution function $\lambda(t)$. Denoting by T_e the time to the first effective event, the survival function of T_e at time t is

$$P(T_e \geq t) = \exp\left\{-\int_0^t \nu(u)\lambda(t-u)\, du\right\}.$$

More details about the general case of the effective model are provided by [132–134].

Let $S_{[1]}$ be the instant at which, for the first time, the deterioration level of a given degradation process exceeds the preventive threshold M,

$$S_{[1]} = \min\{S_i,\, i = 1, 2, \ldots\}$$

Using Corollary 3.1, the distribution function of $S_{[1]}$ is obtained in the following result.

Lemma 3.2. *The survival function of $S_{[1]}$ is*

$$\bar{F}_{S_{[1]}}(t) = \exp\left\{-\int_0^t m(v)F_{\sigma_M}(t-v)dv\right\}, \quad t \geq 0, \tag{3.4}$$

where $m(t)$ denotes the arrival intensity of the degradation processes and F_{σ_M} is given in (2.4).

Proof. By Corollary 3.1, the counting process $\{N_M(t), t \geq 0\}$ follows an NHPP with intensity

$$m_M(t) = (m * f_{\sigma_M})(t) = \int_0^t m(u)f_{\sigma_M}(t-u)du, \quad t \geq 0.$$

Therefore,

$$\begin{aligned}\bar{F}_{S_{[1]}}(t) &= P[N_M(t) = 0] \\ &= \exp\left\{-\int_0^t m_M(u)du\right\} \\ &= \exp\left\{-\int_0^t du \int_0^u m(v)f_{\sigma_M}(u-v)dv\right\} \\ &= \exp\left\{-\int_0^t m(v)F_{\sigma_M}(t-v)dv\right\},\end{aligned}$$

and the result holds. □

Let $W_{[1]}$ be the instant at which, for the first time, the deterioration level of a given degradation process exceeds the breakdown threshold L. That is, $W_{[1]}$ denotes the time to a degradation failure where

$$W_{[1]} = \min\{W_i,\, i = 1, 2, \ldots\}.$$

Analogously to $\bar{F}_{S_{[1]}}(t)$, the survival function of $W_{[1]}$ is obtained by Lemma 3.2 as

$$\bar{F}_{W_{[1]}}(t) = \exp\left\{-\int_0^t m(v)F_{\sigma_L}(t-v)dv\right\}, \quad t \geq 0. \tag{3.5}$$

Considering Corollary 3.1, the failure rate function of the time to the degradation failure is

$$m_L(t) = \int_0^t m(u)f_{\sigma_L}(t-u)du, \quad t \geq 0.$$

3.1.3 Time to the system failure

Let Z be the time to a system failure. Since the failure of the system is provoked by the competing causes of degradation and shocks we have

$$P[Z > t] = P[\min\{W_{[1]}, Y\} > t],$$

being $W_{[1]}$ the time to a degradation failure and Y the time to a sudden shock. Since the shock process follows an NHPP, the survival function of Y is

$$\bar{F}_Y(t) = \exp\{-\Lambda(t)\}, \quad t \geq 0, \tag{3.6}$$

where $\Lambda(t)$ is given in (3.1) and under the independence assumption of degradation and shocks,

$$P[Z > t] = \bar{F}_{W_{[1]}}(t)\bar{F}_Y(t), \quad t \geq 0, \tag{3.7}$$

where $\bar{F}_{W_{[1]}}$ denotes the survival function of $W_{[1]}$ given in (3.5) and $\bar{F}_Y$ is given in (3.6).

Using (3.7), the failure rate of the system at time t is

$$r(t) = m_L(t) + \lambda(t), \quad t \geq 0, \tag{3.8}$$

where $\lambda(t)$ corresponds to the shocks intensity and $m_L(t)$ denotes the failure rate function of the time to a degradation failure. Since $\lambda(t)$ and $m_L(t)$ are non-decreasing in t, the failure rate function $r(t)$ given in (3.8) is also non-decreasing in t.

In the remainder of this chapter, we shall use the survival probability of the system at time t conditioned to the number of degradation processes that exceed the deterioration level M at time t. Let $(S_{[1]}, S_{[2]}, \ldots S_{[n]})$ be the random vector that represents the first n arrival times from the NHPP $N_M(t)$ given in (3.2). By [36], the joint probability density of $(S_{[1]}, S_{[2]}, \ldots, S_{[n]}, N_M(t) = n)$ is

$$f_{S_{[1]},S_{[2]},\ldots,S_{[n]},N_M(t)=n}(s_1, s_2, \ldots, s_n) = \exp\left[-\int_0^t m_M(u)du\right]\prod_{i=1}^{n} m_M(s_i), \tag{3.9}$$

for $0 < s_1 < s_2 < \ldots < s_n < t$, where m_M is given in (3.3). Using (3.9), we have that $P[W_{[1]} > t, N_M(t) = n]$ is equal to

$$\exp\left\{-\int_0^t m_M(u)du\right\}\int_0^t\int_{u_1}^t \cdots \int_{u_{n-1}}^t \prod_{i=1}^{n} m_M(u_i)\bar{F}_{\sigma_L-\sigma_M}(t-u_i)du_i. \tag{3.10}$$

Before starting with the maintenance strategy for this model, we shall give a result related to the derivative of the second term of (3.10) that will allow us to reduce the expression for the probabilities of the different maintenance actions.

Lemma 3.3. *For $u_1 > 0$ fixed, let $Q_{u_1,i}(t)$ be the function*

$$Q_{u_1,i}(t) = \int_{u_1}^t q_t(u_2)du_2 \int_{u_2}^t q_t(u_3)du_3 \ldots \int_{u_{i-1}}^t q_t(u_i)du_i \int_{u_i}^t q_t(u_{i+1})du_{i+1}, \quad t \geq 0, \tag{3.11}$$

for $i = 2, 3, \ldots$, and where

$$q_t(u) = m_M(u)\bar{F}_{\sigma_L-\sigma_M}(t-u), \quad u \leq t.$$

Then, $Q'_{u_1,i}(t) = Q'_{u_1,1}(t)Q_{u_1,i-1}(t)$ for $i = 2, 3, \ldots$ where

$$Q_{u_1,1}(t) = \int_{u_1}^t q_t(u_2)du_2, \quad t \geq u_1.$$

Proof. In a general framework, we consider $Q_{u_1,i}(t)$ given in (3.11) where $q_t(u) \geq 0$, $\forall t$ and $\forall u$. We assume that $\frac{\partial Q_{u_1,i}(t)}{\partial t} = Q'_{u_1,i}(t)$ and $\frac{\partial q_t(u)}{\partial t} = q'_t(u)$. Then

$$\begin{aligned}
Q'_{u_1,i}(t) = &\int_{u_1}^{t} q'_t(u_2)du_2 \int_{u_2}^{t} q_t(u_3)du_3 \ldots \int_{u_{i-1}}^{t} q_t(u_i)du_i \int_{u_i}^{t} q_t(u_{i+1})du_{i+1} \\
&+ \int_{u_1}^{t} q_t(u_2)du_2 \int_{u_2}^{t} q'_t(u_3)du_3 \ldots \int_{u_{i-1}}^{t} q_t(u_i)du_i \int_{u_i}^{t} q_t(u_{i+1})du_{i+1} \\
&\vdots \\
&+ \int_{u_1}^{t} q_t(u_2)du_2 \int_{u_2}^{t} q_t(u_3)du_2 \ldots \int_{u_{i-1}}^{t} q'_t(u_i)du_i \int_{u_i}^{t} q_t(u_{i+1})du_{i+1} \\
&+ \int_{u_1}^{t} q_t(u_2)du_2 \int_{u_2}^{t} q_t(u_3)du_3 \ldots \int_{u_{i-1}}^{t} q_t(u_i)du_i \left(q_t(t) + \int_{u_i}^{t} q'_t(u_{i+1})du_{i+1} \right),
\end{aligned}$$

for $i = 2, 3, \ldots$ Notice that

$$\begin{aligned}
&\int_{u_1}^{t} q'_t(u_2)du_2 \int_{u_2}^{t} q_t(u_3)du_3 \ldots \int_{u_{i-1}}^{t} q_t(u_i)du_i \int_{u_i}^{t} q_t(u_{i+1})du_{i+1} \\
= &\int_{u_1}^{t} q_t(u_2)du_1 \int_{u_2}^{t} q_t(u_3)du_3 \ldots \int_{u_{i-1}}^{t} q_t(u_i)du_i \int_{u_1}^{u_2} q'_t(u_{i+1})du_{i+1},
\end{aligned}$$

and for $j = 3, 4, \ldots, i$

$$\begin{aligned}
&\int_{u_1}^{t} q_t(u_2)du_2 \int_{u_2}^{t} q_t(u_3)du_3 \ldots \int_{u_{j-1}}^{t} q'_t(u_j)du_j \ldots \int_{u_{i-1}}^{t} q_t(u_i)du_i \int_{u_i}^{t} q_t(u_{i+1})du_{i+1} \\
= &\int_{u_1}^{t} q_t(u_2)du_2 \int_{u_2}^{t} q_t(u_3)du_3 \ldots \int_{u_{i-1}}^{t} q_t(u_i)du_i \int_{u_{j-1}}^{u_j} q'_t(u_{i+1})du_{i+1}.
\end{aligned}$$

Hence,

$$Q'_{u_1,i}(t) = \int_{u_1}^{t} q_t(u_2)du_2 \int_{u_2}^{t} q_t(u_3)du_3 \ldots \int_{u_{i-1}}^{t} q_t(u_i)du_i \left(q_t(t) + \int_{u_1}^{t} q'_t(u_{i+1})du_{i+1} \right),$$

and therefore

$$Q'_{u_1,i}(t) = Q'_{u_1,1}(t) Q_{u_1,i-1}(t), \quad i = 2, 3, \ldots$$

and the result holds. □

3.2 Condition-based maintenance

In this chapter, a CBM with periodic inspections is analysed for a system subject to multiple degradation processes and sudden shocks, where both causes of failure are considered independent. The system is inspected each T units of time to measure the deterioration levels of the degradation processes evolving simultaneously in the system. A PM is performed when the deterioration level of a given degradation process exceeds a preventive threshold M in an inspection time. A CM is performed when the system is down in an inspection time.

3.2.1 Maintenance action probability

Let $D_1, D_2, \ldots$ be the time between successive replacements of the system. By Assumption 5 (Subsection 3.1.1), after each maintenance task, the system is replaced by another with the same characteristics. Then, the new process will be an independent and identical replica of the previous one. By theory about renewal processes detailed in Subsection 2.2.2, $D_1, D_2, \ldots, D_n$ are $i.i.d.$ random variables. Let $R_1 = D_1$ be the time to the first maintenance action, *i.e.*, the time to the first CM or to the first PM. The random variable R_1 is

$$R_1 = kT\left[\mathbf{1}_{\{(k-1)T<S_{[1]}\leq kT, kT<\min\{W_{[1]},Y\}\}} + \mathbf{1}_{\{(k-1)T<S_{[1]},(k-1)T<\min\{W_{[1]},Y\}\leq kT\}}\right],$$

for $T > 0$, where $W_{[1]}$ is the time to a a degradation failure and Y is the time to a sudden shock. Let $P^M_{R_1}(kT)$ be the probability of a maintenance action at the k-th inspection for a time between inspections T and preventive threshold M, with $k = 1, 2, \ldots$ Then

$$P^M_{R_1}(kT) = P^M_{R_1,p}(kT) + P^M_{R_1,c}(kT) \quad k = 1, 2, \ldots \tag{3.12}$$

where $P^M_{R_1,p}(kT)$ and $P^M_{R_1,c}(kT)$ denote the probability of a preventive and corrective maintenance action at time kT, respectively.

3.2.2 Preventive maintenance probability

A PM action is performed at time kT for $k = 1, 2, \ldots$, when the system is working at time kT and the preventive threshold M is exceeded for the first time in $((k-1)T, kT]$. That is, a preventive replacement is performed at time kT when the following event occurs

$$\{(k-1)T < S_{[1]} \leq kT, \quad \min\{W_{[1]}, Y\} < kT\}.$$

Hence, the PM probability at time kT for $k = 1, 2, \ldots$ is

$$P^M_{R_1,p}(kT) = P[(k-1)T < S_{[1]} \leq kT < W_{[1]}, \quad Y > kT].$$

Due to the independence of degradation processes and shock processes, we have

$$P^M_{R_1,p}(kT) = P[(k-1)T < S_{[1]} \leq kT < W_{[1]}]P[Y > kT],$$

and, since the shock process follows an NHPP, then

$$P[Y > kT] = \exp\left\{-\int_0^{kT} \lambda(u)du\right\} = \exp\{-\Lambda(kT)\}.$$

Let $P^M_{R_1,p,1}(kT)$ be the PM probability in absence of shocks given by

$$P^M_{R_1,p,1}(kT) = P[(k-1)T < S_{[1]} \leq kT < W_{[1]}].$$

By the law of the total probability, we get

$$\begin{aligned}
P^M_{R_1,p,1}(kT) =& \sum_{i=1}^{\infty} P[(k-1)T < S_{[1]} \leq kT < W_{[1]}, \quad N_M(kT) = i] \\
=& P[(k-1)T < S_{[1]} \leq kT < W_{[1]}, \quad N_M(kT) = 1] \\
&+ \sum_{i=2}^{\infty} P[(k-1)T < S_{[1]} \leq kT < W_{[1]}, \quad N_M(kT) = i].
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&P[(k-1)T < S_{[1]} \leq kT < W_{[1]}, \quad N_M(kT) = 1] \\
&= \int_{(k-1)T}^{kT} P[(k-1)T < S_{[1]} \leq kT < W_{[1]} | S_{[1]} = u_1] P[N_M(u_1, kT) = 0] du_1 \\
&= \int_{(k-1)T}^{kT} A_{u_1}(kT) \exp\left\{-\int_{u_1}^{kT} m_M(z)dz\right\} du_1,
\end{aligned}
$$

where

$$
A_{u_1}(kT) = f_{S_{[1]}}(u_1)\bar{F}_{\sigma_L - \sigma_M}(kT - u_1), \tag{3.13}
$$

that represents the probability the preventive threshold M is exceeded for the first time at u_1 for a given degradation process whose deterioration level at kT is less than L.

On the other hand, by (3.10)

$$
\begin{aligned}
&P[(k-1)T < S_{[1]} \leq kT < W_{[1]}, \quad N_M(kT) = i] \\
&= \int_{(k-1)T}^{kT} f_{S_{[1]}}(u_1)\bar{F}_{\sigma_L - \sigma_M}(kT - u_1) P\left[N_M(u_1, kT) = i-1, W_{[1]} > kT | S_{[1]} = u_1\right] \\
&= \int_{(k-1)T}^{kT} A_{u_1}(kT) \exp\left\{-\int_{u_1}^{kT} m_M(z)dz\right\} Q_{u_1, i-1}(kT) du_1,
\end{aligned}
$$

where $Q_{u_1,i}$ is given in (3.11). Hence

$$
P^M_{R_1,p,1}(kT) = \int_{(k-1)T}^{kT} A_{u_1}(kT) Z_{u_1}(kT) du_1,
$$

where $Z_{u_1}(kT)$ is

$$
Z_{u_1}(kT) = \exp\left\{-\int_{u_1}^{kT} m_M(z)dz\right\} \left(1 + \sum_{i=1}^{\infty} Q_{u_1,i}(kT)\right), \tag{3.14}
$$

and represents the probability that the failure threshold L is not exceeded at kT for degradation processes whose deterioration levels exceed M after u_1.

We shall obtain a compact expression of (3.14) using the following result.

Lemma 3.4. *The function*

$$
Z_{u_1}(t) = \exp\left\{-\int_{u_1}^{t} m_M(z)dz\right\} \left(1 + \sum_{i=1}^{\infty} Q_{u_1,i}(t)\right), \quad u_1 \leq t,
$$

can be expressed as

$$
Z_{u_1}(t) = \exp\left\{-\int_{u_1}^{t} m_M(x) F_{\sigma_L - \sigma_M}(t - x)dx\right\}, \tag{3.15}
$$

where $\bar{F}_{\sigma_L - \sigma_M}$ is given in (2.6) and m_M is given in (3.3).

Proof. We have

$$Z'_{u_1}(t) = -Z_{u_1}(t)m_M(t) + \exp\left\{-\int_{u_1}^{t} m_M(z)dz\right\}\left(Q'_{u_1,1}(t) + \sum_{i=2}^{\infty} Q'_{u_1,i}(t)\right).$$

By Lemma 3.3, $Q'_{u_1,i}(t) = Q'_{u_1,1}(t)Q_{u_1,i-1}(t)$, for $i = 2, 3, \ldots$ Then

$$\begin{aligned}
Z'_{u_1}(t) =& - Z_{u_1}(t)m_M(t) + \exp\left\{-\int_{u_1}^{t} m_M(z)dz\right\}\left(Q'_{u_1,1}(t) + \sum_{i=2}^{\infty} Q'_{u_1,1}(t)Q_{u_1,i-1}(t)\right)\\
=& - Z_{u_1}(t)m_M(t) + \exp\left\{-\int_{u_1}^{t} m_M(z)dz\right\}\left(Q'_{u_1,1}(t)\left[1 + \sum_{i=2}^{\infty} Q_{u_1,i-1}(t)\right]\right)\\
=& - Z_{u_1}(t)m_M(t) + Z_{u_1}(t)Q'_{u_1,1}(t)\\
=& Z_{u_1}(t)\left(Q'_{u_1,1}(t) - m_M(t)\right).
\end{aligned}$$

Using that

$$Q'_{u_1,1}(t) = m_M(t) - \int_{u_1}^{t} m_M(v)f_{\sigma_L-\sigma_M}(t-v)dv,$$

we have

$$\begin{aligned}
Z'_{u_1,1}(t) =& Z_{u_1}(t)\left(-\int_{u_1}^{t} m_M(v)f_{\sigma_L-\sigma_M}(t-v)dv\right)\\
=& Z_{u_1}(t)g_{u_1}(t).
\end{aligned} \tag{3.16}$$

On the other hand, the function

$$Z_{u_1}(t) = C\exp\left\{\int_{u_1}^{t} g_{u_1}(v)dv\right\},$$

where $C = Z_{u_1}(u_1)$ fulfils the differential equation given in (3.16). Finally, since $C = 1$ by (3.14), we obtain that

$$\begin{aligned}
Z_{u_1}(t) =& \exp\left\{-\int_{u_1}^{t} dv \int_{u_1}^{v} m_M(x)f_{\sigma_L-\sigma_M}(v-x)dx\right\}\\
=& \exp\left\{-\int_{u_1}^{t} m_M(x)F_{\sigma_L-\sigma_M}(t-x)dx\right\},
\end{aligned}$$

and the result holds. □

Using Lemma 3.4, the PM probability at time kT is reduced to

$$P^M_{R_1,p}(kT) = \exp\{-\Lambda(kT)\}\int_{(k-1)T}^{kT} A_{u_1}(kT)Z_{u_1}(kT)du_1, \tag{3.17}$$

where $\Lambda(t)$ is given in (3.1), and

$$A_{u_1}(kT)Z_{u_1}(kT),$$

represents the probability that the system survives to a degradation failure at time kT if $S_{[1]} = u_1$ for $(k-1)T < u_1 \leq kT$, where $A_{u_1}(kT)$, and $Z_{u_1}(kT)$ are given in (3.13) and (3.15), respectively.

3.2.3 Corrective maintenance probability

A CM action is performed at time kT for $k = 1, 2, \ldots$ when the system fails in $((k-1)T, kT]$ and the preventive threshold M is exceeded for the first time after $(k-1)T$. That is, a CM action is performed at time kT when one of the following mutually exclusive events occurs

$$\{(k-1)T < S_{[1]} < Y \leq kT,\ Y < W_{[1]}\}, \quad \{(k-1)T < Y \leq kT,\ Y < S_{[1]}\},$$
$$\{(k-1)T < S_{[1]} < W_{[1]} \leq kT,\, W_{[1]} < Y\}.$$

Let $P^M_{R_1,c}(kT)$ be the CM probability at time kT for $k = 1, 2, \ldots$ Therefore,

$$\begin{aligned} P^M_{R_1,c}(kT) =& P^M_{R_1,c,1}(kT) + P^M_{R_1,c,2}(kT) + P^M_{R_1,c,3}(kT) \\ =& P[(k-1)T < S_{[1]} < Y \leq kT,\, Y < W_{[1]}] \\ &+ P[(k-1)T < Y \leq kT,\, Y < S_{[1]}] \\ &+ P[(k-1)T < S_{[1]} < W_{[1]} \leq kT,\, W_{[1]} < Y]. \end{aligned}$$

Firstly,

$$\begin{aligned} P^M_{R_1,c,1}(kT) =& P\left[(k-1)T < S_{[1]} < Y \leq kT,\, Y < W_{[1]}\right] \\ =& \int_{(k-1)T}^{kT} f_{S_{[1]}}(u_1) P\left[u_1 < Y \leq kT,\, Y < W_{[1]} | S_{[1]} = u_1\right] du_1 \\ =& \int_{(k-1)T}^{kT} du_1 \int_{u_1}^{kT} A_{u_1}(v) Z_{u_1}(v) f_Y(v) dv, \end{aligned} \tag{3.18}$$

since $A_{u_1}(v)Z_{u_1}(v)$ represents the probability that the system survives to a degradation failure at time v ($v > u_1$) if $S_{[1]} = u_1$. On the other hand,

$$\begin{aligned} P^M_{R_1,c,2}((k-1)T) =& P[(k-1)T < Y \leq kT,\, Y < S_{[1]}] \\ =& \int_{(k-1)T}^{kT} f_Y(v) \bar{F}_{S_{[1]}}(v) dv. \end{aligned} \tag{3.19}$$

Finally,

$$\begin{aligned} P^M_{R_1,c,3}(kT) =& P[(k-1)T < S_{[1]} < W_{[1]} \leq kT, W_{[1]} < Y] \\ =& \int_{(k-1)T}^{kT} f_{S_{[1]}}(u_1) P\left[u_1 < W_{[1]} \leq kT,\, W_{[1]} < Y | S_{[1]} = u_1\right]. \end{aligned}$$

Notice that, by (3.18), the probability that the system survives to a degradation failure at time v ($v > u_1$) conditioned to $S_{[1]} = u_1$ is given in $A_{u_1}(v)Z_{u_1}(v)$. Hence

$$P^M_{R_1,c,3}(kT) = \int_{(k-1)T}^{kT} du_1 \int_{u_1}^{kT} \frac{\partial}{\partial v}\left[A(u_1, v) Z_{u_1}(v)\right] \bar{F}_Y(v) dv. \tag{3.20}$$

Then, the CM probability at time kT is

$$\begin{aligned} P^M_{R_1,c}(kT) =& \int_{(k-1)T}^{kT} f_Y(v) \bar{F}_{S_{[1]}}(v) dv \\ &+ \int_{(k-1)T}^{kT} du_1 \int_{u_1}^{kT} -\frac{\partial}{\partial v}\left[A_{u_1}(v) Z_{u_1}(v) \bar{F}_Y(v)\right] dv. \end{aligned} \tag{3.21}$$

3.2.4 Expected downtime

Let $W_T^M((k-1)T, kT)$ be the downtime of the system in the interval $((k-1)T, kT]$ for a time between inspections T and preventive threshold M. That is, $W_T^M((k-1)T, kT)$ is equal to

$$\begin{cases} kT - Y & \text{if} \quad (k-1)T < Y \leq kT, \quad Y < W_{[1]}, \quad (k-1)T < S_{[1]} \\ kT - W_{[1]} & \text{if} \quad (k-1)T < S_{[1]} < W_{[1]} \leq kT, \quad W_{[1]} < Y \end{cases},$$

for $T > 0$. Then

$$\begin{aligned} E[W_T^M((k-1)T, kT)] =& E\left[(kT-Y)\mathbf{1}_{\{(k-1)T<S_{[1]}<Y<kT,\, Y<W_{[1]}\}}\right] \\ &+E\left[(kT-Y)\mathbf{1}_{\{(k-1)T<Y<kT,\, Y<S_{[1]}\}}\right] \\ &+E\left[(kT-W_{[1]})\mathbf{1}_{\{(k-1)T<S_{[1]}<W_{[1]}<kT,\, W_{[1]}<Y\}}\right]. \end{aligned}$$

Based on calculations of the CM probability shown in Subsection 3.2.3, the expected downtime in $((k-1), kT]$ is

$$\begin{aligned} E[W_d^M((k-1)T, kT)] = & \int_{(k-1)T}^{kT} f_Y(u_1)\bar{F}_{S_{[1]}}(u_1)\,(kT-u_1)\,du_1 \\ & + \int_{(k-1)T}^{kT} du_1 \int_{u_1}^{kT} -\frac{\partial}{\partial v}\left[A_{u_1}(v)Z_{u_1}(v)\bar{F}_Y(v)\right](kT-v)dv. \end{aligned} \tag{3.22}$$

3.3 Asymptotic expected cost rate

Let C^∞ be the asymptotic cost rate for a time between inspections T and preventive threshold M. In this chapter, we consider as objective cost function the asymptotic expected cost rate. Based on Theorem 2.1, $C^\infty(T, M)$ is

$$\begin{aligned} C^\infty(T,M) = & \frac{\sum_{k=1}^{\infty}\left[C_c P_{R_1,c}^M(kT) + C_d E\left[W_T^M((k-1)T, kT)\right]\right]}{\sum_{k=1}^{\infty} kT\, P_{R_1}^M(kT)} \\ & + \frac{\sum_{k=1}^{\infty}\left[C_p P_{R_1,p}^M(kT) + C_I k P_{R_1}^M(kT)\right]}{\sum_{k=1}^{\infty} kT\, P_{R_1}^M(kT)}, \end{aligned} \tag{3.23}$$

where $P_{R_1}^M(kT)$, $P_{R_1,p}^M(kT)$, $P_{R_1,c}^M(kT)$, and $E\left[W_T^M((k-1)T, kT)\right]$ are given in (3.12), (3.17), (3.21), and (3.22), respectively. An aim of this chapter is to assess an optimal maintenance strategy for this model. By optimal, we mean the values of T and M which minimise the objective function given in (3.23). Due to the analytical complexity of $C^\infty(T, M)$, the optimisation of the asymptotic expected cost rate $C^\infty(T, M)$ for a dataset is performed in the next section using numerical methods.

3.4 Numerical examples

In order to illustrate this maintenance model, we suppose a system subject to multiple degradation processes that arrive at the system according to an NHPP following a power-law process. The power-law process is a flexible model for an NHPP (see [97] for more details of the use and estimation of the power-law process). In this example, we assume the degradation processes arrive at the system following a power-law process with intensity

$$m(t) = abt^{b-1}, \qquad t \geq 0,$$

where $a = 0.01$ and $b = 11$. That is,

$$m(t) = 0.11t^{10}, \qquad t \geq 0.$$

The process of growth of the degradation processes is modelled according to an homogeneous gamma process with parameters $\alpha = \beta = 0.1$. A degradation failure is produced when the deterioration level of a degradation process exceeds $L = 20$.

The system is also subject to sudden shocks. Sudden shocks arrive at the system according to an NHPP with intensity

$$\lambda(t) = \frac{5}{3}\left(\frac{t}{30}\right)^{49}, \quad t \geq 0.$$

Under these specifications, the expected time to a degradation failure is 23.26 $t.u.$ and the expected time to a sudden shock is 29.68 $t.u.$

We assume the following sequence of costs: $C_d = 2\,m.u/t.u.$, $C_I = 5\,m.u$, $C_p = 400\,m.u$ and $C_c = 800\,m.u.$ R software, in its version 3.1.0, was used for the following examples. The code was run on an Intel Core i5-2500 processor with 8GB DDR3 RAM under Windows 7 Professional.

Considering the previous specifications, the optimisation problem for the asymptotic expected cost rate given in (3.23) is computed as follows:

1. A grid of size 20 is obtained by discretising the set $[11.20, 15]$ into 20 equally spaced points from 11.20 to 15 for T. Let T_i be the i-th value of the grid obtained previously, for $i = 1, 2, \ldots, 20$.

2. A grid of size 20 is obtained by discretising the set $[5.35, 12]$ into 20 equally spaced points from 5.35 to 12 for M. Let M_j be the j-th value of the grid obtained previously, for $j = 1, 2, \ldots, 20$.

3. For each combination (T_i, M_j) fixed, we obtain 50000 simulations of (R_1, I_1, W_d), where R_1 corresponds to the time to a replacement (corrective or preventive) in the first renewal cycle, I_1 the nature of the first maintenance action performed (corrective or preventive), and W_d the downtime up to the first maintenance action. With these simulations, and applying the Monte Carlo method, we obtain $\tilde{P}_{R_1}^{M_j}(kT_i)$, $\tilde{P}_{R_{1,p}}^{M_j}(kT_i)$, $\tilde{P}_{R_{1,c}}^{M_j}(kT_i)$, and $\tilde{E}\left[W_{T_i}^{M_j}((k-1)T_i, kT_i))\right]$ corresponding to the estimations of $P_{R_1}^{M_j}(kT_i))$, $P_{R_{1,p}}^{M_j}(kT_i))$, $P_{R_{1,c}}^{M_j}(kT_i))$, and $E\left[W_{T_i}^{M_j}((k-1)T_i, kT_i))\right]$ for $k = 1, 2, 3, \ldots$ given in Section 3.2 by (3.12), (3.17), (3.21), and (3.22), respectively. Figure 3.4 shows the procedure of the Monte Carlo simulation method for variable T and M.

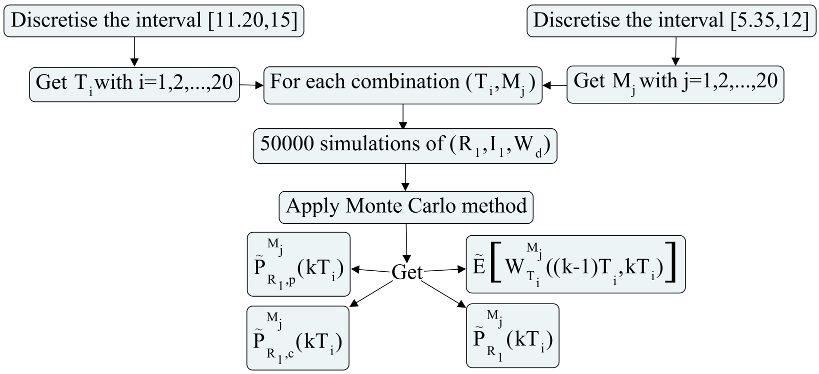


Figure 3.4: Procedure of the Monte Carlo simulation method for variable T and M.

4. Quantity $\tilde{C}^{\infty}(T, M)$, representing the asymptotic expected cost rate is calculated by using Equation (3.23), replacing the corresponding probabilities by their estimations calculated in Step 3.

5. The optimisation problem is reduced to find the values T_{opt} and M_{opt} which minimise the asymptotic expected cost rate $\tilde{C}^{\infty}(T, M)$. That is

$$\tilde{C}^{\infty}(T_{opt}, M_{opt}) = \min_{\substack{T>0 \\ 0<M\leq L}} \left\{\tilde{C}^{\infty}(T, M)\right\}.$$

Figure 3.5 shows the asymptotic expected cost rate versus T and M. The convexity of the cost surface shows the existence of an optimal setting of parameters T and M. The values of T and M which minimise the asymptotic expected cost rate are reached at $T_{opt} = 14$ $t.u.$ and $M_{opt} = 11.65$ degradation units ($d.u.$) with an asymptotic expected cost rate of 25.9011 $m.u./t.u.$

3.5 Conclusions and further extensions

In this chapter, a CBM strategy in a DTS model for a deteriorating system with competing causes of failure, internal degradation and sudden shocks, is analysed. This system is subject to multiple degradation processes arriving at the system following an NHPP and whose growth is modelled under a gamma process. In addition, the system is also subject to sudden shocks arriving at the system following an NHPP. Under these assumptions, a closed-form expression for the survival time of the system is provided. Also the analytical formulas for the different probabilities involved in the maintenance strategy are obtained. Additionally, the asymptotic expected cost rate as objective function is analysed and optimised. To this end, the numerical search of the optimal maintenance strategy is calculated throughout a procedure based on strictly the Monte Carlo simulation.

In this chapter, the maintenance strategy is studied assuming that the competing causes of failure are independent. A possible further extension of this chapter is the maintenance strategy analysis assuming dependent causes of failure. This dependence can be analysed considering that the intensity of the sudden shocks would increase with the deterioration levels of the degradation processes. This extension will be studied in Chapter 4.

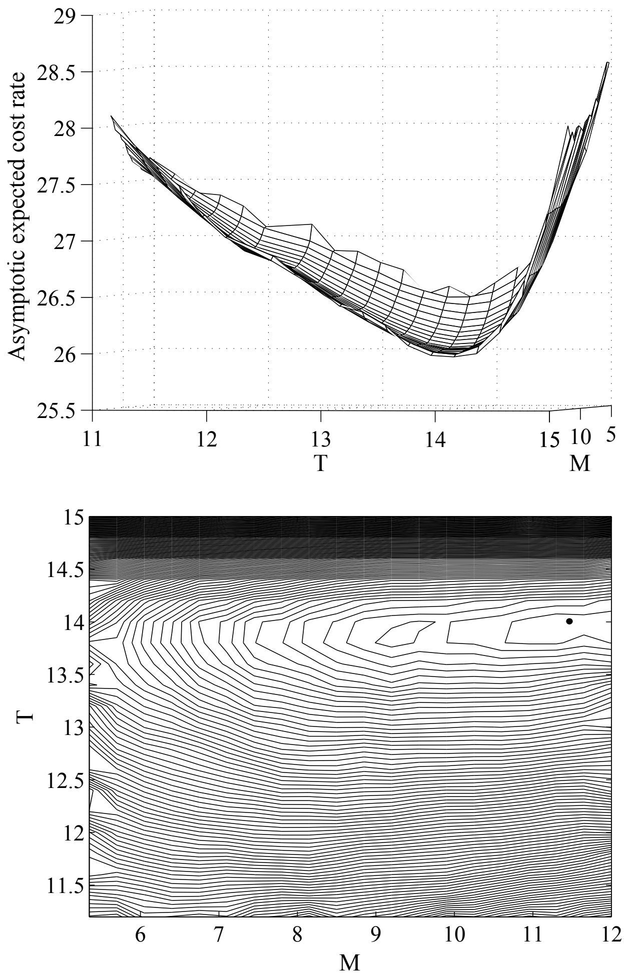


Figure 3.5: Mesh and contour plots for the asymptotic expected cost rate.

CHAPTER

4

Dependent DTS model with multiple degradation processes

This chapter[1] deals with a CBM strategy for a system subject to two causes of failure: internal degradation and sudden shocks. This chapter extends the model developed in Chapter 3 considering that both causes of failure are dependent. The internal degradation is reflected by the presence of multiple degradation processes in the system. Degradation processes start at random times following an NHPP and their growths are modelled by using a gamma process. When the deterioration level of a degradation process exceeds a predetermined value, we assume that a degradation failure occurs. Furthermore, the system is subject to sudden shocks that arrive at the system following a DSPP. A sudden shock provokes the failure of the system. Under these assumptions, the asymptotic expected cost rate function is obtained. Numerical examples are provided to illustrate this complex maintenance model.

In short, the main aspects covered in this chapter are:

1. Considering multiple degradation processes.

2. Considering two dependent causes of failure, internal degradation and sudden shocks.

3. Analysing the asymptotic expected cost rate implementing a CBM strategy in a DTS model.

4. Optimising the asymptotic expected cost rate in a bivariate case.

5. Analysing the robustness of some parameters of the maintenance model.

[1]This chapter is based on the works by Caballé *et al.* [62, 135].

This chapter is structured as follows. In Section 4.1 the general framework of the model is described. Section 4.2 details the maintenance strategy used in this model. Section 4.3 analyses the asymptotic expected cost rate. Numerical examples are shown in Section 4.4. Section 4.5 presents the conclusions and shows further possible extensions of this chapter.

4.1 Framework of the model

A maintenance model for a system subject to two competing causes of failure, internal degradation and sudden shocks, is considered.

4.1.1 General assumptions

The general assumptions of this model are:

1. The system is subject to multiple internal degradation processes. The successive degradation processes start at random times following an NHPP $\{N_d(t), t \geq 0\}$, where $N_d(t)$ denotes the number of degradation processes in the system at time t. We assume that $\{N_d(t), t \geq 0\}$ has intensity $m(t)$ and cumulative intensity $M(t)$. The sequential initiation points of the degradation processes are denoted by $0 \leq T_1 \leq T_2 \leq \ldots$, where T_i denotes the initiation point of the i-th degradation process.

2. We assume that degradation processes grow independently each other. The deterioration level of the degradation processes is developed depending on the environment and the component conditions according to a homogeneous gamma process. Let $X_i(t)$, $i = 1, 2, \ldots$ be the deterioration level of the i-th degradation process t time units after its initiation. We assume, $\{X_i(t),\, t \geq 0\}$ is distributed by a gamma process with parameters αt and β ($\alpha, \beta > 0$) and density function given in (2.3). Let $X_i^*(t)$ be the deterioration level of the i-th degradation process at time t, that is, $X_i^*(t) = X_i(t - T_i)$ for $t \geq T_i$ and let $X^*(t) = \left(X_1^*(t), X_2^*(t), \ldots, X_{N_d(t)}^*(t)\right)$ the vector containing the values of $X_i^*(t)$ for $i = 1, 2, \ldots, N_d(t)$. We assume that a degradation failure occurs when the deterioration level of a degradation process $X_i(t)$ exceeds a failure threshold L.

3. The system is also subject to sudden shocks occurring randomly in time, provoking the failure of the system. Sudden shocks arrive according to a counting process $\{N_s(t), t \geq 0\}$. This shock process shows the dependence between the two competing causes of failure in the following way: the intensity of the shock process at time t depends on the deterioration level of the existing degradation processes in the system. This dependence is reflected in that the system is more susceptible to a sudden shock when the deterioration level of a degradation process exceeds a certain level M_s. Then, $\{N_s(t), t \geq 0\}$, given $X^*(t)$, is a Poisson process with random intensity

$$\lambda\left(t, X^*(t)\right) = \lambda_1(t) \prod_{i=1}^{N_d(t)} \mathbf{1}_{\{X_i^*(t) \leq M_s\}} + \lambda_2(t) \left[1 - \prod_{i=1}^{N_d(t)} \mathbf{1}_{\{X_i^*(t) \leq M_s\}}\right], \tag{4.1}$$

for $t > 0$, where λ_1 and λ_2 denote two failure rate functions verifying that $\lambda_1(t) \leq \lambda_2(t)$ for all $t \geq 0$[1].

[1]By convection $\prod_{i=1}^{0} = 1$.

4. The system is inspected each T time units to check its state. In an inspection time, if the system is down, a CM is performed and the system is replaced by a new one. Figure 3.1 shows a CM event simulation. If the system is not down at the inspection time, the deterioration level of each degradation process is checked. Let M be the deterioration level from which the system is considered as too worn ($M < L$). If the deterioration level of a degradation process exceeds a certain threshold M, a preventive maintenance is performed and the system is replaced by a new one. Figure 3.2 shows a PM event simulation. Otherwise, no maintenance task is performed. We assume the time necessary to perform a maintenance action is negligible.

5. The costs of a corrective and PM are $C_c\ m.u.$ and $C_p\ m.u.$, respectively. The cost of each inspection is $C_I\ m.u.$ and the cost incurred by the system inactivity is $C_d\ m.u./t.u.$ We assume $C_c > C_p > C_I > 0$.

4.1.2 Distribution of counting processes

Let $S_{[1]}$ be the instant at which, for the first time, the deterioration level of a degradation process exceeds the preventive threshold M,

$$S_{[1]} = \min_{i=1,2,\ldots} \{S_i\}$$

where S_i denotes the time from the origin of the i-th degradation process to reach the deterioration level M.

By Lemma 3.2, the survival function of $S_{[1]}$ is

$$\bar{F}_{S_{[1]}}(t) = exp\left\{-\int_0^t m(v)F_{\sigma_M}(t-v)\,dv\right\}, \quad t \geq 0, \tag{4.2}$$

where $m(t)$ denotes the intensity of $\{N_d(t), t \geq 0\}$ and F_{σ_M} is given in (2.4).

Analogously, by Lemma 3.2, the survival function of the time to a degradation failure $W_{[1]}$ is

$$\bar{F}_{W_{[1]}}(t) = \exp\left\{-\int_0^t m(v)F_{\sigma_L}(t-v)dv\right\}, \quad t \geq 0.$$

The threshold M_s is defined in Subsection 4.1.1 as the deterioration level from which the system is more susceptible to a sudden shock. For $i = 1, 2, \ldots$, let σ_{M_s} be the first time at which the deterioration level of the i-th degradation process exceeds the threshold M_s since its initiation. Let V_i be the time where the deterioration level of the i-th degradation process exceeds the threshold M_s. Hence

$$V_i = T_i + \sigma_{M_s}, \quad i = 1, 2, \ldots$$

Let $V_{[1]}$ be the instant at which, for the first time, the deterioration level of a degradation process exceeds the threshold M_s. That is,

$$V_{[1]} = \min_{i=1,2,\ldots} \{V_i\}.$$

Let $N_{M_s}(t)$ be the number of degradation processes whose deterioration level exceed M_s at time t. That is

$$N_{M_s}(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{V_i \leq t\}}.$$

Using Lemma 3.1, $\{N_{M_s}(t),\, t \geq 0\}$ is an NHPP with intensity

$$m_{M_s}(t) = \int_0^t m(u) f_{\sigma_{M_s}}(t-u)\, du.$$

By Lemma 3.2, the survival function of $V_{[1]}$ is

$$\bar{F}_{V_{[1]}}(t) = exp\left\{-\int_0^t m(v) F_{\sigma_{M_s}}(t-v)\, dv\right\}, \quad t \geq 0, \tag{4.3}$$

where $F_{\sigma_{M_s}}$ is given in (2.4).

4.1.3 Time to a sudden shock

As previously mentioned, the system is also subject to sudden shocks occurring randomly in time. A shock provokes the total failure of the system. Let Y be the time to a sudden shock. Using Equation (4.1), the stochastic failure rate of Y is

$$\lambda(t, X^*(t)) = \lambda_1(t)\mathbf{1}_{\{V_{[1]}>t\}} + \lambda_2(t)\mathbf{1}_{\{V_{[1]}\leq t\}}, \quad t \geq 0.$$

Hence, we have

$$I(v,t) = P\left[Y > t | V_{[1]} = v\right] = \exp\left\{-\int_0^t \lambda(z, X^*(z))\, dz\right\} = \frac{\bar{F}_1(v)}{\bar{F}_1(0)} \frac{\bar{F}_2(t)}{\bar{F}_2(v)}, \tag{4.4}$$

for $v \leq t$, where

$$\bar{F}_j(t) = \exp\left\{-\int_0^t \lambda_j(u)\, du\right\}, \quad j = 1, 2, \tag{4.5}$$

with density functions $f_j(t)$ for $j = 1, 2$.

4.2 Condition-based maintenance

In this chapter, a CBM with periodic inspections is analysed for a system subject to multiple degradation processes and sudden shocks, where both causes of failure are considered dependent. The system is inspected each T time units to measure the deterioration level of the degradation processes. A PM action is performed when the deterioration level of a degradation process exceeds the preventive threshold M in an inspection time and the system is still working. A CM action is performed when the system is down at an inspection time.

4.2.1 Maintenance action probability

By Assumption 4 of Subsection 4.1.1, $D_1, D_2, \ldots$ are $i.i.d.$ random variables. Let $R_1 = D_1$ be the replacement cycle length and let $P_{R_1}^M(kT)$ be the probability

$$P_{R_1}^M(kT) = P[R_1 = kT],$$

for a time between inspections T and preventive threshold M, with $k = 1, 2, 3, \ldots$ The probability $P_{R_1}^M(kT)$ is

$$P_{R_1}^M(kT) = P_{R_1,p}^M(kT) + P_{R_1,c}^M(kT), \tag{4.6}$$

where $P_{R_1,p}^M(kT)$ and $P_{R_1,c}^M(kT)$ denote the probability that a preventive and CM action at time kT with $k = 1, 2, 3, \ldots$, respectively.

4.2.2 Preventive maintenance probability

A PM action is performed at time kT for $k = 1, 2, 3, \dots$ if the system is working at time kT and the preventive threshold M is exceeded for the first time in $((k-1)T, kT]$. Let $P^M_{R_1,p}(kT)$ be the probability of a PM action at time kT for $k = 1, 2, 3, \dots$ is

$$P^M_{R_1,p}(kT) = P^M_{R_1,p,1}(kT)\mathbf{1}_{\{M \leq M_s\}} + P^M_{R_1,p,2}(kT)\mathbf{1}_{\{M > M_s\}},$$

where $P^M_{R_1,p,1}(kT)$ denotes the probability of a PM action at time kT if $M \leq M_s$ and $P^M_{R_1,p,2}(kT)$ denotes the probability of a PM action at time kT if $M > M_s$.

First, we assume $M \leq M_s$. A PM action is performed at time kT if one of the following mutually exclusive events occur

$$\left\{(k-1)T < S_{[1]} < kT < V_{[1]}, Y > kT\right\},$$

and

$$\left\{(k-1)T < S_{[1]} < V_{[1]} < kT < W_{[1]}, Y > kT\right\}.$$

Hence,

$$P^M_{R_1,p,1}(kT) = P^M_{R_1,p,1,1}(kT) + P^M_{R_1,p,1,2}(kT),$$

where

$$P^M_{R_1,p,1,1}(kT) = P\left[(k-1)T < S_{[1]} < kT < V_{[1]}, Y > kT\right],$$

and

$$P^M_{R_1,p,1,2}(kT) = P\left[(k-1)T < S_{[1]} < V_{[1]} < kT < W_{[1]}, Y > kT\right].$$

Then,

$$\begin{aligned} P^M_{R_1,p,1,1}(kT) =& P\left[(k-1)T < S_{[1]} < kT < V_{[1]}, Y > kT\right] \\ =& \int_{(k-1)T}^{kT} f_{S_{[1]}}(u) P\left[kT < V_{[1]}, Y > kT | S_{[1]} = u\right]\, du \\ =& \int_{(k-1)T}^{kT} f_{S_{[1]}}(u) \bar{F}_{\sigma_{M_s} - \sigma_M}(kT - u) \\ & \sum_{i=0}^{\infty} P\left[kT < V_{[1]}, N_M(u, kT) = i, Y > kT\right]\, du. \end{aligned}$$

On the other hand, following the reasoning given in Lemma 3.3

$$\sum_{i=0}^{\infty} P[kT < V_{[1]}, N_M(u, kT) = i] = \exp\left\{-\int_u^{kT} m_M(z)dz\right\}\left(1 + \sum_{i=1}^{\infty} Q_{u_1,i}(kT)\right),$$

being

$$Q_{u_1,i}(t) = \int_{u_1}^{t} q_t(u_2)\, du_2 \int_{u_2}^{t} q_t(u_3)\, du_3 \dots \int_{u_{i-1}}^{t} q_t(u_i)\, du_i \int_{u_i}^{t} q_t(u_{i+1})\, du_{i+1}, \tag{4.7}$$

where

$$q_t(u) = m_M(u)\bar{F}_{\sigma_{M_s} - \sigma_M}(t - u), \quad u \leq t.$$

Let $Z_{M,M_s}(u_1,t)$ be the function

$$Z_{M,M_s}(u_1,t) = \exp\left\{-\int_{u_1}^{t} m_M(z)dz\right\}\left(1+\sum_{i=1}^{\infty} Q_{u_1,i}(t)\right),$$

where $Q_{u_1,i}(t)$ is given in (4.7). Using Lemma 3.4 given in Subsection 3.2.2, $Z_{M,M_s}(u_1,t)$ can be simplified as

$$Z_{M,M_s}(u_1,t) = \exp\left\{-\int_{u_1}^{t} m_M(z)F_{\sigma_{M_s}-\sigma_M}(t-z)dz\right\}. \tag{4.8}$$

Hence,

$$P^{M}_{R_1,p,1,1}(kT) = \bar{F}_1(kT)\int_{(k-1)T}^{kT} f_{S_{[1]}}(u)\bar{F}_{\sigma_{M_s}-\sigma_M}(kT-u)Z_{M,M_s}(u,kT)\,du.$$

Let $A_{M,M_s}(u,v)$ be the function representing the probability that the deterioration level M_s is not exceeded at time v by a degradation process, given that the preventive threshold M was exceeded for the first time at u. That is,

$$A_{M,M_s}(u,v) = P\left[V_{[1]}>v|S_{[1]}=u\right] = \bar{F}_{\sigma_{M_s}-\sigma_M}(v-u)Z_{M,M_s}(u,v), \tag{4.9}$$

where $Z_{M,M_s}(u,v)$ is given in (4.8). Thus

$$P^{M}_{R_1,p,1,1}(kT) = \bar{F}_1(kT)\int_{(k-1)T}^{kT} f_{S_{[1]}}(u)A_{M,M_s}(u,kT)\,du.$$

Let $a_{M,M_s}(u,v)$ be the derivative function of (4.9). That is,

$$a_{M,M_s}(u,v) = \frac{-\partial}{\partial v}A_{M,M_s}(u,v). \tag{4.10}$$

Hence,

$$\begin{aligned}
P^{M}_{R_1,p,1,2}(kT) =& P\left[(k-1)T<S_{[1]}<V_{[1]}<kT<W_{[1]},Y>kT\right]\\
=& \int_{(k-1)T}^{kT} f_{S_{[1]}}(u)P\left[u<V_{[1]}<kT<W_{[1]},Y>kT|S_{[1]}=u\right]\,du\\
=& \int_{(k-1)T}^{kT} f_{S_{[1]}}(u)\int_{u}^{kT}\frac{-\partial}{\partial v}\Big[\bar{F}_{\sigma_{M_s}-\sigma_M}(v-u)\\
& \sum_{i=0}^{\infty}P\left[u<V_{[1]}<kT,N_M(u,kT)=i|S_{[1]}=u\right]\Big]\\
& P\left[kT<W_{[1]},Y>kT|V_{[1]}=u\right]\,dv\,du\\
=& \int_{(k-1)T}^{kT} f_{S_{[1]}}(u)\int_{u}^{kT} a_{M,M_s}(u,v)\bar{F}_{\sigma_L-\sigma_{M_s}}(kT-v)\\
& \sum_{i=0}^{\infty}P\left[kT<W_{[1]},N_M(u,kT)=i|V_{[1]}=v\right]P\left[Y>kT|V_{[1]}=u\right]\,dv\,du\\
=& \int_{(k-1)T}^{kT} f_{S_{[1]}}(u)\int_{u}^{kT} a_{M,M_s}(u,v)A_{M_s,L}(v,kT)I(v,kT)\,dv\,du,
\end{aligned}$$

where $f_{S_{[1]}}$ denotes the density function of (4.2) and $I(u,t)$, $A_{M_s,L}(v,kT)$, and $a_{M,M_s}(u,v)$ are given in (4.4), (4.9), and (4.10), respectively.

Therefore, the PM probability for $M \leq M_s$ is

$$\begin{aligned}P^M_{R_1,p,1}(kT) =& \bar{F}_1(kT)\int_{(k-1)T}^{kT} f_{S_{[1]}}(u)A_{M,M_s}(u,kT)\,du \\ &+ \int_{(k-1)T}^{kT} f_{S_{[1]}}(u)\int_u^{kT} a_{M,M_s}(u,v)A_{M_s,L}(v,kT)I(v,kT)\,dv\,du.\end{aligned}$$

Considering $M > M_s$, a PM action is performed at time kT if one of the following mutually exclusive events occurs

$$\left\{(k-1)T < V_{[1]} < S_{[1]} < kT < W_{[1]}, Y > kT\right\},$$

and

$$\left\{V_{[1]} < (k-1)T < S_{[1]} < kT < W_{[1]}, Y > kT\right\}.$$

Thus,

$$P^M_{R_1,p,2}(kT) = P^M_{R_1,p,2,1}(kT) + P^M_{R_1,p,2,2}(kT),$$

where

$$P^M_{R_1,p,2,1}(kT) = P\left[(k-1)T < V_{[1]} < S_{[1]} < kT < W_{[1]}, Y > kT\right],$$

and $P^M_{R_1,p,2,2}(kT)$ is

$$P^M_{R_1,p,2,2}(kT) = P\left[V_{[1]} < (k-1)T < S_{[1]} < kT < W_{[1]}, Y > kT\right].$$

Hence,

$$\begin{aligned}P^M_{R_1,p,2,1}(kT) &= \int_{(k-1)T}^{kT} f_{V_{[1]}}(v)P\left[v < S_{[1]} < kT < W_{[1]}, Y > kT | V_{[1]} = v\right]\,dv \\ &= \int_{(k-1)T}^{kT} f_{V_{[1]}}(v)I(v,kT)P\left[v < S_{[1]} < kT < W_{[1]}\right]\,dv \\ &= \int_{(k-1)T}^{kT} f_{V_{[1]}}(v)I(v,kT)\int_v^{kT} a_{M_s,M}(v,u)P\left[kT < W_{[1]} | S_{[1]} = u\right]\,du\,dv \\ &= \int_{(k-1)T}^{kT} f_{V_{[1]}}(v)I(v,kT)\int_v^{kT} a_{M_s,M}(v,u)A_{M,L}(u,kT)\,du\,dv,\end{aligned}$$

and

$$\begin{aligned}P^M_{R_1,p,2,2}(kT) &= \int_0^{(k-1)T} f_{V_{[1]}}(v)P\left[(k-1)T < S_{[1]} < kT < W_{[1]}, Y > kT | V_{[1]} = v\right]\,dv \\ &= \int_0^{(k-1)T} f_{V_{[1]}}(v)I(v,kT)P\left[(k-1)T < S_{[1]} < kT < W_{[1]} | V_{[1]} = v\right]\,dv \\ &= \int_0^{(k-1)T} f_{V_{[1]}}(v)I(v,kT)\int_{(k-1)T}^{kT} a_{M_s,M}(v,u)P\left[kT < W_{[1]} | S_{[1]} = u\right]\,du\,dv \\ &= \int_0^{(k-1)T} f_{V_{[1]}}(v)I(v,kT)\int_{(k-1)T}^{kT} a_{M_s,M}(v,u)A_{M,L}(u,kT)\,du\,dv.\end{aligned}$$

Therefore, the expression for the PM probability when $M > M_s$ is

$$P^M_{R_1,p,2}(kT) = \int_{(k-1)T}^{kT} f_{V_{[1]}}(v) \int_{v}^{kT} a_{M_s,M}(v,u) A_{M,L}(u,kT) I(v,kT)\, du\, dv + \int_{0}^{(k-1)T} f_{V_{[1]}}(v) \int_{(k-1)T}^{kT} a_{M_s,M}(v,u) A_{M,L}(u,kT) I(v,kT)\, du\, dv.$$

Then, the PM probability at time kT is

$$\begin{aligned} P^M_{R_1,p}(kT) = &\Bigg[\bar{F}_1(kT) \int_{(k-1)T}^{kT} f_{S_{[1]}}(u) A_{M,M_s}(u,kT)\, du \\ &+ \int_{(k-1)T}^{kT} f_{S_{[1]}}(u) \int_{u}^{kT} a_{M,M_s}(u,v) A_{M_s,L}(v,kT) I(v,kT)\, dv\, du \Bigg] \mathbf{1}_{\{M \le M_s\}} \\ &+ \Bigg[\int_{(k-1)T}^{kT} f_{V_{[1]}}(v) \int_{v}^{kT} a_{M_s,M}(v,u) A_{M,L}(u,kT) I(v,kT)\, du\, dv \\ &+ \int_{0}^{(k-1)T} f_{V_{[1]}}(v) \int_{(k-1)T}^{kT} a_{M_s,M}(v,u) A_{M,L}(u,kT) I(v,kT)\, du\, dv \Bigg] \mathbf{1}_{\{M > M_s\}}, \end{aligned} \tag{4.11}$$

where $f_{S_{[1]}}$ and $f_{V_{[1]}}$ denote the density function of (3.4) and (4.3), respectively, and $I(u,t)$, $A_{x,y}(u,v)$, and $a_{x,y}(v,u)$ are given in (4.4), (4.9), and (4.10), respectively.

4.2.3 Corrective maintenance probability

A CM action is performed at time kT for $k = 1, 2, 3, \ldots$ if the system is down at time kT and the preventive threshold M is exceeded for the first time after $(k-1)T$. Let $P^M_{R_1,c}(kT)$ be the probability of a CM action at time kT for $k = 1, 2, 3, \ldots$ is

$$P^M_{R_1,c}(kT) = P^M_{R_1,c,1}(kT) \mathbf{1}_{\{M \le M_s\}} + P^M_{R_1,c,2}(kT) \mathbf{1}_{\{M > M_s\}},$$

where $P^M_{R_1,c,1}(kT)$ denotes the probability of a CM action at time kT if $M \le M_s$ and $P^M_{R_1,c,2}(kT)$ the probability of a CM action at time kT if $M > M_s$.

For $M \le M_s$, a CM action due to degradation occurs in kT if

$$\left\{(k-1)T < S_{[1]} < V_{[1]} < W_{[1]} < kT, W_{[1]} < Y\right\},$$

or equivalently by the occurrence of a shock in $((k-1), kT]$ given by the following mutually exclusive events

$$\left\{(k-1)T < S_{[1]} < Y < kT, Y < V_{[1]}\right\},$$

$$\left\{(k-1)T < S_{[1]} < V_{[1]} < Y < kT, Y < W_{[1]}\right\},$$

and

$$\left\{(k-1)T < Y < kT, Y < S_{[1]}\right\}.$$

Hence,

$$P^M_{R_1,c,1}(kT) = \sum_{i=1}^{4} P^M_{R_1,c,1,i}(kT),$$

where the probabilities $P^M_{R_1,c,1,i}(kT)$ for $i = 1, 2, 3, 4$ are

$$P^M_{R_1,c,1,1}(kT) = P\left[(k-1)T < S_{[1]} < V_{[1]} < W_{[1]} < kT, W_{[1]} < Y\right],$$

$$P^M_{R_1,c,1,2}(kT) = P\left[(k-1)T < S_{[1]} < Y < kT, Y < V_{[1]}\right],$$

$$P^M_{R_1,c,1,3}(kT) = P\left[(k-1)T < S_{[1]} < V_{[1]} < Y < kT, Y < W_{[1]}\right],$$

and

$$P^M_{R_1,c,1,4}(kT) = P\left[(k-1)T < Y < kT, Y < S_{[1]}\right],$$

respectively.

Following the development of the PM probability calculation, the CM probability at time kT when $M \leq M_s$ is

$$\begin{aligned} P^M_{R_1,c,1}(kT) = & \int_{(k-1)T}^{kT} f_{S_{[1]}}(u) \int_u^{kT} a_{M,M_s}(u,v) \int_v^{kT} -\frac{\partial}{\partial w}\left[A_{M_s,L}(v,w)I(v,w)\right] dw\, dv\, du \\ & + \int_{(k-1)T}^{kT} f_{S_{[1]}}(u) \int_u^{kT} f_1(x) A_{M,M_s}(u,x)\, dx\, du \\ & + \int_{(k-1)T}^{kT} f_1(x)\bar{F}_{S_{[1]}}(x)\, dx. \end{aligned}$$

For $M > M_s$, a CM action due to degradation is performed at time kT if one of the following mutually exclusive events takes place

$$\left\{V_{[1]} < (k-1)T < S_{[1]} < W_{[1]} < kT, Y > W_{[1]}\right\},$$

and

$$\left\{(k-1)T < V_{[1]} < W_{[1]} < kT, Y > W_{[1]}\right\},$$

and due to a sudden shock

$$\left\{V_{[1]} < (k-1)T < S_{[1]} < Y < kT, Y < W_{[1]}\right\},$$

$$\left\{V_{[1]} < (k-1)T < Y < kT, Y < S_{[1]}\right\},$$

$$\left\{(k-1)T < V_{[1]} < Y < kT, Y < W_{[1]}\right\},$$

and

$$\left\{(k-1)T < Y < kT, Y < V_{[1]}\right\}.$$

Hence,

$$P^M_{R_1,c,2}(kT) = \sum_{i=1}^{6} P^M_{R_1,c,2,i}(kT),$$

where $P^M_{R_1,c,2,i}(kT)$ for $i = 1, 2, \ldots, 6$ are

$$P^M_{R_1,c,2,1}(kT) = P\left[V_{[1]} < (k-1)T < S_{[1]} < W_{[1]} < kT, Y > W_{[1]}\right],$$

$$P^M_{R_1,c,2,2}(kT) = P\left[(k-1)T < V_{[1]} < W_{[1]} < kT, Y > W_{[1]}\right],$$

$$P^M_{R_1,c,2,3}(kT) = P\left[V_{[1]} < (k-1)T < S_{[1]} < Y < kT, Y < W_{[1]}\right],$$

$$P^{M}_{R_1,c,2,4}(kT) = P\left[V_{[1]} < (k-1)T < Y < kT, Y < S_{[1]}\right],$$

$$P^{M}_{R_1,c,2,5}(kT) = P\left[(k-1)T < V_{[1]} < Y < kT, Y < W_{[1]}\right],$$

and

$$P^{M}_{R_1,c,2,6}(kT) = P\left[(k-1)T < Y < kT, Y < V_{[1]}\right],$$

respectively.

Thus, based on previous calculations, $P^{M}_{R_1,c,2}(kT)$ is

$$\begin{aligned}
P^{M}_{R_1,c,2}(kT) = &\int_{0}^{kT} f_{V_{[1]}}(v) \int_{(k-1)T}^{kT} a_{M_s,M}(v,u) \int_{u}^{kT} -\frac{\partial}{\partial w}\Big[A_{M,L}(u,w)I(v,w)\Big]\, dw\, du\, dv \\
&+ \int_{(k-1)T}^{kT} f_{V_{[1]}}(v) \int_{v}^{kT} -\frac{\partial}{\partial w}\Big[A_{M_s,L}(v,w)I(v,w)\Big]\, dw\, dv \\
&+ \int_{0}^{(k-1)T} f_{V_{[1]}}(v) \int_{(k-1)T}^{kT} \Big[-\frac{\partial}{\partial w} I(v,x)\Big] A_{M_s,M}(v,x)\, dx\, dv \\
&+ \int_{(k-1)T}^{kT} f_1(x) \bar{F}_{V_{[1]}}(x)\, dx.
\end{aligned}$$

Hence, the CM probability at time kT is

$$\begin{aligned}
P^{M}_{R_1,c}(kT) = &\Bigg[\int_{(k-1)T}^{kT} f_{S_{[1]}}(u) \int_{u}^{kT} a_{M,M_s}(u,v) \int_{v}^{kT} -\frac{\partial}{\partial w}\Big[A_{M_s,L}(v,w)I(v,w)\Big]\, dw\, dv\, du \\
&+ \int_{(k-1)T}^{kT} f_{S_{[1]}}(u) \int_{u}^{kT} f_1(x) A_{M,M_s}(u,x)\, dx\, du \\
&+ \int_{(k-1)T}^{kT} f_1(x) \bar{F}_{S_{[1]}}(x)\, dx\Bigg] \mathbf{1}_{\{M \le M_s\}} \\
+ &\Bigg[\int_{0}^{kT} f_{V_{[1]}}(v) \int_{(k-1)T}^{kT} a_{M_s,M}(v,u) \int_{u}^{kT} -\frac{\partial}{\partial w}\Big[A_{M,L}(u,w)I(v,w)\Big]\, dw\, du\, dv \\
&+ \int_{(k-1)T}^{kT} f_{V_{[1]}}(v) \int_{v}^{kT} -\frac{\partial}{\partial w}\Big[A_{M_s,L}(v,w)I(v,w)\Big]\, dw\, dv \\
&+ \int_{0}^{(k-1)T} f_{V_{[1]}}(v) \int_{(k-1)T}^{kT} \Big[-\frac{\partial}{\partial w} I(v,x)\Big] A_{M_s,M}(v,x)\, dx\, dv \\
&+ \int_{(k-1)T}^{kT} f_1(x) \bar{F}_{V_{[1]}}(x)\, dx\Bigg] \mathbf{1}_{\{M > M_s\}},
\end{aligned} \tag{4.12}$$

where $f_{S_{[1]}}$ and $f_{V_{[1]}}$ denote the density function of (3.4) and (4.3), respectively, $f_1(t)$ is the density function associated with (4.5) and $I(u,t)$, $A_{x,y}(u,v)$, and $a_{x,y}(u,v)$ are given in (4.4), (4.9), and (4.10), respectively.

4.2.4 Expected downtime

Let $W_T^M((k-1)T, kT)$ be the time that the system is down in the interval $((k-1)T, kT]$ for T and M fixed. That is $W_T^M((k-1)T, kT)$ is equal to

$$\begin{cases} kT - Y & \text{if} \quad (k-1)T < Y \leq kT, \quad Y < W_{[1]}, \quad (k-1)T < S_{[1]} \\ kT - W_{[1]} & \text{if} \quad (k-1)T < S_{[1]} < W_{[1]} \leq kT, \quad W_{[1]} < Y \end{cases},$$

for $T > 0$. Based on calculations of the CM probability shown in Subsection 4.2.3, the expected downtime in $((k-1), kT]$ is

$$\begin{aligned} E[W_T^M((k-1)T, kT)] = & E[W_{T,1}^M((k-1)T, kT)]\mathbf{1}_{\{M \leq M_s\}} \\ & + E[W_{T,2}^M((k-1)T, kT)]\mathbf{1}_{\{M > M_s\}}, \end{aligned}$$

being

$$\begin{aligned} W_{T,1}^M((k-1)T, kT) = & \,(kT - Y)\,\mathbf{1}_{\left\{(k-1)T < S_{[1]} < Y < kT, Y < W_{[1]}\right\}} \\ & + (kT - Y)\,\mathbf{1}_{\left\{(k-1)T < S_{[1]} < V_{[1]} < Y < kT, Y < W_{[1]}\right\}} \\ & + (kT - Y)\,\mathbf{1}_{\left\{(k-1)T < Y < kT, Y < W_{[1]}\right\}} \\ & + \left(kT - W_{[1]}\right)\mathbf{1}_{\left\{(k-1)T < S_{[1]} < V_{[1]} < W_{[1]} < kT, W_{[1]} < Y\right\}}, \end{aligned}$$

and

$$\begin{aligned} W_{T,2}^M((k-1)T, kT) = & \,(kT - Y)\,\mathbf{1}_{\left\{V_{[1]} < (k-1)T < S_{[1]} < Y < kT, Y < W_{[1]}\right\}} \\ & + (kT - Y)\,\mathbf{1}_{\left\{V_{[1]} < (k-1)T < Y < kT, Y < S_{[1]}\right\}} \\ & + (kT - Y)\,\mathbf{1}_{\left\{(k-1)T < V_{[1]} < Y < kT, Y < W_{[1]}\right\}} \\ & + (kT - Y)\,\mathbf{1}_{\left\{(k-1)T < Y < kT, Y < V_{[1]}\right\}} \\ & + \left(kT - W_{[1]}\right)\mathbf{1}_{\left\{V_{[1]} < (k-1)T < S_{[1]} < W_{[1]} < kT, W_{[1]} < Y\right\}} \\ & + \left(kT - W_{[1]}\right)\mathbf{1}_{\left\{(k-1)T < V_{[1]} < W_{[1]} < kT, W_{[1]} < Y\right\}}. \end{aligned}$$

That is

$$
\begin{aligned}
E[W_T^M((k-1)T,kT)] =& \Bigg[\int_{(k-1)T}^{kT} f_{S_{[1]}}(u) \int_u^{kT} a_{M,M_s}(u,v) \\
& \int_v^{kT} -\frac{\partial}{\partial w}\Big[A_{M_s,L}(v,w)I(v,w)\Big](kT-w)\;dw\;dv\;du \\
& + \int_{(k-1)T}^{kT} f_{S_{[1]}}(u) \int_u^{kT} f_1(x)A_{M,M_s}(u,x)(kT-x)\;dx\;du \\
& + \int_{(k-1)T}^{kT} f_1(x)\bar{F}_{S_{[1]}}(x)(kT-x)\;dx\Bigg]\mathbf{1}_{\{M\le M_s\}} \\
+& \Bigg[\int_0^{kT} f_{V_{[1]}}(v) \int_{(k-1)T}^{kT} a_{M_s,M}(v,u) \\
& \int_u^{kT} -\frac{\partial}{\partial w}\Big[A_{M,L}(u,w)I(v,w)\Big](kT-w)\;dw\;du\;dv \\
& + \int_{(k-1)T}^{kT} f_{V_{[1]}}(v) \int_v^{kT} -\frac{\partial}{\partial w}\Big[A_{M_s,L}(v,w)I(v,w)\Big] \\
& (kT-w)\;dw\;dv \\
& + \int_0^{(k-1)T} f_{V_{[1]}}(v) \int_{(k-1)T}^{kT} \Big[-\frac{\partial}{\partial w}I(v,x)\Big] \\
& A_{M_s,M}(v,x)(kT-x)\;dx\;dv \\
& + \int_{(k-1)T}^{kT} f_1(x)\bar{F}_{V_{[1]}}(x)(kT-x)\;dx\Bigg]\mathbf{1}_{\{M>M_s\}}.
\end{aligned}
\tag{4.13}
$$

4.3 Asymptotic expected cost rate

As in Chapter 3, we consider as objective cost function the asymptotic expected cost rate. Based on Theorem 2.1,

$$
\begin{aligned}
C^\infty(T,M) =& \frac{\sum_{k=1}^{\infty}\Big[C_c P_{R_1,c}^M(kT) + C_d E\left[W_T^M((k-1)T,kT)\right]\Big]}{\sum_{k=1}^{\infty} kT\; P_{R_1}^M(kT)} \\
&+ \frac{\sum_{k=1}^{\infty}\Big[C_p P_{R_1,p}^M(kT) + C_I k P_{R_1}^M(kT)\Big]}{\sum_{k=1}^{\infty} kT\; P_{R_1}^M(kT)},
\end{aligned}
\tag{4.14}
$$

where $P_{R_1}^M(kT)$, $P_{R_1,p}^M(kT)$, $P_{R_1,c}^M(kT)$, and $E\left[W_T^M((k-1)T,kT)\right]$ are given in (4.6), (4.11), (4.12), and (4.13), respectively. Due to the analytical complexity of $C^\infty(T,M)$, the optimisation of the expected cost rate $C^{\infty}(T,M)$ for a data set is performed in the next section using numerical methods.

4.4 Numerical examples

We assume a system subject to multiple degradation processes where the initiation of degradation processes is distributed according to an NHPP with power-law intensity

$$m(t)=\frac{b}{a^b}t^{b-1}, \qquad t\geq 0,$$

where a and b ($a,b>0$) are the scale and shape parameters of the process, respectively. For this example, we consider $a=7$ and $b=5$. The growth of degradation processes is modelled according to a homogeneous gamma process with parameters $\alpha=\beta=0.2$.

The system fails due to two competing causes of failure: degradation and sudden shocks. A degradation failure occurs when the deterioration level of a degradation process exceeds the deterioration threshold $L=20$. A dependence model is considered between the two competing causes of failure. This dependence is reflected in the failure rate function of the sudden shock process. We assume that the sudden shock process follows a DSPP with intensity

$$\lambda(t,X^*(t))=0.01\prod_{i=1}^{N_d(t)}\mathbf{1}_{\{X_i^*(t)\leq 15\}}+0.10\left[1-\prod_{i=1}^{N_d(t)}\mathbf{1}_{\{X_i^*(t)\leq 15\}}\right], \quad t\geq 0,$$

where $X_i^*(t)$ denotes the deterioration level of the i-th degradation process at time t and $N_d(t)$ is the number of degradation processes in the system at time t. Under these specifications, the expected time until a degradation failure is 18.96 $t.u.$ and the expected time to a sudden shock is 24.58 $t.u.$

We assume that the sequence of costs is $C_I=7\ m.u.$, $C_d=46\ m.u.$, $C_p=281\ m.u.$, and $C_c=442\ m.u.$ MATLAB software, in its version R2014a, was used for the following examples. The code was run on an Intel Core i5-2500 processor with 8GB DDR3 RAM under Windows 7 Professional.

Considering the previous specifications, the optimisation problem for the asymptotic expected cost rate given in (4.14) is computed as follows:

1. A grid of size 50 is obtained by discretising the set $[0.2,10]$ into 50 equally spaced points from 0.2 to 10 for T. Let T_i be the i-th value of the grid obtained previously, for $i=1,2,\ldots,50$.

2. A grid of size 50 is obtained by discretising the set $[0.4,20]$ into 50 equally spaced points from 0.4 to 20 for M. Let M_j be the j-th value of the grid obtained previously, for $j=1,2,\ldots,50$.

3. For each combination (T_i,M_j) fixed, we obtain 25000 simulations of (R_1,I_1,W_d), where R_1 corresponds to the length of the first renewal cycle, I_1 the nature of the first maintenance action performed (corrective or preventive), and W_d the downtime up to the first maintenance action. With these simulations, and applying Monte Carlo method, we obtain $\tilde{P}_{R_1}^{M_j}(kT_i)$, $\tilde{P}_{R_{1,p}}^{M_j}(kT_i)$, $\tilde{P}_{R_{1,c}}^{M_j}(kT_i)$, and $\tilde{E}\left[W_{T_i}^{M_j}((k-1)T_i,kT_i))\right]$ corresponding to the estimations of $P_{R_1}^{M_j}(kT_i))$, $P_{R_{1,p}}^{M_j}(kT_i))$, $P_{R_{1,c}}^{M_j}(kT_i))$, and

$E\left[W_{T_i}^{M_j}((k-1)T_i, kT_i))\right]$ for $k = 1, 2, 3, \ldots$ given in Section 4.2 by (4.6), (4.11), (4.12), and (4.13), respectively (see Figure 3.4).

4. Quantity $\tilde{C}^{\infty}(T, M)$, representing the asymptotic expected cost rate, is calculated by using Equation (4.14), replacing the corresponding probabilities by their estimations calculated in Step 3.

5. The optimisation problem is reduced to find the values T_{opt} and M_{opt} which minimise the asymptotic expected cost rate $\tilde{C}^{\infty}(T, M)$. That is

$$\tilde{C}^{\infty}(T_{opt}, M_{opt}) = \min_{\substack{T>0 \\ 0<M\leq L}} \left\{\tilde{C}^{\infty}(T, M)\right\}.$$

Figure 4.1 shows the asymptotic expected cost rate versus T and M.

The convexity of the cost surface evidences the existence of an optimal setting of parameters T and M. The values of T and M which minimise the asymptotic expected cost rate are reached at $T_{opt} = 3.60$ $t.u.$ and $M_{opt} = 12.40$ $d.u.$, with an asymptotic expected cost rate of 27.1331 $m.u./t.u.$

Now, we focus on the influence of the main model parameters on the asymptotic expected cost rate. Firstly, a sensitivity analysis of the gamma process parameter is performed. Later, the sensitivity analysis of the power-law intensity parameters is considered.

The values of the gamma process parameters are modified according to the following specifications:

$$\alpha_{(v_i\%)} = \alpha\left[1+\frac{v_i}{100}\right] \quad \text{and} \quad \beta_{(v_j\%)} = \beta\left[1+\frac{v_j}{100}\right],$$

where v_i and v_j are, respectively, the i-th and j-th position of the vector $\mathbf{v} = (-10, -5, -1, 0, 1, 5, 10)$. Then, the parameter values for α and β can be simultaneous and independently modified both for increasing and decreasing changes.

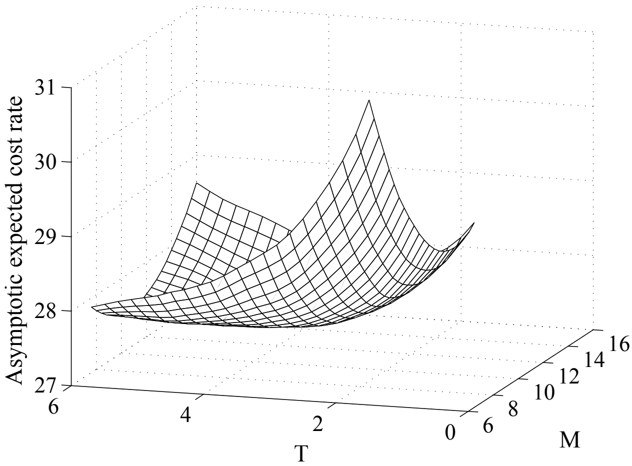
Let $\tilde{C}^{M,\infty}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}$ be the minimal expected cost rate obtained by varying the gamma process parameters simultaneously. Then, a relative measure is defined as

$$V^{M,\infty}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}} = \frac{|\tilde{C}^{\infty}(T_{opt}, M_{opt}) - \tilde{C}^{M,\infty}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}|}{\tilde{C}^{\infty}(T_{opt}, M_{opt})},$$

where $\tilde{C}^{\infty}(T_{opt}, M_{opt})$ is the minimal asymptotic expected cost rate previously calculated with the original parameter values.

For i and j fixed, $V^{M,\infty}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}$ measures the relative difference between the current optimal cost and the optimal cost that were calculated by using the modified parameter values. The values closer to zero represent a lower influence on the solution.

Table 4.1 shows the relative variation percentages with a shaded grey scale. Each cell represents $V^{M,\infty}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}$ multiplied by 100. Darker colours of cells denote a higher relative variation percentage. By modifying $\pm 5\%$ around $\alpha = 0.2$ and $\beta = 0.2$, the relative variation percentages are small. The results also show that the relative variation percentages are lower in the diagonal of the table. That is, when the parameters α and β are modified in the same direction and magnitude.



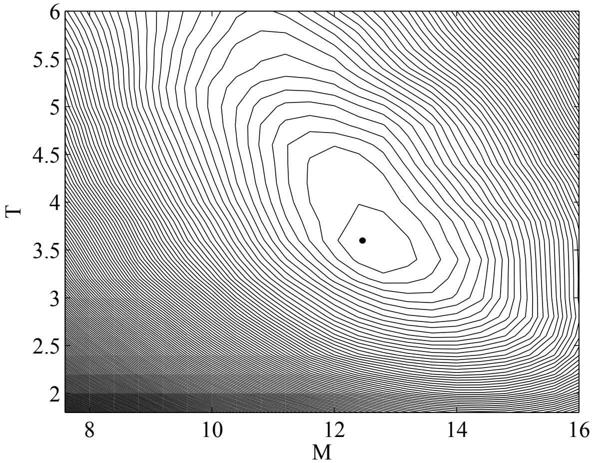


Figure 4.1: Mesh and contour plots for the asymptotic expected cost rate.

	$\beta_{(-10\%)}$	$\beta_{(-5\%)}$	$\beta_{(-1\%)}$	β	$\beta_{(1\%)}$	$\beta_{(5\%)}$	$\beta_{(10\%)}$
$\alpha_{(-10\%)}$	1.3154	1.7764	4.6644	5.4867	5.6558	8.6687	10.6825
$\alpha_{(-5\%)}$	3.6085	0.4349	1.9725	2.8729	3.1102	5.6672	9.0845
$\alpha_{(-1\%)}$	5.6754	2.3691	0.1224	0.4699	1.5877	3.5827	6.2433
α	5.9838	2.6031	0.3590	0.0000	0.6763	2.9193	5.9754
$\alpha_{(1\%)}$	6.3446	3.4205	1.3157	0.2215	0.2270	2.5415	5.5062
$\alpha_{(5\%)}$	8.4384	5.2887	2.4793	2.2482	1.4543	0.7338	4.0128
$\alpha_{(10\%)}$	11.1060	7.6585	4.8535	4.4031	3.8389	1.4492	1.2767

Table 4.1: Relative variation percentages for the gamma process parameters for $T = 3.6\ t.u.$ and $M = 12.40\ d.u.$ fixed.

	$b_{(-10\%)}$	$b_{(-5\%)}$	$b_{(-1\%)}$	b	$b_{(1\%)}$	$b_{(5\%)}$	$b_{(10\%)}$
$a_{(-10\%)}$	6.1626	7.4020	9.0060	9.1608	9.4140	10.5111	11.9452
$a_{(-5\%)}$	1.7469	2.6333	4.2221	4.3998	4.5034	5.4340	6.6690
$a_{(-1\%)}$	1.0238	0.2377	0.4463	0.7629	1.0334	1.7270	3.0284
a	2.2533	1.2188	0.3741	0.0000	0.2237	0.5311	1.6463
$a_{(1\%)}$	2.9492	2.2567	1.5155	0.8046	0.9630	0.2215	0.9767
$a_{(5\%)}$	5.9057	5.1229	3.9074	4.0102	4.0301	3.7158	2.7789
$a_{(10\%)}$	8.7546	8.2092	7.9796	8.2751	7.2229	7.1153	6.3340

Table 4.2: Relative variation percentages for the power-law intensity parameters for $T = 3.6\ t.u.$ and $M = 12.40\ d.u.$ fixed.

Similarly, the values of the power-law intensity parameters are modified according to the following specifications:

$$a_{(v_i\%)} = a\left[1 + \frac{v_i}{100}\right] \quad \text{and} \quad b_{(v_j\%)} = b\left[1 + \frac{v_j}{100}\right].$$

Let $C^{M,\infty}_{T,a_{(v_i\%)},b_{(v_j\%)}}$ be the minimal asymptotic expected cost rate obtained by varying the power-law intensity parameters simultaneously as in the previous scheme. Now, the relative measure is

$$V^{M,\infty}_{T,a_{(v_i\%)},b_{(v_j\%)}} = \frac{|C^\infty(T_{opt}, M_{opt}) - C^{M,\infty}_{T,a_{(v_i\%)},b_{(v_j\%)}}|}{C^\infty(T_{opt}, M_{opt})}.$$

The relative variation percentages are presented in Table 4.2. Again, by modifying $\pm 5\%$ around $a = 7$ and $b = 5$, the relative variation percentages are small. The results show that the parameter a has greater effects on $V^{M,\infty}_{T,a_{(v_i\%)},b_{(v_j\%)}}$ than the parameter b.

4.5 Conclusion and further extensions

In this chapter, a CBM maintenance strategy in a DTS model for a deteriorating system with dependent competing causes of failure, internal degradation and sudden shocks, is analysed. This system is subject to multiple degradation processes arriving at the system following an NHPP and whose growth is modelled under a homogeneous gamma process. On the other hand, the system is also subject to sudden shocks arriving at the system following a DSPP. The dependence between both causes of failure is reflected in that the system is more susceptible to external shocks when the deterioration level of a degradation process reaches a certain threshold M_s. Under these assumptions, the asymptotic expected cost rate as objective function is analysed and optimised. To this end, the numerical search of the optimal maintenance strategy is calculated throughout a procedure based on strictly Monte Carlo simulation. In addition, the robustness of the gamma process parameters and power-law intensity parameters is analysed.

In this chapter, the maintenance strategy under an asymptotic approach is analysed and optimised. To the best of our knowledge, DTS models are commonly studied under an asymptotic approach. However, most systems actually have a finite life cycle. Technology does not remain constant and the product life cycles take advantages of the improved technology. This fact makes some systems cannot always be replaced by a new one with the same charasteristics as the previous one an infinite number of times, and hence, the steady state assumption seems to be questionable. This has motivated the development of maintenance optimisation models for systems which experience a finite life cycle. This further extension is considered in Chapters 5 and 6.

CHAPTER

5

Transient approach for an independent DTS model with a degradation process

This chapter[1] deals with the analysis of a CBM strategy for a deteriorating system subject to two different causes of failure: internal continuous degradation and sudden shocks. In this chapter we assume only a degradation process modelled as a homogeneous gamma process. The system is regarded to fail when the deterioration level reaches a critical threshold. Furthermore, sudden shocks arrive at the system at random times following an NHPP. When a sudden shock takes place, the system fails. Under this functioning scheme, a CBM model with periodic inspection times is developed for controlling the system reliability. Traditionally, the search for the optimal maintenance strategy is evaluated over an infinite time span. However, some systems have a finite life cycle and the application of the asymptotic approach is questionable. In this chapter, we change the perspective assumed in Chapters 3 and 4 by developing a CBM strategy considering a finite life cycle, that is a transient approach. A recursive method which combines numerical integration and Monte Carlo simulation is developed to obtain the expected cost rate in the life cycle of the system, its associated standard deviation, and some availability measures. Additionally, the expected cost based on the recursive method is analysed under a random finite life cycle. Numerical examples are provided to illustrate this complex maintenance model.

In short, the main aspects covered in this chapter are:

1. Analysing the expected cost in the finite life cycle implementing a CBM strategy in a DTS model.

2. Developing a recursive method which combines numerical integration and Monte Carlo simulation to obtain the expected cost in the finite life cycle.

[1] This chapter is based on the works by Caballé and Castro [74, 136–140].

3. Comparing the results obtained using this recursive method to the results obtained by using strictly Monte Carlo simulation.

4. Analysing the standard deviation cost associated with the expected cost in the finite life cycle provided through this recursive method.

5. Comparing the asymptotic expected cost rate and the expected cost in the finite life cycle.

6. Analysing the expected cost based on the recursive method considering a random finite life cycle.

7. Analysing the robustness of some parameters of the maintenance model.

8. Analysing the reliability, the point availability, the joint availability, the interval reliability, and the joint interval reliability.

This chapter is structured as follows. In Section 5.1 the general framework of the model is described. In Section 5.2 the CBM of the system is studied. Section 5.3 analyses the expected cost in the finite life cycle and the standard deviation associated. Section 5.4 develops different availability measures for this maintenance model. Numerical examples are provided in Section 5.5. Section 5.6 presents the conclusions and shows further possible extensions of this chapter.

5.1 Framework of the model

A CBM model for a system subject to two competing causes of failure, internal degradation and sudden shocks is considered. The assumptions of this maintenance model are similar to those considered in Chapters 3 and 4 with the difference that the system is functioning over a finite life cycle and only one degradation process is considered.

5.1.1 General assumptions

The general assumptions of this model are:

1. The system starts working at time $t = 0$ and it is subject to a continuous degradation process. Let $X(t)$ be the deterioration level of the system at time t. We assume that $\{X(t), t \geq 0\}$ follows a homogeneous gamma process with parameters αt and β $(\alpha, \beta > 0)$ and probability density function given in (2.3). We assume that the system fails when its deterioration level exceeds the breakdown threshold L.

2. The system is also subject to sudden shocks arriving at the system following an NHPP $\{N_s(t), t \geq 0\}$ with intensity $\lambda(t)$, where $\lambda(t)$ is a non-decreasing function in t. A sudden shock arrival provokes the failure of the system.

3. We assume that the sudden shock process intensity at time t is independent of the degradation of the system.

4. The system is inspected each T $(T > 0)$ time units to check if it is working or down. If the system is down, a CM is performed and the system is replaced by a new one. On the other hand, if the system is still working at the inspection time, the deterioration level is checked. Let M be the deterioration level from which the system is considered too much worn and the system must be replaced in a preventive way $(M < L)$. If the system is

working and the deterioration level in the inspection exceeds the threshold M, a PM is performed and the system is replaced by a new one (see Figures 3.1 and 3.2, respectively). Otherwise, no maintenance task is performed. After a replacement, a new cycle starts with an identical time between inspections T. We suppose that the time necessary to perform a maintenance task is negligible.

5. Each maintenance action implies a cost. A cost of C_p and C_c $m.u.$ are associated with a PM and a CM, respectively. An inspection implies a cost of C_I $m.u.$ In addition, if the system fails, the system is down until the next inspection with a down cost of C_d $m.u./t.u.$ We assume that $C_c > C_p > C_I > 0$.

6. We assume that the life cycle ends at time $t_f > 0$. It means that, if the calendar time exceeds t_f, the system can no longer be replaced by a new one with the same characteristics. We also assume that, at time t_f, the system is scrapped regardless of its true state.

5.1.2 Time to the system failure

Let Y be the time to a failure due to a sudden shock in degradation absence. Then, the survival distribution of Y is

$$\bar{F}_Y(t) = \exp\left\{-\int_0^t \lambda(u)du\right\} = \exp\left\{-\Lambda(t)\right\}, \quad t \geq 0, \tag{5.1}$$

where $\Lambda(t)$ is the cumulative intensity of $\lambda(t)$.

Let Z be the time to the system failure. Then, the survival function of Z for $t \geq 0$ is

$$\bar{F}_Z(t) = P\left[\sigma_L > t,\ Y > t\right] = \bar{F}_{\sigma_L}(t)\bar{F}_Y(t),$$

where σ_L is given in (2.4).

5.2 Condition-based maintenance

In this chapter, a CBM with periodic inspections is analysed under a transient approach for a system subject to a degradation process and sudden shocks, where both causes of failure are independent. The system is inspected each T time units to measure the deterioration level of the degradation process. A PM is performed when the deterioration level exceeds a preventive threshold M in an inspection time. A CM is performed when the system is down in an inspection time.

5.2.1 Maintenance action probability

Let $D_1, D_2, \ldots, D_n$ the times between two successive replacements of the system. By Assumption 4 of Subsection 5.1.1, $D_1, D_2, \ldots, D_n$ are $i.i.d.$ random variables. For T and M fixed, let $P_{R_1}^M(kT)$ be the probability

$$P_{R_1}^M(kT) = P[R_1 = kT],$$

with $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$. The probability of a replacement at time kT for $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$ is

$$P_{R_1}^M(kT) = P_{R_{1,p}}^M(kT) + P_{R_{1,c}}^M(kT), \tag{5.2}$$

where $\lfloor \cdot \rfloor$ is the floor function[1] and $P^M_{R_{1,p}}(kT)$ and $P^M_{R_{1,c}}(kT)$ denote the probability of a preventive and corrective replacement at time kT for $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$, respectively.

5.2.2 Preventive maintenance probability

A PM action is performed at time kT for $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$ when the system is working at time kT and the preventive threshold M is exceeded in $((k-1)T, kT]$. That is, a PM is performed at time kT when the following event occurs

$$\{(k-1)T < \sigma_M \leq kT < \sigma_L, \quad Y > kT\}.$$

Hence, the probability of a preventive replacement at time kT for $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$ is

$$P^M_{R_{1,p}}(kT) = P[(k-1)T < \sigma_M \leq kT < \sigma_L, \quad Y > kT].$$

Due to the independence of degradation and shock process, we have

$$P^M_{R_{1,p}}(kT) = P[(k-1)T < \sigma_M \leq kT < \sigma_L] P[Y > kT].$$

Then, by (5.1),

$$P^M_{R_{1,p}}(kT) = \exp\{-\Lambda(kT)\} \int_{(k-1)T}^{kT} f_{\sigma_M}(u) \bar{F}_{\sigma_L - \sigma_M}(kT - u)\, du, \tag{5.3}$$

where f_{σ_M} denotes the density function of $\bar{F}_{\sigma_M}$ given in (2.4) and $\bar{F}_{\sigma_L - \sigma_M}$ is given in (2.6) .

5.2.3 Corrective maintenance probability

A CM action is performed at time kT for $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$ when the system fails in $((k-1)T, kT]$ and $\sigma_M > (k-1)T$. That is, when one of the following mutually exclusive events occurs

$$\{(k-1)T < \sigma_M < Y \leq kT,\ Y < \sigma_L\}, \quad \{(k-1)T < Y \leq kT,\ Y < \sigma_M\},$$
$$\{(k-1)T < \sigma_M < \sigma_L \leq kT,\ \sigma_L < Y\}.$$

Hence, the probability of a corrective replacement at time kT for $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$ is

$$\begin{aligned} P^M_{R_{1,c}}(kT) =& P[(k-1)T < \sigma_M < Y \leq kT,\ Y < \sigma_L] \\ &+ P[(k-1)T < Y \leq kT,\ Y < \sigma_M] \\ &+ P[(k-1)T < \sigma_M < \sigma_L \leq kT,\ \sigma_L < Y]. \end{aligned}$$

Then,

$$\begin{aligned} P^M_{R_{1,c}}(kT) =& \int_{(k-1)T}^{kT} f_Y(u) \bar{F}_{\sigma_M}(u)\, du \\ &+ \int_{(k-1)T}^{kT} f_{\sigma_M}(u) \int_u^{kT} -\frac{\partial}{\partial v}\left[\bar{F}_Y(v) \bar{F}_{\sigma_L - \sigma_M}(v-u)\right] dv\, du, \end{aligned} \tag{5.4}$$

where f_{σ_M} denotes the density function of $\bar{F}_{\sigma_M}$ given in (2.4), and $f_{\sigma_L - \sigma_M}$ and f_Y are the density functions of $\bar{F}_{\sigma_L - \sigma_M}$ and $\bar{F}_Y$ given in (2.6) and (5.1), respectively.

[1] The floor function takes as its value the greatest preceding integer. That is, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

5.2.4 Expected downtime

For T and M fixed, let $W_T^M((k-1)T, kT)$ be the time that the system is down in $((k-1)T, kT]$ for $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$. That is, $W_T^M((k-1)T, kT)$ is equal to

$$\begin{cases} kT - Y & \text{if} \quad (k-1)T < Y \leq kT, \quad Y < \sigma_L, \quad (k-1)T < \sigma_M \\ kT - \sigma_L & \text{if} \quad (k-1)T < \sigma_M < \sigma_L \leq kT, \quad \sigma_L < Y \end{cases},$$

for $T > 0$. Hence, the expected downtime in $((k-1), kT]$ is

$$\begin{aligned} E[W_d^M((k-1)T, kT)] =& E\left[(kT - Y)\mathbf{1}_{\{(k-1)T<\sigma_M<Y\leq kT,\ Y<\sigma_L\}}\right] \\ &+ E\left[(kT - Y)\mathbf{1}_{\{(k-1)T<Y\leq kT,\ Y<\sigma_M\}}\right] \\ &+ E\left[(kT - \sigma_L)\mathbf{1}_{\{(k-1)T<\sigma_M<\sigma_L\leq kT,\, \sigma_L<Y\}}\right]. \end{aligned}$$

Then, based on calculations of the CM probability shown in Subsection 5.2.3

$$\begin{aligned} E\left[W_T^M((k-1)T, kT)\right] =& \int_{(k-1)T}^{kT} f_Y(u)\bar{F}_{\sigma_M}(u)(kT-u)\, du \\ &+ \int_{(k-1)T}^{kT} f_{\sigma_M}(u) \int_u^{kT} -\frac{\partial}{\partial v}\left[\bar{F}_Y(v)\bar{F}_{\sigma_L-\sigma_M}(v-u)\right] \\ &(kT - v)\, dv\, du. \end{aligned} \tag{5.5}$$

5.3 Analysis of the expected cost in the finite life cycle

For T and M fixed, let $C_T^M(t)$ be the total cost of the system in the interval $(0, t]$ calculated as the sum of the incurred costs in the different $N_T^M(t)$ complete renewal cycles up to t and the cost incurred between $R_{N_T^M(t)}$ and t. Figure 5.1 shows a process realisation.

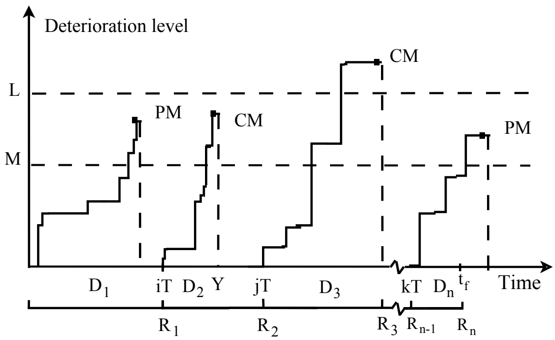


Figure 5.1: A realisation of a system finite life cycle.

Next, we obtain a recursive formula for the expected cost at time t $E\left[C_T^M(t)\right]$.

Theorem 5.1. *For $t < T$, we get*

$$E\left[C_T^M(t)\right] = C_d \int_0^t \left(1 - \bar{F}_Y(u)\bar{F}_{\sigma_L}(u)\right) du. \tag{5.6}$$

For $t \geq T$, the function $E\left[C_T^M(t)\right]$ fulfils the following recursive equation[1]

$$E\left[C_T^M(t)\right] = \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C_T^M(t-kT)\right] P_{R_1}^M(kT) + G_T^M(t), \tag{5.7}$$

with initial condition $E\left[C_T^M(0)\right] = 0$, where

$$\begin{aligned} G_T^M(t) = & \sum_{k=1}^{\lfloor t/T \rfloor} \Big(C_p + C_I(k-1)\Big) P_{R_{1,p}}^M(kT) \\ & + \sum_{k=1}^{\lfloor t/T \rfloor} \Big(C_c + C_I(k-1) + C_d E\left[W_T^M((k-1)T, kT)\right]\Big) P_{R_{1,c}}^M(kT) \\ & + \Big(\lfloor t/T \rfloor C_I + C_d E\left[W_T^M(\lfloor t/T \rfloor T, t)\right]\Big)\Big(1 - \sum_{k=1}^{\lfloor t/T \rfloor} P_{R_1}^M(kT)\Big), \end{aligned} \tag{5.8}$$

where $P_{R_1}^M(kT)$, $P_{R_{1,p}}^M(kT)$, and $P_{R_{1,c}}^M(kT)$ are the probabilities given in (5.2), (5.3), and (5.4), respectively, and $E\left[W_T^M(x,t)\right]$ is the expected downtime in $(x,t]$ given in (5.5).

Proof. For $t < T$,

$$\begin{aligned} E\left[C_T^M(t)\right] = & C_d E\left[(t-Y)\mathbf{1}_{\{Y<t,\ Y<\sigma_L\}}\right] + C_d E\left[(t-\sigma_L)\mathbf{1}_{\{\sigma_L<t,\ \sigma_L<Y\}}\right] \\ = & C_d \int_0^t (t-u)\left[-\frac{d}{du}\Big(\bar{F}_{\sigma_L}(u)\bar{F}_Y(u)\Big)\right] du \\ = & C_d \int_0^t \Big(1 - \bar{F}_Y(u)\bar{F}_{\sigma_L}(u)\Big) du. \end{aligned}$$

Based on the reasoning detailed in Section 2.5, conditioning to the first renewal R_1, $E\left[C_T^M(t)\right]$ is written as

$$E\left[C_T^M(t)\right] = E\left[C_T^M(t)|R_1 \leq t\right] + E\left[C_T^M(t)|R_1 > t\right].$$

Thus, if $R_1 > t$

$$E\left[C_T^M(t)|R_1 > t\right] = \Big(\lfloor t/T \rfloor C_I + C_d E\left[W_T^M(\lfloor t/T \rfloor T, t)\right]\Big)\left(1 - \sum_{k=1}^{\lfloor t/T \rfloor} P_{R_1}^M(kT)\right),$$

where $P_{R_1}^M$ is given in (5.2).

If $R_1 \leq t$, $E\left[C_T^M(t)\right]$ can be split into two terms: the cost in the first renewal cycle $\left(C_T^M(R_1)\right)$ and the cost in the remaining time horizon $\left(C_T^M(R_1,t)\right)$. Since $C_T^M(R_1)$ and $C_T^M(R_1,t)$ are independent, we get

$$E\left[C_T^M(t)|R_1 \leq t\right] = E\left[C_T^M(R_1)|R_1 \leq t\right] + E\left[C_T^M(R_1,t)|R_1 \leq t\right].$$

For T fixed

$$E\left[C_T^M(R_1)|R_1 \leq t\right] = \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C_T^M(R_1)|R_{1,c} = kT\right] + \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C_T^M(R_1)|R_{1,p} = kT\right],$$

[1]By convection $\sum_i^j = 0$, for $j < i$.

being

$$E\left[C_T^M(R_1)|R_{1,c}=kT\right]=\Big(C_c+C_I(k-1)+C_dE\left[W_T^M((k-1)T,kT)\right]\Big)P_{R_{1,c}}^M(kT),$$

and

$$E\left[C_T^M(R_1)|R_{1,p}=kT\right]=\Big(C_p+C_I(k-1)\Big)P_{R_{1,p}}^M(kT),$$

where $P_{R_1}^M$, $P_{R_{1,p}}^M$, and $P_{R_{1,c}}^M$ are given in (5.2), (5.3), and (5.4), respectively, and $E\left[W_T^M((k-1)T,kT)\right]$ denotes the expected downtime in $((k-1)T,kT]$ given in (5.5).

Since $C_T^M(R_1,t)$ is stochastically the same as $C_T^M(t-R_1)$,

$$E\left[C_T^M(R_1,t)|R_1=kT\right]=E\left[C_T^M(t-kT)\right]P_{R_1}^M(kT).$$

Then $E\left[C_T^M(t)\right]$ verifies the following recursive equation

$$E\left[C_T^M(t)\right]=\sum_{k=1}^{\lfloor t/T\rfloor}E\left[C_T^M(t-kT)\right]P_{R_1}^M(kT)+G_T^M(t),$$

being

$$\begin{aligned}G_T^M(t)=&\sum_{k=1}^{\lfloor t/T\rfloor}\Big(C_p+C_I(k-1)\Big)P_{R_{1,p}}^M(kT)\\&+\sum_{k=1}^{\lfloor t/T\rfloor}\Big(C_c+C_I(k-1)+C_dE\left[W_T^M((k-1)T,kT)\right]\Big)P_{R_{1,c}}^M(kT)\\&+\Big(\lfloor t/T\rfloor C_I+C_dE\left[W_T^M(\lfloor t/T\rfloor T,t)\right]\Big)\Big(1-\sum_{k=1}^{\lfloor t/T\rfloor}P_{R_1}^M(kT)\Big),\end{aligned}$$

and the result holds. □

In addition to the expected cost in the finite life cycle, it is necessary to analyse its standard deviation since the uncertainty associated with the estimated cost must be known. For T and M fixed, let $\Big(S_T^M(t)\Big)^2$ be the variance of the expected cost at time t. Thus,

$$\Big(S_T^M(t)\Big)^2=E\left[C_T^M(t)^2\right]-\left(E\left[C_T^M(t)\right]\right)^2. \tag{5.9}$$

Next, we obtain a recursive formula that verifies the expected square cost at time t.

Theorem 5.2. *For $t<T$, we get*

$$E\left[C_T^M(t)^2\right]=C_d^2\int_0^t(t-u)^2\left[-\frac{d}{du}\Big(\bar{F}_{\sigma_L}(u)\bar{F}_Y(u)\Big)\right]du.$$

For $t\geq T$, the function $E\left[C_T^M(t)^2\right]$ fulfils the following recursive equation

$$E\left[C_T^M(t)^2\right]=\sum_{k=1}^{\lfloor t/T\rfloor}E\left[C_T^M(t-kT)^2\right]P_{R_1}^M(kT)+H_T^M(t),$$

with initial condition $E\left[C_T^M(0)^2\right]=0$*, where* $\lfloor\cdot\rfloor$ *denotes the floor function and*

$$\begin{aligned}
H_T^M(t) =& \sum_{k=1}^{\lfloor t/T\rfloor} \left(C_p + C_I(k-1)\right)^2 P_{R_{1,p}}^M(kT) \\
&+ \sum_{k=1}^{\lfloor t/T\rfloor} \left(C_c + C_I(k-1) + C_d E\left[W_T^M((k-1)T,kT)\right]\right)^2 P_{R_{1,c}}^M(kT) \\
&+2 \sum_{k=1}^{\lfloor t/T\rfloor} \left(C_c + C_I(k-1)\right) E\left[C_T^M(t-kT)\right] P_{R_{1,c}}^M(kT) \\
&+2 \sum_{k=1}^{\lfloor t/T\rfloor} C_d E\left[W_T^M((k-1)T,kT)\right] E\left[C_T^M(t-kT)\right] P_{R_{1,c}}^M(kT) \\
&+2 \sum_{k=1}^{\lfloor t/T\rfloor} \left(C_p + C_I(k-1)\right) E\left[C_T^M(t-kT)\right] P_{R_{1,p}}^M(kT) \\
&+\Big(\lfloor t/T\rfloor C_I + C_d E\left[W_T^M(\lfloor t/T\rfloor T,t)\right]\Big)^2 \left(1 - \sum_{k=1}^{\lfloor t/T\rfloor} P_{R_1}^M(kT)\right),
\end{aligned} \tag{5.10}$$

where $P_{R_1}^M(kT)$*,* $P_{R_{1,p}}^M(kT)$*, and* $P_{R_{1,c}}^M(kT)$ *are the probabilities given in (5.2), (5.3), and (5.4), respectively, and* $E\left[W_T^M(x,t)\right]$ *denotes the expected downtime in* $(x,t]$ *given in (5.5).*

Proof. For $t<T$, the expected square cost is

$$\begin{aligned}
E\left[C_T^M(t)^2\right] =& C_d^2 E\left[(t-Y)^2 \mathbf{1}_{\{Y<t,\ Y<\sigma_L\}}\right] + C_d^2 E\left[(t-\sigma_L)^2 \mathbf{1}_{\{\sigma_L<t,\ \sigma_L<Y\}}\right] \\
=& C_d^2 \int_0^t (t-u)^2 \left[-\frac{d}{du}\Big(\bar{F}_{\sigma_L}(u)\bar{F}_Y(u)\Big)\right] du.
\end{aligned}$$

Based on the reasoning of Section 2.5, for $t \geq T$, conditioning to the first renewal R_1, $E\Big[C_T^M(t)^2\Big]$ is written as

$$E\left[C_T^M(t)^2\right] = E\left[C_T^M(t)^2 | R_1 \leq t\right] + E\left[C_T^M(t)^2 | R_1 > t\right].$$

Thus, if $R_1 > t$

$$E\left[C_T^M(t)^2 | R_1 > t\right] = \Big(\lfloor t/T\rfloor C_I + C_d E\left[W_T^M(\lfloor t/T\rfloor T,t)\right]\Big)^2 \left(1 - \sum_{k=1}^{\lfloor t/T\rfloor} P_{R_1}^M(kT)\right),$$

where $P_{R_1}^M$ is given in (5.2).

If $R_1 \leq t$,

$$E\left[C_T^M(t)^2 | R_1 \leq t\right] = E\left[\left(C_T^M(R_1) + C_T^M(R_1,t)\right)^2 | R_1 \leq t\right].$$

Since $C_T^M(R_1)$ and $C_T^M(R_1,t)$ are independent

$$\begin{aligned}
&E\left[\left(C_T^M(R_1) + C_T^M(R_1,t)\right)^2 | R_1 \leq t\right] \\
&= E\left[\left(C_T^M(R_1)\right)^2 | R_1 \leq t\right] + E\left[C_T^M(R_1,t)^2 | R_1 \leq t\right] \\
&+ 2\, E\left[C_T^M(R_1) C_T^M(R_1,t) | R_1 \leq t\right].
\end{aligned}$$

Thus,

$$E\left[\left(C_T^M(R_1)\right)^2 | R_1 \leq t\right] = \sum_{k=1}^{\lfloor t/T \rfloor} E\left[\left(C_T^M(R_1)\right)^2 | R_{1,c} = kT\right] + \sum_{k=1}^{\lfloor t/T \rfloor} E\left[\left(C_T^M(R_1)\right)^2 | R_{1,p} = kT\right],$$

being

$$E\left[\left(C_T^M(R_1)\right)^2 | R_{1,c} = kT\right] = \left(C_c + C_I(k-1) + C_d E\left[W_T^M((k-1)T, kT)\right]\right)^2 P_{R_{1,c}}^M(kT),$$

where $E\left[W_T^M((k-1)T, kT)\right]$ denotes the expected downtime in $((k-1)T, kT]$ and

$$E\left[\left(C_T^M(R_1)\right)^2 | R_{1,p} = kT\right] = \left(C_c + C_I(k-1)\right)^2 P_{R_{1,p}}^M(kT).$$

Based on Theorem 5.1,

$$\begin{aligned} E\left[C_T^M(R_1) | R_1 \leq t\right] = &\sum_{k=1}^{\lfloor t/T \rfloor} \left(C_c + C_I(k-1)\right) P_{R_{1,c}}^M(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} C_d E\left[W_T^M((k-1)T, kT)\right] P_{R_{1,c}}^M(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_p + C_I(k-1)\right) P_{R_{1,p}}^M(kT). \end{aligned}$$

Since $C_T^M(R_1, t)$ is stochastically the same as $C_T^M(t - R_1)$, we get

$$\begin{aligned} E\left[C_T^M(R_1, t) | R_1 \leq t\right] &= \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C_T^M(R_1, t) | R_1 = kT\right] P_{R_1}^M(kT) \\ &= \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C_T^M(t - kT)\right] P_{R_1}^M(kT). \end{aligned}$$

Thus,

$$\begin{aligned} &E\left[C_T^M(R_1) C_T^M(R_1, t) | R_1 \leq t\right] \\ &= \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_p + C_I(k-1)\right) E\left[C_T^M(t - kT)\right] P_{R_{1,p}}^M(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} \left(C_c + C_I(k-1)\right) E\left[C_T^M(t - kT)\right] P_{R_{1,c}}^M(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} C_d E\left[W_T^M((k-1)T, kT)\right] E\left[C_T^M(t - kT)\right] P_{R_{1,c}}^M(kT). \end{aligned}$$

Since $C_T^M(R_1,t)^2$ is stochastically the same as $C_T^M(t-R_1)^2$, we get

$$E\left[C_T^M(R_1,t)^2|R_1=kT\right]=E\left[C_T^M(t-kT)^2\right].$$

In this way, $E\left[C_T^M(t)^2\right]$ verifies the following recursive equation

$$E\left[C_T^M(t)^2\right]=\sum_{k=1}^{\lfloor t/T\rfloor}E\left[C_T^M(t-kT)^2\right]P_{R_1}^M(kT)+H_T^M(t),$$

being

$$\begin{aligned}
H_T^M(t)=&\sum_{k=1}^{\lfloor t/T\rfloor}\left(C_p+C_I(k-1)\right)^2P_{R_{1,p}}^M(kT)\\
&+\sum_{k=1}^{\lfloor t/T\rfloor}\left(C_c+C_I(k-1)+C_dE\left[W_T^M((k-1)T,kT)\right]\right)^2P_{R_{1,c}}^M(kT)\\
&+2\sum_{k=1}^{\lfloor t/T\rfloor}\left(C_c+C_I(k-1)\right)E\left[C_T^M(t-kT)\right]P_{R_{1,c}}^M(kT)\\
&+2\sum_{k=1}^{\lfloor t/T\rfloor}C_dE\left[W_T^M((k-1)T,kT)\right]E\left[C_T^M(t-kT)\right]P_{R_{1,c}}^M(kT)\\
&+2\sum_{k=1}^{\lfloor t/T\rfloor}\left(C_p+C_I(k-1)\right)E\left[C_T^M(t-kT)\right]P_{R_{1,p}}^M(kT)\\
&+\Big(\lfloor t/T\rfloor C_I+C_dE\left[W_T^M(\lfloor t/T\rfloor T,t)\right]\Big)^2\left(1-\sum_{k=1}^{\lfloor t/T\rfloor}P_{R_1}^M(kT)\right),
\end{aligned}$$

and the result holds. □

Based on (5.9), the standard deviation of the cost at time t is

$$S_T^M(t)=\sqrt{E\left[C_T^M(t)^2\right]-\left(E\left[C_T^M(t)\right]\right)^2}. \tag{5.11}$$

In addition, some systems are designed to experience a finite but uncertain random life cycle. Hence, in the cost analysis of this kind of systems, it is important to make use of a random planning life cycle to obtain the expected cost.

Let t_{f_r} be the random life cycle of a system with density function $f_{t_{f_r}}$ and survival function $\bar{F}_{t_{f_r}}$ and let $E\left[C_T^M(t_{f_r})\right]$ be the expected cost in a random life cycle, for $t_{f_r}\in[0,\infty)$. Next, based on Theorem 5.1, the following result is obtained for $E\left[C_T^M(t_{f_r})\right]$.

Corollary 5.1. *The expected cost over a random life cycle t_{f_r} with density function $f_{t_{f_r}}$ and survival function $\bar{F}_{t_{f_r}}$ is*

$$\begin{aligned}
E\left[C_T^M(t_{f_r})\right]=&C_d\int_0^T f_{t_{f_r}}(u)\int_0^u\Big(1-\bar{F}_Y(v)\bar{F}_{\sigma_L}(v)\Big)\,dv\,du\\
&+\sum_{k=1}^{\infty}\left[\bar{F}_{t_{f_r}}(kT)G_T^M(kT)+\left(P_{R_1}^M(kT)\int_{kT}^{\infty}E\left[C_T^M(u-kT)\right]f_{t_{f_r}}(u)\,du\right)\right],
\end{aligned}$$

where $G_T^M(t)$ is given in (5.8).

Proof. By Theorem 5.1, for $t_{f_r} < T$, we get

$$
\begin{aligned}
E\left[C_T^M(t_{f_r})\right] =& \int_0^T E\left[C_T^M(u)\right] f_{t_{f_r}}(u)\, du \\
=& C_d \int_0^T f_{t_{f_r}}(u) \int_0^u \left(1 - \bar{F}_Y(v)\bar{F}_{\sigma_L}(v)\right) dv\, du,
\end{aligned}
$$

where $\bar{F}_{\sigma_L}$ is the survival function of (2.4) and $\bar{F}_Y$ is given in (5.1).

For $t_{f_r} \geq T$, the function $E\left[C_T^M(t_{f_r})\right]$ fulfils the following recursive equation

$$
\begin{aligned}
E\left[C_T^M(t_{f_r})\right] =& \sum_{k=1}^{\infty} \int_{kT}^{\infty} \left(E\left[C_T^M(u-kT)\right] P_{R_1}^M(kT) + G_T^M(u)\right) f_{t_{f_r}}(u)\, du \\
=& \sum_{k=1}^{\infty} P_{R_1}^M(kT) \int_{kT}^{\infty} E\left[C_T^M(u-kT)\right] f_{t_{f_r}}(u)\, du \\
&+ \sum_{k=1}^{\infty} \int_{kT}^{\infty} G_T^M(u) f_{t_{f_r}}(u)\, du.
\end{aligned}
$$

The second term is expressed as

$$
\sum_{k=1}^{\infty} \int_{kT}^{\infty} G_T^M(u) f_{t_{f_r}}(u)\, du = \sum_{k=1}^{\infty} G_T^M(kT) \int_{kT}^{\infty} f_{t_{f_r}}(u)\, du = \sum_{k=1}^{\infty} G_T^M(kT) \bar{F}_{t_{f_r}}(kT).
$$

Then, the expected cost over a random life cycle is

$$
\begin{aligned}
E\left[C_T^M(t_{f_r})\right] =& E\left[C_T^M(t_{f_r}) | t_{f_r} < T\right] + E\left[C_T^M(t_{f_r}) | t_{f_r} \geq T\right] \\
=& C_d \int_0^T f_{t_{f_r}}(u) \int_0^u \left(1 - \bar{F}_Y(v)\bar{F}_{\sigma_L}(v)\right) dv\, du \\
&+ \sum_{k=1}^{\infty} \left[\bar{F}_{t_{f_r}}(kT) G_T^M(kT) + \left(P_{R_1}^M(kT) \int_{kT}^{\infty} E\left[C_T^M(u-kT)\right] f_{t_{f_r}}(u)\, du\right)\right],
\end{aligned}
$$

and the result holds. □

If we assume that t_{f_r} is bounded by a constant value t_{max}, that is $P[t_{f_r} < t_{max}] = 1$, then, based on Corollary 5.1, the expected cost over a random life cycle $t_{f_r} \in [0, t_{max}]$ is

$$
\begin{aligned}
E\left[C_T^M(t_{f_r})\right] =& C_d \int_0^T f_{t_{f_r}}(u) \int_0^u \left(1 - \bar{F}_Y(v)\bar{F}_{\sigma_L}(v)\right) dv\, du \\
&+ \sum_{k=1}^{\lfloor t_{max}/T \rfloor} P_{R_1}^M(kT) \int_{kT}^{t_{max}} E\left[C_T^M(u-kT)\right] f_{t_{f_r}}(u)\, du \\
&+ \sum_{k=1}^{\lfloor t_{max}/T \rfloor} \bar{F}_{t_{f_r}}(kT) G_T^M(kT),
\end{aligned} \tag{5.12}
$$

5.4 Availability measures of the system

Next, some availability measures of the system are evaluated. Markov renewal equations are provided for the point availability, the joint availability, the reliability, the interval reliability, and the joint interval reliability of the system.

Let $\{I(t), t \geq 0\}$ be the continuous-time stochastic process defined in (2.10) as

$$I(t) = \begin{cases} 1 & \text{if the system is up at time } t \\ 0 & \text{otherwise} \end{cases}.$$

The point availability of the system

Let $A_T^M(t)$ be the point availability defined as the probability that the system is up at time t. That is,

$$A_T^M(t) = P\left[I(t) = 1\right], \quad t \geq 0.$$

The following result provides the recursive formula for $A_T^M(t)$.

Theorem 5.3. *For $t < T$, we get*

$$A_T^M(t) = \bar{F}_Y(t)\bar{F}_{\sigma_L}(t). \tag{5.13}$$

For $t \geq T$, the function $A_T^M(t)$ fulfils the following Markov renewal equation

$$A_T^M(t) = \sum_{k=1}^{\lfloor t/T \rfloor} A_T^M(t-kT)P_{R_1}^M(kT) + J_T^M(t), \tag{5.14}$$

with initial condition $A_T^M(0) = 1$ and where

$$J_T^M(t) = \bar{F}_{\sigma_M}(t)\bar{F}_Y(t) + \int_{\lfloor t/T \rfloor T}^{t} f_{\sigma_M}(u)\bar{F}_{\sigma_L-\sigma_M}(t-u)\bar{F}_Y(t)\, du, \tag{5.15}$$

where $P_{R_{1,p}}^M(kT)$ is the probability given in (5.3), f_{σ_M} denotes the density function of $\bar{F}_{\sigma_M}$ given in (2.4), and $\bar{F}_{\sigma_L-\sigma_M}$ and $\bar{F}_Y$ are the survival functions given in (2.6) and (5.1), respectively.

Proof. For $t < T$,

$$A_T^M(t) = P[I(t) = 1] = P[Y > t, \sigma_L > t] = \bar{F}_Y(t)\bar{F}_{\sigma_L}(t).$$

For $t \geq T$, conditioning to the first renewal R_1, the point availability is written as

$$A_T^M(t) = P\left[I(t) = 1 | R_1 \leq t\right] + P\left[I(t) = 1 | R_1 > t\right].$$

If $R_1 > t$, we have

$$\begin{aligned} P\left[I(t) = 1 | R_1 > t\right] =& P\left[\sigma_M > t,\, Y > t\right] + P\left[\lfloor t/T \rfloor T < \sigma_M \leq t,\, t < \min\{Y, \sigma_L\}\right] \\ =& \bar{F}_{\sigma_M}(t)\bar{F}_Y(t) + \int_{\lfloor t/T \rfloor T}^{t} f_{\sigma_M}(u)\bar{F}_{\sigma_L-\sigma_M}(t-u)\bar{F}_Y(t)\, du \\ =& J_T^M(t), \end{aligned}$$

On the other hand, if $R_1 \leq t$

$$P\left[I(t) = 1 | R_1 \leq t\right] = \sum_{k=1}^{\lfloor t/T \rfloor} P_{R_1}^M(kT) P\left[I(t-kT) = 1\right] = \sum_{k=1}^{\lfloor t/T \rfloor} P_{R_1}^M(kT) A_T^M(t-kT),$$

and the result holds. □

The joint availability of the system

Let $JA_T^{M,(2)}(t_1,t_2)$ be the joint availability in t_1 and t_2 defined as the probability that the system is up at t_1 and t_2 with $0 \leq t_1 < t_2$. That is,

$$JA_T^{M,(2)}(t_1,t_2) = P\left[I(t_1)=1,\, I(t_2)=1\right].$$

The joint availability in t_1 and t_2 for $t_1 < t_2$ fulfils the following recursive equation.

Theorem 5.4. *For $t_2 < T$, we get*

$$JA_T^{M,(2)}(t_1,t_2) = A_T^M(t_2), \quad t_1 < t_2,$$

where $A_T^M(t)$ is given in (5.13). For $t_2 \geq T$, the joint availability fulfils the following Markov renewal equation

$$\begin{aligned} JA_T^{M,(2)}(t_1,t_2) =& J_T^M(t_2) + \sum_{k=1}^{\lfloor t_1/T \rfloor} P_{R_1}(kT) JA_T^{M,(2)}(t_1-kT, t_2-kT) \\ &+ \sum_{k=\lfloor t_1/T\rfloor+1}^{\lfloor t_2/T \rfloor} P_{R_1}(kT) A_T^M(t_2-kT) - g_T^M(t_1) A_T^M(t_2-(\lfloor t_1/T\rfloor+1)T), \end{aligned} \tag{5.16}$$

where $A_T^M(t)$ is given in Theorem 5.3, $J_T^M(t)$ is given in (5.15) and

$$\begin{aligned} g_T^M(t_1) =& \int_{\lfloor t_1/T\rfloor T}^{t_1} f_{\sigma_M}(u)du \int_u^{t_1} \frac{-\partial}{\partial v}\left[\bar{F}_Y(v)\bar{F}_{\sigma_L-\sigma_M}(v-u)\right] dv \\ &+ \int_{\lfloor t_1/T\rfloor T}^{t_1} f_Y(u)\bar{F}_{\sigma_M}(u)du. \end{aligned} \tag{5.17}$$

If $\lfloor t_1/T \rfloor = \lfloor t_2/T \rfloor$, then

$$JA_T^{M,(2)}(t_1,t_2) = J_T^M(t_2) + \sum_{k=1}^{\lfloor t_1/T \rfloor} P_{R_1}(kT) JA_T^{M,(2)}(t_1-kT, t_2-kT).$$

Proof. For $t_2 < T$, no maintenance action is performed and

$$JA_T^{M,(2)}(t_1,t_2) = A_T^M(t_2).$$

For $t_2 \geq T$, conditioning to the first renewal R_1, the joint availability is written as

$$\begin{aligned} JA_T^{M,(2)}(t_1,t_2) =& P\left[I(t_1)=1,\, I(t_2)=1 | R_1 \leq t_1\right] \\ &+ P\left[I(t_1)=1,\, I(t_2)=1 | t_1 < R_1 \leq t_2\right] \\ &+ P\left[I(t_1)=1,\, I(t_2)=1 | R_1 > t_2\right]. \end{aligned}$$

We have,

$$\begin{aligned} P\left[I(t_1)=1,\, I(t_2)=1 | R_1 > t_2\right] =& P\left[\sigma_M > t_2,\, Y > t_2\right] \\ &+ P\left[\lfloor t_2/T \rfloor T < \sigma_M \leq t_2,\, t_2 < \min\{Y, \sigma_L\}\right] \\ =& J_T^M(t_2), \end{aligned}$$

where J_T^M is given in (5.15). On the other hand,

$$\begin{aligned}
&P\left[I(t_1)=1,\, I(t_2)=1|t_1<R_1\le t_2\right] \\
&\quad = P\left[I(t_2)=1|t_1<R_1\le t_2\right]-P\left[I(t_1)=0,\, I(t_2)=1|t_1<R_1\le t_2\right] \\
&\quad = P\left[I(t_2)=1|t_1<R_1\le t_2\right]-P\left[I(t_2)=1, \lfloor t_1/T\rfloor T<\sigma_M<t_1, \min\{Y,\sigma_L\}<t_1\right] \\
&\quad - P\left[I(t_2)=1, \lfloor t_1/T\rfloor T<Y<t_1,\, Y<\sigma_M\right] \\
&\quad = \sum_{k=\lfloor t_1/T\rfloor+1}^{\lfloor t_2/T\rfloor} P_{R_1}^M(kT)P\left[I(t_2)=1|R_1=kT\right] \\
&\quad - \sum_{k=\lfloor t_1/T\rfloor+1}^{\lfloor t_2/T\rfloor} P\left[I(t_2)=1|R_{1,c}=(\lfloor t_1/T\rfloor+1)T\right]\int_{\lfloor t_1/T\rfloor T}^{t_1} f_Y(v)\bar{F}_{\sigma_M}(v)dv \\
&\quad - \sum_{k=\lfloor t_1/T\rfloor+1}^{\lfloor t_2/T\rfloor} \Bigg(P\left[I(t_2)=1|R_{1,c}=(\lfloor t_1/T\rfloor+1)T\right]\int_{\lfloor t_1/T\rfloor T}^{t_1} f_{\sigma_M}(u)du \\
&\qquad\qquad \int_u^{t_1}\frac{-\partial}{\partial v}\left[\bar{F}_Y(v)\bar{F}_{\sigma_L-\sigma_M}(v-u)\right]dv\Bigg).
\end{aligned}$$

Since

$$P\left[I(t_2)=1|R_1=kT\right]=A_T^M(t_2-kT),$$

and

$$P\left[I(t_2)=1|R_{1,c}=(\lfloor t_1/T\rfloor+1)T\right]=A_T^M(t_2-\lfloor t_1/T\rfloor+1)T),$$

then

$$\begin{aligned}
&P\left[I(t_1)=1,\, I(t_2)=1|t_1<R_1\le t_2\right] \\
&\quad = \sum_{k=\lfloor t_1/T\rfloor+1}^{\lfloor t_2/T\rfloor} P_{R_1}^M(kT)A_T^M(t_2-kT)-A_T^M(t_2-\lfloor t_1/T\rfloor+1)T)g_T^M(t_1),
\end{aligned}$$

where $g_T^M(t_1)$ is given in (5.17). Finally,

$$\begin{aligned}
P\left[I(t_1)=1,\, I(t_2)=1|R_1\le t_1\right] &= \sum_{k=1}^{\lfloor t_1/T\rfloor} P_{R_1}^M(kT)P\left[I(t_1-kT)=1,\, I(t_2-kT)=1\right] \\
&= \sum_{k=1}^{\lfloor t_1/T\rfloor} P_{R_1}^M(kT)JA_T^{M,(2)}(t_1-kT,t_2-kT),
\end{aligned}$$

and the result holds. □

The *joint availability* $JA_T^{M,(n)}(t_1,t_2)$ can be extended to a finite number of points n. Let $JA_T^{M,(n)}(t_1,t_2,t_3,\ldots,t_n)$ be the *joint availability* in n points $t_1,t_2,\ldots,t_n$ defined as the probability that the system is up at $t_1,t_2,\ldots,t_n$. That is,

$$JA_T^{M,(n)}(t_1,t_2,t_3,\ldots,t_n)=P\left[I(t_1)=1,\, I(t_2)=1,\, I(t_3)=1,\,\ldots,I(t_n)=1\right].$$

The function $JA_T^{M,(n)}(t_1,t_2,t_3,\ldots,t_n)$ fulfils the following result.

Theorem 5.5. *For $t_n < T$, we get*

$$JA_T^{M,(n)}(t_1, t_2, t_3, \ldots, t_n) = A_T^M(t_n), \quad t_1 < t_2 < \ldots < t_n,$$

where $A_T^M(t)$ is given in (5.13). For $t_n \geq T$, the joint availability fulfils the following Markov renewal equation

$$\begin{aligned}
&JA_T^{M,(n)}(t_1, t_2, t_3, \ldots, t_n) \\
=&J_T^M(t_n) + \sum_{k=1}^{\lfloor t_1/T \rfloor} P_{R_1}(kT) JA_T^{M,(n)}(t_1 - kT, t_2 - kT, t_3 - kT, \ldots, t_n - kT) \\
&+ \sum_{i=1}^{n-1} \sum_{k=\lfloor t_i/T \rfloor + 1}^{\lfloor t_{i+1}/T \rfloor} P_{R_1}(kT) JA_T^{M,(n-i)}(t_{i+1} - kT, t_{i+2} - kT, \ldots, t_n - kT) \\
&- \sum_{i=1}^{n-1} g_T^M(t_i) JA_T^{M,(n-i)}(t_{i+1} - (\lfloor t_i/T \rfloor + 1)\, T, \ldots, t_n - (\lfloor t_i/T \rfloor + 1)\, T),
\end{aligned}$$

where $JA_T^{M,(1)}(t) = A_T^M(t)$ is given in Theorem 5.3, $JA_T^{M,(n-i)}(t_{i+1} - kT, t_{i+2} - kT, \ldots, t_n - kT)$ denotes the joint availability at times $t_{i+1} - kT, t_{i+2} - kT, \ldots, t_n - kT$, $J_T^M(t)$ is given in (5.15) and g_T^M is given in (5.17).

Proof. For $t_n < T$, no maintenance action is performed and

$$JA_T^{M,(n)}(t_1, t_2, t_3, \ldots, t_n) = A_T^M(t_n).$$

For $t_n \geq T$, conditioning to the first renewal R_1, the joint availability is written as

$$\begin{aligned}
&JA_T^{M,(n)}(t_1, t_2, t_3, \ldots, t_n) \\
=&P\left[I(t_1) = 1,\, I(t_2) = 1,\, I(t_3) = 1,\, \ldots, I(t_n) = 1 | R_1 \leq t_1\right] \\
&+ \sum_{i=1}^{n-1} P\left[I(t_1) = 1,\, I(t_2) = 1,\, I(t_3) = 1,\, \ldots, I(t_{i+1}) = 1 | t_i < R_1 \leq t_{i+1}\right] \\
&+ P\left[I(t_1) = 1,\, I(t_2) = 1,\, I(t_3) = 1,\, \ldots, I(t_n) = 1 | R_1 > t_n\right].
\end{aligned}$$

We have,

$$\begin{aligned}
&P\left[I(t_i) = 1,\, i = 1, 2, \ldots, n | R_1 > t_n\right] \\
=&P\left[\sigma_M > t_n,\, Y > t_n\right] + P\left[\lfloor t_n/T \rfloor T < \sigma_M \leq t_n,\, t_n < \min\{Y, \sigma_L\}\right] \\
=&J_T^M(t_n),
\end{aligned}$$

where J_T^M is given in (5.15). For $i = 1, 2, \ldots, n-1$ we get

$$\begin{aligned}
&P\left[I(t_1)=1,\, I(t_2)=1,\ldots,I(t_n)=1|t_i<R_1\le t_{i+1}\right]\\
&\quad= P\left[I(t_{i+1})=1,\, I(t_{i+2})=1,\,\ldots,I(t_n)=1|t_i<R_1\le t_{i+1}\right]\\
&\quad- P\left[I(t_i)=0,\, I(t_{i+1})=1,\, I(t_{i+2})=1,\,\ldots,I(t_n)=1|t_i<R_1\le t_{i+1}\right]\\
&\quad= P\left[I(t_{i+1})=1,\, I(t_{i+2})=1|t_i<R_1\le t_{i+1}\right]\\
&\quad- P\left[I(t_{i+1})=1,\, I(t_{i+2})=1,\,\ldots,I(t_n)=1,\, \lfloor t_i/T\rfloor T<\sigma_M<t_i, \min\{Y,\sigma_L\}<t_i\right]\\
&\quad- P\left[I(t_{i+1})=1,\, I(t_{i+2})=1,\,\ldots,I(t_n)=1\; \lfloor t_i/T\rfloor T<Y<t_i,\, Y<\sigma_M\right]\\
&\quad= \sum_{k=\lfloor t_i/T\rfloor+1}^{\lfloor t_{i+1}/T\rfloor} P_{R_1}^M(kT)P\left[I(t_{i+1})=1,\, I(t_{i+2})=1,\ldots,I(t_n)=1|R_1=kT\right]\\
&\quad- P\left[I(t_{i+1})=1,\, I(t_{i+2})=1,\ldots,I(t_n)=1|R_{1,c}=(\lfloor t_i/T\rfloor+1)T\right]\\
&\quad\left(\int_{\lfloor t_i/T\rfloor T}^{t_i} f_Y(v)\bar{F}_{\sigma_M}(v)dv+\int_{\lfloor t_i/T\rfloor T}^{t_i} f_{\sigma_M}(u)du\int_u^{t_i}\frac{-\partial}{\partial v}\left[\bar{F}_Y(v)\bar{F}_{\sigma_L-\sigma_M}(v-u)\right]dv\right)\\
&\quad= \sum_{k=\lfloor t_i/T\rfloor+1}^{\lfloor t_{i+1}/T\rfloor} P_{R_1}^M(kT)JA_T^{M,(n-i)}(t_{i+1}-kT, t_{i+2}-kT,\ldots,t_n-kT)\\
&\quad- JA_T^{M,(n-i)}(t_{i+1}-(\lfloor t_i/T\rfloor+1)T, t_{i+2}-(\lfloor t_i/T\rfloor+1)T,\ldots,t_n-(\lfloor t_i/T\rfloor+1)T)g_T^M(t_i).
\end{aligned}$$

Finally,

$$\begin{aligned}
&P\left[I(t_i)=1,\, i=1,2,\ldots,n|R_1\le t_1\right]\\
&\quad= \sum_{k=1}^{\lfloor t_1/T\rfloor} P_{R_1}^M(kT)P\left[I(t_i-kT)=1,\, i=1,2,\ldots,n\right]\\
&\quad= \sum_{k=1}^{\lfloor t_1/T\rfloor} P_{R_1}^M(kT)JA_T^{M,(n)}(t_1-kT, t_2-kT,\ldots,t_n-kT),
\end{aligned}$$

and the result holds[1]. □

The reliability of the system

Often, also of interest is the probability that the system is working at a specified time and continues operating for a specified time interval. This is the case of the reliability of the system. Let $O(t)$ be the deterioration level of the maintained system at time t, that is

$$O(t)=\sum_{j=0}^{\infty}\mathbf{1}_{\{R_j\le t<R_{j+1}\}}X(t-R_j), \tag{5.18}$$

where R_j is the time instant of the j-th renewal.

Let $R_T^M(t)$ be the *reliability* of the system at time t, that is, the probability that the system is up in the interval $(0,t]$

$$R_T^M(t)=P\left[O(u)<L,\forall u\in(0,t], N_s(0,t)=0\right].$$

Next, we obtain a recursive formula for $R_T^M(t)$.

[1]Note that if $\lfloor t_i/T\rfloor=\lfloor t_{i+1}/T\rfloor$, then $JA_T^{M,(n)}(t_1,t_2,t_3,\ldots,t_n)=0$.

Theorem 5.6. *For $t < T$, we get*

$$R_T^M(t) = A_T^M(t),$$

where $A_T^M(t)$ is given in (5.13). For $t \geq T$, the function $R_T^M(t)$ fulfils the following Markov renewal equation

$$R_T^M(t) = \sum_{k=1}^{\lfloor t/T \rfloor} R_T^M(t-kT) P_{R_{1,p}}^M(kT) + J_T^M(t), \tag{5.19}$$

with initial condition $R_T^M(0) = 1$ and where $J_T^M(t)$ is given in (5.15).

Proof. For $t < T$, $R_T^M(t)$ is

$$R_T^M(t) = P[Y > t, \sigma_L > t] = \bar{F}_Y(t)\bar{F}_{\sigma_L}(t).$$

For $t \geq T$, conditioning to the first renewal R_1, the reliability is written as

$$\begin{aligned} R_T^M(t) =& P\left[O(u) < L, \forall u \in (0,t], N_s(0,t) = 0 | R_1 \leq t\right] \\ &+ P\left[O(u) < L, \forall u \in (0,t], N_s(0,t) = 0 | R_1 > t\right]. \end{aligned}$$

If $R_1 > t$

$$R_T^M(t) = P\left[\sigma_M > t, Y > t\right] + P\left[\lfloor t/T \rfloor T < \sigma_M < t < \sigma_L, Y > t\right] = J_T^M(t).$$

If $R_1 \leq t$, conditioning to the first preventive replacement $R_{1,p}$, the reliability is written as

$$\begin{aligned} R_T^M(t) =& P\left[O(u) < L,\ \forall u \in (0,t], N_s(0,t) = 0 | R_1 \leq t\right] \\ =& \sum_{k=1}^{\lfloor t/T \rfloor} P_{R_{1,p}}^M(kT) P\left[O(u-kT) < L, \forall u \in (0, t-kT], N_s(0, t-kT) = 0\right] \\ =& \sum_{k=1}^{\lfloor t/T \rfloor} R_T^M(t-kT) P_{R_{1,p}}^M(kT). \end{aligned}$$

Then, for $t \geq T$, $R_T^M(t)$ verifies the following recursive equation

$$R_T^M(t) = \sum_{k=1}^{\lfloor t/T \rfloor} R_T^M(t-kT) P_{R_{1,p}}^M(kT) + J_T^M(t),$$

and the result holds. □

The reliability of the system can be calculated by an iterative formula provided in the following result.

Corollary 5.2. *Setting $R_T^{M(i)}(t) = R_T^M(t)$, for all $(i-1)T \leq t < iT$ with $i = 1, 2, \ldots, \lfloor t/T \rfloor$ the reliability function can be expressed in an iterative way as follows*

$$R_T^{M(1)}(t) = R_T^M(t),$$

and for $i \geq 1$

$$R_T^{M(i+1)}(t) = \sum_{k=1}^{i} R_T^{M(i+1-k)}(t-kT) P_{R_{1,p}}^M(kT) + J_T^{M(i)}(t),$$

where, by (5.15)

$$J_T^{M(i)}(t) = \bar{F}_{\sigma_M}(t)\bar{F}_Y(t) + \int_{iT}^{t} f_{\sigma_M}(u)\bar{F}_{\sigma_L-\sigma_M}(t-u)\bar{F}_Y(t)\,du.$$

Let $F_t(s)$ be the residual life of the system defined as

$$F_t(s) = P\left[Z < t+s | Z > t\right].$$

Then, we get the following result.

Corollary 5.3. *If $t+s < T$*

$$F_t(s) = 1 - P\left[Z \geq t+s|Z > t\right] = 1 - \frac{R_T^M(t+s)}{R_T^M(t)}.$$

For $t+s \geq T$, the distribution of the residual life fulfils the following Markov renewal equation

$$F_t(s) = 1 - \frac{J_T^M(t+s)}{R_T^M(t)} - \sum_{k=1}^{\lfloor (t+s)/T \rfloor} F_t(s-kT)P_{R_{1,p}}^M(kT).$$

The interval reliability of the system

A measure that extends the reliability is the interval reliability defined as the probability that the system is working at time t, and will continue to work over a finite time interval of length h. The interval reliability is applied when there are periods in the life cycle when a failure should be avoided with high probability. Let $IR_T^M(t, t+h)$ be the interval reliability of the system in $(t, t+h]$

$$IR_T^M(t,t+h) = P\left[O(u) < L, \forall u \in (t,t+h], N_s(t,t+h) = 0\right].$$

Next, we obtain a recursive formula for the interval reliability of the system in $(t, t+h]$.

Theorem 5.7. *For $t+h < T$, we get*

$$IR_T^M(t,t+h) = R_T^M(t+h).$$

For $t \geq T$, the function $IR_T^M(t)$ fulfils the following Markov renewal equation

$$\begin{aligned} IR_T^M(t,t+h) = &\sum_{k=\lfloor t/T \rfloor+1}^{\lfloor (t+h)/T \rfloor} R_T^M(t+h-kT)P_{R_{1,p}}^M(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} IR_T^M(t-kT, t+h-kT)P_{R_1}^M(kT) + J_T^M(t+h), \end{aligned} \tag{5.20}$$

with initial condition $IR_T^M(0,0) = 1$ and where $J_T^M(t+h)$ is given in (5.15).

Proof. For $(t+h) < T$, $IR_T^M(t,t+h)$ is equal to $R_T^M(t+h)$. That is

$$IR_T^M(t,t+h) = \bar{F}_Y(t+h)\bar{F}_{\sigma_L}(t+h).$$

For $t+h \geq T$, conditioning to the first renewal R_1, the interval reliability is written as

$$\begin{aligned} IR_T^M(t,t+h) =& P\left[O(u) < L, \forall u \in (t,t+h],\ N_s(t,t+h)=0 | R_1 \leq t\right] \\ &+ P\left[O(u) < L,\ \forall u \in (t,t+h], N_s(t,t+h)=0 | t < R_1 < t+h\right] \\ &+ P\left[O(u) < L,\ \forall u \in (t,t+h], N_s(t,t+h)=0 | R_1 \geq t+h\right]\Big]. \end{aligned}$$

If $R_1 \geq t+h$

$$IR_T^M(t,t+h) = J_T^M(t+h).$$

If $t < R_1 < t+h$

$$\begin{aligned} IR_T^M(t,t+h) &= P\left[O(u) < L,\ \forall u \in (t,t+h], N_s(t,t+h)=0 | t < R_1 < t+h\right] \\ &= \sum_{k=\lfloor t/T \rfloor+1}^{\lfloor (t+h)/T \rfloor} P_{R_{1,p}}^M(kT) P\left[O(u-kT) < L, \forall u \in (0,t+h-kT], N_s(0,t+h-kT)=0\right] \\ &= \sum_{k=\lfloor t/T \rfloor+1}^{\lfloor (t+h)/T \rfloor} R_T^M(t+h-kT) P_{R_{1,p}}^M(kT). \end{aligned}$$

If $R_1 \leq t$

$$\begin{aligned} IR_T^M(t,t+h) &= P\left[O(u) < L, \forall u \in (t,t+h],\ N_s(t,t+h)=0 | R_1 \leq t\right] \\ &= \sum_{k=1}^{\lfloor t/T \rfloor} P_{R_1}^M(kT) P\left[O(u-kT) < L, \forall u \in (t-kT, t+h-kT],\right. \\ &\quad \left. N_s(t-kT, t+h-kT)=0\right] \\ &= \sum_{k=1}^{\lfloor t/T \rfloor} IR_T^M(t-kT, t+h-kT) P_{R_1}^M(kT). \end{aligned}$$

Then, for $t+h \geq T$, $IR_T^M(t,t+h)$ verifies the following recursive equation

$$\begin{aligned} IR_T^M(t,t+h) =& \sum_{k=\lfloor t/T \rfloor+1}^{\lfloor (t+h)/T \rfloor} R_T^M(t+h-kT) P_{R_{1,p}}^M(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} IR_T^M(t-kT, t+h-kT) P_{R_1}^M(kT) + J_T^M(t+h), \end{aligned}$$

and the result holds[1]. □

The joint interval reliability of the system

Let $JIR_T^M(t_1, t_1+h_1, t_2, t_2+h_2, \ldots, t_n, t_n+h_n)$ be the joint interval reliability defined as the probability that the system is working in $[t_1, t_1+h_1]$, $[t_2, t_2+h_2]$, …, $[t_n, t_n+h_n]$ with $t_i, h_i \geq 0$, $\forall i = 1, 2, \ldots, n$, that is,

$$\begin{aligned} & JIR_T^M(t_1, t_1+h_1, t_2, t_2+h_2, \ldots, t_n, t_n+h_n) \\ &\quad = P\left[O(u) < L, \forall u \in \cup_{i=1}^n [t_i, t_i+h_i],\ N_s(t_i, t_i+h_i)=0,\ \forall i = 1, 2, \ldots, n\right]. \end{aligned}$$

[1]Note that if $\lfloor t/T \rfloor = \lfloor (t+h)/T \rfloor$, then $IR_T^M(t,t+h)=0$.

To ease the calculation, we consider $n = 2$. Then

$$\begin{aligned}
&JIR_T^M(t_1, h_1, t_2, h_2) \\
&\qquad = P\left[O(u) < L, \forall u \in [t_1, t_1 + h_1] \cup [t_2, t_2 + h_2], N_s(t_1, t_1 + h_1) = 0,\right. \\
&\qquad \left. N_s(t_2, t_2 + h_2) = 0\right].
\end{aligned}$$

If the intervals $[t_1, t_1 + h_1]$ and $[t_2, t_2 + h_2]$ overlap, the joint interval reliability can be written in terms of the interval reliability as [141]

$$JIR_T^M(t_1, h_1, t_2, h_2) = IR_T^M(\min\{t_1, t_2\}, \max\{t_1 + h_1, t_2 + h_2\}).$$

The joint interval reliability for two disjoint intervals ($[t_1, t_1 + h_1] \cap [t_2, t_2 + h_2] = \emptyset$) is given by the following result.

Theorem 5.8. *If $t_2 + h_2 < T$, the joint interval reliability in $[t_1, t_1 + h_1]$ and $[t_2, t_2 + h_2]$ is*

$$JIR_T^M(t_1, t_1 + h_1, t_2, t_2 + h_2) = R_T^M(t_2 + h_2),$$

where $R_T^M(t)$ is given in (5.6). If $t_2 + h_2 \geq T$, the joint interval reliability fulfils the following recursive equation

$$\begin{aligned}
JIR_T^M(t_1, t_1 + h_1, t_2, t_2 + h_2) &= J_T^M(t_2 + h_2) + \sum_{k=\lfloor t_2/T \rfloor + 1}^{\lfloor (t_2+h_2)/T \rfloor} P_{R_1,p}^M(kT) R_T^M(t_2 + h_2 - kT) \\
&+ \sum_{k=\lfloor (t_1+h_1)/T \rfloor + 1}^{\lfloor t_2/T \rfloor} P_{R_1}^M(kT) IR_T^M(t_2 - kT, t_2 + h_2 - kT) \\
&- g_T^M(t_1 + h_1) IR_T^M\left(t_2 - (\lfloor (t_1 + h_1)/T \rfloor + 1)T, t_2 + h_2 - (\lfloor (t_1 + h_1)/T \rfloor + 1)T\right) \\
&+ \sum_{k=\lfloor t_1/T \rfloor + 1}^{\lfloor (t_1+h_1)/T \rfloor} P_{R_1,p}^M(kT) JIR_T^M(0, t_1 + h_1 - kT, t_2 - kT, t_2 + h_2 - kT) \\
&+ \sum_{k=1}^{\lfloor t_1/T \rfloor} P_{R_1}^M(kT) JIR_T^M(t_1 - kT, t_1 + h_1 - kT, t_2 - kT, t_2 + h_2 - kT),
\end{aligned} \tag{5.21}$$

where J_T^M is given in (5.15), $R_T^M(t)$ is given in Theorem 5.6, $IR_T^M(t, t+h)$ is given in Theorem 5.7, $g_T^M(t)$ is given in (5.17), and $IR_T^M(t_1, t_2) = 0$ if $t_1 < 0$.

Proof. If $t_2 + h_2 < T$, no maintenance action is performed and the result holds. For $t_2 + h_2 \geq T$, conditioning to the first renewal R_1, the joint interval reliability is written as

$$JIR_T^M(t_1, h_1, t_2, h_2) = P\left[O(u) < L, \forall u \in Q, N_s(Q) = 0\right],$$

where $Q = [t_1, t_1 + h_1] \cup [t_2, t_2 + h_2]$. So, we get

$$\begin{aligned}
JIR_T^M(t_1, h_1, t_2, h_2) = &P\left[O(u) < L, \forall u \in Q, N_s(Q) = 0 | R_1 > t_2 + h_2\right] \\
&+ P\left[O(u) < L, \forall u \in Q, N_s(Q) = 0 | t_2 < R_1 \leq t_2 + h_2\right] \\
&+ P\left[O(u) < L, \forall u \in Q, N_s(Q) = 0 | t_1 + h_1 < R_1 \leq t_2\right] \\
&+ P\left[O(u) < L, \forall u \in Q, N_s(Q) = 0 | t_1 < R_1 \leq t_1 + h_1\right] \\
&+ P\left[O(u) < L, \forall u \in Q, N_s(Q) = 0 | R_1 \leq t_1\right].
\end{aligned}$$

Some quantities are obtained in a similar way to the previous theorems. So, we get

$$P\left[O(u) < L, \forall u \in Q,\, N_s(Q) = 0 | R_1 > t_2 + h_2\right] = J_T^M(t_2 + h_2).$$

$$\begin{aligned} &P\left[O(u) < L, \forall u \in Q,\, N_s(Q) = 0 | t_2 < R_1 \leq t_2 + h_2\right] \\ &= \sum_{k=\lfloor t_2/T \rfloor + 1}^{\lfloor (t_2+h_2)/T \rfloor} P_{R_{1,p}}(kT) R_T^M(t_2 + h_2 - kT). \end{aligned}$$

Similar to the reasoning of the joint availability and the interval reliability,

$$\begin{aligned} &P\left[O(u) < L, \forall u \in Q,\, N_s(Q) = 0 | t_1 + h_1 < R_1 \leq t_2\right] \\ &= \sum_{k=\lfloor (t_1+h_1)/T) \rfloor + 1}^{\lfloor t_2/T \rfloor} P_{R_1}^M(kT) IR_T^M(t_2 - kT, t_2 + h_2 - kT) \\ &- g_T^M(t_1 + h_1) IR_T^M(t_2 - (\lfloor (t_1 + h_1/T) \rfloor + 1)T, t_2 + h_2 - (\lfloor (t_1 + h_1/T) \rfloor + 1)T). \end{aligned}$$

Furthermore,

$$\begin{aligned} &P\left[O(u) < L, \forall u \in Q,\, N_s(Q) = 0 | t_1 < R_1 \leq t_1 + h_1\right] \\ &= \sum_{k=\lfloor t_1/T \rfloor + 1}^{\lfloor (t_1+h_1)/T \rfloor} P_{R_{1,p}}^M(kT) JIR_T^M(0, t_1 + h_1 - kT, t_2 - kT, t_2 + h_2 - kT). \end{aligned}$$

Finally,

$$\begin{aligned} &P\left[O(u) < L, \forall u \in Q,\, N_s(Q) = 0 | R_1 \leq t\right] \\ &= \sum_{k=1}^{\lfloor t_1/T \rfloor} P_{R_1}^M(kT) JIR_T^M(t_1 - kT, t_1 + h_1 - kT, t_2 - kT, t_2 + h_2 - kT), \end{aligned}$$

and the result holds. □

5.5 Numerical examples

In this section, some numerical examples are provided to illustrate the analytical results. To this end, we consider a system subject to a degradation process whose growth is distributed following a homogeneous gamma process with parameters $\alpha = 0.3$ and $\beta = 0.15$. The system fails when the deterioration level of the system exceeds the threshold $L = 40$.

The system is also subject to sudden shocks that arrive at the system following an NHPP with intensity modelled by a Weibull distribution with scale parameter $c = 23$ and shape parameter $d = 15$.

Under these specifications, the expected time to a degradation failure is $E\left[\sigma_L\right] = 20.7173$ $t.u.$, and the expected time to a sudden shock is $E\left[Y\right] = 22.2052$ $t.u.$

Furthermore, we assume the cost sequence $C_c = 400$ $m.u.$, $C_p = 200$ $m.u.$, $C_d = 50$ $m.u./t.u.$, and $C_I = 5$ $m.u.$

First, we analyse the expected cost in the system life cycle $(0, t_f]$. In these examples, we assume that the life cycle is finite and equal to $t_f = 50$ $t.u.$ At time t_f the system is scrapped regardless of the true state of the system.

MATLAB software, in its version R2014a, was used for the following examples. The code was run on an Intel Core i5-2500 processor with 8GB DDR3 RAM under Windows 7 Professional.

5.5.1 Analysis of the expected cost rate in the finite life cycle for T fixed

We consider a time between inspections $T = 10\ t.u.$ The optimisation problem for the expected cost in the life cycle based on the recursive formula given in (5.7) is computed as follows:

1. A grid of size 40 is obtained by discretising the set $[1, 40]$ into 40 equally spaced points from 1 to 40 for the preventive threshold M. Let M_i be the i-th value of the grid, for $i = 1, 2, \ldots, 40$.

2. For $T = 10$ fixed and for each M_i fixed, we obtain 50000 simulation of (R_1, I_1, W_d), where R_1 is the length of the first renewal cycle, I_1 is to the nature of the first maintenance action performed (corrective or preventive), and W_d is the downtime in the first replacement cycle. With these simulations and applying Monte Carlo method, we obtain $\tilde{P}^{M_i}_{R_{1,p}}(10k)$, $\tilde{P}^{M_i}_{R_{1,c}}(10k)$, $\tilde{P}^{M_i}_{R_1}(10k)$, and $\tilde{E}\left[W^{M_i}_{10}(10(k-1), 10k)\right]$ corresponding to the estimations of the quantities $P^{M_i}_{R_{1,p}}(10k)$, $P^{M_i}_{R_{1,c}}(10k)$, $P^{M_i}_{R_1}(10k)$, and $E\left[W^{M_i}_{10}(10(k-1), 10k)\right]$ for $k = 1, 2, \ldots, 5$ given in Section 5.2 by (5.2), (5.3), (5.4), and (5.5), respectively (see Figure 5.2).

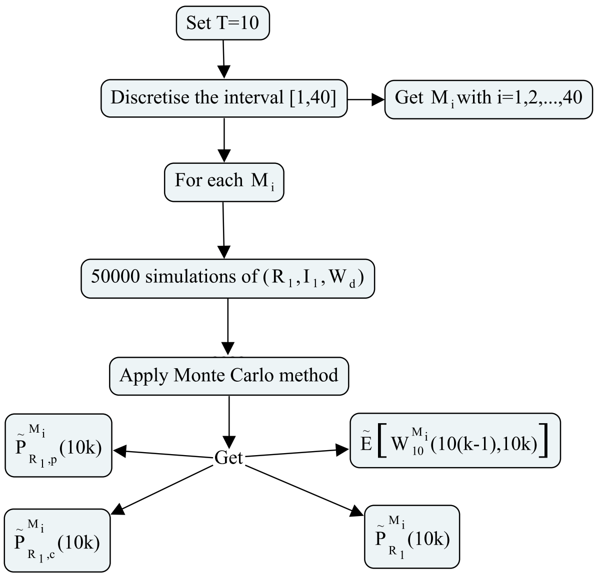


Figure 5.2: Procedure of the Monte Carlo simulation method for $T = 10$ and variable M.

3. If $t < 10$, the expression given by (5.6) is evaluated by using the Simpson's rule in 12 equally spaced points in the interval $[0, t]$.

4. The expected cost in the system life cycle $\tilde{E}\left[C^{M_i}_{10}(50)\right]$ is calculated by using the recursive formula given in (5.7), replacing the corresponding probabilities by their estimations calculated in Step 2, with initial condition $\tilde{E}\left[C^{M_i}_{10}(0)\right] = 0$.

5. For $T = 10\ t.u.$ fixed, the search of an optimal maintenance strategy is reduced to find the value M_{opt} minimising the expected cost in the life cycle $\tilde{E}\left[C_{10}^{M}(50)\right]$. That is

$$\tilde{E}\left[C_{10}^{M_{opt}}(50)\right] = \min_{0<M\leq L}\left\{\tilde{E}\left[C_{10}^{M}(50)\right]\right\}.$$

Now, we compare the expected cost rate in the life cycle, $\tilde{E}\left[C_{10}^{M}(50)\right]/50$, based on the recursive method to the expected cost rate in the life cycle based on strictly Monte Carlo simulation. Secondly, the expected cost rate in a life cycle is compared to the asymptotic expected cost rate. The results are detailed below.

Figure 5.3 shows the expected cost rate in the life cycle calculated using the recursive method and the expected cost rate in the life cycle calculated using strictly Monte Carlo simulation. The expected cost rate in the life cycle based on strictly Monte Carlo simulation was calculated for 40 equally spaced points in the interval $[1, 40]$ with 50000 simulations for each point. Based on Figure 5.3, the expected cost rate based in the life cycle on the recursive method reaches its minimum value at $M_{opt} = 18\ d.u.$, with an expected cost rate in the life cycle of 18.6310 $m.u./t.u.$ On the other hand, the expected cost rate in the life cycle based on strictly Monte Carlo simulation reaches its minimum value at $M = 18\ d.u.$, with an expected cost rate in the life cycle of $18.6621\ m.u./t.u.$

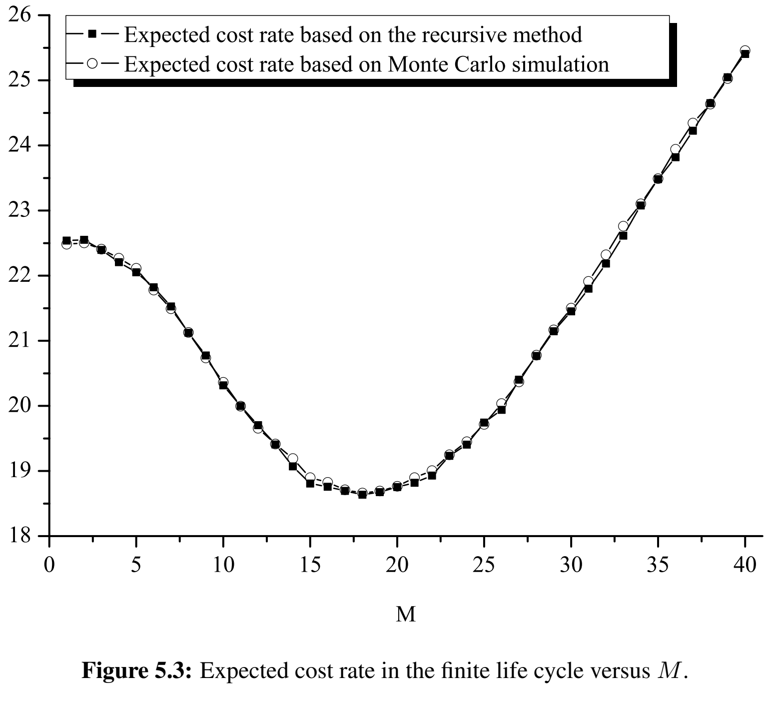


Figure 5.3: Expected cost rate in the finite life cycle versus M.

For $T = 10\ t.u.$, Table 5.1 shows the average number of completed renewal cycles at time $t_f = 50\ t.u.$ As we can observe in Table 5.1, the average number of complete renewal cycles is not too high. For this particular example, different calculation times are obtained. The total time necessary to compute the method based strictly on Monte Carlo simulation is 3.6946 hours against 1.0166 hours necessaries to compute the recursive method.

M	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{E}\left[\mathbf{N}_{10}^{\mathbf{M}}(\mathbf{50})\right]$	4.9989	4.9906	4.9693	4.9250	4.8637	4.7785	4.6813	4.5650	4.4372	4.2988

M	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
$\mathbf{E}\left[\mathbf{N}_{10}^{\mathbf{M}}(\mathbf{50})\right]$	4.1588	4.0157	3.8785	3.7473	3.6030	3.4801	3.3567	3.2379	3.1246	3.0205

M	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
$\mathbf{E}\left[\mathbf{N}_{10}^{\mathbf{M}}(\mathbf{50})\right]$	2.9235	2.8249	2.7415	2.6551	2.5763	2.5039	2.4430	2.3745	2.3148	2.2581

M	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
$\mathbf{E}\left[\mathbf{N}_{10}^{\mathbf{M}}(\mathbf{50})\right]$	2.2085	2.1598	2.1153	2.0679	2.0264	1.9904	1.9515	1.9149	1.8807	1.8549

Table 5.1: Average number of complete renewal cycles up to $t_f = 50\ t.u.$ for different values of M and $T = 10\ t.u.$ fixed.

For $T = 10\ t.u.$, Figure 5.4 shows the expected cost rate in the life cycle calculated by using the recursive method and the asymptotic expected cost rate. The asymptotic expected cost rate was obtained for 40 equally spaced points in the interval $[1, 40]$ with 50000 simulations for each point. As we said previously, the value of M minimising the expected cost in the life cycle is

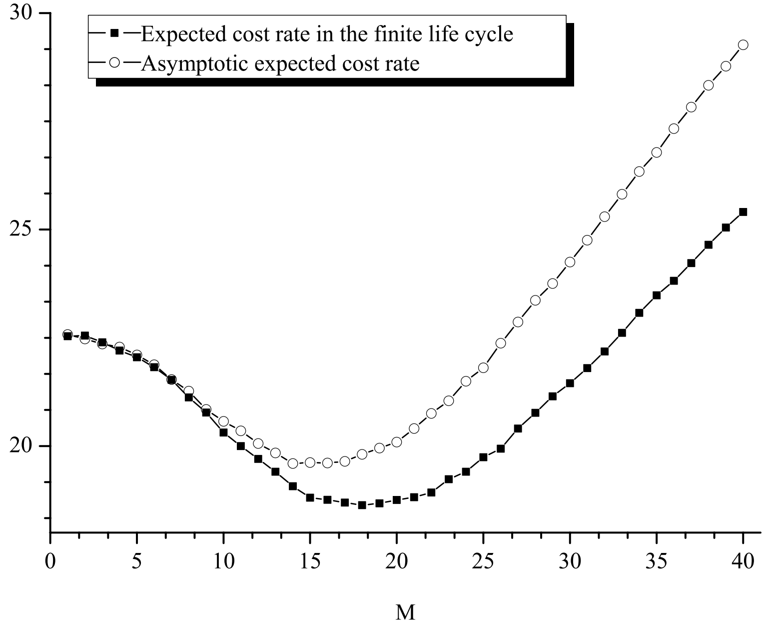


Figure 5.4: Expected cost rate in the finite life cycle and asymptotic expected cost rate versus M.

reached at $M_{opt} = 18\ d.u.$, with an expected cost rate in the life cycle of 18.6310 $m.u./t.u.$ On the other hand, the asymptotic expected cost rate reaches its minimum value at $M_{opt} = 14\ d.u.$, with an asymptotic expected cost rate of 19.5978 $m.u./t.u.$

In order to analyse the variation between the transient and the asymptotic approach, a grid of size 10 is obtained by discretising the set $[10, 100]$ into 10 equally spaced points from 10 to 100 for the finite time t_f. Let t_{f_n} be the n-th value of the grid obtained previously, for $n = 1, 2, \ldots, 10$. Let $VP_{10}^{M_{opt},16}(t_{f_n})$ be the relative variation percentage for the expected cost rate at time t_{f_n} defined as

$$VP_{10}^{M_{opt},14}(t_{f_n}) = \frac{\tilde{C}^{\infty}(10,14) - \tilde{E}_{opt,10}^{M_{opt}}(t_{f_n})}{\tilde{C}^{\infty}(10,14)} \cdot 100,$$

for $n = 1, 2, \ldots, 10$, where $\tilde{E}_{opt,10}^{M_{opt}}(t_{f_n})$ denotes the minimal expected cost rate in the life cycle at time t_{f_n} for a time between inspections $T = 10\ t.u.$ and preventive threshold M_{opt} and $\tilde{C}^{\infty}(10,14)$ denotes the minimal asymtotic cost rate for a time between inspections $T = 10$ $t.u.$ fixed, that is, when $M = 14\ d.u.$ Figure 5.5 shows the relative variation percentage of the expected cost rate.

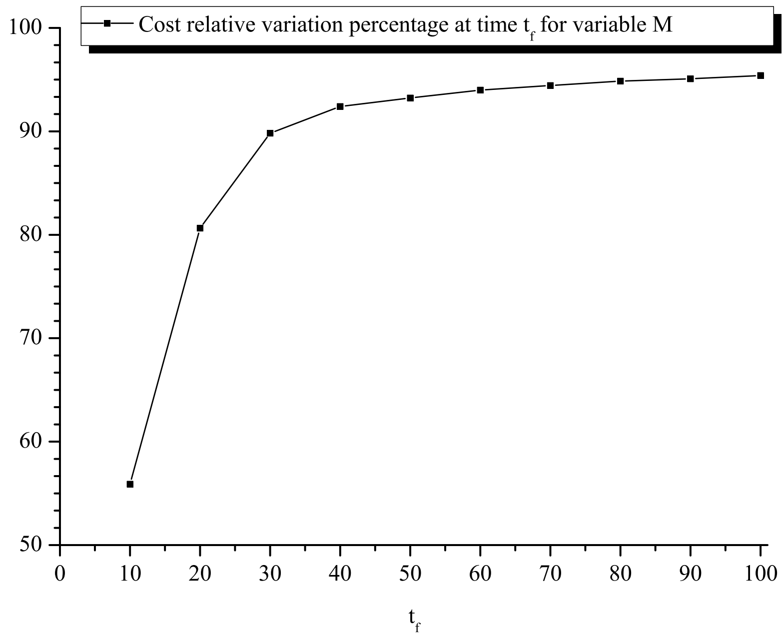


Figure 5.5: Relative variation percentage for the expected cost rate for a random finite life cycle and $T = 10\ t.u.$

Besides the expected cost in the life cycle, it is necessary to evaluate its standard deviation. Figure 5.6 shows the expected cost rate in the life cycle calculated by using the recursive method with its standard deviation associated. The corresponding deviation was calculated for 40 equally spaced points in the interval $[1, 40]$ throughout the recursive formula based on (5.11) and following the steps detailed in 5.5.1.

Now, we focus on the influence of the main model parameters on the expected cost in the life cycle. First, a sensitivity analysis of the gamma process parameter is performed.

The values of the gamma process parameters are modified according to the following specifications:

$$\alpha_{(v_i\%)} = \alpha \left[1 + \frac{v_i}{100}\right] \quad \text{and} \quad \beta_{(v_j\%)} = \beta \left[1 + \frac{v_j}{100}\right], \tag{5.22}$$

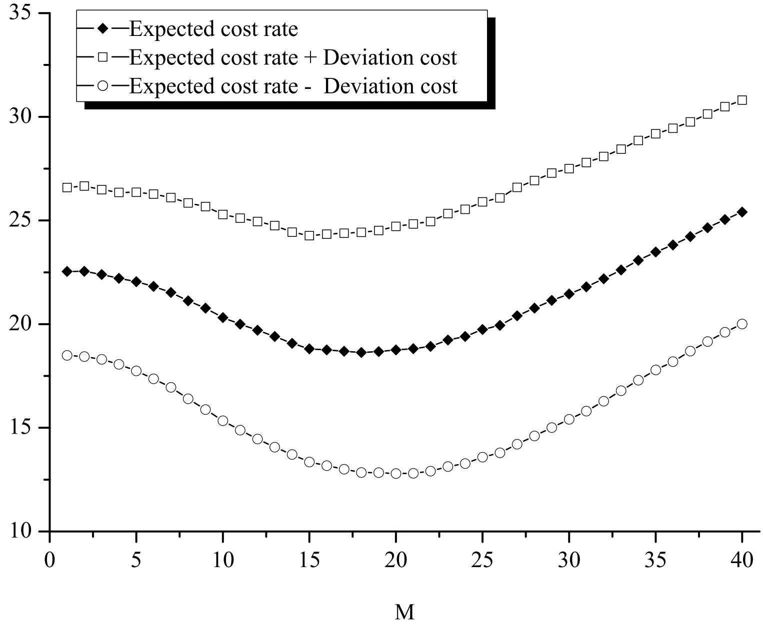


Figure 5.6: Expected cost rate in the finite life cycle and its associated standard deviation versus M.

where v_i and v_j are, respectively, the i-th and j-th position of the vector $\mathbf{v} = (-10, -5, -1, 0, 1, 5, 10)$. Then, the parameter values for α and β are simultaneous and independently modified both for increasing and decreasing changes.

Let $\tilde{E}\left[C^M_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(t_f)\right]$ be the minimal expected cost rate in the life cycle obtained when the gamma process parameters α and β vary according to the specifications given in (5.22). The expected cost in the life cycle for each combination of $\alpha_{(v_i\%)}$ and $\beta_{(v_j\%)}$ were calculated following the steps detailed in 5.5.1. The relative measure $V^M_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ is defined as

$$V^M_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50) = \frac{\left|\tilde{E}\left[C^{M_{opt}}_{10}(50)\right] - \tilde{E}\left[C^M_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)\right]\right|}{\tilde{E}\left[C^{M_{opt}}_{10}(50)\right]}, \tag{5.23}$$

where $\tilde{E}\left[C^{M_{opt}}_{10}(50)\right]$ is the minimal expected cost rate in the life cycle calculated in 5.5.1.

For i and j fixed, $V^M_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ measures the relative difference between the expected cost in the life cycle with the original parameter values and the minimal expected cost in the life cycle calculated using the modified parameter values for M and $T = 10\, t.u.$ fixed. Values closer to zero have a lower influence on the expected cost rate in the life cycle.

Table 5.2 shows the relative variation percentages with a shaded grey scale. Each cell represents $V^M_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ expressed in percentage. Darker colours of cells denote a higher relative variation percentage. The results obtained show that $V^M_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ grows when α increases and β decreases and $V^M_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ decreases when α decreases and β increases.

	$\beta_{(-10\%)}$	$\beta_{(-5\%)}$	$\beta_{(-1\%)}$	β	$\beta_{(1\%)}$	$\beta_{(5\%)}$	$\beta_{(10\%)}$
$\alpha_{(-10\%)}$	1.7227	2.4799	5.2504	6.2096	7.0398	9.0155	11.3808
$\alpha_{(-5\%)}$	5.6246	0.8200	2.6973	3.0862	4.1563	6.5277	9.8321
$\alpha_{(-1\%)}$	9.1302	3.7448	0.2358	0.8907	1.1872	4.6431	7.6524
α	9.9997	4.2054	0.9112	0.0000	0.7065	3.7942	7.0842
$\alpha_{(1\%)}$	10.8585	5.0557	1.4441	0.7544	0.1074	3.6036	7.1393
$\alpha_{(5\%)}$	14.7767	8.7330	4.5634	3.6443	2.5358	0.9741	4.6922
$\alpha_{(10\%)}$	19.6825	12.8122	8.3895	7.4019	0.1817	2.5717	1.6827

Table 5.2: Relative variation percentages for the expected cost rate in the finite life cycle for the gamma process parameters for $T = 10\ t.u.$ fixed.

In this way, $V^M_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ reaches its minimum value when α is minimum and β is maximum and its maximum value when α is maximum and β is minimum.

Similarly, the values of the Weibull distribution parameters are modified according to the following specifications:

$$c_{(v_i\%)} = c\left[1 + \frac{v_i}{100}\right] \quad \text{and} \quad d_{(v_j\%)} = d\left[1 + \frac{v_j}{100}\right]. \tag{5.24}$$

Let $\tilde{E}\left[C^M_{10,c_{(v_i\%)},d_{(v_j\%)}}(t_f)\right]$ be the minimal expected cost rate in the life cycle obtained when the Weibull distribution parameters (c and d) vary simultaneously according to the specifications given in (5.24) for different values of M for $T = 10\ t.u.$ The expected cost rate in the life cycle for each combination of $c_{(v_i\%)}$ and $d_{(v_j\%)}$ were calculated following the steps detailed in 5.5.1. Now, the relative measure $V^M_{10,c_{(v_i\%)},d_{(v_j\%)}}(50)$ is

$$V^M_{10,c_{(v_i\%)},d_{(v_j\%)}}(50) = \frac{\left|\tilde{E}\left[C^{M_{opt}}_{10}(50)\right] - \tilde{E}\left[C^M_{10,c_{(v_i\%)},d_{(v_j\%)}}(50)\right]\right|}{\tilde{E}\left[C^{M_{opt}}_{10}(50)\right]}. \tag{5.25}$$

The relative variation percentages are presented in Table 5.3. The results show that the parameter c has a greater effect on $V^M_{10,c_{(v_i\%)},d_{(v_j\%)}}(50)$ than the parameter d, reaching the lowest values when the variation for c is maximised, that is $\pm 10\%$.

5.5.2 Analysis of the expected cost rate in the finite life cycle for M fixed

We now analyse the influence of the time between inspections T on the expected cost in the life cycle $t_f = 50\ t.u.$ for a preventive threshold $M = 20\ d.u.$ The optimisation problem for the expected cost in the life cycle based on the recursive formula given in (5.7) is computed throughout the following steps:

1. A grid of size 16 is obtained by discretising the set $[3, 48]$ into 16 equally spaced points from 3 to 48 for T. Let T_i be the i-th value of the grid, for $i = 1, 2, \ldots, 16$.

	$d_{(-10\%)}$	$d_{(-5\%)}$	$d_{(-1\%)}$	d	$d_{(1\%)}$	$d_{(5\%)}$	$d_{(10\%)}$
$c_{(-10\%)}$	10.6760	10.1071	9.5059	9.6025	9.3771	9.1839	8.8940
$c_{(-5\%)}$	4.3617	3.7390	3.6215	3.3275	3.2088	2.8553	2.3229
$c_{(-1\%)}$	1.2366	1.1433	0.3590	0.4833	0.5284	0.3006	0.4629
c	0.9166	0.1397	0.1545	0.0000	0.1774	0.2099	0.9162
$c_{(1\%)}$	0.1830	0.3904	0.6361	0.5326	0.4693	1.2679	0.9988
$c_{(5\%)}$	1.4259	1.6798	2.0418	2.0641	2.2912	2.1884	2.6672
$c_{(10\%)}$	2.9402	3.4337	3.4041	3.5633	3.6862	3.7134	3.6401

Table 5.3: Relative variation percentages for the expected cost rate in the finite life cycle for the sudden shock process parameters for $T = 10\ t.u.$ fixed.

2. For $M = 20\ d.u.$ fixed and for each T_i fixed, we obtain 50000 simulation of (R_1, I_1, W_d). With these simulations and applying Monte Carlo method, we obtain the estimations of the probabilities $\tilde{P}^{20}_{R_{1,p}}(kT_i)$, $\tilde{P}^{20}_{R_{1,c}}(kT_i)$, $\tilde{P}^{20}_{R_1}(kT_i)$, and $\tilde{E}\left[W^{20}_{T_i}((k-1)T_i, kT_i)\right]$ for $k = 1, 2, 3$, respectively. Figure 5.7 shows the procedure of the Monte Carlo simulation method for $M = 20$ and variable T.

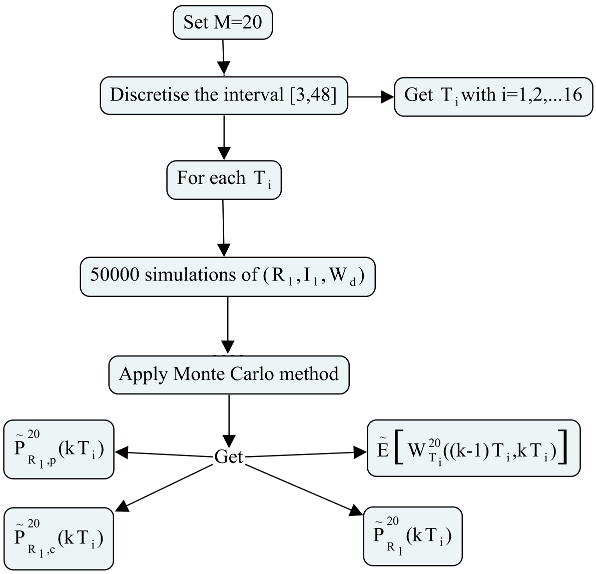


Figure 5.7: Procedure of the Monte Carlo simulation method for $M = 20$ and variable T.

3. If $t < T$, the expression given in (5.6) is evaluated by using the Simpson's rule in 12 equally spaced points in the interval $[0, t]$.

4. The expected cost in the life cycle $\tilde{E}\left[C_{T_i}^{20}(50)\right]$ is calculated by using the recursive formula given in (5.7), replacing the corresponding probabilities by their estimations calculated in Step 2, with initial condition $\tilde{E}\left[C_{T_i}^{20}(0)\right]=0$.

5. For $M=20\ d.u.$ fixed, the search of the optimal maintenance strategy is reduced to find the value T_{opt} minimising the expected cost in the life cycle $\tilde{E}\left[C_{T}^{20}(50)\right]$. That is

$$\tilde{E}\left[C_{T_{opt}}^{20}(50)\right]=\min_{T>0}\left\{\tilde{E}\left[C_{T}^{20}(50)\right]\right\}.$$

Now, we compare the expected cost rate in the life cycle, $\tilde{E}\left[C_{T_i}^{20}(50)\right]/50$, based on the recursive method to the method based on strictly Monte Carlo simulation. Secondly, the expected cost rate in a life cycle is compared to the expected cost rate. The results are detailed below.

Figure 5.8 shows the expected cost rate in the life cycle of the system, $\tilde{E}\left[C_{T}^{20}(50)\right]/50$, calculated using the recursive formula and the expected cost rate in the life cycle calculated based on strictly Monte Carlo simulation. The expected cost rate in the life cycle calculated based on strictly Monte Carlo simulation was obtained for 16 equally spaced points in $[3,48]$ with 50000 simulations for each point. Based on Figure 5.8, the expected cost rate in the life cycle using the recursive method reaches its minimum value at $T_{opt}=18\ t.u.$, with an expected cost rate in the life cycle of 16.4890 $m.u./t.u.$ On the other hand, using strictly Monte Carlo simulation, the expected cost rate in the life cycle reaches its minimum value at $T_{opt}=18\ t.u.$ with an expected cost rate in the life cycle of 15.9406 $m.u./t.u.$

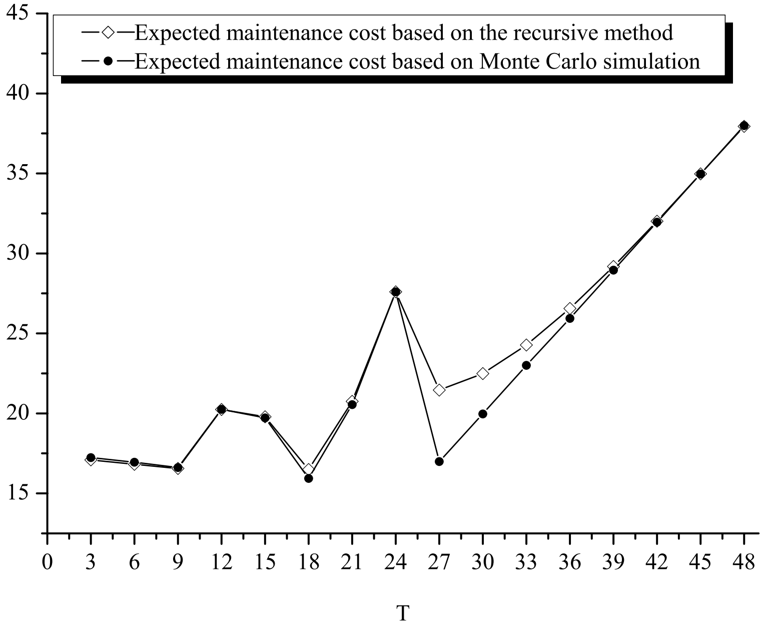


Figure 5.8: Expected cost rate in the finite life cycle versus T.

For $M=20\ d.u.$, Table 5.4 shows the average number of completed renewal cycles up to $t_f=50\ t.u.$ for each value of T. As we can observe in Table 5.4, the average number of completed renewal cycles is not too high. For this particular example, different calculation times are obtained. The total time necessary to compute the expected cost rate in the life cycle using strictly on Monte Carlo simulation is 77.2434 minutes against 30.7836 minutes necessaries using the recursive method.

T	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**
$\mathbf{E}\left[\mathbf{N_T^{20}(50)}\right]$	3.5716	3.2430	2.7779	2.7808	2.4149	1.7878	1.9215	1.9925
T	**27**	**30**	**33**	**36**	**39**	**42**	**45**	**48**
$\mathbf{E}\left[\mathbf{N_T^{20}(50)}\right]$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 5.4: Average number of complete renewal cycles up to $t_f = 50\ t.u.$ for different values of T and $M = 20\ d.u.$ fixed.

For $M = 20\ d.u.$ fixed, Figure 5.9 shows the expected cost rate in the life cycle calculated by using the recursive method and the asymptotic expected cost rate. The asymptotic expected cost rate was obtained by using strictly Monte Carlo simulation for 16 equally spaced points in the interval $[3, 48]$ with 50000 simulations for each point. As we said previously, the expected cost

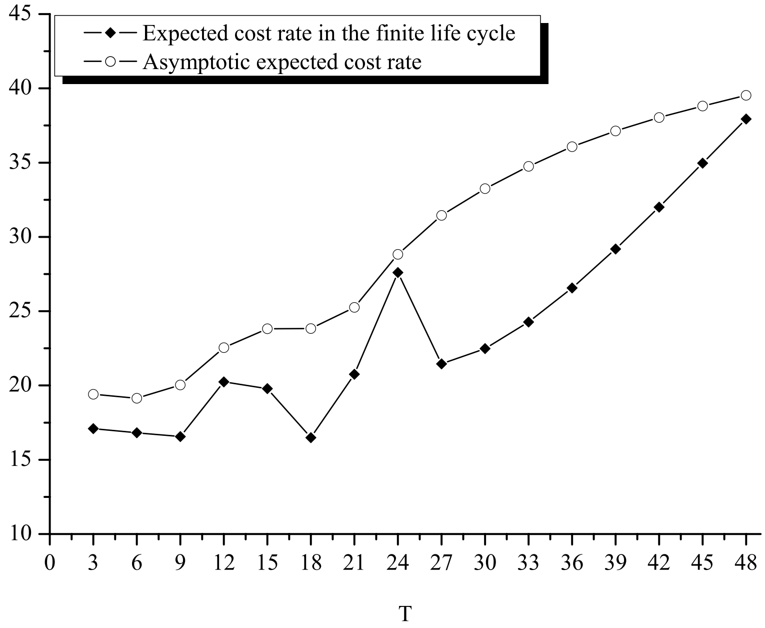


Figure 5.9: Expected cost rate in the finite life cycle and asymptotic expected cost rate versus T.

rate in the life cycle of the system reaches its minimum value at $T = 18\ t.u.$ with an expected cost rate in the life cycle of $16.4890\ m.u./t.u.$ The asymptotic expected cost rate reaches its minimum value at $T = 6\ t.u.$, with an asymptotic expected cost rate of $19.1390\ m.u./t.u.$ The expected cost rate in the life cycle shows a smoother behaviour when is compared to the asymptotic expected cost rate.

Let $VP^{20}_{T_{opt},6}(t_{f_n})$ be the relative variation percentage for the expected cost rate at time t_{f_n} defined as

$$VP^{20}_{T_{opt},6}(t_{f_n}) = \frac{\tilde{C}^{\infty}(6,20) - \tilde{E}^{20}_{opt,T_{opt}}(t_{f_n})}{\tilde{C}^{\infty}(6,20)} \cdot 100,$$

for $n = 1, 2, \ldots, 10$, where $\tilde{E}^{20}_{opt,T_{opt}}(t_{f_n})$ denotes the minimal expected cost rate in the life cycle at time t_{f_n} for a preventive threshold $M = 20\ d.u.$ and time between inspections T_{opt} and $\tilde{C}^{\infty}(6,20)$ denotes the minimal asymtotic cost rate for a preventive threshold $M = 20\ d.u.$, that is, when $T = 6\ t.u.$ Figure 5.10 shows the relative variation percentage of the expected cost rate.

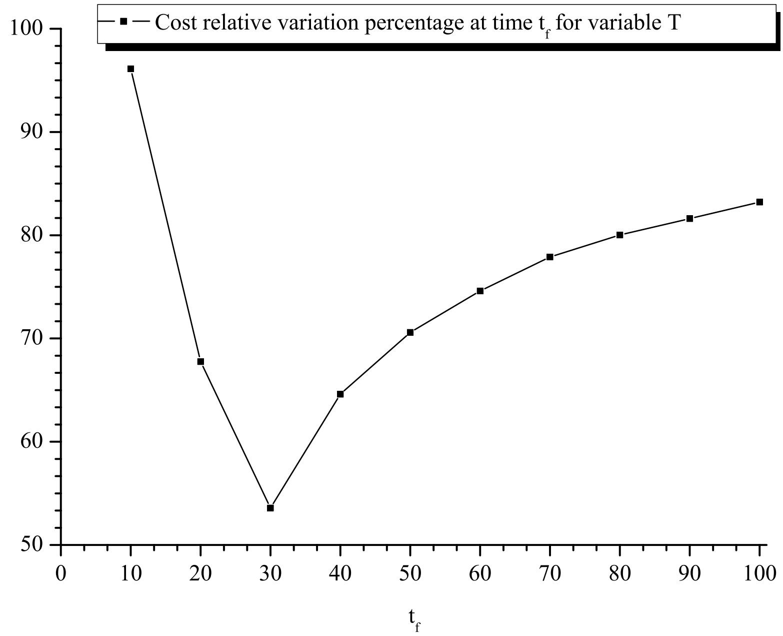


Figure 5.10: Relative variation percentage for the expected cost rate for a random finite life cycle and $M = 6\ t.u.$

Besides the expected cost in the life cycle, it is necessary to evaluate its standard deviation. Figure 5.11 shows the expected cost rate in the life cycle calculated using the recursive method with its associated standard deviation rate. The corresponding deviation was calculated for 16 equally spaced points in the interval $[3, 48]$ by using the recursive formula based on Equation (5.11) and following the steps detailed in 5.5.2.

Focusing now on the influence of the main model parameters on the solution, we analyse first the gamma process parameter sensitivity.

Let $\tilde{E}\left[C^{20}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)\right]$ be the minimal expected cost in the life cycle of the system obtained when the gamma process parameters α and β vary according to the specifications given in (5.22) for $M = 20$ fixed. Based on (5.23), $V^{20}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ denotes the relative variation between the minimal expected cost in the life cycle with the original parameter values and the minimal expected cost in the life cycle calculated by using the modified parameter values for T

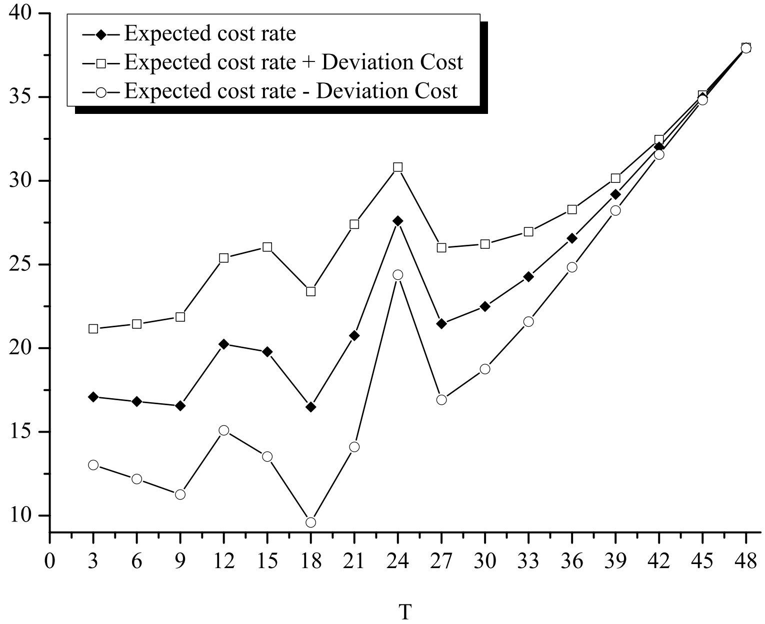


Figure 5.11: Expected cost rate in the finite life cycle and its associated standard deviation versus T.

and $M = 20$ fixed. Table 5.5 shows the values obtained for $V^{20}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ expressed in percentage. By modifying $\pm 1\%$ around $\alpha = 0.3$ and $\beta = 0.15$, the relative variation percentages are small. The results obtained also show that $V^{20}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ is lower in the diagonal of the table, that means, when the parameters α and β are modified in the same direction and magnitude.

Let $\tilde{E}\left[C^{20}_{T,c_{(v_i\%)},d_{(v_j\%)}}(50)\right]$ be the minimal expected cost in the life cycle of the system obtained when the Weibull distribution parameters c and d vary according to the specifications given in (5.24).

The relative variation $V^{20}_{T,c_{(v_i\%)},d_{(v_j\%)}}(50)$ calculated based on (5.25) are presented in Table 5.6. The results show that the parameter c has a greater effect on $V^{20}_{T,c_{(v_i\%)},d_{(v_j\%)}}(50)$ than the parameter d, reaching the lowest values when the variation for c is minimal, that is $\pm 1\%$, and the highest values when the variation for c is $+10\%$.

5.5.3 Analysis of two-dimensional expected cost rate in the finite life cycle

The expected cost in the life cycle based on the recursive formula given in 5.7 versus M and T is analysed. The optimisation problem is computed as follows:

1. A grid of size 8 is obtained by discretising the set $[6, 48]$ into 8 equally spaced points from 6 to 48 for the time between inspections T. Let T_i be the i-th value of the grid, for $i = 1, 2, \ldots, 8$.

	$\beta_{(-10\%)}$	$\beta_{(-5\%)}$	$\beta_{(-1\%)}$	β	$\beta_{(1\%)}$	$\beta_{(5\%)}$	$\beta_{(10\%)}$
$\alpha_{(-10\%)}$	1.8551	3.2107	6.5888	8.1476	8.8151	11.6468	13.7099
$\alpha_{(-5\%)}$	5.9496	0.7601	3.6188	4.3947	5.4969	8.8912	12.3448
$\alpha_{(-1\%)}$	8.6753	4.0542	0.1573	1.5787	1.7070	5.7289	10.2296
α	9.7731	4.5497	0.9700	0.0000	1.6012	5.1911	9.4719
$\alpha_{(1\%)}$	10.7052	5.1593	1.9387	0.7945	0.5656	4.7908	8.4794
$\alpha_{(5\%)}$	13.7844	8.1395	3.9495	4.2929	2.5108	0.8475	6.0115
$\alpha_{(10\%)}$	18.2417	12.7589	8.2003	7.5419	6.4367	2.3394	1.7980

Table 5.5: Relative variation percentages for the expected cost rate in the finite life cycle for the gamma process parameters for $M = 20\ d.u.$ fixed.

	$d_{(-10\%)}$	$d_{(-5\%)}$	$d_{(-1\%)}$	d	$d_{(1\%)}$	$d_{(5\%)}$	$d_{(10\%)}$
$c_{(-10\%)}$	4.8827	3.9630	3.9103	4.2142	3.9213	3.4161	3.0901
$c_{(-5\%)}$	1.7596	2.0377	1.4683	1.5498	1.0908	1.0754	1.1823
$c_{(-1\%)}$	0.4878	0.5031	0.2592	0.3003	0.0982	0.1044	0.2660
c	0.0671	0.0155	0.4093	0.0000	0.2809	0.2577	0.1621
$c_{(1\%)}$	0.1577	0.2728	0.1936	0.4307	0.2970	0.6293	1.1227
$c_{(5\%)}$	1.3489	1.0044	1.3308	1.3730	1.6873	1.4668	1.5381
$c_{(10\%)}$	2.7644	2.9502	3.2001	3.4924	3.5279	3.6381	3.0841

Table 5.6: Relative variation percentages for the expected cost rate in the finite life cycle for the sudden shock process parameters for $M = 20\ d.u.$ fixed.

2. A grid of size 40 is obtained by discretising the set $[1, 40]$ into 40 equally spaced points from 1 to 40 for the preventive threshold M. Let M_j be the j-th value of the grid, for $j = 1, 2, \ldots, 40$.

3. For each combination (T_i, M_j), we obtain 50000 simulations of (R_1, I_1, W_d). With these simulations and applying Monte Carlo method, $\tilde{P}_{R_{1,p}}^{M_j}(kT_i)$, $\tilde{P}_{R_{1,c}}^{M_j}(kT_i)$, $\tilde{P}_{R_1}^{M_j}(kT_i)$, and $\tilde{E}\left[W_{T_i}^{M_j}((k-1)T_i, kT_i))\right]$ for $k = 1, 2, \ldots, \lfloor 50/T_i \rfloor$ are obtained (see Figures 5.2 and 5.7).

4. The expected cost in the life cycle of the system $\tilde{E}\left[C_{T_i}^{M_j}(50)\right]$ is calculated by using the recursive formula given in (5.7), replacing the corresponding probabilities by their estimations calculated in Step 2, with initial condition $\tilde{E}\left[C_{T_i}^{M_j}(0)\right] = 0$.

5. The search of an optimal maintenance strategy is reduced to find the values T_{opt} and M_{opt} minimising the expected cost rate in the life cycle $\tilde{E}\left[C_{T_i}^{M_j}(50)\right]$. That is

$$\tilde{E}\left[C_{T_{opt}}^{M_{opt}}(50)\right] = \min_{\substack{T>0 \\ 0<M\leq L}} \left\{\tilde{E}\left[C_T^M(50)\right]\right\}.$$

The values of T and M minimising the expected cost rate in the life cycle $\tilde{E}\left[C_{T_i}^{M_j}(50)\right]/50$ are reached for $M_{opt} = 3\ d.u.$ and $T_{opt} = 18\ t.u.$ with an expected cost rate in the life cycle of 15.7020 $m.u./t.u.$

5.5.4 Analysis of the expected cost rate for a random finite life cycle

A random t_f is considered in this section. To this end, we assume that t_f follows a gamma distribution with parameters $\alpha = 80$ y $\beta = 0.5$. In this case, $E[t_f] = 40$ and $P[20 < t_f < 60] = 1$. The optimal maintenance strategy in this case is obtained considering the values T_{opt} and M_{opt} such that

$$\tilde{E}\left[C_{T_{opt}}^{M_{opt}}(t_f)\right] = \min\left\{T > 0;\ 0 < M \leq L;\ \tilde{E}\left[C_T^M(t_f)\right]\right\}.$$

Figure 5.13 shows the expected cost per time unit in a random finite cycle $\tilde{E}\left[C_{T_i}^{M_j}(t_f)\right]/t_f$ versus M and T. The values of this plot has been obtained considering 25 equispaced values for M and 25 values for T (from 1 to 25). The value of the integral given by (5.12) is calculated applying the Simpson's rule with 16 points from 20 to 60. The values of T and M minimising the expected cost rate in the life cycle $\tilde{E}\left[C_{T_i}^{M_j}(t_f)\right]/t_f$ are reached for $T_{opt} = 1\ t.u.$ and $M_{opt} = 12.3750\ d.u.$ with an expected cost rate in the random finite life cycle of 12.5832 $m.u./t.u.$

5.5.5 Numerical analysis of the availability measures for T and M fixed

We now analyse the point availability, the joint availability, the reliability, the interval reliability, and the joint interval reliability of the system for a time between inspections $T = 10\ t.u.$ and a preventive threshold $M = 20\ d.u.$ The methods used in the calculus are based on the recursive

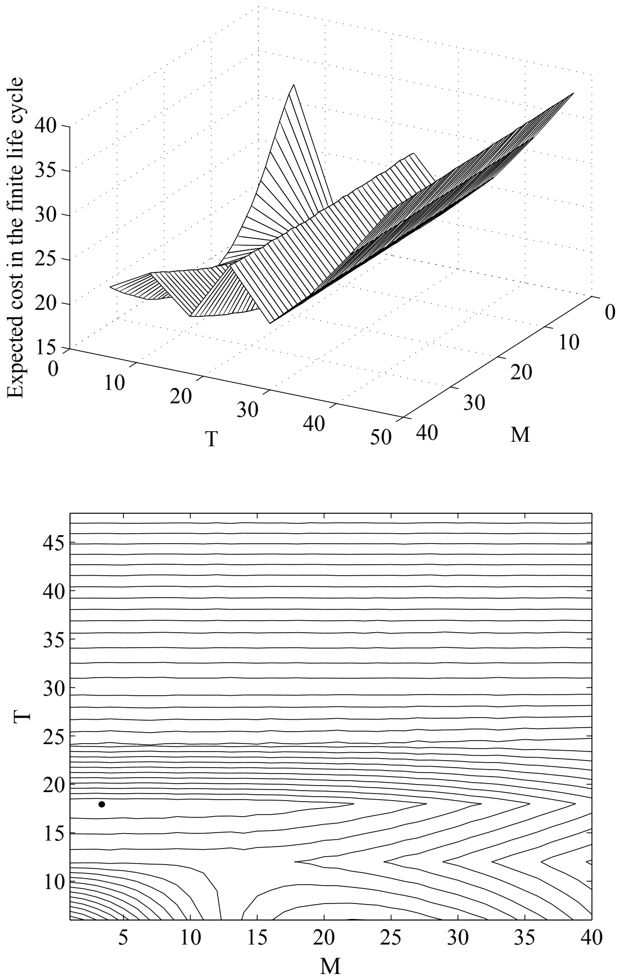


Figure 5.12: Mesh and contour plots for the expected cost rate in the finite life cycle.

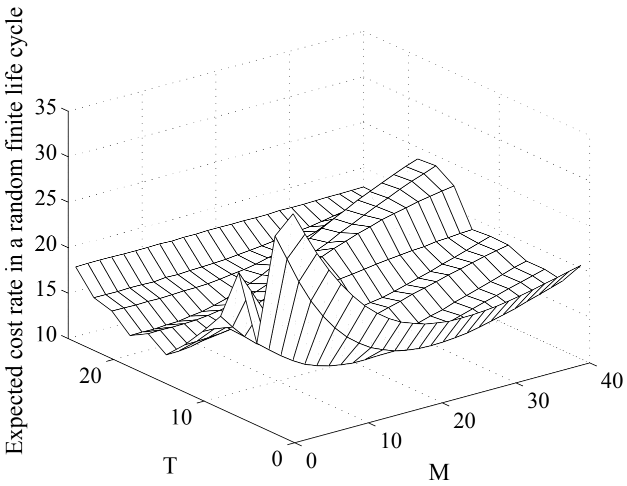


Figure 5.13: Expected cost rate in the random finite life cycle versus T and M.

methods exposed in Section 5.4 and are compared to the results obtained throughout strictly Monte Carlo simulation.

The point availability of the system based on the recursive formula given in (5.14) is computed throughout the following steps:

1. A grid of size 50 is obtained for t by discretising $[1, 50]$ into 50 points. Let t_n be the n-th value of the grid, for $n = 1, 2, \ldots, 50$.

2. For $T = 10$ and $M = 20$, 50000 simulations of (R_1, I_1, W_d) are performed and an estimation of $\tilde{P}_{R_1}^{20}(10k)$ is obtained.

3. The system availability $\tilde{A}_{10}^{20}(t_n)$ is calculated by using the recursive formula given in (5.14), replacing $P_{R_1}^{20}(10k)$ by its estimation $\tilde{P}_{R_1}^{20}(10k)$ and initial condition $\tilde{A}_{10}^{20}(0) = 1$.

Quantity $\tilde{A}_{10}^{20}(t_n)$ is also computed using Monte Carlo simulation with 50000 simulations for each value t_n. Figure 5.14 shows the availability of the system calculated using the recursive method and using Monte Carlo simulation versus t. We can conclude that, for $T = 10$ and $M = 20$ fixed, the probability that the system is working at any time instant of the life cycle is, at least, of the 84%.

The joint availability of the system at time t_1 and t_2, $JA_{10}^{20}(t_1, t_2)$ is computed throughout the following steps:

1. A grid of size 10 is obtained by discretising the set $[10, 20]$ into 10 equally spaced points from 10 to 20 for t_1. Let t_{1i} be the i-th value of the grid, for $i = 1, 2, \ldots, 10$.

2. A grid of size 10 is obtained by discretising the set $[30, 40]$ into 10 equally spaced points from 30 to 40 for t_2. Let t_{2j} be the j-th value of the grid, for $j = 1, 2, \ldots, 10$.

3. For $T = 10$ $t.u.$ and $M = 20$ $d.u.$, we obtain 50000 simulations of (R_1, I_1, W_d). With these simulations and applying Monte Carlo method, $\tilde{P}_{R_{1,p}}^{20}(10k)$ and $\tilde{P}_{R_1}^{20}(10k)$ are obtained.

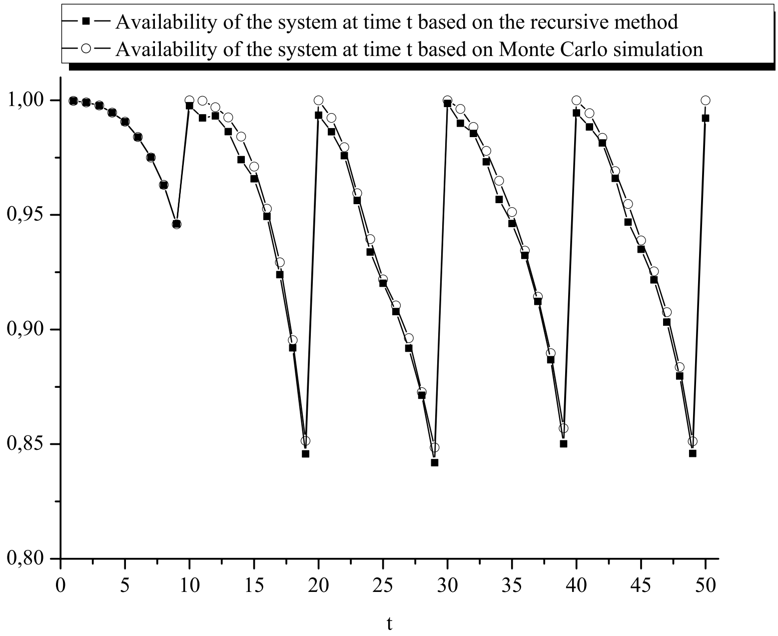


Figure 5.14: Availability of the system versus t.

4. The joint availability $\tilde{JA}_{10}^{20(2)}(t_{1i}, t_{2j})$ is calculated by using the recursive formula given in (5.16), replacing $P_{R_1}^{20}(10k)$ by its estimation $\tilde{P}_{R_1}^{20}(10k)$ and initial condition $\tilde{JA}_{10}^{20}(0,0) = 1$.

Figure 5.15 shows the joint availability versus t_1 and t_2. We can conclude that, for $T = 10\ t.u.$ and $M = 20\ d.u.$ fixed, the probability that the system is working at any time instants t_1 and t_2 is, at least, of the 61%.

The reliability of the system based on the recursive formula given in (5.19) is computed throughout the following steps:

1. A grid of size 50 is obtained by discretising the set $[1, 50]$ into 50 equally spaced points from 1 to 50 for the time instant t. Let t_n be the n-th value of the grid, for $n = 1, 2, \ldots, 50$.

2. For $T = 10\ t.u.$ and $M = 20\ d.u.$, we obtain 50000 simulations of (R_1, I_1, W_d). With these simulations and applying Monte Carlo method $\tilde{P}_{R_{1,p}}^{20}(10k)$ is obtained.

3. The system reliability $\tilde{R}_{10}^{20}(t_n)$ is calculated by using the recursive formula given in (5.19), replacing $P_{R_{1,p}}^{20}(10k)$ by its estimation $\tilde{P}_{R_{1,p}}^{20}(10k)$ calculated in Step 2, with initial condition $\tilde{R}_{10}^{14}(0) = 1$.

Quantity $\tilde{R}_{10}^{20}(t_n)$ is also computed using Monte Carlo simulation with 50000 simulations for each value t_n. Figure 5.16 shows the reliability of the system calculated using the recursive method and using Monte Carlo simulation versus t. We can conclude that, for $T = 10$ and $M = 20$ fixed, the probability that the system does not fail in its life cycle is, at least, of the 33%.

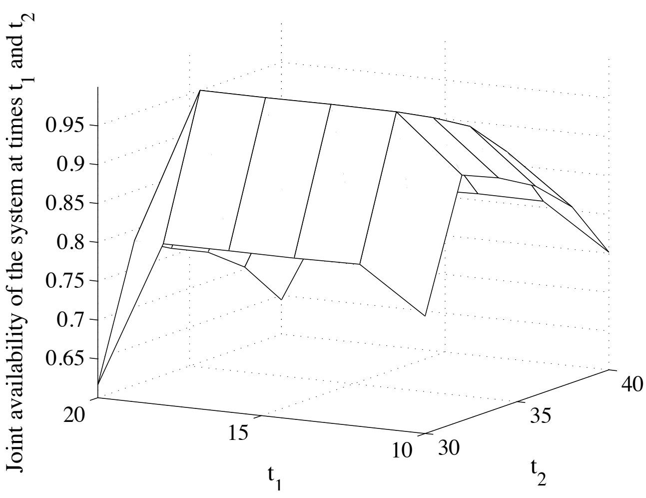


Figure 5.15: Joint availability of the system versus t_1 and t_2.

Next, the interval reliability of the system $IR_T^M(t_n, t_n + h)$ is evaluated. The method based on the recursive formula given in (5.20) is computed throughout the following steps:

1. A grid of size 10 is obtained by discretising the set $[10, 30]$ into 10 equally spaced points from 10 to 30 for the time instant t.

2. For $T = 10\ t.u.$ and $M = 20\ d.u.$, we obtain 50000 simulations of (R_1, I_1, W_d). With these simulations and applying Monte Carlo method $\tilde{P}_{R_{1,p}}^{20}(10k)$ and $\tilde{P}_{R_1}^{20}(10k)$ are obtained.

3. For $h = 5$, let $\tilde{IR}_{10}^{14}(t_n, t_n + h)$ be the system interval reliability estimation for a time between inspections $T_{opt} = 10\ t.u.$ and a preventive threshold $M_{opt} = 14\ d.u.$ at time t_n. The interval reliability of the system is calculated by using the recursive formula given in (5.17), replacing $P_{R_{1,p}}^{14}(10k)$ and $P_{R_1}^{14}(10k)$ by the estimations $\tilde{P}_{R_{1,p}}^{14}(10k)$ and $\tilde{P}_{R_1}^{14}(10k)$, respectively, calculated in Step 2 of the procedure detailed in Subsection 5.5.3, with initial condition $\tilde{IR}_{10}^{14}(0, 0) = 1$.

Quantity $\tilde{IR}_{10}^{14}(t_n, t_n + 5)$ is also computed using Monte Carlo simulation with 50000 simulations for each value t_n. Figure 5.17 shows the interval reliability of the system versus t. As we can observe, the results provided for both methods are very similar. Furthermore, we can conclude that, for $T = 10\ t.u.$ and $M = 20\ d.u.$ fixed, the probability that the system does not fail in the interval $[t, t + 5]$ is, at least, of the 71% for $10 \leq t \leq 30$.

Finally, the joint interval reliability of the system based on (5.21) is computed throughout the following steps:

1. A grid of size 10 is obtained by discretising the set $[10, 20]$ into 10 equally spaced points from 10 to 20 for t_1. Let t_{1i} be the i-th value of the grid, for $i = 1, 2, \ldots, 10$.

2. A grid of size 10 is obtained by discretising the set $[30, 40]$ into 10 equally spaced points from 30 to 40 for t_2. Let t_{2j} be the j-th value of the grid, for $j = 1, 2, \ldots, 10$.

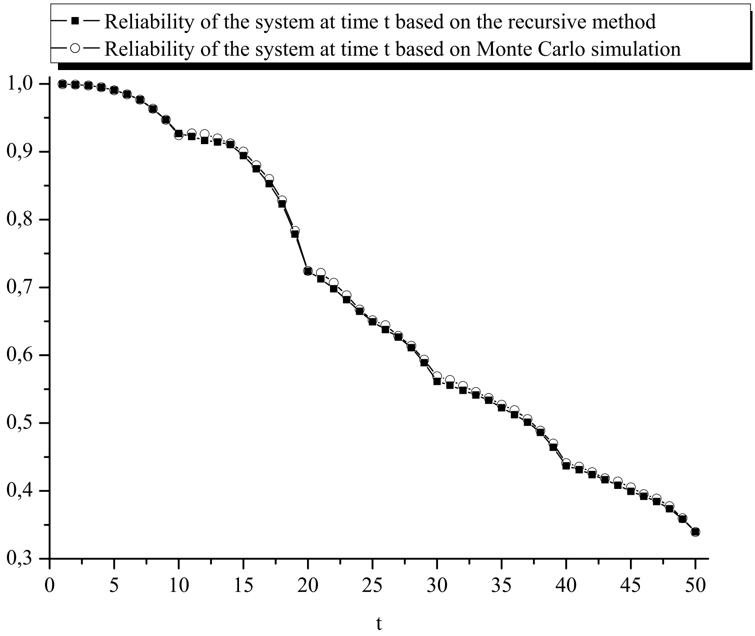


Figure 5.16: Reliability of the system versus t.

3. For $T = 10$ $t.u.$ and $M = 20$ $d.u.$, we obtain 50000 simulations of (R_1, I_1, W_d). With these simulations and applying Monte Carlo method $\tilde{P}^{20}_{R_{1,p}}(10k)$ and $\tilde{P}^{20}_{R_1}(10k)$ are obtained.

4. For $h_1 = 5$ and $h_2 = 2$, the joint interval reliability of the system $\tilde{JIR}^{20}_{10}(t_{1i}, 5, t_{2j}, h_2)$ is calculated by using the recursive formula given in (5.21), replacing $P^{20}_{R_{1,p}}(10k)$ and $P^{20}_{R_1}(10k)$ by the estimations $\tilde{P}^{20}_{R_{1,p}}(10k)$ and $\tilde{P}^{20}_{R_1}(10k)$, respectively, calculated in Step 2, with initial condition $\tilde{JIR}^{20}_{10}(0, 0, 0, 0) = 1$.

Figure 5.18 shows the joint interval reliability of the system versus t_1 and t_2. Furthermore, we can conclude that for $T = 10$ $t.u.$ and $M = 20$ $d.u.$ fixed, the probability that the system does not fail in the interval $[t_1, t_1 + 5]$ nor in the interval $[t_2, t_2 + 2]$ is, at least, of the 60% for $10 \leq t_1 \leq 20$ and $30 \leq t_2 \leq 40$.

5.6 Conclusions and further extensions

In this chapter, a DTS model is analysed considering a CBM strategy under a finite life cycle for a system subject to two competing causes of failure, degradation and sudden shocks. We consider in this chapter the expected cost rate in the finite life cycle as the objective cost function to be optimised. A numerical method based on recursive computation is developed to provide the expected cost rate in the finite life cycle and its associated deviation cost. This numerical method shows similar results to the obtained considering strictly the Monte Carlo simulation. Besides, the time necessary to compute this numerical method is lower than when strictly Monte Carlo method is considered. Furthermore, the results obtained from a finite life cycle scheme are

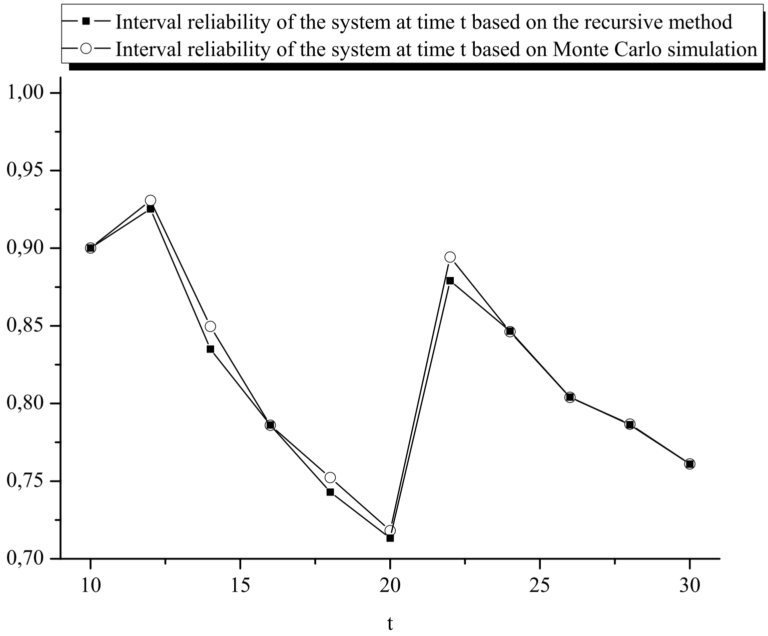


Figure 5.17: Interval reliability of the system for different values of t.

compared with the results obtained under an infinite life cycle. As we expect, we obtain different results for the expected cost rate in the finite life cycle. As Pandey *et al.* [72] concluded, we highlight that the asymptotic cost rate can be a rough approximation of the expected cost rate in the finite life cycle. Finally, some availability measures that describe the functioning of the system are obtained. It is showed that these availability measures fulfil some renewal equations. Numerical methods are developed to evaluate numerically these availability measures. However, the recursive method is really complex for some compound measures such as the joint interval reliability and, in these cases, a Monte Carlo is used to calculate the availability measure.

In this chapter, we consider a unique degradation process. A possible further extension is assuming that the system is subject to multiple degradation processes. In addition, we also consider the independence between the two different competing causes of failure. Along with this line, another further extension is the analysis of maintenance strategies under a finite time assuming dependent causes of failure. This dependence can be considered through the shock process intensity, *i.e.*, the shock process intensity increases with the deterioration level of the degradation process. This further extension is considered in Chapter 6.

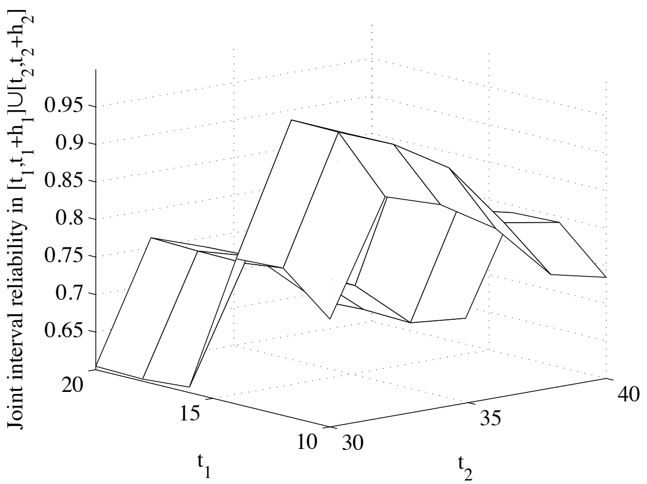


Figure 5.18: Joint interval reliability of the system for different values of t_1 and t_2.

CHAPTER

6

Transient approach for a dependent DTS model with a degradation process

This chapter[1] deals with the transient approach for a system subject to two causes of failure: internal degradation and sudden shocks. This chapter extends Chapter 5 considering that both causes of failure are dependent. In this chapter we assume only a degradation process modelled under a homogeneous gamma process. When the deterioration level of the degradation process exceeds a predetermined value, we assume that a degradation failure occurs. Sudden shocks arrive at the system under a DSPP. A sudden shock provokes the total breakdown of the system. In this chapter, we develop a CBM strategy with periodic inspection times considering a finite life cycle. In order to analyse some transient measures of the system, we develop recursive methods combining numerical integration and Monte Carlo simulation to obtain the expected cost rate in the finite life cycle, its standard deviation associated, and some availability measures such as the point availability, the reliability, and the interval reliability of the system. Numerical examples are provided to illustrate the analytical results.

In short, the main aspects covered in this chapter are:

1. Considering dependent both causes of failure, degradation and sudden shocks.

2. Analysing the expected cost in the finite life cycle implementing a CBM strategy in a DTS model.

3. Developing a recursive method combining numerical integration and Monte Carlo method to compute the expected cost in the finite life cycle.

4. Comparing the results obtained from this recursive method to the results obtained by using strictly Monte Carlo simulation.

[1]This chapter is based on the works by Caballé and Castro [75, 138, 139, 142].

5. Analysing the standard deviation cost associated with the expected cost in the finite life cycle provided through this recursive method.

6. Comparing the asymptotic expected cost rate and the expected cost in the finite life cycle in bivariate and univariate cases.

7. Optimising the expected cost rate in the finite life cycle in a bivariate case.

8. Analysing the robustness of some parameters of the maintenance model.

9. Analysing the point availability, reliability, and interval reliability of the system considering the optimal maintenance strategy.

This chapter is structured as follows. In Section 6.1 the general framework of the model is described. In Section 6.2 the CBM of the system is studied. Expected cost in the finite life cycle and its standard deviation associated are analysed in Section 6.3 Availability measures of the system are exposed in Section 6.4. Numerical examples are shown in Section 6.5. Conclusions and further possible extensions of this chapter are provided in Section 6.6.

6.1 Framework of the problem

A system subject to two dependent competing causes of failure, degradation and sudden shocks, is considered in this chapter. The assumptions of this maintenance model are similar to those considered in Chapter 5 with the difference that, in this chapter, the causes of failure are considered dependent.

6.1.1 General assumptions

The general assumptions of this model are:

1. The system starts working at time $t = 0$ and it is subject to a continuous degradation process. Let $X(t)$ be the deterioration level of the system at time t. We assume that $\{X(t), t \geq 0\}$ follows a homogeneous gamma process with parameters αt and β $(\alpha, \beta > 0)$ and probability density function given in (2.3). We assume that the system fails when its deterioration level exceeds the breakdown threshold L.

2. The system is subject to sudden shocks arriving at the system according to a DSPP $\{N_s(t), t \geq 0\}$ with intensity (see Chapter 4)
$$\lambda(t, X(t)) = \lambda_1(t)\mathbf{1}_{\{X(t)\leq M_s\}} + \lambda_2(t)\mathbf{1}_{\{X(t)>M_s\}}, \quad t \geq 0, \tag{6.1}$$
where λ_1 and λ_2 denote two failure rate functions verifying that $\lambda_1(t) \leq \lambda_2(t)$, for all $t \geq 0$. The arrival of a sudden shock provokes the system failure.

3. The system is inspected each T $(T > 0)$ time units to check if it is working or down. If the system is down, a CM is performed and the system is replaced by a new one. On the other hand, if the system is still working at the inspection time, the deterioration level is checked. Let M be the deterioration level from which the system is considered too much worn and the system must be replaced in a preventive way $(M < L)$. If the system is working and the deterioration level in the inspection exceeds the threshold M, a PM is performed and the system is replaced by a new one (see Figures 3.1 and 3.2, respectively).

Otherwise, no maintenance task is performed. After a replacement, a new cycle starts with an identical time between inspections T. We suppose that the time necessary to perform a maintenance task is negligible.

4. All maintenance actions imply a cost. A CM and a PM have associated a cost of C_c and C_p $m.u.$, respectively, and each inspection implies a cost of C_I $m.u.$ In addition, if the system fails, the system is down until the next inspection. Each time unit that the system is down, a cost of C_d $m.u./t.u.$ is incurred. We assume $C_c > C_p > C_I$.

5. We assume that the life cycle of the system is finite and equals to t_f. It means that, if the calendar time exceeds t_f, the system can no longer be replaced by a new one with the same characteristics. We assume that, at time t_f, the system is scrapped regardless of its true state.

6.1.2 Time to a sudden shock

In absence of maintenance, let Y be the time to a sudden shock with failure rate function given in (6.1). Let $I(v,t)$ be the survival function of Y conditioned to $\sigma_{M_s} = v$ for $t \geq v$. That is

$$I(v,t) = P\left[Y > t | \sigma_{M_s} = v\right] = \exp\left\{-\int_0^t \lambda\left(z, X(z)\right)\, dz\right\} = \frac{\bar{F}_1(v)}{\bar{F}_1(0)}\frac{\bar{F}_2(t)}{\bar{F}_2(v)}, \quad t \geq v, \tag{6.2}$$

where

$$\bar{F}_j(t) = \exp\left\{-\int_0^t \lambda_j(u)du\right\}, \quad j = 1,2, \tag{6.3}$$

with density function $f_j(t)$, for $j = 1,2$.

6.2 Condition-based maintenance

6.2.1 Maintenance action probability

By Assumption 3 of Subsection 6.1.1, $D_1, D_2, \ldots, D_n$ are $i.i.d.$ random variables. Let $R_1 = D_1$ be the replacement cycle length and let $P^M_{R_1}(kT)$ be the probability

$$P^M_{R_1}(kT) = P[R_1 = kT],$$

for a time between inspections T and preventive threshold M, with $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$. The probability of a maintenance replacement at time kT, $P^M_{R_1}(kT)$, for $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$ is

$$P^M_{R_1}(kT) = P^M_{R_{1,p}}(kT) + P^M_{R_{1,c}}(kT), \tag{6.4}$$

where $P^M_{R_{1,p}}(kT)$ and $P^M_{R_{1,c}}(kT)$ denote the probability of a preventive and corrective replacement at time kT for $k = 1, 2, \ldots, \lfloor t_f/T \rfloor$, respectively.

6.2.2 Preventive maintenance probability

A PM action is performed at time kT for $k = 1, 2, 3, \ldots, \lfloor t_f/T \rfloor$ if the system is working at time kT and the preventive threshold M is exceeded for the first time in $((k-1)T, kT]$. Let $P^M_{R_{1,p}}(kT)$ be the probability of a PM action at time kT for $k = 1, 2, 3, \ldots, \lfloor t_f/T \rfloor$

$$P^M_{R_{1,p}}(kT) = P^M_{R_{1,p,1}}(kT)\mathbf{1}_{\{M \leq M_s\}} + P^M_{R_{1,p,2}}(kT)\mathbf{1}_{\{M > M_s\}}, \tag{6.5}$$

where $P^M_{R_1,p,1}(kT)$ and $P^M_{R_1,p,2}(kT)$ denote the probability of a PM action at time kT if $M \leq M_s$ and if $M > M_s$, respectively.

First, we assume $M \leq M_s$. A PM action is performed at time kT if one of the following mutually exclusive events occur

$$\{(k-1)T < \sigma_M < kT < \sigma_{M_s}, Y > kT\},$$

and

$$\{(k-1)T < \sigma_M < \sigma_{M_s} < kT < \sigma_L, Y > kT\}.$$

Hence,

$$\begin{aligned}
&P^M_{R_1,p,1}(kT) = P\left[(k-1)T < \sigma_M < kT < \sigma_{M_s}, Y > kT\right] \\
&+P\left[(k-1)T < \sigma_M < \sigma_{M_s} < kT < \sigma_L, Y > kT\right] \\
&= \int_{(k-1)T}^{kT} f_{\sigma_M}(u)\bar{F}_{\sigma_{M_s}-\sigma_M}(kT-u)\bar{F}_1(kT)\,du \\
&+ \int_{(k-1)T}^{kT} f_{\sigma_M}(u)\int_u^{kT} f_{\sigma_{M_s}-\sigma_M}(w-u)\bar{F}_{\sigma_L-\sigma_{M_s}}(kT-w)I(w,kT)\,dw\,du.
\end{aligned}$$

Considering $M > M_s$, a PM action is performed at time kT if one of the following mutually exclusive events occurs

$$\{(k-1)T < \sigma_{M_s} < \sigma_M < kT < \sigma_L, Y > kT\},$$

and

$$\{\sigma_{M_s} < (k-1)T < \sigma_M < kT < \sigma_L, Y > kT\}.$$

Hence,

$$\begin{aligned}
&P^M_{R_1,p,2}(kT) = P\left[(k-1)T < \sigma_{M_s} < \sigma_M < kT < \sigma_L, Y > kT\right] \\
&+P\left[\sigma_{M_s} < (k-1)T < \sigma_M < kT < \sigma_L, Y > kT\right] \\
&= \int_{(k-1)T}^{kT} f_{\sigma_{M_s}}(u)\int_u^{kT} f_{\sigma_M-\sigma_{M_s}}(w-u)\bar{F}_{\sigma_L-\sigma_M}(kT-w)I(u,kT)\,dw\,du \\
&+ \int_0^{(k-1)T} f_{\sigma_{M_s}}(u)\int_{(k-1)T}^{kT} f_{\sigma_M-\sigma_{M_s}}(w-u)\bar{F}_{\sigma_L-\sigma_M}(kT-w)I(u,kT)\,dw\,du,
\end{aligned}$$

where f_{σ_M} and $f_{\sigma_{M_s}}$ denote the density function of (2.4), $f_{\sigma_{z_2}-\sigma_{z_1}}$ denotes the densify funcion of $\bar{F}_{\sigma_{z_2}-\sigma_{z_1}}$ given in (2.6), and $I(u,t)$ is given in (4.4).

6.2.3 Corrective maintenance probability

A CM action is performed at time kT for $k = 1, 2, 3, \ldots, \lfloor t_f/T \rfloor$ if the system is down at time kT and the preventive threshold M is exceeded for the first time after $(k-1)T$. Let $P^M_{R_1,c}(kT)$ be the probability of a CM action at time kT for $k = 1, 2, 3, \ldots, \lfloor t_f/T \rfloor$

$$P^M_{R_1,c}(kT) = P^M_{R_1,c,1}(kT)\mathbf{1}_{\{M \leq M_s\}} + P^M_{R_1,c,2}(kT)\mathbf{1}_{\{M > M_s\}}, \tag{6.6}$$

where $P^M_{R_1,c,1}(kT)$ and $P^M_{R_1,c,2}(kT)$ denote the probability of a CM action at time kT if $M \leq M_s$ and if $M > M_s$, respectively.

For $M \leq M_s$, a CM action due to degradation occurs in kT if

$$\{(k-1)T < \sigma_M < \sigma_{M_s} < \sigma_L < kT, \sigma_L < Y\},$$

and a CM action due to a sudden shock in kT is given by the occurrence of one of the following mutually exclusive events

$$\{(k-1)T < \sigma_M < Y < kT, Y < \sigma_{M_s}\},$$

$$\{(k-1)T < \sigma_M < \sigma_{M_s} < Y < kT, Y < \sigma_L\},$$

and

$$\{(k-1)T < Y < kT, Y < \sigma_M\}.$$

Hence

$$P^M_{R_1,c,1}(kT) = \int_{(k-1)T}^{kT} f_{\sigma_M}(u) \int_u^{kT} f_1(v)\bar{F}_{\sigma_{M_s}-\sigma_M}(v-u)\, dv\, du$$
$$+ \int_{(k-1)T}^{kT} f_{\sigma_M}(u) \int_u^{kT} f_{\sigma_{M_s}-\sigma_M}(w-u) \int_w^{kT} \left[-\frac{\partial}{\partial x}\left(I(w,x)\bar{F}_{\sigma_L-\sigma_{M_s}}(x-w)\right)\right] dx\, dw\, du$$
$$+ \int_{(k-1)T}^{kT} f_1(u)\bar{F}_{\sigma_M}(u)\, du.$$

For $M > M_s$, a CM action due to degradation is performed at time kT if one of the following mutually exclusive events takes place

$$\{\sigma_{M_s} < (k-1)T < \sigma_M < \sigma_L < kT, Y > \sigma_L\},$$

and

$$\{(k-1)T < \sigma_{M_s} < \sigma_L < kT, Y > \sigma_L\},$$

and due to a sudden shock

$$\{\sigma_{M_s} < (k-1)T < \sigma_M < Y < kT, Y < \sigma_L\},$$

$$\{\sigma_{M_s} < (k-1)T < Y < kT, Y < \sigma_M\},$$

$$\{(k-1)T < \sigma_{M_s} < Y < kT, Y < \sigma_L\},$$

and

$$\{(k-1)T < Y < kT, Y < \sigma_{M_s}\}.$$

That is

$$P^M_{R_1,c,2}(kT) = \int_0^{(k-1)T} f_{\sigma_{M_s}}(u) \int_{(k-1)T}^{kT} \left[-\frac{\partial}{\partial v} I(u,v)\right] \bar{F}_{\sigma_M-\sigma_{M_s}}(v-u)\, dv\, du$$
$$+ \int_0^{(k-1)T} f_{\sigma_{M_s}}(u) \int_{(k-1)T}^{kT} f_{\sigma_M-\sigma_{M_s}}(w-u)$$
$$\int_w^{kT} \left[-\frac{\partial}{\partial x}\left(I(u,x)\bar{F}_{\sigma_L-\sigma_M}(x-w)\right)\right] dx\, dw\, du$$
$$+ \int_{(k-1)T}^{kT} f_{\sigma_{M_s}}(u) \int_u^{kT} \left[-\frac{\partial}{\partial v}\left(I(u,v)\bar{F}_{\sigma_L-\sigma_{M_s}}(v-u)\right)\right] dv\, du$$
$$+ \int_{(k-1)T}^{kT} f_1(u)\bar{F}_{\sigma_{M_s}}(u)\, du,$$

where f_{σ_M} and $f_{\sigma_{M_s}}$ denote the density function of (2.4), $f_{\sigma_{z_2}-\sigma_{z_1}}$ denotes the densify funcion of $\bar{F}_{\sigma_{z_2}-\sigma_{z_1}}$ given in (2.6), $f_1(t)$ is the density function associated with (4.5), and $I(u,t)$ is given in (4.4).

6.2.4 Expected downtime

For T and M fixed, let $W_T^M((k-1)T,kT)$ be the time that the system is down in $((k-1)T,kT]$. That is, $W_T^M((k-1)T,kT)$ is equal to

$$\left\{\begin{array}{lll} kT-Y & \text{if} & (k-1)T<Y\leq kT, \quad Y<\sigma_L, \quad (k-1)T<\sigma_M \\ kT-\sigma_L & \text{if} & (k-1)T<\sigma_M<\sigma_L\leq kT, \quad \sigma_L<Y \end{array}\right. ,$$

for $T>0$. Based on calculations of the CM probability shown in Subsection 4.2.3, the expected downtime in $((k-1),kT]$ is

$$\begin{aligned} E[W_T^M((k-1)T,kT)] =& E[W_{T,1}^M((k-1)T,kT)]\mathbf{1}_{\{M\leq M_s\}} \\ &+E[W_{T,2}^M((k-1)T,kT)]\mathbf{1}_{\{M>M_s\}}, \end{aligned}$$

being

$$\begin{aligned} W_{T,1}^M((k-1)T,kT) =& (kT-Y)\,\mathbf{1}_{\{(k-1)T<\sigma_M<Y<kT,Y<\sigma_L\}} \\ &+(kT-Y)\,\mathbf{1}_{\{(k-1)T<\sigma_M<\sigma_{M_s}<Y<kT,Y<\sigma_L\}} \\ &+(kT-Y)\,\mathbf{1}_{\{(k-1)T<Y<kT,Y<\sigma_L\}} \\ &+(kT-\sigma_L)\,\mathbf{1}_{\{(k-1)T<\sigma_M<\sigma_{M_s}<\sigma_L<kT,\sigma_L<Y\}}, \end{aligned}$$

and

$$\begin{aligned} W_{T,2}^M((k-1)T,kT) =& (kT-Y)\,\mathbf{1}_{\{\sigma_{M_s}<(k-1)T<\sigma_M<Y<kT,Y<\sigma_L\}} \\ &+(kT-Y)\,\mathbf{1}_{\{\sigma_{M_s}<(k-1)T<Y<kT,Y<\sigma_M\}} \\ &+(kT-Y)\,\mathbf{1}_{\{(k-1)T<\sigma_{M_s}<Y<kT,Y<\sigma_L\}} \\ &+(kT-Y)\,\mathbf{1}_{\{(k-1)T<Y<kT,Y<\sigma_{M_s}\}} \\ &+(kT-\sigma_L)\,\mathbf{1}_{\{\sigma_{M_s}<(k-1)T<\sigma_M<\sigma_L<kT,\sigma_L<Y\}} \\ &+(kT-\sigma_L)\,\mathbf{1}_{\{(k-1)T<\sigma_{M_s}<\sigma_L<kT,\sigma_L<Y\}}. \end{aligned}$$

That is

$$
\begin{aligned}
E\left[W_T^M((k-1)T,kT)\right] &= \Bigg[\int_{(k-1)T}^{kT} f_{\sigma_M}(u)\int_u^{kT} f_1(v)\bar{F}_{\sigma_{M_s}-\sigma_M}(v-u)(kT-v)\,dv\,du \\
&+ \int_{(k-1)T}^{kT} f_{\sigma_M}(u)\int_u^{kT} f_{\sigma_{M_s}-\sigma_M}(w-u) \\
&\int_w^{kT}\left[-\frac{\partial}{\partial x}\left(I(w,x)\bar{F}_{\sigma_L-\sigma_{M_s}}(x-w)\right)\right](kT-x)\,dx\,dw\,du \\
&+ \int_{(k-1)T}^{kT} f_1(u)\bar{F}_{\sigma_M}(u)(kT-u)\,du\Bigg]\mathbf{1}_{\{M\le M_s\}} \\
&+ \Bigg[\int_0^{(k-1)T} f_{\sigma_{M_s}}(u)\int_{(k-1)T}^{kT} f_{\sigma_M-\sigma_{M_s}}(w-u) \\
&\int_w^{kT}\left[-\frac{\partial}{\partial x}\left(I(u,x)\bar{F}_{\sigma_L-\sigma_M}(x-w)\right)\right](kT-x)\,dx\,dw\,du \\
&+ \int_0^{(k-1)T} f_{\sigma_{M_s}}(u)\int_{(k-1)T}^{kT}\left[-\frac{\partial}{\partial v}I(u,v)\right]\bar{F}_{\sigma_M-\sigma_{M_s}}(v-u)(kT-v)\,dv\,du \\
&+ \int_{(k-1)T}^{kT} f_{\sigma_{M_s}}(u)\int_u^{kT}\left[-\frac{\partial}{\partial v}\left(I(u,v)\bar{F}_{\sigma_L-\sigma_{M_s}}(v-u)\right)\right](kT-v)\,dv\,du \\
&+ \int_{(k-1)T}^{kT} f_1(u)\bar{F}_{\sigma_{M_s}}(u)(kT-u)\,du\Bigg]\mathbf{1}_{\{M>M_s\}}.
\end{aligned} \tag{6.7}
$$

6.3 Analysis of the expected cost in the finite life cycle

For T and M fixed, let $E\left[C_T^M(t)\right]$ be the expected cost in the finite life cycle at time $t>0$. Next result provides the Markov renewal equation that fulfils $E\left[C_T^M(t)\right]$.

Theorem 6.1. *For $t<T$, the expected cost in the finite life cycle is*

$$
\begin{aligned}
E\left[C_T^M(t)\right] = C_d\int_0^t f_{\sigma_{M_s}}(u)\int_u^t\left[-\frac{\partial}{\partial v}\left(I(u,v)\bar{F}_{\sigma_L-\sigma_{M_s}}(v-u)\right)\right](t-v)\,dv\,du \\
+ C_d\int_0^t f_1(u)\bar{F}_{\sigma_{M_s}}(u)(t-u)du,
\end{aligned}
$$

where $\bar{F}_{\sigma_{M_s}}(u)$ and $f_{\sigma_{M_s}}(u)$ denote the survival and density functions of σ_{M_s} given in (2.4), $\bar{F}_{\sigma_L-\sigma_{M_s}}$ and $I(x,y)$ the survival functions given in (2.6) and (6.2), respectively, and $f_1(x)$ the density function of the survival function given in (6.3).

For $t\ge T$, the expected cost in the finite life cycle fulfils the following recursive equation

$$
E\left[C_T^M(t)\right] = \sum_{k=1}^{\lfloor t/T\rfloor} E\left[C_T^M(t-kT)\right]P_{R_1}^M(kT) + G_T^M(t), \tag{6.8}
$$

where $G_T^M(t)$ is given in (5.8) with initial condition $E\left[C_T^M(0)\right]=0$.

Proof. For $t < T$,

$$\begin{aligned} E\left[C_T^M(t)\right] =& C_d E\left[(t-Y)\mathbf{1}_{\{\sigma_{M_s}<Y<t,\, Y<\sigma_L\}}\right] \\ &+ C_d E\left[(t-\sigma_L)\mathbf{1}_{\{\sigma_{M_s}<\sigma_L<t,\, \sigma_L<Y\}}\right] \\ &+ C_d E\left[(t-Y)\mathbf{1}_{\{Y<t,\, Y<\sigma_{M_s}\}}\right]. \end{aligned}$$

That is

$$\begin{aligned} E\left[C_T^M(t)\right] =& C_d \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v} I(u,v)\right] \bar{F}_{\sigma_L-\sigma_{M_s}}(v-u)(t-v)\, dv\, du \\ &+ C_d \int_0^t f_{\sigma_{M_s}}(u) \int_u^t I(u,v) f_{\sigma_L-\sigma_{M_s}}(v-u)(t-v)\, dv\, du \\ &+ C_d \int_0^t f_1(u) \bar{F}_{\sigma_{M_s}}(u)(t-u) du. \end{aligned}$$

For $t \geq T$, $E\left[C_T^M(t)\right]$ is developed following Theorem 5.1, and the result holds[1]. $\square$

Corollary 6.1. *Setting* $E\left[C_T^{M(i)}(t)\right] = E\left[C_T^M(t)\right]$, *for all* $(i-1)T < t \leq iT$ *with* $i = 1, 2, \ldots, \lfloor t_f/T \rfloor$ *the expected cost in the finite life cycle is*

$$E\left[C_T^{M(1)}(t)\right] = E\left[C_T^M(t)\right],$$

and for $i \geq 1$

$$E\left[C_T^{M(i+1)}(t)\right] = \sum_{k=1}^{i} E\left[C_T^{M(i+1-k)}(t-kT)\right] P_{R_1}^M(kT) + G_T^{M(i)}(t),$$

where $G_T^{M(i)}(t)$ *is given in (5.8) replacing* $\lfloor t/T \lfloor$ *by* i *with initial condition* $E\left[C_T^{M(i)}(0)\right] = 0.$

Let $\left(S_T^M(t)\right)^2$ be the variance of the expected cost in the finite life cycle at time t associated to $E\left[C_T^M(t)\right]$. Based on Theorem 5.2, the following result is obtained.

Theorem 6.2. *For* $t < T$, *the expected square cost at time* $t > 0$ *is*

$$\begin{aligned} E\left[C_T^M(t)^2\right] =& C_d^2 \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v}\left(I(u,v)\bar{F}_{\sigma_L-\sigma_{M_s}}(v-u)\right)\right](t-v)^2\, dv\, du \\ &+ C_d^2 \int_0^t f_1(u)\bar{F}_{\sigma_{M_s}}(u)(t-u)^2 du. \end{aligned}$$

For $t \geq T$, *the mean square fulfils the following recursive equation*

$$E\left[C_T^M(t)^2\right] = \sum_{k=1}^{\lfloor t/T \rfloor} E\left[C_T^M(t-kT)^2\right] P_{R_1}^M(kT) + H_T^M(t), \tag{6.9}$$

where $H_T^M(t)$ *is given in (5.10) with initial condition* $E\left[C_T^M(0)^2\right] = 0.$

[1]Note that the structure of this theorem is similar to the one of Theorem 5.1, but the probabilities are different.

Proof. For $t < T$, the expected square cost is

$$E\left[C_T^M(t)^2\right] = C_d^2 E\left[(t-Y)^2 \mathbf{1}_{\{\sigma_{M_s}<Y<t,\ Y<\sigma_L\}}\right] + C_d^2 E\left[(t-\sigma_L)^2 \mathbf{1}_{\{\sigma_{M_s}<\sigma_L<t,\ \sigma_L<Y\}}\right] + C_d^2 E\left[(t-Y)^2 \mathbf{1}_{\{Y<t,\ Y<\sigma_{M_s}\}}\right].$$

That is

$$E\left[C_T^M(t)^2\right] = C_d^2 \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v} I(u,v)\right] \bar{F}_{\sigma_L-\sigma_{M_s}}(v-u)(t-v)^2\, dv\, du + C_d^2 \int_0^t f_{\sigma_{M_s}}(u) \int_u^t I(u,v) f_{\sigma_L-\sigma_{M_s}}(v-u)(t-v)^2\, dv\, du + C_d^2 \int_0^t f_1(u)\bar{F}_{\sigma_{M_s}}(u)(t-u)^2 du.$$

For $t \geq T$, $E\left[C_T^M(t)^2\right]$ is developed following Theorem 5.2, and the result holds[1]. □

Corollary 6.2. *Setting $E\left[C_T^{M(i)}(t)^2\right] = E\left[C_T^M(t)^2\right]$, for all $(i-1)T < t \leq iT$ with $i = 1, 2, \ldots, \lfloor t_f/T \rfloor$ the expected square cost is*

$$E\left[C_T^{M(1)}(t)^2\right] = C_d^2 \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v}\left(I(u,v)\bar{F}_{\sigma_L-\sigma_{M_s}}(v-u)\right)\right](t-v)^2\, dv\, du + C_d^2 \int_0^t f_1(u)\bar{F}_{\sigma_{M_s}}(u)(t-u)^2 du,$$

and for $i \geq 1$

$$E\left[C_T^{M(i+1)}(t)^2\right] = \sum_{k=1}^{i} E\left[C_T^{M(i+1-k)}(t-kT)^2\right] P_{R_1}^M(kT) + H_T^{M(i)}(t),$$

where $H_T^{M(i)}(t)$ is given in (5.10), replacing $\lfloor t/T \rfloor$ by i, and with initial condition $E\left[C_T^{M(i)}(0)^2\right] = 0$.

Hence, by (5.9) the standard deviation of the expected cost in the finite life cycle at time t is

$$S_T^M(t) = \sqrt{E\left[C_T^M(t)^2\right] - \left(E\left[C_T^M(t)\right]\right)^2}. \tag{6.10}$$

6.4 Availability measures of the system

In addition to the expected cost in the finite life cycle and its standard deviation associated, the point availability, the reliability, and the interval reliability of the system are computed. Let $I(t)$ be the function given in (2.10).

[1]Note that the structure of this theorem is similar to the one of Theorem 5.2, but the probabilities are different.

The point availability of the system

For T and M fixed, let $A_T^M(t)$ be the point availability of the system at time $t > 0$. That is, the probability that the system is working at time t given by

$$A_T^M(t) = P\left[I(t) = 1\right], \quad t \geq 0.$$

Based on Theorem 5.3, the following result is obtained.

Theorem 6.3. *For $t < T$,*

$$A_T^M(t) = \bar{F}_{\sigma_{M_s}}(t)\bar{F}_1(t) + \int_0^t f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_L - \sigma_{M_s}}(t-u)I(u,t)du.$$

For $t \geq T$, $A_T^M(t)$ fulfils the following recursive equation

$$A_T^M(t) = \sum_{k=1}^{\lfloor t/T \rfloor} A_T^M(t-kT)P_{R_1}^M(kT) + J_{T,1}^M(t)\mathbf{1}_{\{M \leq M_s\}} + J_{T,2}^M(t)\mathbf{1}_{\{M > M_s\}}, \tag{6.11}$$

where

$$\begin{aligned} J_{T,1}^M(t) = &\int_{\lfloor t/T \rfloor T}^t f_{\sigma_M}(u)\int_u^t f_{\sigma_{M_s}-\sigma_M}(v-u)\bar{F}_{\sigma_L-\sigma_{M_s}}(t-v)I(v,t)\,dv\,du \\ &+\int_{\lfloor t/T \rfloor T}^t f_{\sigma_M}(u)\bar{F}_{\sigma_{M_s}-\sigma_M}(t-u)\bar{F}_1(t)\,du \\ &+\bar{F}_{\sigma_M}(t)\bar{F}_1(t), \end{aligned} \tag{6.12}$$

and

$$\begin{aligned} J_{T,2}^M(t) = &\int_0^{\lfloor t/T \rfloor T} f_{\sigma_{M_s}}(u)\int_{\lfloor t/T \rfloor T}^t f_{\sigma_M-\sigma_{M_s}}(v-u)\bar{F}_{\sigma_L-\sigma_M}(t-v)I(u,t)\,dv\,du \\ &+\int_0^{\lfloor t/T \rfloor T} f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_M-\sigma_{M_s}}(t-u)I(u,t)\,du \\ &+\int_{\lfloor t/T \rfloor T}^t f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_L-\sigma_{M_s}}(t-u)I(u,t)\,du \\ &+\bar{F}_{\sigma_{M_s}}(t)\bar{F}_1(t), \end{aligned} \tag{6.13}$$

with initial condition $A_T^M(0) = 1$, and where $P_{R_1}^M(kT)$ is given in (6.4).

Proof. For $t < T$,

$$\begin{aligned} P\left[I(t)=1\right] =&P\left[t < \sigma_{M_s},\ Y > t\right] + P\left[\sigma_{M_s} < t < \sigma_L,\ Y > t\right] \\ =&\bar{F}_{\sigma_{M_s}}(t)\bar{F}_1(t) + \int_0^t f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_L-\sigma_{M_s}}(t-u)I(u,t)du. \end{aligned}$$

For $t \geq T$, conditioning to the first renewal R_1, the point availability is written as

$$A_T^M(t) = P\left[I(t)=1|R_1 \leq t\right] + P\left[I(t)=1|R_1 > t\right].$$

If $R_1 > t$

$$\begin{aligned}
A_T^M(t) =& \Big[P\left[\lfloor t/T \rfloor T < \sigma_M < \sigma_{M_s} < t < \sigma_L,\ Y > t\right] \\
& + P\left[\lfloor t/T \rfloor T < \sigma_M < t < \sigma_{M_s},\ Y > t\right] \\
& + P\left[t < \sigma_M,\ Y > t\right] \Big] \mathbf{1}_{\{M \le M_s\}} \\
& + \Big[P\left[\sigma_{M_s} < \lfloor t/T \rfloor T < \sigma_M < t < \sigma_L,\ Y > t\right] \\
& + P\left[\sigma_{M_s} < \lfloor t/T \rfloor T < t < \sigma_M,\ Y > t\right] \\
& + P\left[\lfloor t/T \rfloor T < \sigma_{M_s} < t < \sigma_L,\ Y > t\right] \\
& + P\left[t < \sigma_{M_s},\ Y > t\right] \Big] \mathbf{1}_{\{M > M_s\}}.
\end{aligned}$$

That is

$$\begin{aligned}
A_T^M(t) =& \Big[\int_{\lfloor t/T \rfloor T}^t f_{\sigma_M}(u) \int_u^t f_{\sigma_{M_s}-\sigma_M}(v-u) \bar{F}_{\sigma_L - \sigma_{M_s}}(t-v) I(v,t)\, dv\, du \\
& + \int_{\lfloor t/T \rfloor T}^t f_{\sigma_M}(u) \bar{F}_{\sigma_{M_s}-\sigma_M}(t-u) \bar{F}_1(t)\, du \\
& + \bar{F}_{\sigma_M}(t) \bar{F}_1(t) \Big] \mathbf{1}_{\{M \le M_s\}} \\
& + \Big[\int_0^{\lfloor t/T \rfloor T} f_{\sigma_{M_s}}(u) \int_{\lfloor t/T \rfloor T}^t f_{\sigma_M - \sigma_{M_s}}(v-u) \bar{F}_{\sigma_L - \sigma_M}(t-v) I(u,t)\, dv\, du \\
& + \int_0^{\lfloor t/T \rfloor T} f_{\sigma_{M_s}}(u) \bar{F}_{\sigma_M - \sigma_{M_s}}(t-u) I(u,t)\, du \\
& + \int_{\lfloor t/T \rfloor T}^t f_{\sigma_{M_s}}(u) \bar{F}_{\sigma_L - \sigma_{M_s}}(t-u) I(u,t)\, du \\
& + \bar{F}_{\sigma_{M_s}}(t) \bar{F}_1(t) \Big] \mathbf{1}_{\{M > M_s\}} \\
=& J_{T,1}^M(t) \mathbf{1}_{\{M \le M_s\}} + J_{T,2}^M(t) \mathbf{1}_{\{M > M_s\}}.
\end{aligned}$$

If $R_1 \le t$,

$$P\left[I(t) = 1 | R_1 \le t\right] = \sum_{k=1}^{\lfloor t/T \rfloor} P_{R_1}^M(kT) P\left[I(t-kT) = 1\right] = \sum_{k=1}^{\lfloor t/T \rfloor} A_T^M(t-kT) P_{R_1}^M(kT),$$

and the result holds. □

The reliability of the system

Let $R_T^M(t)$ be the reliability of the system at time $t > 0$. That is, the probability that the system is working in $(0, t]$ given by

$$R_T^M(t) = P\left[O(u) < L, \forall u \in (0, t], N_s(0, t) = 0\right],$$

where $O(t)$ is given in (5.18). Based on Theorem 5.6, the following result is obtained.

Theorem 6.4. *For $t < T$,*

$$R_T^M(t) = A_T^M(t).$$

For $t \geq T$, $R_T^M(t)$ fulfils the following recursive equation

$$R_T^M(t) = \sum_{k=1}^{\lfloor t/T \rfloor} R_T^M(t-kT)P_{R_{1,p}}^M(kT) + J_{T,1}^M(t)\mathbf{1}_{\{M \leq M_s\}} + J_{T,2}^M(t)\mathbf{1}_{\{M > M_s\}}, \tag{6.14}$$

with initial condition $R_T^M(0) = 1$, where $J_{T,1}^M$ and $J_{T,2}^M$ are given in (6.12) and (6.13), respectively.

Proof. For $t < T$, $R_T^M(t)$ is equal to $A_T^M(t)$. That is

$$R_T^M(t) = \bar{F}_{\sigma_{M_s}}(t)\bar{F}_1(t) + \int_0^t f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_L - \sigma_{M_s}}(t-u)I(u,t)du.$$

For $t \geq T$, conditioning to the first renewal R_1, the reliability is written as

$$\begin{aligned} R_T^M(t) =& P\left[O(u) < L,\ \forall u \in (0,t],\ N_s(0,t) = 0 | R_1 \leq t\right] \\ &+ P\left[O(u) < L,\ \forall u \in (0,t],\ N_s(0,t) = 0 | R_1 > t\right]. \end{aligned}$$

If $R_1 > t$

$$R_T^M(t) = A_T^M(t|R_1 > t) = J_{T,1}^M(t)\mathbf{1}_{\{M \leq M_s\}} + J_{T,2}^M(t)\mathbf{1}_{\{M > M_s\}}.$$

If $R_1 \leq t$,

$$\begin{aligned} R_T^M(t) =& P\left[O(u) < L,\ \forall u \in (0,t], N_s(0,t) = 0 | R_1 \leq t\right] \\ =& \sum_{k=1}^{\lfloor t/T \rfloor} P_{R_{1,p}}^M(kT)P\left[O(u-kT) < L, \forall u \in (0,t-kT], N_s(0,t-kT) = 0\right] \\ =& \sum_{k=1}^{\lfloor t/T \rfloor} R_T^M(t-kT)P_{R_{1,p}}^M(kT), \end{aligned}$$

and the result holds. □

The interval reliability of the system

Let $IR_T^M(t,t+h)$ be the interval reliability in $(t,t+h]$. That is

$$IR_T^M(t,t+h) = P\left[O(u) < L, \forall u \in (t,t+h], N_s(t,t+h) = 0\right].$$

Based on Theorem 5.7, the following result is obtained.

Theorem 6.5. *For $t+h < T$,*

$$IR_T^M(t,t+h) = R_T^M(t+h).$$

For $t+h \geq T$, $IR_T^M(t,t+h)$ fulfils the following recursive equation

$$\begin{aligned} IR_T^M(t,t+h) =& \sum_{k=\lfloor t/T \rfloor + 1}^{\lfloor (t+h)/T \rfloor} R_T^M(t+h-kT)P_{R_{1,p}}^M(kT) \\ &+ \sum_{k=1}^{\lfloor t/T \rfloor} IR_T^M(t-kT,t+h-kT)P_{R_1}^M(kT) \\ &+ J_{T,1}^M(t+h)\mathbf{1}_{\{M \leq M_s\}} + J_{T,2}^M(t+h)\mathbf{1}_{\{M > M_s\}}, \end{aligned} \tag{6.15}$$

with initial conditions $R_T^M(0) = 1$ and $IR_T^M(0,0) = 1$.

Proof. For $(t+h) < T$, $IR_T^M(t,t+h)$ is equal to $R_T^M(t+h)$. That is

$$
\begin{aligned}
IR_T^M(t,t+h) =& \bar{F}_{\sigma_{M_s}}(t+h)\bar{F}_1(t+h) \\
& + \int_0^{t+h} f_{\sigma_{M_s}}(u)\bar{F}_{\sigma_L-\sigma_{M_s}}(t+h-u)I(u,t+h)du.
\end{aligned}
$$

For $t+h \geq T$, conditioning to the first renewal R_1, the interval reliability is written as

$$
\begin{aligned}
IR_T^M(t,t+h) =& P\left[O(u) < L, \forall u \in (t,t+h],\ N_s(t,t+h) = 0 | R_1 \leq t\right] \\
& + P\left[O(u) < L,\ \forall u \in (t,t+h], N_s(t,t+h) = 0 | t < R_1 < t+h\right] \\
& + P\left[O(u) < L,\ \forall u \in (t,t+h], N_s(t,t+h) = 0 | R_1 \geq t+h\right]\Big].
\end{aligned}
$$

If $R_1 \geq t+h$

$$
IR_T^M(t,t+h) = J_{T,1}^M(t+h)\mathbf{1}_{\{M \leq M_s\}} + J_{T,2}^M(t+h)\mathbf{1}_{\{M > M_s\}}.
$$

If $t < R_1 < t+h$

$$
\begin{aligned}
IR_T^M(t,t+h) =& P\left[O(u) < L,\ \forall u \in (t,t+h], N_s(t,t+h) = 0 | t < R_1 < t+h\right] \\
=& \sum_{k=\lfloor t/T \rfloor + 1}^{\lfloor (t+h)/T \rfloor} P_{R_{1,p}}^M(kT) P\left[O(u-kT) < L, \forall u \in (0, t+h-kT],\right. \\
& \left. N_s(0,t+h-kT) = 0\right] \\
=& \sum_{k=\lfloor t/T \rfloor + 1}^{\lfloor (t+h)/T \rfloor} R_T^M(t+h-kT) P_{R_{1,p}}^M(kT).
\end{aligned}
$$

If $R_1 \leq t$

$$
\begin{aligned}
IR_T^M(t,t+h) =& P\left[O(u) < L, \forall u \in (t,t+h],\ N_s(t,t+h) = 0 | R_1 \leq t\right] \\
=& \sum_{k=1}^{\lfloor t/T \rfloor} P_{R_1}^M(kT) P\left[O(u-kT) < L, \forall u \in (t-kT, t+h-kT],\right. \\
& \left. N_s(t-kT, t+h-kT) = 0\right] \\
=& \sum_{k=1}^{\lfloor t/T \rfloor} IR_T^M(t-kT, t+h-kT) P_{R_1}^M(kT),
\end{aligned}
$$

and the result holds.

□

6.5 Numerical examples

Some numerical examples are provided to illustrate the analytical results. We consider a system subject to a degradation process modelled using a homogeneous gamma process with parameters $\alpha = \beta = 0.1$. The system fails when the deterioration level of the system reaches the breakdown threshold $L = 30\ d.u.$

The system is also subject to sudden shocks. We assume that the sudden shock process is modelled under a DSPP with intensity

$$\lambda(t, X(t)) = 0.01 \cdot \mathbf{1}_{\{X(t) \leq M_s\}} + 0.1 \cdot \mathbf{1}_{\{X(t) > M_s\}}, \quad t \geq 0.$$

In addition, we assume the cost sequence $C_c = 300\ m.u.$, $C_p = 150\ m.u.$, $C_I = 45\ m.u.$, and $C_d = 25\ m.u./t.u.$ We assume that the life cycle is finite and equal to $t_f = 50\ t.u.$

This dataset represents a case of high variance in deterioration increment $\left(\bar{x} = \frac{\alpha}{\beta} = 1, \sigma_x^2 = \frac{\alpha}{\beta^2} = 10\right)$ and high intensity to shock failure.

Under this specifications, the expected time to a deterioration failure is $E\left[\sigma_L\right] = 34.0335$ $t.u.$ and the expected time to a sudden shock failure is $E\left[Y\right] = 28.3556\ t.u.$

MATLAB software, in its version R2014a, was used for the following examples. The code was run on an Intel Core i5-2500 processor with 8GB DDR3 RAM under Windows 7 Professional.

6.5.1 Asymptotic expected cost rate analysis

Based on Chapters 3 and 4, the optimisation problem for the asymptotic expected cost is computed as follows:

1. A grid of size 10 is obtained by discretising the set $[5, 50]$ into 10 equally spaced points from 5 to 50 for T. Let T_i be the i-th value of the grid, for $i = 1, 2, \ldots, 10$.

2. A grid of size 30 is obtained by discretising the set $[1, 30]$ into 30 equally spaced points from 1 to 30 for M. Let M_j be the j-th value of the grid, for $j = 1, 2, \ldots, 30$.

3. For each combination (T_i, M_j) fixed, we obtain 50000 simulations of (R_1, I_1, W_d). With these simulations, and applying Monte Carlo method, we obtain $\tilde{P}_{R_{1,p}}^{M_j}(kT_i)$, $\tilde{P}_{R_{1,c}}^{M_j}(kT_i)$, $\tilde{P}_{R_1}^{M_j}(kT_i)$, and $\tilde{E}\left[W_{T_i}^{M_j}((k-1)T_i, kT_i))\right]$ corresponding to the estimations of $P_{R_{1,p}}^{M_j}(kT_i))$, $P_{R_{1,c}}^{M_j}(kT_i))$, $P_{R_1}^{M_j}(kT_i))$, and $E\left[W_{T_i}^{M_j}((k-1)T_i, kT_i))\right]$ for $k = 1, 2, \ldots, \lfloor 50/T_i \rfloor$ given in Section 6.2 (see Figure 3.4).

4. Quantity $\tilde{C}^{\infty}(T, M)$, representing the asymptotic expected cost rate, is calculated by using Equation (4.14), replacing the corresponding probabilities by their estimations calculated in Step 3.

5. The search of the optimal maintenance strategy is reduced to find the values T_{opt} and M_{opt} minimising the asymptotic expected cost rate $\tilde{C}^{\infty}(T, M)$. That is

$$\tilde{C}^{\infty}(T_{opt}, M_{opt}) = \min_{\substack{T>0 \\ 0<M\leq L}} \left\{\tilde{C}^{\infty}(T, M)\right\}.$$

Figure 6.1 shows the value of $\tilde{C}^{\infty}(T, M)$ versus T and M. The values of T and M minimising $\tilde{C}^{\infty}(T, M)$ are reached at $M_{opt} = 14\ d.u.$ and $T_{opt} = 10\ t.u.$, with $\tilde{C}^{\infty}(10, 14) = 15.3819\ m.u./t.u.$ Later, the expected cost in the life cycle will be compared to the asymptotic expected cost using the values T_{opt} and M_{opt}.

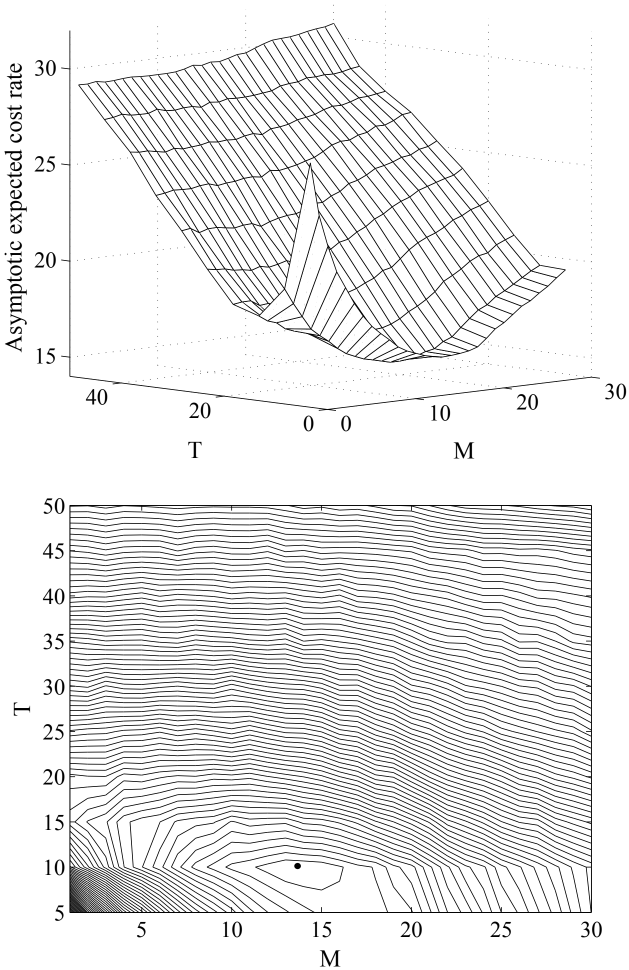


Figure 6.1: Mesh and contour plots for the asymptotic expected cost rate.

6.5.2 Analysis of the expected cost rate in the finite life cycle for T fixed

We consider a time between inspections $T = 10\ t.u.$ The optimisation problem for the expected cost in the life cycle based on the recursive formula given in (6.8) is computed as follows:

1. A grid of size 30 is obtained by discretising the set $[1, 30]$ into 30 equally spaced points from 1 to 30 for M.

2. For $T = 10\ t.u.$ fixed, 50000 simulations of (R_1, I_1, W_d) are calculated. With these simulations, and applying Monte Carlo method, we obtain the estimations $\tilde{P}^{M_j}_{R_{1,p}}(10k)$, $\tilde{P}^{M_j}_{R_{1,c}}(10k)$, $\tilde{P}^{M_j}_{R_1}(10k)$, and $\tilde{E}\left[W^{M_j}_{10}(10(k-1), 10k)\right]$ from (6.4), (6.5), (6.6), and (6.7), respectively (see Figure 3.4).

3. The expected cost in the life cycle $\tilde{E}\left[C^{M_j}_{10}(50)\right]$ is calculated by using the recursive formula given in (6.8), replacing the corresponding probabilities by their estimations calculated in Step 2, with initial condition $\tilde{E}\left[C^{M_j}_{10}(0)\right] = 0$.

4. The search of the optimal maintenance strategy is reduced to find the value M_{opt} minimising $\tilde{E}\left[C^{M}_{10}(50)\right]$. That is

$$\tilde{E}\left[C^{M_{opt}}_{10}(50)\right] = \min_{0<M\leq L}\left\{\tilde{E}\left[C^{M}_{10}(50)\right]\right\}.$$

Now, we compare the expected cost rate in the life cycle, $\tilde{E}\left[C^{M}_{10}(50)\right]/50$, based on the recursive method to the expected cost rate in the life cycle based on strictly Monte Carlo simulation. Secondly, the expected cost rate in a life cycle is compared to the expected cost rate. The results are detailed below.

Figure 6.2 shows the expected cost rate in the life cycle calculated using the recursive method and the expected cost rate in the life cycle calculated using strictly Monte Carlo simulation. The expected cost rate in the life cycle based on strictly Monte Carlo simulation was calculated for 30 equally spaced points in $[1, 30]$ with 50000 simulations for each point. Based on Figure 6.2, for the recursive method, the expected cost rate in the life cycle based on the recursive method reaches its minimum value at $M_{opt} = 14\ d.u.$, with an expected cost rate in the life cycle of $14.7639\ m.u./t.u.$ On the other hand, using strictly Monte Carlo simulation, the expected cost rate in the life cycle reaches its minimum value at $M_{opt} = 14\ d.u.$, with an expected cost rate in the life cycle of $15.2096\ m.u./t.u.$ For $T = 10\ t.u.$, Table 6.1 shows the average number of completed renewal cycles at time $t_f = 50\ t.u.$

M	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{E}\left[\mathbf{N^M_{10}(50)}\right]$	4.6678	4.3307	4.0400	3.7683	3.5316	3.3157	3.1261	2.9485	2.7957	2.6585

M	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
$\mathbf{E}\left[\mathbf{N^M_{10}(50)}\right]$	2.5253	2.4146	2.3046	2.2052	2.1092	2.0310	1.9403	1.8688	1.7999	1.7386

M	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
$\mathbf{E}\left[\mathbf{N^M_{10}(50)}\right]$	1.6874	1.6520	1.6247	1.5947	1.5676	1.5415	1.5234	1.4948	1.4825	1.4612

Table 6.1: Average number of complete renewal cycles up to $t_f = 50\ t.u.$ versus M.

For $T = 10\ t.u.$, Figure 6.3 shows the expected cost rate in the life cycle calculated by using the recursive method and the asymptotic expected cost rate versus M. As we said

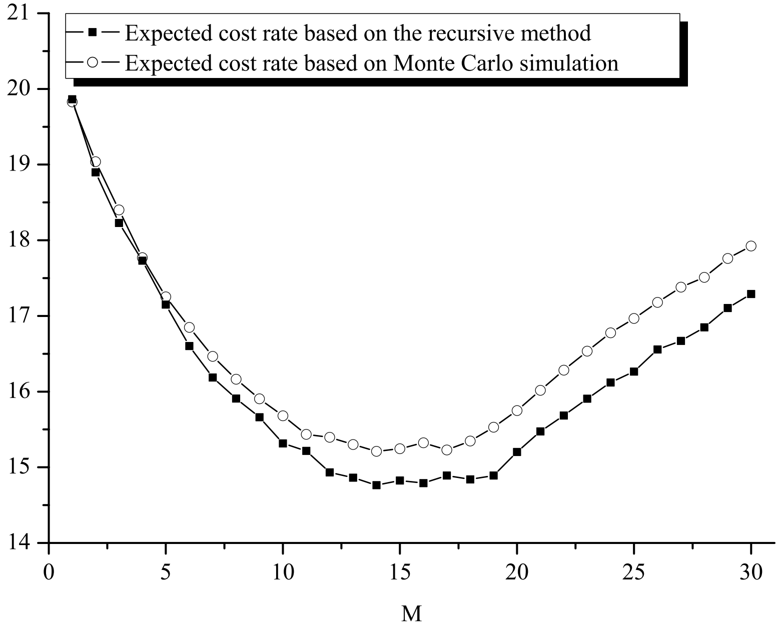


Figure 6.2: Expected cost rate in the finite life cycle versus M.

previously, the value of M minimising $\tilde{E}\left[C_{10}^{M}(50)\right]/50$ is reached at $M_{opt} = 14\ d.u.$, with an expected cost rate in the life cycle of $\tilde{E}\left[C_{10}^{14}(50)\right]/50 = 14.7639\ m.u./t.u.$ On the other hand, $\tilde{C}^{\infty}(10, M)$ reaches its minimum value at $M_{opt} = 14\ d.u.$, with an asymptotic expected cost rate of $\tilde{C}^{\infty}(10, 14) = 15.3819\ m.u./t.u.$

Now, we calculate the standard deviation of the cost. Figure 6.4 shows the expected cost rate in the life cycle with its standard deviation associated given by Equation (6.10). Both quantities were calculated for 30 equally spaced points in $[1, 30]$ by using the recursive formula given in (6.9) throughout the steps detailed in 6.5.2.

Now, we focus on the main model parameters influence on the expected cost in the life cycle. Firstly, a sensitivity analysis of the gamma process parameters is performed.

The values of the gamma process parameters are modified according to the following specifications:

$$\alpha_{(v_i\%)} = \alpha\left[1 + \frac{v_i}{100}\right] \quad \text{and} \quad \beta_{(v_j\%)} = \beta\left[1 + \frac{v_j}{100}\right], \tag{6.16}$$

where v_i and v_j are, respectively, the i-th and j-th position of the vector $\mathbf{v} = (-10, -5, -1, 0, 1, 5, 10)$.

Let $\tilde{E}\left[C_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^{M}(t_f)\right]$ be the minimal expected cost in the life cycle obtained when α and β vary according to the specifications given in (6.16). The expected cost in the life cycle for each combination of $\alpha_{(v_i\%)}$ and $\beta_{(v_j\%)}$ are calculated based on the recursive method following the steps detailed in 6.5.2. The relative measure $V_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^{M}(50)$ is defined as

$$V_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^{M}(50) = \frac{\left|\tilde{E}\left[C_{10}^{M_{opt}}(50)\right] - \tilde{E}\left[C_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^{M}(50)\right]\right|}{\tilde{E}\left[C_{10}^{M_{opt}}(50)\right]}, \tag{6.17}$$

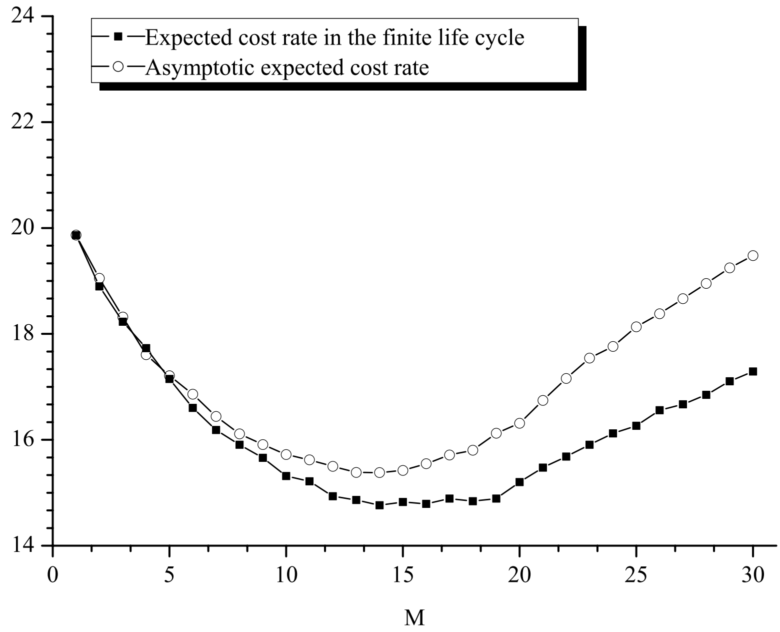


Figure 6.3: Expected cost rate in the finite life cycle and asymptotic expected cost rate versus M.

where $\tilde{E}\left[C_{10}^{M_{opt}}(50)\right]$ is the minimal expected cost in the life cycle calculated in 6.5.2.

For i and j fixed, $V_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^{M}(50)$ measures the relative difference between the minimal expected cost in the life cycle with the original parameter values and the minimal expected cost in the life cycle calculated using the modified parameter values for M and $T = 10\ t.u.$ fixed. Values closer to zero have a lower influence on the expected cost rate in the life cycle.

Table 6.2 shows the relative variation percentages with a shaded grey scale. Each cell represents $V_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^{M}(50)$ expressed in percentage. Darker colours of cells denote a higher relative variation percentage. The results obtained show that $V_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^{M}(50)$ grows when α increases and β decreases and $V_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^{M}(50)$ decreases when α decreases and β increases. In this way, $V_{10,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^{M}(50)$ reaches its minimum value when α is minimum and β is maximum and its maximum value when α is maximum and β is minimum.

By modifying $\pm 1\%$ around $\alpha = \beta = 0.1$, the relative variation percentages are small. The results also show that the relative variation percentages are lower in the diagonal of the table, that is, when the parameters α and β are modified in the same direction and magnitude.

Similarly, the values of the parameters λ_1 and λ_2 are modified according to the following specifications:

$$\lambda_{1,(v_i\%)} = \lambda_1\left[1 + \frac{v_i}{100}\right] \quad \text{and} \quad \lambda_{2,(v_j\%)} = \lambda_2\left[1 + \frac{v_j}{100}\right]. \tag{6.18}$$

Let $\tilde{E}^*\left[C_{10,\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}^{M}(t_f)\right]$ be the minimal expected cost in the life cycle obtained when the parameters λ_1 and λ_2 vary simultaneously as in the scheme given in (6.18). Now, the

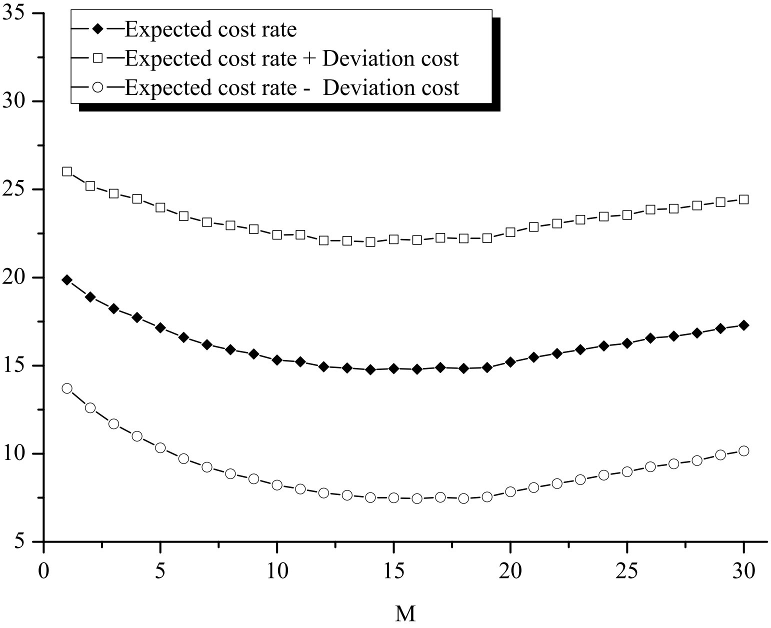


Figure 6.4: Expected cost rate in the finite life cycle and standard deviation versus M.

relative variation $V^M_{10,\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50)$ is

$$V^M_{10,\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50) = \frac{\left|\tilde{E}^*\left[C_{10}^{M_{opt}}(50)\right] - \tilde{E}^*\left[C^M_{10,\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50)\right]\right|}{\tilde{E}^*\left[C_{10}^{M_{opt}}(50)\right]}. \tag{6.19}$$

The relative variation percentages are presented in Table 6.3. The results show that the parameter λ_1 has greater effects on $V^M_{10,\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50)$ than the parameter λ_2. Additionally, the relative variation percentages reach the lowest values when the variation for $\lambda_1 = 0.01$ is minimal, that is $\pm 1\%$, and the highest values when the variation for λ_1 is maximised, that is $\pm 10\%$.

6.5.3 Analysis of the expected cost rate in the finite life cycle for M fixed

We now analyse the influence of T on the expected cost in the life cycle for $M = 14\ d.u.$ As in Subsection 6.5.2, the optimisation problem for the expected cost in the life cycle based on the recursive formula given in (6.8) is computed throughout the following steps:

1. A grid of size 10 is obtained by discretising the set $[5, 50]$ into 10 equally spaced points from 5 to 50 for the time between inspections T.

2. For $M = 14\ d.u.$ fixed and for each T_i fixed, we obtain 50000 simulations of (R_1, I_1, W_d). With these simulations and applying Monte Carlo method, we obtain the estimations $\tilde{P}^{14}_{R_{1,p}}(kT_i)$, $\tilde{P}^{14}_{R_{1,c}}(kT_i)$, $\tilde{P}^{14}_{R_1}(kT_i)$, and $\tilde{E}\left[W^{14}_{T_i}((k-1)T_i, kT_i)\right]$ for $k = 1, 2, \ldots, \lfloor 50/T_i \rfloor$ (see Figure 3.4).

	$\beta_{(-10\%)}$	$\beta_{(-5\%)}$	$\beta_{(-1\%)}$	β	$\beta_{(1\%)}$	$\beta_{(5\%)}$	$\beta_{(10\%)}$
$\alpha_{(-10\%)}$	1.1862	2.2865	5.8303	5.7383	6.4372	8.6707	11.0707
$\alpha_{(-5\%)}$	4.1067	0.9778	1.9635	2.7403	3.2077	6.0850	9.0580
$\alpha_{(-1\%)}$	7.0083	2.7336	0.6373	0.3666	1.0904	3.6113	6.1380
α	8.0187	3.8252	1.0523	0.0000	0.2624	2.8032	6.0328
$\alpha_{(1\%)}$	8.7105	4.5726	1.4507	1.0696	0.0267	2.6336	5.1357
$\alpha_{(5\%)}$	11.3000	7.0853	4.0552	3.6551	2.7247	0.2344	2.7598
$\alpha_{(10\%)}$	15.2929	10.5769	7.2532	6.6684	5.7666	3.0604	0.0958

Table 6.2: Relative variation percentages for the expected cost in the finite life cycle for the gamma process parameters for $T = 10\ t.u.$ fixed.

	$\lambda_{2,(-10\%)}$	$\lambda_{2,(-5\%)}$	$\lambda_{2,(-1\%)}$	λ_2	$\lambda_{2,(1\%)}$	$\lambda_{2,(5\%)}$	$\lambda_{2,(10\%)}$
$\lambda_{1,(-10\%)}$	3.0794	2.4761	2.3266	1.9374	1.8849	2.0284	1.3703
$\lambda_{1,(-5\%)}$	1.6095	1.2419	1.3581	0.8946	0.9002	0.3058	0.3755
$\lambda_{1,(-1\%)}$	0.6159	0.5227	0.2235	0.0000	0.5353	0.0241	0.9365
λ_1	0.4598	0.2900	0.2235	0.0000	0.5353	0.0241	0.9365
$\lambda_{1,(1\%)}$	0.4598	0.2451	0.4946	0.0118	0.3289	1.0549	0.9833
$\lambda_{1,(5\%)}$	0.2698	0.3685	0.5846	0.7321	0.4718	0.8365	1.1554
$\lambda_{1,(10\%)}$	1.3381	2.3405	2.7045	2.2538	2.7964	2.5593	3.2779

Table 6.3: Relative variation percentages for the expected cost in the finite life cycle for parameters λ_1 and λ_2 for $T = 10\ t.u.$ fixed.

3. The expected cost in the life cycle $\tilde{E}\left[C_{T_i}^{14}(50)\right]$ is calculated by using the recursive formula given in (6.8), replacing the corresponding probabilities by their estimations calculated in Step 2, with initial condition $\tilde{E}\left[C_{T_i}^{14}(0)\right] = 0$.

4. For $M = 14\ d.u.$ fixed, the search of the optimal maintenance strategy is reduced to find the value of T_{opt} minimising $\tilde{E}\left[C_T^{14}(50)\right]$. That is

$$\tilde{E}\left[C_{T_{opt}}^{14}(50)\right] = \min_{T>0}\left\{\tilde{E}\left[C_T^{14}(50)\right]\right\}.$$

Now, we compare the expected cost rate in the life cycle, $\tilde{E}\left[C_T^{14}(50)\right]/50$, based on the recursive method to the expected cost rate in the life cycle based on strictly Monte Carlo simulation. Secondly, the expected cost rate in a life cycle is compared to the expected cost rate. The results are detailed below.

The expected cost rate in the life cycle $\tilde{E}\left[C_{T_i}^{14}(50)\right]/50$ calculated using the recursive formula and the expected cost rate in the life cycle calculated based strictly on Monte Carlo simulation are shown in Figure 6.5. The expected cost rate in the life cycle based on strictly Monte Carlo simulation was calculated for 10 equally spaced points in the interval $[5, 50]$ with 50000 realizations for each point.

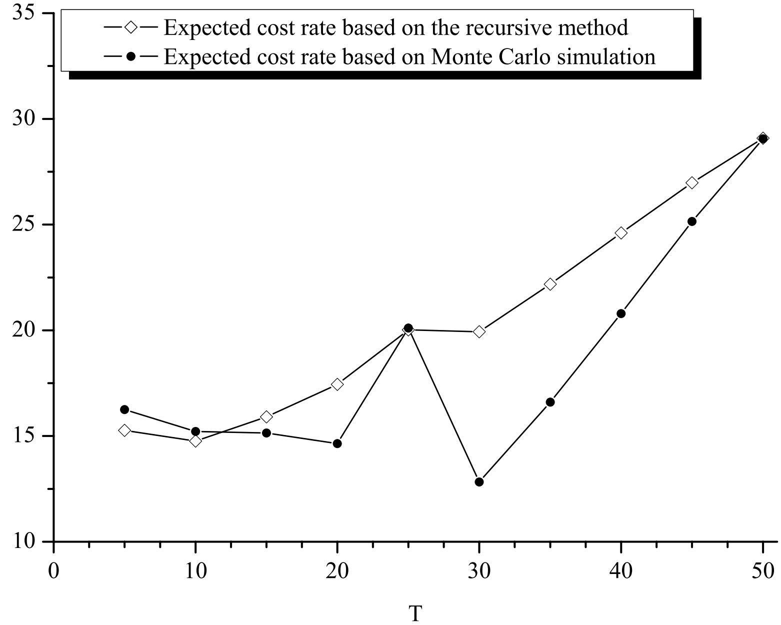


Figure 6.5: Expected cost rate in the finite life cycle versus T.

Based on Figure 6.5, for the recursive method, the expected cost rate in the life cycle reaches its minimum values for $T_{opt} = 10\ t.u.$, with an expected cost rate in the life cycle of $\tilde{E}\left[C_{10}^{14}(50)\right]/50 = 14.7637\ m.u./t.u.$ Using strictly Monte Carlo simulation, the expected cost rate in the life cycle reaches its minimum value at $T_{opt} = 30\ t.u.$, with an expected cost rate in the life cycle of $\tilde{E}\left[C_{10}^{14}(50)\right]/50 = 12.8252\ m.u./t.u.$ This difference can be explained due to the high variance in deterioration increments of the degradation process. For $M = 14$ $d.u.$ Table 6.4 shows the average number of completed renewal cycles at time $t_f = 50$ $t.u.$ for each value of T.

T	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**
$\mathbf{E}\left[\mathbf{N}_{\mathbf{T}}^{\mathbf{14}}(\mathbf{50})\right]$	2.4650	2.2007	1.7772	1.4327	1.6419	0.8884	0.93964	0.9682	0.9833	0.9926

Table 6.4: Average number of complete renewal cycles up to $t_f = 50$ $t.u.$ versus T.

For $M = 14$ $d.u.$ fixed, Figure 6.6 shows the expected cost rate in the life cycle calculated by using the recursive method and the asymptotic expected cost rate calculated throughout the procedure detailed in Subsection 6.5.1 versus T. The asymptotic expected cost rate reaches its

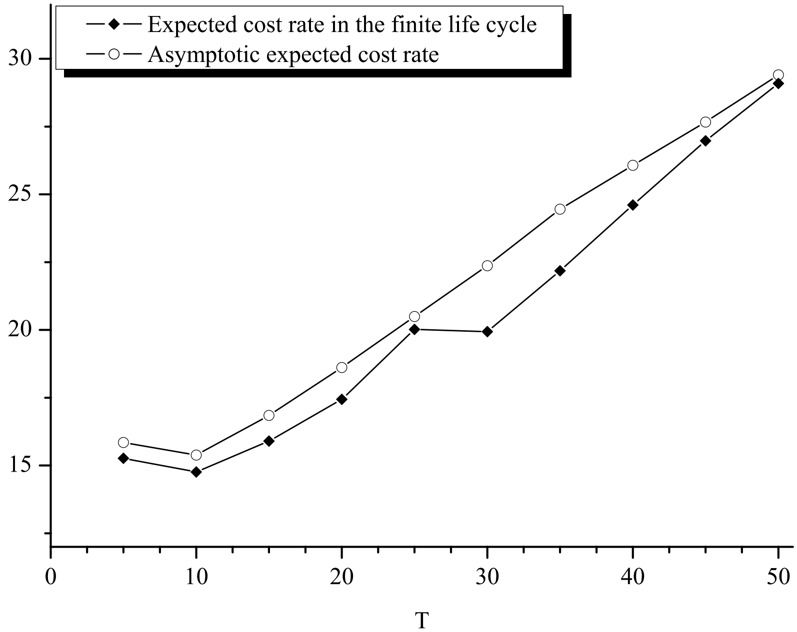


Figure 6.6: Expected cost rate in the finite life cycle and asymptotic expected cost rate versus T.

minimum value at $T_{opt} = 10$ $t.u.$, with an asymptotic expected cost rate of $\tilde{C}^{\infty}(10, 14) = 15.3819$ $m.u./t.u.$ Asymptotic expected cost rate shows a smoother behaviour compared to the expected cost rate in the life cycle.

Next, the standard deviation of the cost is calculated. Figure 6.7 shows the expected cost rate in the life cycle calculated using the recursive method with its standard deviation associated given in (6.10) versus T. This deviation was calculated for 10 equally spaced points in the interval $[5, 50]$ by using the recursive formula given in Equation (6.9) throughout the steps detailed in 6.5.3.

Focusing now on the main model parameters influence on the solution, we analyse the gamma process parameters sensitivity.

Let $\tilde{E}\left[C^{14}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)\right]$ be the minimal expected cost in the life cycle obtained when the gamma process parameters (α and β) vary according to the specifications given in (6.16) for

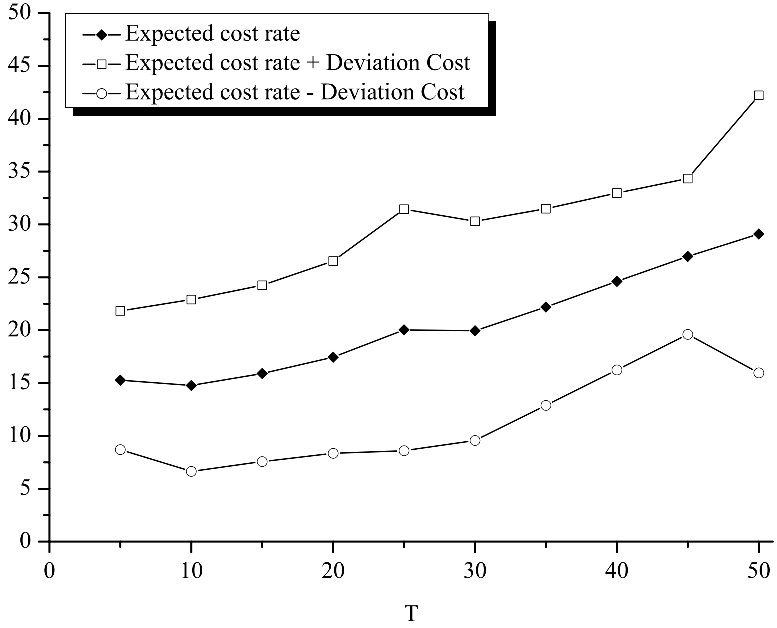


Figure 6.7: Expected cost rate in the finite life cycle and its standard deviation associated versus T.

each value of T with $M = 14\ d.u.$ Based on (6.17), $V^{14}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ denotes the relative variation between the minimal expected cost in the life cycle with the original parameter values and the minimal expected cost in the life cycle calculated by using the parameter values modified according to (6.16) for variable T and $M = 14\ d.u.$

Table 6.5 shows the values obtained for $V^{14}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ expressed in percentage. By modifying $\pm 1\%$ around $\alpha = \beta = 0.1$, the relative variation percentages are small. The results obtained also show that $V^{14}_{T,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(50)$ is lower in the diagonal of the table, that is, when the parameters α and β are modified in the same direction and magnitude.

On the other hand, let $\tilde{E}^*\left[C^{14}_{T,\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(t_f)\right]$ be the minimal expected cost in the life cycle obtained by varying the parameters λ_1 and λ_2 simultaneously as in the scheme given in (6.18). Based on (6.19), $V^{14}_{T,\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50)$ denotes the relative variation between the minimal expected cost in the life cycle with the original parameter values and the minimal expected cost in the life cycle calculated by using the parameter values modified according to (6.18) for variable T and $M = 14\ d.u.$

The relative variation percentages are presented in Table 6.6. The results show that when $\lambda_1 = 0.01$ is modified between -5% and 1%, the relative variation percentages are small. In addition, the parameter λ_1 has greater effects on $V^{14}_{T,\lambda_{1,(v_i\%)},\lambda_{2,(v_j\%)}}(50)$ than the parameter λ_2.

	$\beta_{(-10\%)}$	$\beta_{(-5\%)}$	$\beta_{(-1\%)}$	β	$\beta_{(1\%)}$	$\beta_{(5\%)}$	$\beta_{(10\%)}$
$\alpha_{(-10\%)}$	1.5684	1.8862	4.7069	5.7107	6.1450	8.0609	11.2240
$\alpha_{(-5\%)}$	4.5915	0.9400	1.7235	2.5432	3.3861	5.1360	8.1300
$\alpha_{(-1\%)}$	8.5189	3.4618	0.9003	0.0450	0.6994	3.0001	6.2257
α_1	4.4937	8.2817	1.3744	0.0000	0.1749	2.7793	5.2119
$\alpha_{(1\%)}$	8.9568	4.7309	1.9896	1.8475	0.4936	1.4929	5.0625
$\alpha_{(5\%)}$	11.5044	7.2387	5.0048	3.5146	3.3441	0.8091	2.7099
$\alpha_{(10\%)}$	15.1620	10.7048	7.5764	6.5529	6.7841	3.6827	0.6109

Table 6.5: Relative variation percentages for the expected cost in the finite life cycle for the gamma process parameters for $M = 14\ d.u.$ fixed.

	$\lambda_{2,(-10\%)}$	$\lambda_{2,(-5\%)}$	$\lambda_{2,(-1\%)}$	λ_2	$\lambda_{2,(1\%)}$	$\lambda_{2,(5\%)}$	$\lambda_{2,(10\%)}$
$\lambda_{1,(-10\%)}$	2.1506	1.9893	1.9244	1.2564	1.7375	1.4737	1.7043
$\lambda_{1,(-5\%)}$	0.8525	0.3948	0.7135	0.2861	0.0258	0.2199	0.2733
$\lambda_{1,(-1\%)}$	0.8510	0.5039	0.4442	0.6259	0.4789	0.7896	1.3432
λ_1	0.4986	0.5000	0.4663	0.0000	0.7516	1.0475	1.1239
$\lambda_{1,(1\%)}$	0.0194	0.6036	0.9957	0.9308	0.7760	1.1128	1.9800
$\lambda_{1,(5\%)}$	1.9045	1.5869	2.8829	1.4909	2.1346	2.5697	3.0034
$\lambda_{1,(10\%)}$	2.4267	2.1395	2.8044	2.9864	2.8605	3.3628	3.0087

Table 6.6: Relative variation percentages for the expected cost rate in the finite life cycle for parameters λ_1 and λ_2 for $M = 14\ d.u.$ fixed.

6.5.4 Analysis of the two-dimensional expected cost rate in the finite life cycle

The expected cost in the life cycle based on the recursive formula given in (6.8) considering a bidimensional case is analysed. The optimisation problem is computed as follows:

1. A grid of size 10 is obtained by discretising the set $[5, 50]$ into 10 equally spaced points from 5 to 50 for T. Let T_i be the i-th value of the grid, for $i = 1, 2, \ldots, 10$.

2. A grid of size 30 is obtained by discretising the set $[1, 30]$ into 30 equally spaced points from 1 to 30 for M. Let M_j be the j-th value of the grid, for $j = 1, 2, \ldots, 30$.

3. For each combination (T_i, M_j) fixed, we obtain 50000 simulations of (R_1, I_1, W_d). With these simulations and applying Monte Carlo method, $\tilde{P}_{R_{1,p}}^{M_j}(kT_i)$, $\tilde{P}_{R_{1,c}}^{M_j}(kT_i)$, $\tilde{P}_{R_1}^{M_j}(kT_i)$, and $\tilde{E}\left[W_{T_i}^{M_j}((k-1)T_i, kT_i))\right]$ for $k = 1, 2, \ldots, \lfloor 50/T_i \rfloor$ are obtained (see Figure 3.4).

4. The expected cost in the life cycle $\tilde{E}\left[C_{T_i}^{M_j}(50)\right]$ is calculated by using the recursive formula given in (6.8), replacing the corresponding probabilities by their estimations calculated in Step 2, with initial condition $\tilde{E}\left[C_{T_i}^{M_j}(0)\right] = 0$.

5. The search of the optimal maintenance strategy is reduced to find the values T_{opt} and M_{opt} minimising $\tilde{E}\left[C_T^M(50)\right]$. That is

$$\tilde{E}\left[C_{T_{opt}}^{M_{opt}}(50)\right] = \min_{\substack{T>0 \\ 0<M\leq L}} \left\{\tilde{E}\left[C_T^M(50)\right]\right\}.$$

The expected cost rate in the life cycle $\tilde{E}\left[C_{T_i}^{M_j}(50)\right]/50$ versus T and M is shown in Figure 6.8. The values of T and M minimising the expected cost rate in the life cycle are reached for $M_{opt} = 14$ $d.u.$ and $T_{opt} = 10$ $t.u.$ with an expected cost rate in the life cycle of $\tilde{E}\left[C_{10}^{14}(50)\right]/50 = 14.7637$ $m.u./t.u.$

6.5.5 Numerical analysis of the availability measures for T and M fixed for optimal values of T and M

We now analyse the point availability, the reliability, and the interval reliability of the system considering $T = 10$ $t.u.$ and $M = 14$ $d.u.$, that is, the optimal maintenance strategy.

The point availability of the system $\tilde{A}_T^M(t)$ based on the recursive formula given in (6.11) is computed as follows:

1. For t, a grid of size 50 is obtained by discretising the set $[1, 50]$ into 50 equally spaced points from 1 to 50. Let t_n be the n-th value of the grid, for $n = 1, 2, \ldots, 50$.

2. For $T = 10$ $t.u.$ and $M = 14$ $d.u.$, we obtain 50000 simulations of (R_1, I_1, W_d). With these simulations, and applying Monte Carlo method, $\tilde{P}_{R_1}^{14}(10k)$ is obtained.

3. The point availability of the system $\tilde{A}_{10}^{14}(t_n)$ is calculated by using the recursive formula given in (6.11), replacing $P_{R_1}^{14}(10k)$ by its estimation $\tilde{P}_{R_1}^{14}(10k)$ calculated in Step 2 of the procedure detailed in Subsection 6.5.4, with initial condition $\tilde{A}_{10}^{14}(0) = 1$.

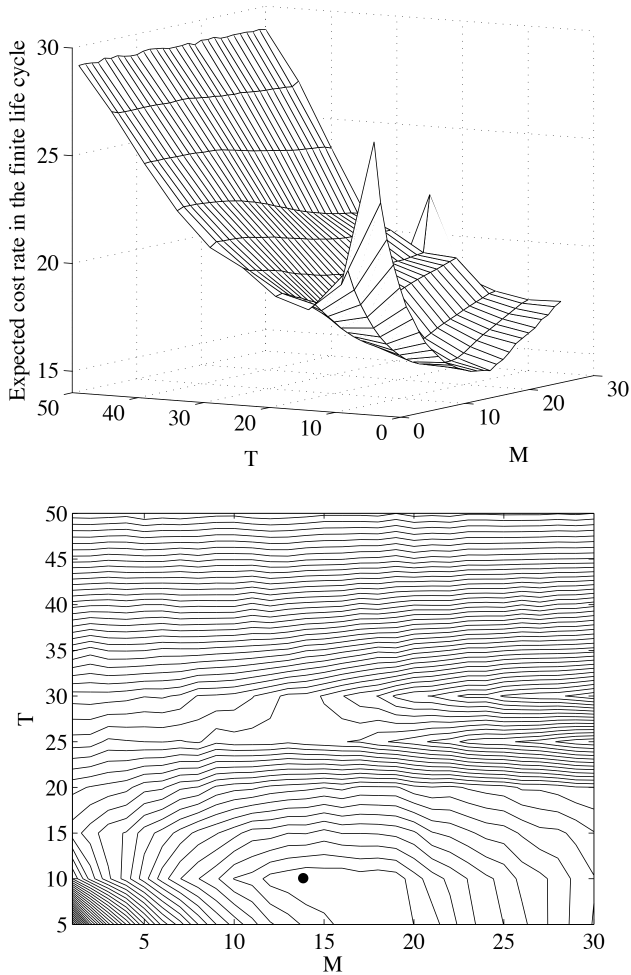


Figure 6.8: Mesh and contour plots for the expected cost rate in the finite life cycle.

Quantity $\tilde{A}_{10}^{14}(t_n)$ is also calculated using strictly Monte Carlo simulation with 50000 realisations for each t_n. Figure 6.9 shows the point availability of the system versus t calculated using the recursive method and using strictly Monte Carlo simulation. We can conclude that, for $T = 10\ t.u.$ and $M = 14\ d.u.$ fixed, the probability that the system is working at any time instant of its life cycle is, at least, of the 82%.

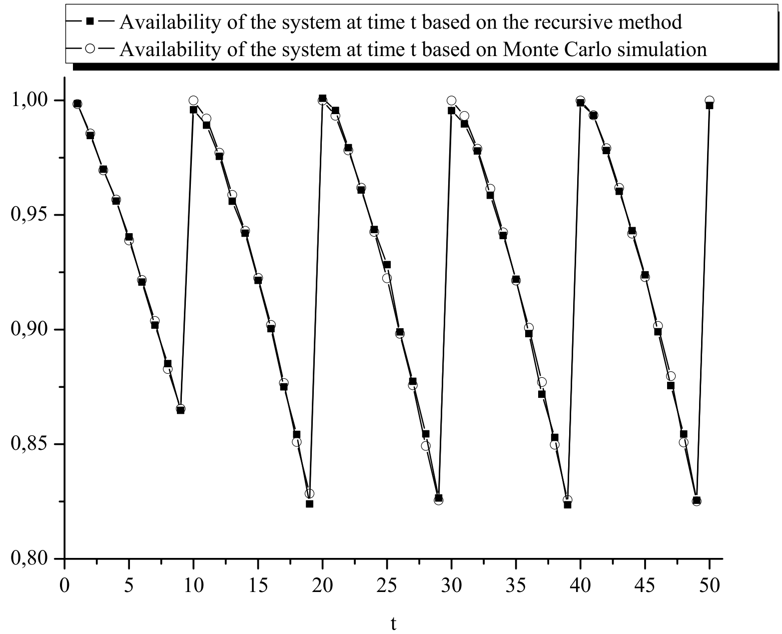


Figure 6.9: Availability of the system versus t.

Next, the reliability of the system $\tilde{R}_T^M(t)$ based on the recursive formula given in (6.14) is computed as follows:

1. For t, a grid of size 50 is obtained by discretising the set $[1, 50]$ into 50 equally spaced points from 1 to 50. Let t_n be the n-th value of the grid, for $n = 1, 2, \ldots, 50$.
2. For $T = 10$ and $M = 14$, 50000 simulations of (R_1, I_1, W_d) are obtained and an estimation of $\tilde{P}_{R_{1,p}}^{14}(10k)$ is given.
3. The reliability of the system $\tilde{R}_{10}^{14}(t_n)$ is calculated by using the recursive formula given in (6.14), replacing $P_{R_{1,p}}^{14}(10k)$ by its estimation $\tilde{P}_{R_{1,p}}^{14}(10k)$ calculated in Step 2 of the procedure detailed in Subsection 6.5.4, with initial condition $\tilde{R}_{10}^{14}(0) = 1$.

Quantity $\tilde{R}_{10}^{14}(t_n)$ is also computed using Monte Carlo simulation with 50000 realisations for each value t_n. Figure 6.10 shows the reliability of the system versus t calculated using the recursive method and using strictly Monte Carlo simulation. We can conclude that, for $T = 10\ t.u.$ and $M = 14\ d.u.$ fixed, the probability that the system does not fail in its life cycle is of the 32%.

Finally, the interval reliability of the system based on (6.15) is computed throughout the following steps:

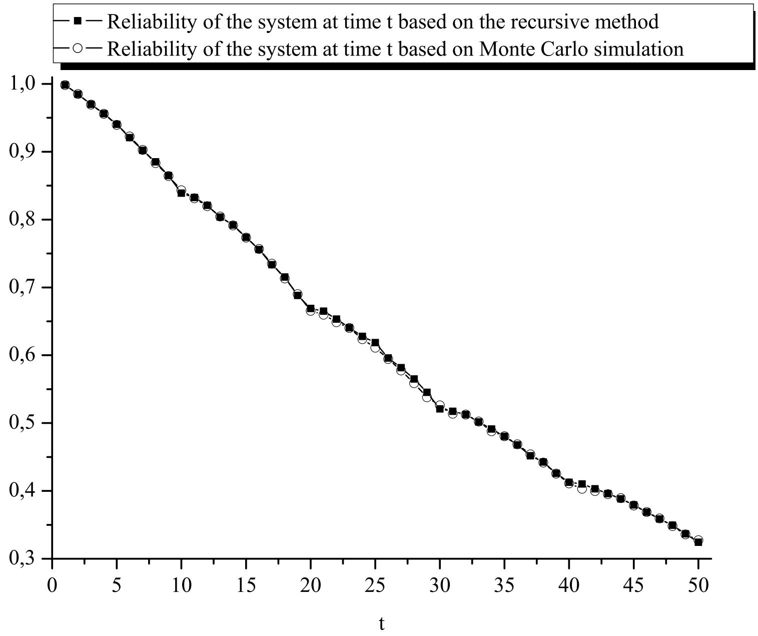


Figure 6.10: Reliability of the system versus t.

1. For t, a grid of size 10 is obtained by discretising the set $[10, 30]$ into 10 equally spaced points from 10 to 30. Let t_n be the n-th value of the grid, for $n = 1, 2, \ldots, 10$.

2. For $T = 10$ $t.u.$ and $M = 14$ $d.u.$, we obtain 50000 simulations of (R_1, I_1, W_d). With these simulations and applying Monte Carlo method $\tilde{P}^{14}_{R_{1,p}}(10k)$ and $\tilde{P}^{14}_{R_1}(10k)$ are obtained.

3. For $h = 5$, the interval reliability of the system $\tilde{IR}^{14}_{10}(t_n, t_n + h)$ is calculated by using the recursive formula given in (6.15), replacing $P^{14}_{R_{1,p}}(10k)$ and $P^{14}_{R_1}(10k)$ by the estimations $\tilde{P}^{14}_{R_{1,p}}(10k)$ and $\tilde{P}^{14}_{R_1}(10k)$, respectively, calculated in Step 2 of the procedure detailed in Subsection 6.5.4, with initial condition $\tilde{IR}^{14}_{10}(0, 0) = 1$.

Quantity $\tilde{IR}^{14}_{10}(t_n, t_n + 5)$ is also computed using Monte Carlo simulation with 50000 simulations for each value t_n. Figure 6.11 shows the interval reliability of the system versus t calculated using the recursive method and using strictly Monte Carlo simulation. As we can observe, the results provided for both methods are very similar. Furthermore, we can conclude that, for $T = 10$ $t.u.$ and $M = 14$ $d.u.$ fixed, the probability that the system does not fail in the interval $(t, t + 5]$ is, at least, of the 72% for $10 \leq t \leq 30$.

6.6 Conclusions and further extensions

In this chapter, a CBM strategy in a DTS model is analysed by considering a finite life cycle. The system is subject to two different causes of failure, a degradation process modelled under

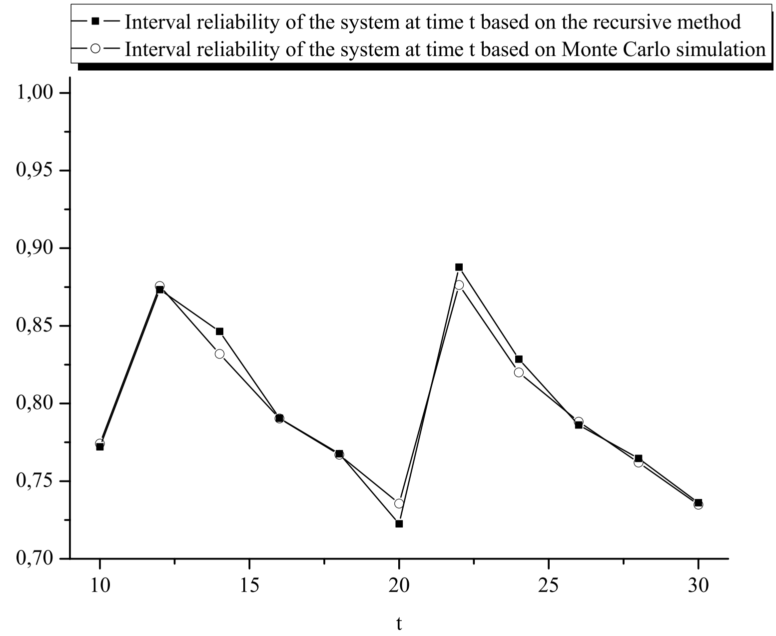


Figure 6.11: Interval reliability of the system versus t.

a gamma process and a sudden shock process following a DSPP. We consider that both causes of failure are dependent. This dependence is reflected in that the system is more susceptible to external shocks when the deterioration level of the system reaches a certain threshold M_s.

Under these assumptions, the optimisation of the expected cost rate in the life cycle as objective function is performed. To this end, a numerical method based on a recursive formula is provided to evaluate the expected cost rate in the life cycle and the standard deviation associated.

The expected cost rate in the life cycle calculated using the recursive formula is compared to both the asymptotic expected cost rate and the expected cost rate in the life cycle calculated using strictly Monte Carlo simulation. In addition, the robustness of the gamma process parameters and DSPP is analysed. Furthermore, for the comparison between transient and asymptotic cost rate the results are also provided under a bivariate case, where the time between inspections and the preventive threshold vary simultaneously. In the comparison between the method based on strictly Monte Carlo simulation and the method based on the recursive formula, we observed similar results for a time between inspections T fixed. For the preventive threshold M, the results presented some differences, which could be explained due to the high variance in the deterioration increments of the degradation process. If the life cycle increases, the recursive method shall tend to be more costly in terms of computation than the method based on strictly Monte Carlo simulation.

Finally, three important availability measures in the maintenance field, namely, the point availability, the reliability, and the interval reliability of the system are analysed under the bivariate optimal maintenance strategy using strictly Monte Carlo simulation and using recursive formulas.

In this chapter, we consider a system subject to a unique degradation process. However, sometimes the system is subject to multiple degradation processes. A possible further extension of this work is to consider a system subject to multiple degradation processes and to analyse the maintenance cost in a finite life cycle. Related to the relationship between both causes of failure, in this chapter, we consider the dependence between both causes of failure based on the dependence of the shock process intensity on the degradation of the system. An interesting extension would be to assume a bidirectional relationship of dependence where the shock process affects also to the degradation process.

CHAPTER

7

Conclusions and further research lines

In this chapter, the main contributions of the preceding chapters and a discussion of possible further lines of this research are detailed.

7.1 Conclusions

In this thesis, different CBM strategies are proposed considering systems subject to two different competing causes of failures: soft failures (described by degradation processes) and sudden failures (described by a counting process). Related to the relationship between both causes of failure, they are considered as independent in Chapters 3 and 5, and dependent in Chapters 4 and 6. About the soft failures, they can be described by one degradation process as in Chapters 5 and 6, or by multiple degradation processes as in Chapters 3 and 4.

Additionally, two approaches are studied considering the nature of the life cycle. Chapters 3 and 4 deal with an infinite life cycle. On the other hand, Chapters 5 and 6 deal with a finite life cycle, representing more realistic situations but with a more complex treatment than the other ones.

Along this thesis, we have discussed the following issues aiming to describe and prevent system failures of these systems:

- Introduction of CBM strategies to return the system in a fully-functioning and to decrease the downtime period. These maintenance strategies were developed considering periodic inspection times and optimal levels of degradation for performing preventive maintenance actions.

- Development of closed-form expressions for the probability of corrective and preventive maintenance actions and for the expected downtime of the systems.

- Analysis and comparison of the objective cost function considering two approaches: asymptotic approach and transient approach. In both approaches, the expression of the objective cost function is obtained based on the renewal processes theory.
- Analysis of the objective cost function considering a random finite life cycle.
- Numerical search of an optimal CBM strategy considering Monte Carlo simulation in the asymptotic approach.
- Numerical search of the optimal CBM strategy considering recursive methods and Monte Carlo simulation in the transient approach.
- Analysis of the standard deviation allowing us to evaluate the fluctuations of the objective cost function in the finite life cycle.
- Study of the robustness of some parameters of the different models.
- Making use of some renewal equations, development of some availability measures throughout the closed-form expressions obtained

The closed-form expressions obtained along this thesis simplify the analytical existing expressions involved in the theory of CBM strategy. In a complex process framework, these closed-form expressions can be generalised, not just for degradation modelled using gamma processes, but for other stochastic models of degradation. The only assumption is that, if we consider multiple degradation processes, they must arrive at the system following an NHPP.

In the comparison between the objective cost function of both approaches, they sometimes provide quite different results. Since the asymptotic approach is an approximation-optimisation of the transient approach, based on these results, we can conclude that the use of the asymptotic approach for optimising the objective cost function is sometimes questionable.

Along with the availability measures, it is found slight differences between the recursive methods and strictly Monte Carlo method. So that, for some complex availability measures, recursive method is not recommended since it is very time-consuming. Then, Monte Carlo simulation method is more appropriate for the availability measure evaluation in these cases.

7.2 Further research lines

According to the knowledge obtained from this Ph. D. Thesis, we can define some different future lines of this research.

The closed-form expressions developed in this thesis assume that the degradation processes start at random times following an NHPP and they grow according to a homogeneous gamma process. An extension of these assumptions is to consider a different framework for the time of arrival and growth of the degradation processes.

The second further extension is to consider a bidirectional relationship of dependence. Up to now, we have assumed that the dependence between both causes of failure is unidirectional in the sense that only the degradation process affects to the shock process. Also, an interesting extension could be that the shocks increases the deterioration level of degradation processes of the system. Thus, if we also assume that the intensity of sudden shocks depends on the degradation process, the dependence between both causes of failure is bidirectional: sudden shocks depend on the degradation process and degradation depends on the sudden shock process.

Other possible extension is to consider imperfect repairs in the maintenance actions. Often, the maintenance of a deteriorating system is imperfect, that is, the system after a maintenance action is not as good as new, but younger. In this sense, when an imperfect repair is performed, the deterioration level is reduced hence the lifetime of the system increases.

Finally, we can consider measurement errors in the evaluation of the condition of the system. CBM strategies analysed in this thesis are based on exact state variable measures. However, these measures can contain disturbances and the measured value does not match with the true value of the variable. In this aspect, it would be of interest to analyse these measurement errors to provide better maintenance strategies.

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Declaration

I herewith declare that I have produced this work without the prohibited assistance of third parties and without making use of aids other than those specified; notions taken over directly or indirectly from other sources have been identified as such. This work has not previously been presented in identical or similar form to any examination board.

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