

A Generalization of the Hyers-Ulam-Aoki Type Stability of Some Banach Lattice-Valued Functional Equation

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Abstract: We obtained a generalization of the stability of some Banach lattice-valued functional equation with the addition replaced in the Cauchy functional equation by lattice operations and their combinations.

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1. INTRODUCTION

All along $(\mathcal{X}, \wedge_{\mathcal{X}}, \vee_{\mathcal{X}})$ will stand for a normed Riesz space and $(\mathcal{Y}, \wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}})$ for a Banach lattice with \mathcal{X}^+ and \mathcal{Y}^+ their respective positive cones. Let us pose the following problem.

PROBLEM 1. Given three numbers $\varepsilon, p, q \in (0, \infty)$, two Riesz spaces G_1 and G_2 with G_2 being endowed with a metric $d(\cdot, \cdot)$, four lattice operations $\Delta_{G_1}^*, \Delta_{G_1}^{**} \in \{\wedge_{G_1}, \vee_{G_1}\}$ and $\Delta_{G_2}^*, \Delta_{G_2}^{**} \in \{\wedge_{G_2}, \vee_{G_2}\}$, does there exist some real number $\delta > 0$ such that, if a mapping $F : G_1 \rightarrow G_2$ satisfies

$$d\left(\left(F((\tau^q|x|)\Delta_{G_1}^*(\eta^q|y|))\right)\Delta_{G_2}^*\left(F((\tau^q|x|)\Delta_{G_1}^{**}(\eta^q|y|))\right), \right. \\ \left. (\tau^p F(|x|))\Delta_{G_2}^{**}(\eta^p F(|y|))\right) \leq \delta \tag{1.1}$$

for all $x, y \in G_1$ and all $\tau, \eta \in [0, \infty)$, then an operation-preserving functional $T : G_1 \rightarrow G_2$ exists with the property that

$$d(T(x), F(x)) \leq \varepsilon$$

for all $x \in G_1$ and all $\tau, \eta \in [0, \infty)$?

If in (1.1) we let $\tau = \eta = 1$, then the above problem reduces to the problem posed and treated in [5].

The study of functional equations and inequalities in lattice environments is motivated by the fact that many addition-related results or theorems can be extended and can be proved *mutatis mutandis*. For more references about the earliest extensions of the kind, we would refer the reader to the papers [1, 2, 3, 4].

The main goal of this paper is to show how Ulam-Hyers-Aoki styled version of perturbation (1.1) leads to the unique solution of the functional equation

$$\begin{aligned} \left(T((\tau^q|x|)\Delta_{\mathcal{X}}^*(\eta^q|y|)) \right) \Delta_{\mathcal{Y}}^* \left(T((\tau^q|x|)\Delta_{\mathcal{X}}^{**}(\eta^q|y|)) \right) \\ = (\tau^p T(|x|)) \Delta_{\mathcal{Y}}^{**} (\eta^p T(|y|)) \end{aligned} \quad (1.2)$$

for all $x, y \in \mathcal{X}$ and all $\tau, \eta \in [0, \infty)$, where $\Delta_{\mathcal{X}}^*, \Delta_{\mathcal{X}}^{**} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\Delta_{\mathcal{Y}}^*, \Delta_{\mathcal{Y}}^{**} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$ are fixed lattice operations.

Remark 1.1. If we let $\eta = \tau$ and $y = x$ in equation (1.2), then we observe that

$$T(\tau^q|x|) = \tau^p T(|x|) \quad (1.3)$$

for all $x \in \mathcal{X}$ and all $\tau \in [0, \infty)$.

The results we obtained are straightforward generalizations of Agbeko [3, 5, 6] and Salahi et al [16]. For an additional reference we would like to mention the paper [7] where we proved separation and stability results for operators mapping a semi-group with values in a Riesz lattice.

We recall that a functional $H : \mathcal{X} \rightarrow \mathcal{Y}$ is *cone-related* if $H(\mathcal{X}^+) = \{H(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$ (see more about this notion in [3, 4]).

Some few references about Hyers-Ulam stability problems and solutions can be found, e.g. in [8, 11, 13, 14, 15].

Our theorems will be deduced from the following Forti's result [10].

THEOREM 1.1. (Forti) *Let (X, d) be a complete metric space and S an appropriate set. Assume some functions $f : S \rightarrow X$, $G : S \rightarrow S$, $H : X \rightarrow X$ and $\delta : S \rightarrow [0, \infty)$ satisfy the inequality*

$$d(H(f(G(x))), f(x)) \leq \delta(x), \quad (1.4)$$

for all $x \in S$. If function H both is continuous and satisfies the inequality

$$d(H(u), H(v)) \leq \varphi(d(u, v)), \quad u, v \in X, \quad (1.5)$$

for a certain non-decreasing subadditive function $\varphi : [0, \infty) \rightarrow [0, \infty)$ and the series

$$\sum_{j=0}^{\infty} \varphi^j(\delta(G^j(x))) \quad (1.6)$$

is convergent for every $x \in S$, then there exists a unique function $F : S \rightarrow X$ solution of the functional equation

$$H(F(G(x))) = F(x), \quad x \in S, \quad (1.7)$$

and satisfying the following inequality:

$$d(F(x), f(x)) \leq \sum_{j=0}^{\infty} \varphi^j(\delta(G^j(x))). \quad (1.8)$$

The function F is given by

$$F(x) = \lim_{n \rightarrow \infty} H^n(f(G^n(x))). \quad (1.9)$$

2. THE MAIN RESULTS

THEOREM 2.1. *Given a pair of real numbers $(p, q) \in (0, \infty) \times (0, \infty)$, consider a cone-related functional $F : \mathcal{X} \rightarrow \mathcal{Y}$ for which there are numbers $\vartheta > 0$ and α with $q\alpha \in (p, \infty)$ such that*

$$\left\| F((\tau^q|x|)\Delta_{\mathcal{X}}^*(\eta^q|y|))\Delta_{\mathcal{Y}}^*F((\tau^q|x|)\Delta_{\mathcal{X}}^{**}(\eta^q|y|)) - (\tau^p F(|x|))\Delta_{\mathcal{Y}}^{**}(\eta^p F(|y|)) \right\| \leq 2^{(p-1)\vartheta}(\|x\|^\alpha + \|y\|^\alpha) \quad (2.1)$$

for all $x, y \in \mathcal{X}$ and all $\tau, \eta \in [0, \infty)$. Then the sequence $(2^{np}F(2^{-nq}|x|))_{n \in \mathbb{N}}$ is a Cauchy sequence for every $x \in \mathcal{X}$. Let the functional $T : \mathcal{X}^+ \rightarrow \mathcal{Y}^+$ be defined by

$$T(|x|) = \lim_{n \rightarrow \infty} 2^{np}F(2^{-nq}|x|) \quad (2.2)$$

for all $x \in \mathcal{X}$. Then T both is the unique solution of (1.2) and satisfies inequality

$$\|T(|x|) - F(|x|)\| \leq \frac{2^p \vartheta}{2^{q\alpha} - 2^p} \|x\|^\alpha \quad (2.3)$$

for every $x \in \mathcal{X}$.

THEOREM 2.2. *Given a pair of real numbers $(p, q) \in (0, \infty) \times (0, \infty)$, consider a cone-related functional $F : \mathcal{X} \rightarrow \mathcal{Y}$ for which there are numbers $\beta \in [0, \infty)$, $\vartheta > 0$ and α with $q\alpha \in (0, p)$ such that*

$$\begin{aligned} & \left\| F((\tau^q|x|)\Delta_{\mathcal{X}}^*(\eta^q|y|))\Delta_{\mathcal{Y}}^*F((\tau^q|x|)\Delta_{\mathcal{X}}^{**}(\eta^q|y|)) \right. \\ & \quad \left. - (\tau^p F(|x|))\Delta_{\mathcal{Y}}^{**}(\eta^p F(|y|)) \right\| \leq \beta + \vartheta 2^{-(p+1)}(\|x\|^\alpha + \|y\|^\alpha) \end{aligned} \quad (2.4)$$

for all $x, y \in \mathcal{X}$ and all $\tau, \eta \in [0, \infty)$. Then the sequence $(2^{-np}F(2^{nq}|x|))_{n \in \mathbb{N}}$ is a Cauchy sequence for every fixed $x \in \mathcal{X}$. Let the functional $T : \mathcal{X}^+ \rightarrow \mathcal{Y}^+$ be defined by

$$T(|x|) = \lim_{n \rightarrow \infty} 2^{-np}F(2^{nq}|x|) \quad (2.5)$$

for all $x \in \mathcal{X}$. Then T both is the unique solution of (1.2) and satisfies inequality

$$\|T(|x|) - F(|x|)\| \leq \frac{\beta 2^p}{2^p - 1} + \frac{\vartheta \|x\|^\alpha 2^{q\alpha}}{2^p - 2^{q\alpha}} \quad (2.6)$$

for every $x \in \mathcal{X}$.

Remark 2.1. If the conditions of Theorem 2.1 or Theorem 2.2 hold true, then $F(0) = 0$.

Proof. The proof is similar to its counterpart in [5, 6] under the conditions of Theorem 2.1 or Theorem 2.2 when $\beta = 0$. Under the condition of Theorem 2.2 with $\beta > 0$, we need to prove that $F(0) = 0$. Suppose in the contrary that $F(0) > 0$ were true. Then by letting $x = y = 0$ and $\eta = \tau$ in (2.4), inequality

$$\|F(0) - \tau^p F(0)\| \leq \beta$$

follows or, equivalently

$$|\tau^p - 1| \leq \frac{\beta}{\|F(0)\|} < \infty,$$

which, as τ tends to infinity, would lead to an absurdity, indeed. Hence the relation $F(0) = 0$ must be true. ■

Remark 2.2. Let Z be a set closed under the scalar multiplication, i.e. $bz \in Z$ whenever $b \in \mathbb{R}$ and $z \in Z$. Given a number $c \in \mathbb{R}$ let the function $\gamma : Z \rightarrow Z$ be defined by $\gamma(z) = cz$. Then $\gamma^j : Z \rightarrow Z$ the j -th iteration of γ is given by $\gamma^j(z) = c^j z$ for every natural number $j \geq 2$.

3. PROOF OF THE MAIN RESULTS

We shall use the technique in [5] to prove the main theorems.

Proof of Theorem 2.1. First, if we choose $\tau = \eta = 2$, $y = x$ and replace x by $2^{-q}x$ in inequality (2.1) then we obviously have

$$\|2^p F(2^{-q}|x|) - F(|x|)\| \leq \vartheta 2^{p-q\alpha} \|x\|^\alpha. \quad (3.1)$$

Next, let us define the following functions:

$$\begin{aligned} G : \mathcal{X}^+ &\rightarrow \mathcal{X}^+, & G(|x|) &= 2^{-q}|x|, & \text{for all } x \in \mathcal{X}, \\ \delta : \mathcal{X}^+ &\rightarrow [0, \infty), & \delta(|x|) &= \vartheta 2^{p-q\alpha} \|x\|^\alpha, & \text{for all } x \in \mathcal{X}, \\ \varphi : [0, \infty) &\rightarrow [0, \infty), & \varphi(t) &= 2^p t, \\ H : \mathcal{Y}^+ &\rightarrow \mathcal{Y}^+, & H(|y|) &= 2^p |y|, & \text{for all } y \in \mathcal{Y}, \\ d(\cdot, \cdot) : \mathcal{Y}^+ \times \mathcal{Y}^+ &\rightarrow [0, \infty), & d(|y_1|, |y_2|) &= \||y_1| - |y_2|\|, & \text{for all } y_1, y_2 \in \mathcal{Y}. \end{aligned}$$

We shall verify the fulfilment of all the three conditions of the Forti's theorem as follows.

(I) From inequality (3.1) we obviously have

$$\begin{aligned} d(H(F(G(|x|))), F(|x|)) &= \|H(F(G(|x|))) - F(|x|)\| \\ &= \|2^p F(2^{-q}|x|) - F(|x|)\| \\ &\leq \vartheta 2^{p-q\alpha} \|x\|^\alpha = \delta(|x|). \end{aligned}$$

(II) $d(H(|y_1|), H(|y_2|)) = 2^p \||y_1| - |y_2|\| = \varphi(d(|y_1|, |y_2|))$ for all $y_1, y_2 \in \mathcal{Y}$.

(III) Clearly, on the one hand φ is a non-decreasing subadditive function on the positive half line, and on the other hand by applying Remark 2.2 on both the iterations G^j and φ^j of G and φ respectively, one can observe that

$$\sum_{j=0}^{\infty} \varphi^j(\delta(G^j(|x|))) = \vartheta 2^{p-q\alpha} \|x\|^\alpha \sum_{j=0}^{\infty} 2^{(p-q\alpha)j} = \vartheta \|x\|^\alpha \frac{2^p}{2^{q\alpha} - 2^p} < \infty.$$

Then in view of Forti's theorem, sequence $(H^n(F(G^n|x|)))_{n \in \mathbb{N}}$ is a Cauchy sequence for every $x \in \mathcal{X}$ and thus so is $(2^{np} F(2^{-nq}|x|))_{n \in \mathbb{N}}$. Furthermore, the mapping (2.2) satisfies inequality (2.3).

Next, we prove that T solves (1.2). In fact, in (2.1) substitute x with $2^{-nq}x$ also y with $2^{-nq}y$, and fix arbitrarily $\tau, \eta \in [0, \infty)$. Then

$$\left\| F\left(\frac{(\tau^q|x|)\Delta_{\mathcal{X}}^*(\eta^q|y|)}{2^{nq}}\right)\Delta_{\mathcal{Y}}^*F\left(\frac{(\tau^q|x|)\Delta_{\mathcal{X}}^{**}(\eta^q|y|)}{2^{nq}}\right) - \left(\tau^p F\left(\frac{|x|}{2^{nq}}\right)\right)\Delta_{\mathcal{Y}}^{**}\left(\eta^p F\left(\frac{|y|}{2^{nq}}\right)\right) \right\| \leq 2^{(p-1)\vartheta} \left(\left\| \frac{x}{2^{nq}} \right\|^\alpha + \left\| \frac{y}{2^{nq}} \right\|^\alpha \right).$$

Multiplying both sides of this last inequality by 2^{np} yields

$$2^{np} \left\| F\left(\frac{(\tau^q|x|)\Delta_{\mathcal{X}}^*(\eta^q|y|)}{2^{nq}}\right)\Delta_{\mathcal{Y}}^*F\left(\frac{(\tau^q|x|)\Delta_{\mathcal{X}}^{**}(\eta^q|y|)}{2^{nq}}\right) - \left(\tau^p F\left(\frac{|x|}{2^{nq}}\right)\right)\Delta_{\mathcal{Y}}^{**}\left(\eta^p F\left(\frac{|y|}{2^{nq}}\right)\right) \right\| \leq \frac{\vartheta}{2^{(1-p)n}} \frac{\|x\|^\alpha + \|y\|^\alpha}{2^{n(q\alpha-p)}}. \quad (3.2)$$

Taking the limit in (3.2) we have via (2.2) that

$$\|T((\tau^q|x|)\Delta_{\mathcal{X}}^*(\eta^q|y|))\Delta_{\mathcal{Y}}^*T((\tau^q|x|)\Delta_{\mathcal{X}}^{**}(\eta^q|y|)) - (\tau^p T(|x|))\Delta_{\mathcal{Y}}^{**}(\eta^p T(|y|))\| = 0$$

for all $\tau, \eta \in [0, \infty)$ and all $x, y \in \mathcal{X}$, which is equivalent to (1.2). Thus T also satisfies (1.3) in Remark 1.1. Finally we show the uniqueness, using a technique in [16]. In fact, assume that there is another functional $S : \mathcal{X} \rightarrow \mathcal{Y}$ which satisfies (1.2) and the inequality $\|S(|x|) - F(|x|)\| \leq \delta_2 \|x\|^{\alpha_2}$ for some numbers $\alpha_2, \delta_2 \in (0, \infty)$ with $q\alpha_2 > p$, and for all $x \in \mathcal{X}$. In (2.3) let $\delta_1 := \frac{2^p \vartheta}{2^{q\alpha - 2p}}$, $\alpha_1 := \alpha$ and by choosing $\tau = 2^{-n}$ in Remark 1.1 one can observe that for all $x \in \mathcal{X}$

$$\begin{aligned} \|S(|x|) - T(|x|)\| &= 2^{np} \|S(2^{-nq}|x|) - T(2^{-nq}|x|)\| \\ &\leq 2^{np} \|F(2^{-nq}|x|) - T(2^{-nq}|x|)\| \\ &\quad + 2^{np} \|S(2^{-nq}|x|) - F(2^{-nq}|x|)\| \\ &\leq 2^{np} \delta_1 \|2^{-nq}x\|^{\alpha_1} + 2^{np} \delta_2 \|2^{-nq}x\|^{\alpha_2} \\ &= 2^{(p-q\alpha_1)n} \delta_1 \|x\|^{\alpha_1} + 2^{(p-q\alpha_2)n} \delta_2 \|x\|^{\alpha_2}. \end{aligned}$$

Hence

$$\|S(|x|) - T(|x|)\| \leq 2^{(p-q\alpha_1)n} \delta_1 \|x\|^{\alpha_1} + 2^{(p-q\alpha_2)n} \delta_2 \|x\|^{\alpha_2}$$

which, in the limit, yields $\|S(|x|) - T(|x|)\| = 0$ or equivalently $S(|x|) = T(|x|)$ for all $x \in \mathcal{X}$.

This was to be proven. ■

Proof of Theorem 2.2. First, if we choose $\tau = \eta = 2^{-1}$, $y = x$ and replace x by $2^q x$ in inequality (2.4) then we obviously have

$$\|2^{-p}F(2^q|x|) - F(|x|)\| \leq \beta + \vartheta 2^{q\alpha-p}\|x\|^\alpha. \quad (3.3)$$

Next, let us define the following functions:

$$\begin{aligned} G : \mathcal{X}^+ &\rightarrow \mathcal{X}^+, & G(|x|) &= 2^q|x|, & \text{for all } x \in \mathcal{X}, \\ \delta : \mathcal{X}^+ &\rightarrow [0, \infty), & \delta(|x|) &= \beta + \vartheta 2^{q\alpha-p}\|x\|^\alpha, & \text{for all } x \in \mathcal{X}, \\ \varphi : [0, \infty) &\rightarrow [0, \infty), & \varphi(t) &= 2^{-p}t, \\ H : \mathcal{Y}^+ &\rightarrow \mathcal{Y}^+, & H(|y|) &= 2^{-p}|y|, & \text{for all } y \in \mathcal{Y}, \\ d(\cdot, \cdot) : \mathcal{Y}^+ \times \mathcal{Y}^+ &\rightarrow [0, \infty), & d(|y_1|, |y_2|) &= \||y_1| - |y_2|\|, & \text{for all } y_1, y_2 \in \mathcal{Y}. \end{aligned}$$

We shall verify the fulfilment of all the three conditions of the Forti's theorem as follows.

(I) From inequality (3.3) we obviously have

$$\begin{aligned} d(H(F(G(|x|))), F(|x|)) &= \|H(F(G(|x|))) - F(|x|)\| \\ &= \|2^{-p}F(2^q|x|) - F(|x|)\| \\ &\leq \beta + \vartheta 2^{q\alpha-p}\|x\|^\alpha = \delta(|x|). \end{aligned}$$

(II) $d(H(|y_1|), H(|y_2|)) = 2^{-p}\||y_1| - |y_2|\| = \varphi(d(|y_1|, |y_2|))$ for all $y_1, y_2 \in \mathcal{Y}$.

(III) Clearly, on the one hand φ is a non-decreasing subadditive function on the positive half line, and on the other hand by applying Remark 2.2 on both the iterations G^j and φ^j of G and φ respectively, one can observe that

$$\begin{aligned} \sum_{j=0}^{\infty} \varphi^j(\delta(G^j(|x|))) &= \beta \sum_{j=0}^{\infty} 2^{-pj} + \vartheta 2^{q\alpha-p}\|x\|^\alpha \sum_{j=0}^{\infty} 2^{(q\alpha-p)j} \\ &= \frac{\beta 2^p}{2^p - 1} + \frac{\vartheta \|x\|^\alpha 2^{q\alpha}}{2^p - 2^{q\alpha}} < \infty. \end{aligned}$$

Then in view of Forti's theorem, sequence $(H^n(F(G^n|x|)))_{n \in \mathbb{N}}$ is a Cauchy sequence for every $x \in \mathcal{X}$ and thus so is $(2^{-np}F(2^{nq}|x|))_{n \in \mathbb{N}}$. Furthermore, the mapping (2.5) satisfies inequality (2.6).

Next, we prove that T solves (1.2). In fact, in (2.4) substitute x with $2^{nq}x$ also y with $2^{nq}y$, and fix arbitrarily $\tau, \eta \in [0, \infty)$. Then

$$\begin{aligned} & \left\| F\left(2^{nq}((\tau^q|x|)\Delta_{\mathcal{X}}^*(\eta^q|y|))\right)\Delta_{\mathcal{Y}}^*F\left(2^{nq}((\tau^q|x|)\Delta_{\mathcal{X}}^{**}(\eta^q|y|))\right) \right. \\ & \quad \left. - (\tau^p F(2^{nq}|x|))\Delta_{\mathcal{Y}}^{**}(\eta^p F(2^{nq}|y|)) \right\| \\ & \leq \beta + 2^{-(p+1)}\vartheta(\|2^{nq}x\|^\alpha + \|2^{nq}y\|^\alpha). \end{aligned}$$

Dividing both sides of this last inequality by 2^{np} yields

$$\begin{aligned} & \left\| \frac{F\left(2^{nq}((\tau^q|x|)\Delta_{\mathcal{X}}^*(\eta^q|y|))\right)\Delta_{\mathcal{Y}}^*F\left(2^{nq}((\tau^q|x|)\Delta_{\mathcal{X}}^{**}(\eta^q|y|))\right)}{2^{np}} \right. \\ & \quad \left. - \frac{(\tau^p F(2^{nq}|x|))\Delta_{\mathcal{Y}}^{**}(\eta^p F(2^{nq}|y|))}{2^{np}} \right\| \quad (3.4) \\ & \leq \beta 2^{-np} + 2^{-(p+1)}\vartheta(\|x\|^\alpha + \|y\|^\alpha)2^{(q\alpha-p)n}. \end{aligned}$$

Taking the limit in (3.4) we have via (2.5) that

$$\left\| T((\tau^q|x|)\Delta_{\mathcal{X}}^*(\eta^q|y|))\Delta_{\mathcal{Y}}^*T((\tau^q|x|)\Delta_{\mathcal{X}}^{**}(\eta^q|y|)) - (\tau^p T(|x|))\Delta_{\mathcal{Y}}^{**}(\eta^p T(|y|)) \right\| = 0$$

for all $\tau, \eta \in [0, \infty)$ and all $x, y \in \mathcal{X}$, which is equivalent to (1.2). Thus T satisfies (1.3) in Remark 1.1. Finally we show the uniqueness, using a technique in [16]. In fact, assume that there is another functional $S : \mathcal{X} \rightarrow \mathcal{Y}$ which satisfies (1.2) and the inequality $\|S(|x|) - F(|x|)\| \leq \beta_2 + \delta_2 \|x\|^{\alpha_2}$ for some numbers $\alpha_2, \delta_2 \in (0, \infty)$, $\beta_2 \in [0, \infty)$ with $q\alpha_2 < p$, and for all $x \in \mathcal{X}$. In (2.6) let $\beta_1 := \frac{\beta 2^p}{2^p - 1}$, $\delta_1 := \frac{\vartheta 2^{2q\alpha}}{2^p - 2q\alpha}$, $\alpha_1 := \alpha$ and by choosing $\tau = 2^n$ in Remark 1.1 one can observe that for all $x \in \mathcal{X}$

$$\begin{aligned} \|S(|x|) - T(|x|)\| &= 2^{-np} \|S(2^{nq}|x|) - T(2^{nq}|x|)\| \\ &\leq 2^{-np} \|F(2^{nq}|x|) - T(2^{nq}|x|)\| \\ &\quad + 2^{-np} \|S(2^{nq}|x|) - F(2^{nq}|x|)\| \\ &\leq 2^{-np} (\beta_1 + \delta_1 \|2^{nq}x\|^{\alpha_1}) + 2^{-np} (\beta_2 + \delta_2 \|2^{nq}x\|^{\alpha_2}) \\ &= 2^{-np} (\beta_1 + \beta_2) + \delta_1 2^{(q\alpha_1-p)n} \|x\|^{\alpha_1} + \delta_2 2^{(q\alpha_2-p)n} \|x\|^{\alpha_2}. \end{aligned}$$

Hence

$$\|S(|x|) - T(|x|)\| \leq 2^{-np} (\beta_1 + \beta_2) + \delta_1 2^{(q\alpha_1-p)n} \|x\|^{\alpha_1} + \delta_2 2^{(q\alpha_2-p)n} \|x\|^{\alpha_2}$$

which, in the limit, yields $\|S(|x|) - T(|x|)\| = 0$ or equivalently $S(|x|) = T(|x|)$ for all $x \in \mathcal{X}$. This completes the proof. \blacksquare

To end our paper we give an example showing that stability fails to occur in general.

EXAMPLE 1. Fix arbitrarily $\tau, \eta \in (0, 2)$ and consider the function

$$F : [0, \infty) \rightarrow [0, \infty), \quad F(x) = x^{\alpha+1}, \quad \alpha = \frac{p}{q}.$$

Since F is increasing the first equality in the chain below is valid, entailing the subsequent relations:

$$\begin{aligned} & \left| F((\tau^q x) \vee (\eta^q y)) - (\tau^p F(x)) \wedge (\eta^p F(y)) \right| \\ &= \left| (\tau^q x)^{\alpha+1} \vee (\eta^q y)^{\alpha+1} - (\tau^p x^{\alpha+1}) \wedge (\eta^p y^{\alpha+1}) \right| \\ &\leq (\tau^q x)^{\alpha+1} \vee (\eta^q y)^{\alpha+1} + (\tau^p x^{\alpha+1}) \wedge (\eta^p y^{\alpha+1}) \\ &\leq (2^q x)^{\alpha+1} \vee (2^q y)^{\alpha+1} + (2^p x^{\alpha+1}) \wedge (2^p y^{\alpha+1}) \\ &\leq 2^{p+q}(x^{\alpha+1} \vee y^{\alpha+1}) + 2^{p+q}(x^{\alpha+1} \wedge y^{\alpha+1}) \\ &= 2^{p+q}(x^{\alpha+1} + y^{\alpha+1}) \end{aligned}$$

for all $x, y \in [0, \infty)$. Now, let $T : [0, \infty) \rightarrow [0, \infty)$ be a function such that $T(\mu^q x) = \mu^p T(x)$ for all $x \in [0, \infty)$ and all $\mu \in [0, \infty)$. Since $x = (x^{1/q})^q$, and α is the ratio of p and q , we can then note that $T(x) = x^\alpha T(1)$ for every $x \in [0, \infty)$. Now,

$$\begin{aligned} \sup_{x \in (0, \infty)} \frac{|F(x) - T(x)|}{2^{p+q}x^{\alpha+1}} &= \sup_{x \in (0, \infty)} \frac{\left| x^{\alpha+1} - T\left(\left(x^{\frac{1}{q}}\right)^q\right) \right|}{2^{p+q}x^{\alpha+1}} \\ &= \sup_{x \in (0, \infty)} \frac{|x^{\alpha+1} - x^\alpha T(1)|}{2^{p+q}x^{\alpha+1}} \\ &= \frac{1}{2^{p+q}} \sup_{x \in (0, \infty)} \left| 1 - \frac{T(1)}{x} \right| = \infty. \end{aligned}$$

The above example about the lack of stability on the real line in lattice environments is the counterpart of the example given by S. Czerwik [9] in the addition environments for quadratic mappings.

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