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# The Geometry of $\mathcal{L}(^3 l_\infty^2)$ and Optimal Constants in the Bohnenblust-Hille Inequality for Multilinear Forms and Polynomials

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*Abstract*: We classify the extreme and exposed 3-linear forms of the unit ball of $\mathcal{L}(^3 l_\infty^2)$. We introduce optimal constants in the Bohnenblust-Hille inequality for symmetric multilinear forms and polynomials and investigate about their relations.

*Key words*: Extreme points, exposed points, the optimal constants in the Bohnenblust-Hille inequality for symmetric multilinear forms and polynomials.

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## 1. Introduction

We write $B_E$ for the closed unit ball of a real Banach space $E$ and the dual space of $E$ is denoted by $E^*$. $x \in B_E$ is called an *extreme point* of $B_E$ if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. $x \in B_E$ is called an *exposed point* of $B_E$ if there is a $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of $B_E$ is an extreme point. We denote by $extB_E$ and $expB_E$ the sets of extreme and exposed points of $B_E$, respectively. Let $n \in \mathbb{N}, n \geq 2$. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous $n$-linear form $L$ on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{L}(^n E)$ the Banach space of all continuous $n$-linear forms on $E$ endowed with the norm $\|L\| = \sup_{\|x_j\|=1, 1\leq j\leq n} |L(x_1, \ldots, x_n)|$. $\mathcal{L}_s(^n E)$ denotes the closed subspace of $\mathcal{L}(^n E)$ consisting all continuous symmetric $n$-linear forms on $E$. $\mathcal{P}(^n E)$ denotes the Banach space of all continuous $n$-homogeneous polynomials from $E$ into $\mathbb{R}$ endowed with the norm $\|P\| =$

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$\sup_{\|x\|=1}|P(x)|$. Note that the spaces $\mathcal{L}(^nE)$, $\mathcal{L}_s(^nE)$, $\mathcal{P}(^nE)$ are very different from a geometric point of view. In particular, for integral multilinear forms and integral polynomials one has ([2], [9], [32])

$$extB_{\mathcal{L}_I(^nE)} = \{\phi_1\phi_2\cdots\phi_n : \phi_i \in extB_{E^*}\}$$

and

$$extB_{\mathcal{P}_I(^nE)} = \{\pm\phi^n : \phi \in E^*, \|\phi\| = 1\},$$

where $\mathcal{L}_I(^nE)$ and $\mathcal{P}_I(^nE)$ are the spaces of integral $n$-linear forms and integral $n$-homogeneous polynomials on $E$, respectively. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [10].

In 1998, Choi *et al.* ([4], [5]) characterized the extreme points of the unit ball of $\mathcal{P}(^2l_1^2)$ and $\mathcal{P}(^2l_2^2)$. Kim [15] classified the exposed 2-homogeneous polynomials on $\mathcal{P}(^2l_p^2)$ $(1 \leq p \leq \infty)$. Kim ([17], [19], [23]) classified the extreme, exposed, smooth points of the unit ball of $\mathcal{P}(^2d_*(1,w)^2)$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm of weight $w$.

In 2009, Kim [16] initiated extremal problems for bilinear forms on a classical finite dimensional real Banach space and classified the extreme, exposed, smooth points of the unit ball of $\mathcal{L}_s(^2l_\infty^2)$. Kim ([18], [20]–[22]) classified the extreme, exposed, smooth points of the unit balls of $\mathcal{L}_s(^2d_*(1,w)^2)$ and $\mathcal{L}(^2d_*(1,w)^2)$.

We refer to ([1]–[9], [11]–[32] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The Bohnenblust-Hille inequality for $n$-linear forms ([3] and references therein) tells us that there exists a sequence of positive scalars $(C(n:\mathbb{K}))_{n=1}^\infty$ in $[1,\infty]$ such that

$$\left(\sum_{j_1,\ldots,j_n=1}^{\infty}\left|T(e_{j_1},\ldots,e_{j_n})\right|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{2n}} \leq C(n:\mathbb{K})\|T\|$$

for all continuous $n$-linear forms $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$. The optimal constant in the Bohnenblust-Hille inequality for $n$-linear forms $C(n:\mathbb{K})$ is defined by

$$C(n:\mathbb{K}) := \sup\left\{\left(\sum_{j_1,\ldots,j_n=1}^{\infty}|T(e_{j_1},\ldots,e_{j_n})|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{2n}} : T \in \mathcal{L}(^nc_0:\mathbb{K}),\ \|T\| = 1\right\}.$$

We introduce the optimal constant in the Bohnenblust-Hille inequality for symmetric $n$-linear forms $C_s(n:\mathbb{K})$ is defined by

$$C_s(n:\mathbb{K}) := \sup\left\{\left(\sum_{j_1,\dots,j_n=1}^{\infty}|T(e_{j_1},\dots,e_{j_n})|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{2n}} : T\in\mathcal{L}_s(^nc_0:\mathbb{K}),\ \|T\|=1\right\}.$$

It is obvious that $C_s(n:\mathbb{K}) \leq C(n:\mathbb{K})$. We also introduce the optimal constant in the Bohnenblust-Hille inequality for $n$-homogeneous polynomials $C_p(n:\mathbb{K})$ is defined by

$$C_p(n:\mathbb{K}) := \sup\left\{\left(\sum_{j=1}^{\infty}|P(e_j)|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{2n}} : P\in\mathcal{P}(^nc_0:\mathbb{K}),\ \|P\|=1\right\}.$$

Recently, Diniz *et al.* [12] showed that $C(2:\mathbb{R})=\sqrt{2}$.

In this paper, we classify the extreme and exposed 3-linear forms of the unit ball of $\mathcal{L}(^3l_\infty^2)$. We introduce optimal constants in the Bohnenblust-Hille inequality for symmetric multilinear forms and polynomials and investigate about their relations.

## 2. The extreme points of the unit ball of $\mathcal{L}(^3l_\infty^2)$

Let $T\in\mathcal{L}(^3l_\infty^2)$ be given by

$$\begin{aligned}T\big((x_1,x_2),(y_1,y_2),(z_1,z_2)\big) &= ax_1y_1z_1+bx_2y_2z_2+c_1x_2y_1z_1+c_2x_1y_2z_1\\ &\quad+c_3x_1y_1z_2+d_1x_1y_2z_2+d_2x_2y_1z_2+d_3x_2y_2z_1\end{aligned}$$

for some $a,b,c_j,d_j\in\mathbb{R}$ and for $j=1,2,3$. For simplicity, we will denote $T=(a,b,c_1,c_2,c_3,d_1,d_2,d_3)$.

Theorem 2.1. *Let $T=(a,b,c_1,c_2,c_3,d_1,d_2,d_3)\in\mathcal{L}(^3l_\infty^2)$. Then*

$$\begin{aligned}\|T\| = \max\big\{&|a+c_1+c_2+d_3|+|b+c_3+d_1+d_2|,\\ &|a-c_2-c_3+d_1|+|b+c_1-d_2-d_3|,\\ &|a-b+c_3-d_3|+|c_1-c_2-d_1+d_2|,\\ &|a+b-c_1-d_1|+|c_2-c_3+d_2-d_3|\big\}.\end{aligned}$$

*Proof.* Note that $extB_{l^2_\infty} = \{(1,1),(1,-1),(-1,1),(-1,-1)\}$. By the Krein-Milman Theorem, $B_{l^2_\infty} = \overline{co}(extB_{l^2_\infty})$. By the continuity and trilinearity of $T$,

$$\begin{aligned}\|T\| = \max\big\{&|T((1,1),(1,1),(1,1))|, |T((1,-1),(1,1),(1,1))|\\ &|T((1,1),(1,-1),(1,1))|, |T((1,1),(1,1),(1,-1))|,\\ &|T((1,-1),(1,-1),(1,1))|, |T((1,-1),(1,1),(1,-1))|,\\ &|T((1,1),(1,-1),(1,-1))|, |T((1,-1),(1,-1),(1,-1))|\big\}\\ = \max\big\{&|a+c_1+c_2+d_3| + |b+c_3+d_1+d_2|,\\ &|a-c_2-c_3+d_1| + |b+c_1-d_2-d_3|,\\ &|a-b+c_3-d_3| + |c_1-c_2-d_1+d_2|,\\ &|a+b-c_1-d_1| + |c_2-c_3+d_2-d_3|\big\}.\end{aligned}$$

∎

Note that if $\|T\| = 1$, then $|a| \leq 1$, $|b| \leq 1$, $|c_j| \leq 1$, $|d_j| \leq 1$, for $j = 1,2,3$.

Theorem 2.2.

$$\begin{aligned}extB_{\mathcal{L}(^3l^2_\infty)} = \big\{&(a,b,c_1,c_2,c_3,d_1,d_2,d_3) :\\ &a = \frac{1}{8}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8),\\ &b = \frac{1}{8}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_7-\epsilon_8),\\ &c_1 = \frac{1}{8}(\epsilon_1-\epsilon_2+\epsilon_3+\epsilon_4-\epsilon_5-\epsilon_6+\epsilon_7-\epsilon_8),\\ &c_2 = \frac{1}{8}(\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4-\epsilon_5+\epsilon_6-\epsilon_7-\epsilon_8),\\ &c_3 = \frac{1}{8}(\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4+\epsilon_5-\epsilon_6-\epsilon_7-\epsilon_8),\\ &d_1 = \frac{1}{8}(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4-\epsilon_5-\epsilon_6+\epsilon_7+\epsilon_8),\\ &d_2 = \frac{1}{8}(\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4-\epsilon_5+\epsilon_6-\epsilon_7+\epsilon_8),\\ &d_3 = \frac{1}{8}(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4+\epsilon_5-\epsilon_6-\epsilon_7+\epsilon_8),\\ &\epsilon_j = \pm 1, \text{ for } j = 1,2,\ldots,8\big\}.\end{aligned}$$

*Proof.* Let $T = (a, b, c_1, c_2, c_3, d_1, d_2, d_3) \in \mathcal{L}(^3l_\infty^2)$ with $\|T\| = 1$. Note that

$$\begin{aligned}
T\big((1,1),(1,1),(1,1)\big) &= a+b+c_1+c_2+c_3+d_1+d_2+d_3,\\
T\big((1,-1),(1,1),(1,1)\big) &= a-b-c_1+c_2+c_3+d_1-d_2-d_3,\\
T\big((1,1),(1,-1),(1,1)\big) &= a-b+c_1-c_2+c_3-d_1+d_2-d_3,\\
T\big((1,1),(1,1),(1,-1)\big) &= a-b+c_1+c_2-c_3-d_1-d_2+d_3,\\
T\big((1,-1),(1,-1),(1,1)\big) &= a+b-c_1-c_2+c_3-d_1-d_2+d_3,\\
T\big((1,-1),(1,1),(1,-1)\big) &= a+b-c_1+c_2-c_3-d_1+d_2-d_3,\\
T\big((1,1),(1,-1),(1,-1)\big) &= a+b+c_1-c_2-c_3+d_1-d_2-d_3,\\
T\big((1,-1),(1,-1),(1,-1)\big) &= a-b-c_1-c_2-c_3+d_1+d_2+d_3.
\end{aligned}$$

Let $A = (a_{ij})_{1\le i,j\le 8}$ be the $8\times 8$ matrix such that

$$\begin{array}{lll}
a_{i1} = 1\ (i = 1,\dots,8), & a_{i2} = 1\ (i = 1,5,6,7), & a_{k2} = -1\ (k = 2,3,4,8),\\
a_{i3} = 1\ (i = 1,3,4,7), & a_{k3} = -1\ (k = 2,5,6,8), & a_{i4} = 1\ (i = 1,2,4,6),\\
a_{k4} = -1\ (k = 3,5,7,8), & a_{i5} = 1\ (i = 1,2,3,5), & a_{k5} = -1\ (k = 4,6,7,8),\\
a_{i6} = 1\ (i = 1,2,7,8), & a_{k6} = -1\ (k = 3,4,5,6), & a_{i7} = 1\ (i = 1,3,6,8),\\
a_{k7} = -1\ (k = 2,4,5,7), & a_{i8} = 1\ (i = 1,4,5,8), & a_{k8} = -1\ (k = 2,3,6,7).
\end{array}$$

By calculation, $det(A) = -2^{12}$, so $A$ is invertible. Note that

$$\begin{aligned}
AT = \Big(&T\big((1,1),(1,1),(1,1)\big),\ T\big((1,-1),(1,1),(1,1)\big),\\
&T\big((1,1),(1,-1),(1,1)\big),\ T\big((1,1),(1,1),(1,-1)\big),\\
&T\big((1,-1),(1,-1),(1,1)\big),\ T\big((1,-1),(1,1),(1,-1)\big),\\
&T\big((1,1),(1,-1),(1,-1)\big),\ T\big((1,-1),(1,-1),(1,-1)\big)\Big)^t
\end{aligned}$$

and $\|AT\|_\infty = \|T\|$. Note also that

$$\begin{aligned}
T = A^{-1}\Big(&T\big((1,1),(1,1),(1,1)\big),\ T\big((1,-1),(1,1),(1,1)\big),\\
&T\big((1,1),(1,-1),(1,1)\big),\ T\big((1,1),(1,1),(1,-1)\big),\\
&T\big((1,-1),(1,-1),(1,1)\big),\ T\big((1,-1),(1,1),(1,-1)\big),\\
&T\big((1,1),(1,-1),(1,-1)\big),\ T\big((1,-1),(1,-1),(1,-1)\big)\Big)^t.
\end{aligned}$$

We claim that $T \in extB_{\mathcal{L}(^3 l_\infty^2)}$ if and only if

$$\begin{aligned} 1 &= \big|T\big((1,1),(1,1),(1,1)\big)\big| = \big|T\big((1,-1),(1,1),(1,1)\big)\big| \\ &= \big|T\big((1,1),(1,-1),(1,1)\big)\big| = \big|T\big((1,1),(1,1),(1,-1)\big)\big| \\ &= \big|T\big((1,-1),(1,-1),(1,1)\big)\big| = \big|T\big((1,-1),(1,1),(1,-1)\big)\big| \\ &= \big|T\big((1,1),(1,-1),(1,-1)\big)\big| = \big|T\big((1,-1),(1,-1),(1,-1)\big)\big|. \end{aligned}$$

($\Rightarrow$): Otherwise. Then we have 8 cases as follows:

$$\begin{aligned} &\text{Case 1}: && \big|T\big((1,1),(1,1),(1,1)\big)\big| < 1 \text{ or} \\ &\text{Case 2}: && \big|T\big((1,-1),(1,1),(1,1)\big)\big| < 1 \text{ or} \\ &\text{Case 3}: && \big|T\big((1,1),(1,-1),(1,1)\big)\big| < 1 \text{ or} \\ &\text{Case 4}: && \big|T\big((1,1),(1,1),(1,-1)\big)\big| < 1 \text{ or} \\ &\text{Case 5}: && \big|T\big((1,-1),(1,-1),(1,1)\big)\big| < 1 \text{ or} \\ &\text{Case 6}: && \big|T\big((1,-1),(1,1),(1,-1)\big)\big| < 1 \text{ or} \\ &\text{Case 7}: && \big|T\big((1,1),(1,-1),(1,-1)\big)\big| < 1 \text{ or} \\ &\text{Case 8}: && \big|T\big((1,-1),(1,-1),(1,-1)\big)\big| < 1. \end{aligned}$$

Case 1: $\big|T\big((1,1),(1,1),(1,1)\big)\big| < 1$. Let

$$\begin{aligned} \epsilon_1 &:= T\big((1,1),(1,1),(1,1)\big), \\ \epsilon_2 &:= T\big((1,-1),(1,1),(1,1)\big), \\ \epsilon_3 &:= T\big((1,1),(1,-1),(1,1)\big), \\ \epsilon_4 &:= T\big((1,1),(1,1),(1,-1)\big), \\ \epsilon_5 &:= T\big((1,-1),(1,-1),(1,1)\big), \\ \epsilon_6 &:= T\big((1,-1),(1,1),(1,-1)\big), \\ \epsilon_7 &:= T\big((1,1),(1,-1),(1,-1)\big), \\ \epsilon_8 &:= T\big((1,-1),(1,-1),(1,-1)\big). \end{aligned}$$

Then,

$$AT = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8)^t.$$

Let $n_0 \in \mathbb{N}$ such that $|\epsilon_1| + \frac{1}{n_0} < 1$. Let $T_1, T_2 \in \mathcal{L}(^3 l_\infty^2)$ be the solutions of

$$AT_1 = \Big(\epsilon_1 + \frac{1}{n_0}, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8\Big)^t, \; AT_2 = \Big(\epsilon_1 - \frac{1}{n_0}, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8\Big)^t.$$

Note that $T_j \neq T, \|T_j\| = \|AT_j\|_\infty = 1$ for $j = 1, 2$. It follows that

$$A\Big(\frac{1}{2}(T_1 + T_2)\Big) = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8)^t = AT,$$

which shows that

$$\frac{1}{2}(T_1 + T_2) = A^{-1}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8)^t = T,$$

so $T$ is not extreme. By the similar argument in the Case 1, if any of Cases 2–8 holds, then we may reach to a contradiction.

($\Leftarrow$): Let $\epsilon_j \in \mathbb{R}$ be given for $j = 1, 2, \ldots, 8$. Consider the following system of 8 simultaneous linear equations: $AT = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8)^t$, ie,

$$\begin{aligned}
a + b + c_1 + c_2 + c_3 + d_1 + d_2 + d_3 &= \epsilon_1,\\
a - b - c_1 + c_2 + c_3 + d_1 - d_2 - d_3 &= \epsilon_2,\\
a - b + c_1 - c_2 + c_3 - d_1 + d_2 - d_3 &= \epsilon_3,\\
a - b + c_1 + c_2 - c_3 - d_1 - d_2 + d_3 &= \epsilon_4,\\
a + b - c_1 - c_2 + c_3 - d_1 - d_2 + d_3 &= \epsilon_5,\\
a + b - c_1 + c_2 - c_3 - d_1 + d_2 - d_3 &= \epsilon_6,\\
a + b + c_1 - c_2 - c_3 + d_1 - d_2 - d_3 &= \epsilon_7,\\
a - b - c_1 - c_2 - c_3 + d_1 + d_2 + d_3 &= \epsilon_8.
\end{aligned} \tag{$*$}$$

We get the unique solution of $(*)$ as follows: $T = A^{-1}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8)^t$, ie,

$$\begin{aligned}
a &= \frac{1}{8}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8),\\
b &= \frac{1}{8}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7 - \epsilon_8),\\
c_1 &= \frac{1}{8}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4 - \epsilon_5 - \epsilon_6 + \epsilon_7 - \epsilon_8),\\
c_2 &= \frac{1}{8}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4 - \epsilon_5 + \epsilon_6 - \epsilon_7 - \epsilon_8),\\
c_3 &= \frac{1}{8}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 - \epsilon_8),\\
d_1 &= \frac{1}{8}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 + \epsilon_7 + \epsilon_8),\\
d_2 &= \frac{1}{8}(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4 - \epsilon_5 + \epsilon_6 - \epsilon_7 + \epsilon_8),\\
d_3 &= \frac{1}{8}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8).
\end{aligned} \tag{$**$}$$

Let $T_1 = (a+\epsilon, b+\delta, c_1+\gamma_1, c_2+\gamma_2, c_3+\gamma_3, d_1+\rho_1, d_2+\rho_2, d_3+\rho_3) \in \mathcal{L}(^3 l_\infty^2)$ and $T_2 = (a-\epsilon, b-\delta, c_1-\gamma_1, c_2-\gamma_2, c_3-\gamma_3, d_1-\rho_1, d_2-\rho_2, d_3-\rho_3) \in \mathcal{L}(^3 l_\infty^2)$ be such that $1 = \|T_1\| = \|T_2\|$ for some $\epsilon, \delta, \gamma_j, \rho_j$ for $j = 1, 2, 3$. Then, for $k = 1, 2$,

$$\begin{aligned}
1 &\geq \big|T_k\big((1,1),(1,1),(1,1)\big)\big| = 1 + |\epsilon+\delta+\gamma_1+\gamma_2+\gamma_3+\rho_1+\rho_2+\rho_3|,\\
1 &\geq \big|T_k\big((1,-1),(1,1),(1,1)\big)\big| = 1 + |\epsilon-\delta-\gamma_1+\gamma_2+\gamma_3+\rho_1-\rho_2-\rho_3|,\\
1 &\geq \big|T_k\big((1,1),(1,-1),(1,1)\big)\big| = 1 + |\epsilon-\delta+\gamma_1-\gamma_2+\gamma_3-\rho_1+\rho_2-\rho_3|,\\
1 &\geq \big|T_k\big((1,1),(1,1),(1,-1)\big)\big| = 1 + |\epsilon-\delta+\gamma_1+\gamma_2-\gamma_3-\rho_1-\rho_2+\rho_3|,\\
1 &\geq \big|T_k\big((1,-1),(1,-1),(1,1)\big)\big| = 1 + |\epsilon+\delta-\gamma_1-\gamma_2+\gamma_3-\rho_1-\rho_2+\rho_3|,\\
1 &\geq \big|T_k\big((1,-1),(1,1),(1,-1)\big)\big| = 1 + |\epsilon+\delta-\gamma_1+\gamma_2-\gamma_3-\rho_1+\rho_2-\rho_3|,\\
1 &\geq \big|T_k\big((1,1),(1,-1),(1,-1)\big)\big| = 1 + |\epsilon+\delta+\gamma_1-\gamma_2-\gamma_3+\rho_1-\rho_2-\rho_3|,\\
1 &\geq \big|T_k\big((1,-1),(1,-1),(1,-1)\big)\big| = 1 + |\epsilon-\delta-\gamma_1-\gamma_2-\gamma_3+\rho_1+\rho_2+\rho_3|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
0 &= \epsilon+\delta+\gamma_1+\gamma_2+\gamma_3+\rho_1+\rho_2+\rho_3,\\
0 &= \epsilon-\delta-\gamma_1+\gamma_2+\gamma_3+\rho_1-\rho_2-\rho_3,\\
0 &= \epsilon-\delta+\gamma_1-\gamma_2+\gamma_3-\rho_1+\rho_2-\rho_3,\\
0 &= \epsilon-\delta+\gamma_1+\gamma_2-\gamma_3-\rho_1-\rho_2+\rho_3,\\
0 &= \epsilon+\delta-\gamma_1-\gamma_2+\gamma_3-\rho_1-\rho_2+\rho_3,\\
0 &= \epsilon+\delta-\gamma_1+\gamma_2-\gamma_3-\rho_1+\rho_2-\rho_3,\\
0 &= \epsilon+\delta+\gamma_1-\gamma_2-\gamma_3+\rho_1-\rho_2-\rho_3,\\
0 &= \epsilon-\delta-\gamma_1-\gamma_2-\gamma_3+\rho_1+\rho_2+\rho_3.
\end{aligned}$$

Hence, $A(\epsilon, \delta, \gamma_1, \gamma_2, \gamma_3, \rho_1, \rho_2, \rho_3)^t = 0$. By $(**)$, $0 = \epsilon = \delta = \gamma_1 = \gamma_2 = \gamma_3 = \rho_1 = \rho_2 = \rho_3$. Hence, $T$ is extreme. Therefore, we complete the proof. ∎

Corollary 2.3. *If* $T = (a, b, c_1, c_2, c_3, d_1, d_2, d_3) \in ext B_{\mathcal{L}(^3 l_\infty^2)}$, *then* $|a|$, $|b|, |c_j|, |d_j| \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ *for* $j = 1, 2, 3$.

THEOREM 2.4. ([26])

$$extB_{\mathcal{L}_s(^3l_\infty^2)} = \Big\{ \pm(1,0,0,0,0,0,0,0),\ \pm(0,1,0,0,0,0,0,0,),\\ \pm\Big(\frac{1}{2},0,0,0,0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\Big),\ \pm\Big(0,\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0\Big),\\ \pm\Big(\frac{1}{4},-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\Big),\ \pm\Big(-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\Big),\\ \pm\Big(\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4}\Big),\\ \pm\Big(\frac{1}{4},\frac{3}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\Big)\Big\}.$$

THEOREM 2.5. $extB_{\mathcal{L}_s(^3l_\infty^2)} = extB_{\mathcal{L}(^3l_\infty^2)} \cap \mathcal{L}_s(^3l_\infty^2)$.

*Proof.* It follows from Theorems 2.2 and 2.4. ∎

*Remarks.* (1) $2^4 = |extB_{\mathcal{L}_s(^3l_\infty^2)}| < |extB_{\mathcal{L}(^3l_\infty^2)}| = 2^8$.
(2) Let $T = (a,b,c_1,c_2,c_3,d_1,d_2,d_3) \in \mathcal{L}(^3l_\infty^2)$. Then, by scaling, we may assume that $d_j \geq 0$ for $j = 1,2,3$.

*Proof.* Let $T_1((x_1,x_2),\ (y_1,y_2),(z_1,z_2)) := T((\epsilon_1 x_1,x_2),\ (\epsilon_2 y_1,y_2)),(\epsilon_3 z_1,z_2)$, where $\epsilon_k = \pm 1$ be given satisfying $\epsilon_j d_j \geq 0$ for $j = 1,2,3$. ∎

QUESTION. Is it true that $extB_{\mathcal{L}_s(^nl_\infty^2)} = extB_{\mathcal{L}(^nl_\infty^2)} \cap \mathcal{L}_s(^nl_\infty^2)$ for $n \geq 4$?

## 3. THE EXPOSED POINTS OF THE UNIT BALL OF $\mathcal{L}(^3l_\infty^2)$

THEOREM 3.1. *Let* $f \in \mathcal{L}(^3l_\infty^2)^*$ *with*

$$\alpha = f(x_1y_1z_1),\quad \beta = f(x_2y_2z_2),\quad \gamma_1 = f(x_2y_1z_1),\quad \gamma_2 = f(x_1y_2z_1),$$
$$\gamma_3 = f(x_1y_1z_2),\quad \delta_1 = f(x_1y_2z_2),\quad \delta_2 = f(x_2y_1z_2),\quad \delta_3 = f(x_2y_2z_1).$$

*Then,*

$$\|f\| = \max\Big\{\Big|a\alpha + b\beta + \sum_{j=1,2,3} c_j\gamma_j + \sum_{j=1,2,3} d_j\delta_j\Big| :$$

$$a = \frac{1}{8}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8),$$
$$b = \frac{1}{8}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7 - \epsilon_8),$$
$$c_1 = \frac{1}{8}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4 - \epsilon_5 - \epsilon_6 + \epsilon_7 - \epsilon_8),$$
$$c_2 = \frac{1}{8}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4 - \epsilon_5 + \epsilon_6 - \epsilon_7 - \epsilon_8),$$
$$c_3 = \frac{1}{8}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 - \epsilon_8),$$
$$d_1 = \frac{1}{8}(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 + \epsilon_7 + \epsilon_8),$$
$$d_2 = \frac{1}{8}(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4 - \epsilon_5 + \epsilon_6 - \epsilon_7 + \epsilon_8),$$
$$d_3 = \frac{1}{8}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8),$$
$$\epsilon_j = \pm 1, \text{ for } j = 1, 2, \ldots, 8\Big\}$$

*Proof. Proof.* It follows from Theorem 2.2 and the Krein-Milman Theorem. ∎

THEOREM 3.2. $expB_{\mathcal{L}(^3 l^2_\infty)} = extB_{\mathcal{L}(^3 l^2_\infty)}$.

*Proof.* We will show that $extB_{\mathcal{L}(^3 l^2_\infty)} \subset expB_{\mathcal{L}(^3 l^2_\infty)}$. By Theorem 2.2, Corollary 2.3 and Remarks(2), it suffices to show that if

$$T = (1,0,0,0,0,0,0,0),\ \Big(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0\Big)$$
$$\Big(-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\Big),\ \Big(\frac{3}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\Big),$$

then $T \in expB_{\mathcal{L}(^3 l^2_\infty)}$.

Claim: $T = (1,0,0,0,0,0,0,0) \in expB_{\mathcal{L}(^3 l^2_\infty)}$.

Let $f \in \mathcal{L}(^3 l^2_\infty)^*$ with $\alpha = 1, 0 = \beta = \gamma_j = \delta_j$ for $j = 1,2,3$. Note that, by Corollary 2.3 and Theorems 2.2 and 3.1, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for all $S \in extB_{\mathcal{L}(^3 l^2_\infty)} \backslash \{\pm T\}$. The claim follows from Theorem 2.3 of [21].

Claim: $T = \big(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\big) \in expB_{\mathcal{L}(^3l^2_\infty)}$.

Let $f \in \mathcal{L}(^3l^2_\infty)^*$ with $-\alpha = \frac{1}{2} = \beta = \gamma_1 = \gamma_2, 0 = \gamma_3 = \delta_j$ for $j = 1, 2, 3$. Note that, by Corollary 2.3 and Theorems 2.2 and 3.1, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for all $S \in extB_{\mathcal{L}(^3l^2_\infty)} \backslash \{\pm T\}$. The claim follows from Theorem 2.3 of [21].

Claim: $T = \big(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\big) \in expB_{\mathcal{L}(^3l^2_\infty)}$.

Let $f \in \mathcal{L}(^3l^2_\infty)^*$ with $\alpha = -\frac{1}{2}, \frac{5}{14} = \beta = \gamma_j = \delta_j$ for $j = 1, 2, 3$. Note that, by Corollary 2.3 and Theorems 2.2 and 3.1, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for all $S \in extB_{\mathcal{L}(^3l^2_\infty)} \backslash \{\pm T\}$. The claim follows from Theorem 2.3 of [21].

Claim: $T = \big(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\big) \in expB_{\mathcal{L}(^3l^2_\infty)}$.

Let $f \in \mathcal{L}(^3l^2_\infty)^*$ with $\alpha = \frac{1}{2}, -\frac{5}{14} = \beta = \gamma_1 = \gamma_2 = -\gamma_3 = -\delta_j$ for $j = 1, 2, 3$. Note that, by Corollary 2.3 and Theorems 2.2 and 3.1, $\|f\| = 1 = f(T)$ and $|f(S)| < 1$ for all $S \in extB_{\mathcal{L}(^3l^2_\infty)} \backslash \{\pm T\}$. The claim follows from Theorem 2.3 of [21]. We complete the proof. ∎

THEOREM 3.3. ([26]) $expB_{\mathcal{L}_s(^3l^2_\infty)} = extB_{\mathcal{L}_s(^3l^2_\infty)}$.

THEOREM 3.4. $expB_{\mathcal{L}_s(^3l^2_\infty)} = expB_{\mathcal{L}(^3l^2_\infty)} \cap \mathcal{L}_s(^3l^2_\infty)$.

*Proof.* It follows from Theorems 3.2 and 3.3. ∎

QUESTION. Is it true that $expB_{\mathcal{L}_s(^nl^2_\infty)} = expB_{\mathcal{L}(^nl^2_\infty)} \cap \mathcal{L}_s(^nl^2_\infty)$ for $n \geq 4$?

## 4. OPTIMAL CONSTANTS IN THE BOHNENBLUST-HILLE INEQUALITY FOR SYMMETRIC MULTILINEAR FORMS AND POLYNOMIALS

THEOREM 4.1. $1 \leq C_p(n : \mathbb{K}) \leq \frac{n^n}{n!} C_s(n : \mathbb{K}) \leq \frac{n^n}{n!} C(n : \mathbb{K})$ *for all* $n \geq 2$.

*Proof.* It is enough to show that $C_p(n : \mathbb{K}) \leq \frac{n^n}{n!} C_s(n : \mathbb{K})$. Let $P \in \mathcal{P}(^nc_0 : \mathbb{K}), \|P\| = 1$. By the polarization formula, $\|\check{P}\| \leq \frac{n^n}{n!}\|P\| = \frac{n^n}{n!}$, where $\check{P}$ is the corresponding symmetric $n$-linear form to $P$. Hence,

$$\left(\sum_{j=1}^{\infty} \Big|\frac{n!}{n^n} P(e_j)\Big|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{2n}} \leq \left(\sum_{j_1, \cdots, j_n = 1}^{\infty} \left|\frac{n!}{n^n} \check{P}(e_{j_1}, \cdots, e_{j_n})\right|^{\frac{2n}{n+1}}\right)^{\frac{n+1}{2n}}$$

$$\leq C_s(n : \mathbb{K}).$$

∎

THEOREM 4.2. $C_s(2:\mathbb{R}) = C(2:\mathbb{R}) = \sqrt{2}$.

*Proof.* It is enough to show that $C_s(2:\mathbb{R}) = \sqrt{2}$. Let

$$T\big((x_1,x_2),(y_1,y_2)\big) = \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1.$$

Then $T \in \mathcal{L}_s(^2l_\infty^2), \|T\| = 1$. By a result of [12], $\sqrt{2} \le (\sum_{i,j=1}^2 |T(e_i,e_j)|^{\frac{4}{3}})^{\frac{3}{4}} \le C_s(2:\mathbb{R}) \le C(2:\mathbb{R}) = \sqrt{2}$. ∎

THEOREM 4.3. ([16])

$$extB_{\mathcal{L}_s(^2l_\infty^2)} = \Big\{\pm(1,0,0,0),\ \pm(0,1,0,0),\ \pm\frac{1}{2}(1,-1,1,1),\ \pm\frac{1}{2}(1,-1,-1,-1)\Big\}.$$

THEOREM 4.4.

$$\sup\bigg\{\bigg(\sum_{i,j=1}^2 |T(e_i,e_j)|^{\frac{4}{3}}\bigg)^{\frac{3}{4}} : T \in \mathcal{L}_s(^2l_\infty^2),\ \|T\| = 1,\ T \notin extB_{\mathcal{L}_s(^2l_\infty^2)}\bigg\}$$
$$= C_s(2:\mathbb{R}).$$

*Proof.* Let

$$l := \sup\bigg\{\bigg(\sum_{i,j=1}^2 |T(e_i,e_j)|^{\frac{4}{3}}\bigg)^{\frac{3}{4}} : T \in \mathcal{L}_s(^2l_\infty^2),\ \|T\| = 1,\ T \notin extB_{\mathcal{L}_s(^2l_\infty^2)}\bigg\}.$$

For $|c| < \frac{1}{2}$, let

$$T_c\big((x_1,x_2),(y_1,y_2)\big) = \frac{1}{2}x_1y_1 - \frac{1}{2}x_2y_2 + cx_1y_2 + cx_2y_1.$$

Then $T_c \in \mathcal{L}_s(^2l_\infty^2), \|T_c\| = 1$. By Theorem 4.3, $T_c \notin extB_{\mathcal{L}_s(^2l_\infty^2)}$. It follows that

$$C_s(2:\mathbb{R}) \ge l \ge \sup\bigg\{\bigg(\sum_{i,j=1}^2 |T_c(e_i,e_j)|^{\frac{4}{3}}\bigg)^{\frac{3}{4}} : |c| < \frac{1}{2}\bigg\}$$
$$= \sup_{|c|<\frac{1}{2}}\bigg(2\Big(\frac{1}{2}\Big)^{\frac{4}{3}} + 2|c|^{\frac{4}{3}}\bigg)^{\frac{3}{4}} = \sqrt{2} = C_s(2:\mathbb{R}).$$

∎

THEOREM 4.5. *Let $n \geq 2$. Then, $2^{\frac{n+1}{2n}} \leq C_p(n : \mathbb{R})$.*

*Proof.* Let

$$w := \sup\left\{ \left( \sum_{j=1}^{\infty} |P(e_j)|^{\frac{2n}{n+1}} \right)^{\frac{n+1}{2n}} : P \in \mathcal{P}(^n c_0),\ \|P\| = 1, \right.$$
$$\left. P\big((x_m)_{m=1}^{\infty}\big) = \sum_{j=1}^{\infty} a_j x_j^n \ \text{ for some } a_j \in \mathbb{R} \right\}.$$

Claim: $w = 2^{\frac{n+1}{2n}}$.

Let $P \in \mathcal{P}(^n c_0), \|P\| = 1, P((x_m)_{m=1}^{\infty}) = \sum_{j=1}^{\infty} a_j x_j^n$ for some $a_j \in \mathbb{R}$. Let $A := \{j \in \mathbb{N} : a_j \geq 0\}$ and $B := \mathbb{N} \backslash A$. Note that, for every $k \in \mathbb{N}$,

$$1 \geq \left| P\Big( \sum_{j \in A,\ j \leq k} e_j \Big) \right| = \sum_{j \in A,\ j \leq k} |a_j|$$

and

$$1 \geq \left| P\Big( \sum_{j \in B,\ j \leq k} e_j \Big) \right| = \sum_{j \in B,\ j \leq k} |a_j|.$$

Hence,

$$\sum_{j \in A} |a_j| \leq 1, \quad \sum_{j \in B} |a_j| \leq 1.$$

It follows that

$$\begin{aligned} \left( \sum_{j=1}^{\infty} |P(e_j)|^{\frac{2n}{n+1}} \right)^{\frac{n+1}{2n}} &= \left( \sum_{j \in A} |a_j|^{\frac{2n}{n+1}} + \sum_{j \in B} |a_j|^{\frac{2n}{n+1}} \right)^{\frac{n+1}{2n}} \\ &\leq \left( \sum_{j \in A} |a_j| + \sum_{j \in B} |a_j| \right)^{\frac{n+1}{2n}} \\ &\leq 2^{\frac{n+1}{2n}}, \end{aligned}$$

which shows that $w \leq 2^{\frac{n+1}{2n}}$. Let $P_0\big((x_m)_{m=1}^{\infty}\big) = x_1^n - x_2^n \in \mathcal{P}(^n c_0)$ for $(x_m)_{m=1}^{\infty} \in c_0$. Then $\|P_0\| = 1$. Hence,

$$2^{\frac{n+1}{2n}} = \left( \sum_{j=1}^{\infty} |P_0(e_j)|^{\frac{2n}{n+1}} \right)^{\frac{n+1}{2n}} \leq w \leq 2^{\frac{n+1}{2n}}.$$

Therefore, $2^{\frac{n+1}{2n}} \leq C_p(n : \mathbb{R})$. We complete the proof. ∎

Corollary 4.6. $C(2:\mathbb{R}) < 2^{\frac{3}{4}} \leq C_p(2:\mathbb{R}) \leq 2\sqrt{2}$.

Theorem 4.7. *Let* $T : l_\infty^2(\mathbb{R}) \times l_\infty^2(\mathbb{R}) \to \mathbb{R}$ *be given by* $T(x,y) = \sum_{i,j=1}^2 a_{ij}x_iy_j$, *with* $a_{ij} \in \mathbb{R}$, $a_{12} = a_{21}$. *Then the symmetric bilinear forms satisfying*

$$\left(\sum_{i,j=1}^{2} |T(e_i,e_j)|^{\frac{4}{3}}\right)^{\frac{3}{4}} = \sqrt{2}\|T\|$$

*are given by* $T(x,y) = a(x_1y_1 - x_2y_2 + x_1y_2 + x_2y_1)$ *or* $T(x,y) = a(-x_1y_1 + x_2y_2 + x_1y_2 + x_2y_1)$ *for all* $a \in \mathbb{R}\backslash\{0\}$.

*Proof.* It follows from Theorem 4.1 of [3]. ∎

Theorem 4.8. *Let* $T \in \mathcal{L}_s({}^2l_\infty^2)$, $\|T\| = 1$, $T(e_j,e_j) \neq 0$ *for* $j = 1,2$. *Then,*

$$\left(\sum_{i,j=1}^{2} |T(e_i,e_j)|^{\frac{4}{3}}\right)^{\frac{3}{4}} = C_s(2:\mathbb{R})$$

*if and only if* $T \in extB_{\mathcal{L}_s({}^2l_\infty^2)}$.

*Proof.* It follows from Theorems 4.2, 4.3 and 4.7. ∎

Theorem 4.9.

$$\sup\{(\sum_{i,j=1}^{2} |T(e_i,e_j)|^{\frac{6}{4}})^{\frac{4}{6}} : T \in extB_{\mathcal{L}({}^3l_\infty^2)}\} = \frac{(7+3^{\frac{3}{2}})^{\frac{2}{3}}}{4} < C(3:\mathbb{R}).$$

*Proof.* Diniz *et al.* [12] showed that $2^{\frac{2}{3}} \leq C(3:\mathbb{R}) \leq 1.782$. By Theorem 2.2 and Corollary 2.3,

$$\sup\left\{\left(\sum_{i,j=1}^{2} |T(e_i,e_j)|^{\frac{6}{4}}\right)^{\frac{4}{6}} : T \in extB_{\mathcal{L}({}^3l_\infty^2)}\right\} = \frac{(7+3^{\frac{3}{2}})^{\frac{2}{3}}}{4} < 2^{\frac{2}{3}} \leq C(3:\mathbb{R}).$$

∎

Questions. (1) Is it true that $C_s(n:\mathbb{R}) = C(n:\mathbb{R})$ for all $n \geq 3$?
(2) Is it true that $C_p(2:\mathbb{R}) = 2^{\frac{3}{4}}$?

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