# On some Inequalities for Strongly Reciprocally Convex Functions 

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Abstract: We establish some Hermite-Hadamard and Fejér type inequalities for the class of strongly reciprocally convex functions.
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## 1. Introduction

Due to its important role in mathematical economics, engineering, management science, and optimization theory, convexity of functions and sets has been studied intensively; see $[1,5,7,8,9,11,13,15,16]$ and the references therein. Let $\mathbb{R}$ be the set of real numbers and $I \subseteq \mathbb{R}$ be a interval. A function $f: I \rightarrow \mathbb{R}$ is said to be convex in the classical sense if it satisfies the following inequality

$$
f(t x+(1-t) y) \leq t f(x)+(1-t) f(y)
$$

for all $x, y \in I$ and $t \in[0,1]$. We say that $f$ is concave if $-f$ is convex.
In recent years several extensions and generalizations have been considered for classical convexity, and the theory of inequalities has made essential contributions in many areas of mathematics. A significant subclass of convex functions is that of strongly convex functions introduced by B.T. Polyak [20]. Strongly convex functions are widely used in applied economics, as well as in nonlinear optimization and other branches of pure and applied mathematics. In this paper we present a new class of strongly convex functions, mainly the class of strongly harmonically convex functions. Our investigation is devoted
to the classical results related to convex functions due to Charles Hermite, Jaques Hadamard [10] and Lipót Fejér [8]. The Hermite-Hadamard inequalities and Fejér inequalities have been the subject of intensive research, and many applications, generalizations and improvements of them can be found in the literature (see, for instance, $[1,7,15,18,19,21,24]$ and the references therein).

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, this asserts that the mean value of a continuous convex functions $f:[a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ lies between the value of $f$ at the midpoint of the interval $[a, b]$ and the arithmetic mean of the values of $f$ at the endpoints of this interval, that is,

$$
\begin{equation*}
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \mathrm{d} x \leq \frac{f(a)+f(b)}{2} . \tag{1.1}
\end{equation*}
$$

Moreover, each side of this double inequality characterizes convexity in the sense that a real-valued continuous function $f$ defined on an interval $I$ is convex if its restriction to each compact subinterval $[a, b] \subseteq I$ verifies the left hand side of (1.1) (equivalently, the right hand side on (1.1)). See [17].

In [8], Lipót Fejér established the following inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1): If $f:[a, b] \rightarrow \mathbb{R}$ is a convex function, then the inequality

$$
\begin{align*}
f\left(\frac{a+b}{2}\right) \int_{a}^{b} p(x) \mathrm{d} x & \leq \frac{1}{b-a} \int_{a}^{b} f(x) p(x) \mathrm{d} x \\
& \leq \frac{f(a)+f(b)}{2} \int_{a}^{b} p(x) \mathrm{d} x \tag{1.2}
\end{align*}
$$

holds, where $p:[a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric about $x=(a+b) / 2$.

Various generalizations have been pointed out in many directions, for recent developments of inequalities (1.1) and (1.2) and its generalizations, see $[5,6,7,4,9,13]$.

In [13], Imdat Iscan gave the definition of harmonically convex functions:
Definition 1.1. [13] Let $I$ be an interval in $\mathbb{R} \backslash\{0\}$. A function $f: I \rightarrow \mathbb{R}$ is said to be harmonically convex on $I$ if the inequality

$$
\begin{equation*}
f\left(\frac{x y}{t x+(1-t) y}\right) \leq t f(y)+(1-t) f(x) \tag{1.3}
\end{equation*}
$$

holds, for all $x, y \in I$ and $t \in[0,1]$.

If the inequality in (1.3) is reversed, then $f$ is said to be harmonically concave.

The following result of the Hermite-Hadamard type for harmonically convex functions holds.

THEOREM 1.2. Let $f: I \subseteq \mathbb{R} \backslash\{0\} \rightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a<b$. If $f \in L[a, b]$, then the following inequalities hold

$$
\begin{equation*}
f\left(\frac{2 a b}{a+b}\right) \leq \frac{a b}{b-a} \int_{a}^{b} \frac{f(x)}{x^{2}} \mathrm{~d} x \leq \frac{f(a)+f(b)}{2} \tag{1.4}
\end{equation*}
$$

In [4], F. Chen and S. Wu proved the following Fejér inequality for harmonically convex functions.

Theorem 1.3. ([4]) Let $f: I \subseteq \mathbb{R} \backslash\{0\} \rightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a<b$. If $f \in L(a, b)$, then one has

$$
\begin{align*}
f\left(\frac{2 a b}{a+b}\right) \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x & \leq \int_{a}^{b} \frac{f(x)}{x^{2}} p(x) \mathrm{d} x  \tag{1.5}\\
& \leq \frac{f(a)+f(b)}{2} \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x
\end{align*}
$$

where $p:[a, b] \rightarrow \mathbb{R}$ is nonnegative and integrable and satisfies

$$
p\left(\frac{a b}{x}\right)=p\left(\frac{a b}{a+b-x}\right)
$$

## 2. Strongly Reciprocally convex functions

In 1966 Polyak [20] introduced the notions of strongly convex and strongly quasi-convex functions. In 1976 Rockafellar [23] studied the strongly convex functions in connection with the proximal point algorithm. They play an important role in optimization theory and mathematical economics. Nikodem et al. have obtained some interesting properties of strongly convex functions (see $[7,12,14]$ ).

Definition 2.1. (See [12, 16, 22]) Let $D$ be a convex subset of $\mathbb{R}$ and let $c>0$. A function $f: D \rightarrow \mathbb{R}$ is called strongly convex with modulus $c$ if

$$
\begin{equation*}
f(t x+(1-t) y) \leq t f(x)+(1-t) f(y)-c t(1-t)(x-y)^{2} \tag{2.1}
\end{equation*}
$$

for all $x, y \in D$ and $t \in[0,1]$.

The usual notion of convex function correspond to the case $c=0$. For instance, if $f$ is strongly convex, then it is bounded from below, its level sets $\{x \in I: f(x) \leq \lambda\}$ are bounded for each $\lambda$ and $f$ has a unique minimum on every closed subinterval of $I$ [18, p. 268]. Any strongly convex function defined on a real interval admits a quadratic support at every interior point of its domain.

The proofs of the next two theorems can be found in [22].
Theorem 2.2. Let $D$ be a convex subset of $\mathbb{R}$ and let $c$ be a positive constant. A function $f: D \rightarrow \mathbb{R}$ is strongly convex with modulus $c$ if and only if the function $g(x)=f(x)-c x^{2}$ is convex.

Theorem 2.3. The following are equivalent:
(i) $f(t x+(1-t) y) \leq t f(x)+(1-t) f(y)-t(1-t) c(x-y)^{2}$, for all $x, y \in(a, b)$ and $t \in[0,1]$.
(ii) For each $x_{0} \in(a, b)$, there is a linear function $T$ such that $f(x) \geq$ $f\left(x_{0}\right)+T\left(x-x_{0}\right)+c\left(x-x_{0}\right)^{2}$ for all $x, y \in(a, b)$.
(iii) For differentiable $f$, for each $x_{0} \in(a, b): f(x) \geq f\left(x_{0}\right)+f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)+$ $c\left(x-x_{0}\right)^{2}$, for all $x, y \in(a, b)$.
(iv) For twice differentiable $f, f^{\prime \prime}(x) \geq 2 c$, for all $x, y \in(a, b)$.

In [3] we proved the following sandwich theorem for harmonically convex functions:

THEOREM 2.4. Let $f, g$ be real functions defined on the interval $(0,+\infty)$. The following conditions are equivalent:
(i) There exists a harmonically convex function $h:(0,+\infty) \rightarrow \mathbb{R}$ such that $f(x) \leq h(x) \leq g(x)$ for all $x \in(0,+\infty)$.
(ii) The following inequality holds

$$
\begin{equation*}
f\left(\frac{x y}{t x+(1-t) y}\right) \leq t g(y)+(1-t) g(x) \tag{2.2}
\end{equation*}
$$

for all $x, y \in(0,+\infty)$ and $t \in[0,1]$.
On the other hand, in [2] we introduced the notion of harmonically strongly convex function as follows:

Definition 2.5. Let $I$ be an interval in $\mathbb{R} \backslash\{0\}$ and let $c \in \mathbb{R}_{+}$. A function $f: I \rightarrow \mathbb{R}$ is said to be harmonically strongly convex with modulus $c$ on $I$, if the inequality

$$
\begin{equation*}
f\left(\frac{x y}{t x+(1-t) y}\right) \leq t f(y)+(1-t) f(x)-c t(1-t)(x-y)^{2} \tag{2.3}
\end{equation*}
$$

holds, for all $x, y \in I$ and $t \in[0,1]$.
The symbol $\mathrm{SHC}_{(I, c)}$ will denote the class of functions that satisfy the inequality (2.3). We also establish some Hermite-Hadamard and Fejér type inequalities for the class of harmonically strongly convex functions.

Next we will explore a generalization of the concept of harmonically convex functions which we will call reciprocally strongly convex functions, it is a concept parallel to the definition presented in the definition 2.5.

Definition 2.6. Let $I$ be an interval in $\mathbb{R} \backslash\{0\}$ and let $c \in(0, \infty)$. A function $f: I \rightarrow \mathbb{R}$ is said to be reciprocally strongly convex with modulus $c$ on $I$, if the inequality

$$
\begin{equation*}
f\left(\frac{x y}{t x+(1-t) y}\right) \leq t f(y)+(1-t) f(x)-c t(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^{2}, \tag{2.4}
\end{equation*}
$$

holds, for all $x, y \in I$ and $t \in[0,1]$.
The symbol $\operatorname{SRC}_{(I, c)}$ will denote the class of functions that satisfy the inequality (2.4).

Theorem 2.7. Let $I \subset \mathbb{R} \backslash\{0\}$ be a real interval and $c \in(0, \infty)$. If $f \in S R C_{(I, c)}$, then $f$ es harmonically convex.

Proof. Since $\operatorname{ct}(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^{2} \geq 0$, it is a immediate consequence of the definition.

For the rest of this paper we will use $I \subset \mathbb{R} \backslash\{0\}$ to denote a real interval and $c \in(0, \infty)$.

Theorem 2.8. Let $f: I \rightarrow \mathbb{R}$ be a function. $f \in S R C_{(I, c)}$ if and only if the function $g: I \rightarrow \mathbb{R}$, defined by $g(x):=f(x)-\frac{c}{x^{2}}$ es harmonically convex.

Proof. Assume that $f \in \operatorname{SRC}_{(I, c)}$, then

$$
\begin{aligned}
& g\left(\frac{x y}{t x+(1-t) y}\right) \\
& =f\left(\frac{x y}{t x+(1-t) y}\right)-c\left(\frac{t x+(1-t) y}{x y}\right)^{2} \\
& \begin{aligned}
& \leq t f(y)+(1-t) f(x)-c t(1-t)\left(\frac{1}{y}-\frac{1}{x}\right)^{2}-c\left(t \frac{1}{y}+(1-t) \frac{1}{x}\right)^{2} \\
&= t f(y)+(1-t) f(x) \\
& \quad-c t(1-t)\left(\frac{1}{y^{2}}-\frac{2}{x y}+\frac{1}{x^{2}}\right) c\left(\frac{t^{2}}{y^{2}}+\frac{2 t(1-t)}{x y}+\frac{(1-t)^{2}}{x^{2}}\right) \\
&= t f(y)+(1-t) f(x)-c\left(\frac{t}{y^{2}}-\frac{2 t}{x y}+\frac{t}{x^{2}}-\frac{t^{2}}{y^{2}}+\frac{2 t^{2}}{x y}\right. \\
&\left.\quad-\frac{t^{2}}{x^{2}}+\frac{t^{2}}{y^{2}}+\frac{2 t}{x y}-\frac{2 t^{2}}{x y}+\frac{1}{x^{2}}-\frac{2 t}{x^{2}}+\frac{t^{2}}{x^{2}}\right) \\
&= t f(y)+(1-t) f(x)-c\left(\frac{t}{y^{2}}+\frac{1}{x^{2}}-\frac{t}{x^{2}}\right) \\
&= t f(y)+(1-t) f(x)-c\left(\frac{t}{y^{2}}+(1-t) \frac{1}{x^{2}}\right) \\
&=t\left(f(y)-\frac{c}{y^{2}}\right)+(1-t)\left(f(x)-\frac{c}{x^{2}}\right) \\
&= t g(y)+(1-t) g(x)
\end{aligned}
\end{aligned}
$$

for all $x, y \in I$ and $t \in[0,1]$. Which proves that $g$ is harmonically convex.
Conversely, if $g$ is harmonically convex, then

$$
\begin{aligned}
& f\left(\frac{x y}{t x+(1-t) y}\right)=g\left(\frac{x y}{t x+(1-t) y}\right)+c\left(\frac{t x+(1-t) y}{x y}\right)^{2} \\
& \leq t g(y)+(1-t) g(x)+c\left(t \frac{1}{y}+(1-t) \frac{1}{x}\right)^{2} \\
&=t g(y)+(1-t) g(x)+c\left(\frac{t^{2}}{y^{2}}+\frac{2 t(1-t)}{x y}+\frac{(1-t)^{2}}{x^{2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =t g(y)+(1-t) g(x)+c\left(\frac{t(1-1+t)}{y^{2}}+\frac{2 t(1-t)}{x y}+\frac{(1-t)(1-t)}{x^{2}}\right) \\
& =\operatorname{tg}(y)+(1-t) g(x)+c\left(\frac{t(1-1+t)}{y^{2}}+\frac{2 t(1-t)}{x y}+\frac{(1-t)(1-t)}{x^{2}}\right) \\
& =t g(y)+(1-t) g(x)+c\left(\frac{t}{y^{2}}-\frac{t(1-t)}{y^{2}}+\frac{2 t(1-t)}{x y}+\frac{1-t}{x^{2}}-\frac{t(1-t)}{x^{2}}\right) \\
& =t\left(g(y)+c \frac{1}{y^{2}}\right)+(1-t)\left(g(x)+c \frac{1}{x^{2}}\right)-c t(1-t)\left(\frac{1}{y^{2}}-\frac{2}{x y}+\frac{1}{x^{2}}\right) \\
& =t f(y)+(1-t) f(x)-c t(1-t)\left(\frac{1}{y}-\frac{1}{x}\right)^{2},
\end{aligned}
$$

for all $x, y \in I$ and $t \in[0,1]$, showing that $f \in \operatorname{SRC}_{(I, c)}$.
Example 2.9. (a) The constant function is harmonically convex but not reciprocally strongly convex.
(b) The function $f:(0,+\infty) \rightarrow \mathbb{R}$ defined by $f(x)=-x^{2}$, is not a harmonically convex function, since $f$ is a not convex and nonincreasing function. Based on Theorem 2.7, we obtain $f \notin \mathrm{SRC}_{(I, c)}$.
(c) Since $g(x)=\log (x)$ is a harmonically convex function, the function $f(x):=\log (x)+\frac{c}{x^{2}}$ is a reciprocally strongly convex function.

Lemma 2.10. If $f$ is a reciprocally strongly convex function, then the function $\varphi=f+\epsilon$ is also a reciprocally strongly convex function, for any constants $\epsilon$. In fact,

$$
\begin{aligned}
\varphi\left(\frac{x y}{t x+(1-t) y}\right) & =f\left(\frac{x y}{t x+(1-t) y}\right)+\epsilon \\
\leq & t f(y)+(1-t) f(x)+c t(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^{2}+\epsilon \\
= & t f(y)+t \epsilon+(1-t) f(x)+(1-t) \epsilon+c t(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^{2} \\
= & t(f(y)+\epsilon)+(1-t)(f(x)+\epsilon)+c t(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^{2} \\
= & t \varphi(y)+(1-t) \varphi(x)+c t(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^{2}
\end{aligned}
$$

ThEOREM 2.11. If $f:[a, b] \subset \mathbb{R} \backslash\{0\} \rightarrow \mathbb{R}$ and if we consider the function $g:\left[\frac{1}{b}, \frac{1}{a}\right] \rightarrow \mathbb{R}$, defined by $g(t)=f\left(\frac{1}{t}\right)$, then $f \in S R C_{([a, b], c)}$ if and only if $g$ is strongly convex in $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Proof. If for all $x, y \in[a, b]$ and $t \in[0,1]$, we have

$$
f\left(\frac{1}{t \frac{1}{y}+(1-t) \frac{1}{x}}\right) \leq t f(y)+(1-t) f(x)-c t(1-t)\left(\frac{1}{x}-\frac{1}{y}\right)^{2}
$$

this last inequality may be changed by another equivalent one:

$$
g(t w+(1-t) u) \leq t g(w)+(1-t) g(u)-c t(1-t)(u-w)^{2}
$$

where $u, w \in\left[\frac{1}{b}, \frac{1}{a}\right]$ and $t \in[0,1]$. To complete the proof.
It is easy to see that the result is also valid for intervals $(a, b) \subset \mathbb{R} \backslash\{0\}$.
Theorem 2.12. The following are equivalent:
(i) $f \in S R C_{((a, b), c)}$.
(ii) For each $x_{0} \in(a, b)$, there is a linear function $T$ such that

$$
\begin{equation*}
f\left(\frac{1}{x}\right) \geq c\left(x-x_{0}\right)^{2}+T\left(x-x_{0}\right)+f\left(\frac{1}{x_{0}}\right), \quad \text { for all } \quad x \in\left(\frac{1}{b}, \frac{1}{a}\right) . \tag{2.5}
\end{equation*}
$$

(iii) For differentiable $f$ and $x_{0} \in(a, b)$,

$$
\begin{equation*}
f\left(\frac{1}{x}\right) \geq f\left(\frac{1}{x_{0}}\right)-f\left(\frac{1}{x_{0}}\right) \frac{x-x_{0}}{x_{2}}+c\left(x-x_{0}\right)^{2} \tag{2.6}
\end{equation*}
$$

for all $x, y \in(a, b)$.
(iv) For twice differentiable $f$,

$$
\frac{1}{x^{4}}\left[f^{\prime \prime}\left(\frac{1}{x}\right)+2 x f^{\prime}\left(\frac{1}{x}\right)\right] \geq 2 c, \quad \text { for all } \quad x \in\left(\frac{1}{b}, \frac{1}{a}\right) .
$$

Proof. (i) $\Rightarrow$ (ii) : Assume that $f \in \mathrm{SRC}_{((a, b), c)}$. Since all the assumptions of Theorem 2.11 are satisfied, then the function $g(x):=f\left(\frac{1}{x}\right)$ is strongly
convex in $\left(\frac{1}{b}, \frac{1}{a}\right)$. Then by Theorem 2.3, for each $x_{0} \in\left(\frac{1}{b}, \frac{1}{a}\right)$, there is a linear function $T$ such that $g(x) \geq g\left(x_{0}\right)+T\left(x-x_{0}\right)+c\left(x-x_{0}\right)^{2}$, for all $x, y \in\left(\frac{1}{b}, \frac{1}{a}\right)$. This is equivalent to the inequality (2.5).
(i) $\Rightarrow$ (iii) : Assume that $f \in \operatorname{SRC}_{((a, b), c)}$. By Theorem 2.11, the function $g(x):=f\left(\frac{1}{x}\right)$ is strongly convex in $\left(\frac{1}{b}, \frac{1}{a}\right)$, then by Theorem 2.3, for each $x_{0} \in\left(\frac{1}{b}, \frac{1}{a}\right), g(x) \geq g\left(x_{0}\right)+g^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)+c\left(x-x_{0}\right)^{2}$, for all $x, y \in(a, b)$. This is equivalent to the inequality (2.6).
(ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (i) are shown using the reciprocals of the theorem and lemma that we have used in the above part.
(i) $\Longleftrightarrow$ (iv): Suppose $f$ is twice differentiable over $(a, b) . f \in \operatorname{SRC}_{((a, b), c)}$ if only if the function $g(x):=f\left(\frac{1}{x}\right)$ is strongly convex in $\left(\frac{1}{b}, \frac{1}{a}\right)$ (by the theorem 2.11). It follows from Theorem 2.3 that $g$ is a strongly convex function with modulus $c$ if only if $g^{\prime \prime}(x) \geq 2 c$. Hence it is equivalent to

$$
\frac{1}{x^{4}}\left[f^{\prime \prime}\left(\frac{1}{x}\right)+2 x f^{\prime}\left(\frac{1}{x}\right)\right] \geq 2 c, \quad \text { for all } \quad x \in\left(\frac{1}{b}, \frac{1}{a}\right)
$$

## 3. Main Results

In this section, we derive our main results.
3.1. Hermite-Hadamard type inequalities The following result is a counterpart of the Hermite-Hadamard inequality for strongly reciprocally convex functions.

Theorem 3.1. Let $I \subset \mathbb{R} \backslash\{0\}$ be a real interval. If $f: I \rightarrow \mathbb{R}$ is a strongly reciprocally convex function with modulus $c, a, b \in I$ with $a<b$ and $f \in L[a, b]$ then

$$
\begin{align*}
f\left(\frac{2 a b}{a+b}\right)+\frac{c}{12}\left(\frac{b-a}{a b}\right)^{2} & \leq \frac{a b}{b-a} \int_{a}^{b} \frac{f(x)}{x^{2}} \mathrm{~d} x \\
& \leq \frac{f(a)+f(b)}{2}-\frac{c}{6}\left(\frac{b-a}{a b}\right)^{2} \tag{3.1}
\end{align*}
$$

Proof. By Theorem 2.11 the function $g: I \rightarrow \mathbb{R}$, defined by $g(x):=$ $f(x)-\frac{c}{x^{2}}$ is harmonically convex, since $f \in \operatorname{SRC}_{(I, c)}$.

Consequently, by the Hermite-Hadamard type inequality for harmonically convex functions (see [13, Theorem 1]), we have

$$
\begin{gathered}
g\left(\frac{2 a b}{a+b}\right) \leq \frac{a b}{b-a} \int_{a}^{b} \frac{g(x)}{x^{2}} \mathrm{~d} x \leq \frac{g(a)+g(b)}{2} \\
f\left(\frac{2 a b}{a+b}\right)-c\left(\frac{a+b}{2 a b}\right)^{2}
\end{gathered} \leq \frac{a b}{b-a} \int_{a}^{b} \frac{f(x)-\frac{c}{x^{2}}}{x^{2}} \mathrm{~d} x \leq \frac{f(a)-\frac{c}{a^{2}}+f(b)-\frac{c}{b^{2}}}{2} .
$$

This last inequality can be simplified to

$$
\begin{aligned}
f\left(\frac{2 a b}{a+b}\right)-c\left(\frac{a+b}{2 a b}\right)^{2} & \leq \frac{a b}{b-a} \int_{a}^{b} \frac{f(x)}{x^{2}} \mathrm{~d} x-\frac{a b c}{3(b-a)}\left[\frac{b^{3}-a^{3}}{a^{3} b^{3}}\right] \\
& \leq \frac{f(a)+f(b)}{2}-\frac{c}{2}\left(\frac{a^{2}+b^{2}}{a^{2} b^{2}}\right)
\end{aligned}
$$

which in turn is equivalent to the inequality

$$
\begin{aligned}
f\left(\frac{2 a b}{a+b}\right)+\frac{c}{12}\left(\frac{b-a}{a b}\right)^{2} & \leq \frac{a b}{b-a} \int_{a}^{b} \frac{f(x)}{x^{2}} \mathrm{~d} x \\
& \leq \frac{f(a)+f(b)}{2}-\frac{c}{6}\left(\frac{b-a}{a b}\right)^{2}
\end{aligned}
$$

Remark 3.2. Letting $c \rightarrow 0^{+}$, in the inequalities (3.1), we obtain (1.4), which is the Hermite-Hadamard type inequalities for harmonically convex functions.

We establish some new inequalities of Hermite-Hadamard type for functions whose derivatives are strongly reciprocally convex.

We need the following lemma, which can be found in [13].
Lemma 3.3. ([13]) Let $f: I \subset \mathbb{R} \backslash\{0\} \rightarrow \mathbb{R}$ is a differentiable function on $I^{\circ}$ and $a, b \in I$ with $a<b$. If $f^{\prime} \in L[a, b]$, then

$$
\begin{aligned}
\frac{f(a)+f(b)}{2}- & \frac{a b}{b-a} \int_{a}^{b} \frac{f(x)}{x^{2}} \mathrm{~d} x \\
& =\frac{a b(b-a)}{2} \int_{0}^{1} \frac{1-2 t}{(t b+(1-t) a)^{2}} f^{\prime}\left(\frac{a b}{t b+(1-t) a}\right) \mathrm{d} t
\end{aligned}
$$

THEOREM 3.4. Let $f: I \subset(0,+\infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^{\circ}, a, b \in I$ with $a<b$, and $f^{\prime} \in L[a, b]$. If $\left|f^{\prime}\right|^{q}$ is strongly reciprocally convex with modulus $c$ on $[a, b]$ for $q \geq 1$, then

$$
\begin{align*}
& \left|\frac{f(a)+f(b)}{2}-\frac{a b}{b-a} \int_{a}^{b} \frac{f(x)}{x^{2}} \mathrm{~d} x\right| \\
& \quad \leq \frac{a b(b-a)}{2} \lambda_{1}^{1-\frac{1}{q}}\left[\lambda_{2}\left|f^{\prime}(a)\right|^{q}+\lambda_{3}\left|f^{\prime}(b)\right|^{q}-c\left(\frac{1}{b}-\frac{1}{a}\right)^{2} \lambda_{4}\right]^{\frac{1}{q}}, \tag{3.2}
\end{align*}
$$

where

$$
\begin{aligned}
\lambda_{1}= & \frac{1}{a b}-\frac{2}{(b-a)^{2}} \ln \left(\frac{(a+b)^{2}}{4 a b}\right) \\
\lambda_{2}= & -\frac{1}{b(b-a)}+\frac{3 a+b}{(b-a)^{3}} \ln \left(\frac{(a+b)^{2}}{4 a b}\right) \\
\lambda_{3}= & \frac{1}{a(b-a)}-\frac{3 b+a}{(b-a)^{3}} \ln \left(\frac{(a+b)^{2}}{4 a b}\right) \\
\lambda_{4}= & -\frac{1}{b(b-a)}+\frac{1}{(b-a)^{4}}\left[[a(a+2 b)+b(b+2 a)] \ln \left(\frac{(a+b)^{2}}{4 a b}\right)\right. \\
& \left.-\frac{(a+b)^{2}(2 a-b)}{2 b}+b^{2}-3 a^{2}\right] .
\end{aligned}
$$

Proof. From Lemma 3.3, and letting $p:=\frac{q}{q-1}$, we get

$$
\begin{align*}
& \left|\frac{f(a)+f(b)}{2}-\frac{a b}{b-a} \int_{a}^{b} \frac{f(x)}{x^{2}} \mathrm{~d} x\right| \\
& =\left|\frac{a b(b-a)}{2} \int_{0}^{1} \frac{1-2 t}{(t b+(1-t) a)^{2}} f^{\prime}\left(\frac{a b}{t b+(1-t) a}\right) \mathrm{d} t\right| \\
& \leq \frac{a b(b-a)}{2} \int_{0}^{1}\left|\frac{1-2 t}{(t b+(1-t) a)^{2}}\right|\left|f^{\prime}\left(\frac{a b}{t b+(1-t) a}\right)\right| \mathrm{d} t  \tag{3.3}\\
& =\frac{a b(b-a)}{2} \int_{0}^{1}\left|\frac{1-2 t}{(t b+(1-t) a)^{2}}\right|^{\frac{1}{p}}\left(\left|\frac{1-2 t}{(t b+(1-t) a)^{2}}\right|^{\frac{1}{q}}\left|f^{\prime}\left(\frac{a b}{t b+(1-t) a}\right)\right|\right) \mathrm{d} t
\end{align*}
$$

We apply Hölder's inequality to the right-hand side of (3.3) and using the hypothesis that $\left|f^{\prime}\right|^{q} \in \operatorname{SRC}_{([a, b, c)}$, we get

$$
\begin{align*}
& \leq \frac{a b(b-a)}{2}\left[\int_{0}^{1}\left(\left|\frac{1-2 t}{[t b+(1-t) a]^{2}}\right|^{\frac{1}{p}}\right)^{p} \mathrm{~d} t\right]^{\frac{1}{p}} \\
&= \frac{a b(b-a)}{2}\left[\int_{0}^{1}\left|\frac{1-2 t}{[t b+(1-t) a]^{2}}\right| \mathrm{d} t\right]^{1-\frac{1}{q}} \\
&\left.\qquad \cdot\left[\int_{0}^{1}\left|\frac{1-2 t}{[t b+(1-t) a]^{2}}\right|^{\frac{1}{q}}\left|f^{\prime}\left(\frac{a b}{t t b+(1-t) a t}\right)\right|\right)^{q} \mathrm{~d} t\right]^{\frac{1}{q}} \\
& \leq \frac{a b(b-a)}{2}\left[\int_{0}^{1}\left|\frac{1-2 t}{[t b+(1-t) a]^{2}}\right| \mathrm{d} t\right]^{1-\frac{1}{q}} \\
& \cdot\left[\int_{0}^{1} \frac{|1-2 t|}{[t b+(1-t) a]^{2}}\left(t\left|f^{\prime}(a)\right|^{q}+(1-t)\left|f^{\prime}(b)\right|^{q}-c t(1-t)\left(\frac{1}{b}-\frac{1}{a}\right)^{2}\right) \mathrm{d} t\right]^{\frac{1}{q}} . \tag{3.4}
\end{align*}
$$

It can be shown that

$$
\begin{aligned}
\lambda_{1} & :=\int_{0}^{1} \frac{|1-2 t|}{[t b+(1-t) a]^{2}} \mathrm{~d} t=\frac{1}{a b}-\frac{2}{(b-a)^{2}} \ln \left(\frac{(a+b)^{2}}{4 a b}\right), \\
\lambda_{2}: & =\int_{0}^{1} \frac{|1-2 t| t}{[t b+(1-t) a]^{2}} \mathrm{~d} t=\int_{0}^{\frac{1}{2}} \frac{(1-2 t) t}{[t b+(1-t) a]^{2}} \mathrm{~d} t-\int_{\frac{1}{2}}^{1} \frac{(1-2 t) t}{[t b+(1-t) a]^{2}} \mathrm{~d} t \\
= & -\frac{1}{b(b-a)}+\frac{b+3 a}{(b-a)^{3}} \ln \left(\frac{(a+b)^{2}}{4 a b}\right), \\
\lambda_{3}: & =\int_{0}^{1} \frac{|1-2 t|(1-t)}{[t b+(1-t) a]^{2}} \mathrm{~d} t=\lambda_{1}+\lambda_{2}, \\
\lambda_{4}:= & \int_{0}^{1} \frac{t(1-t)|1-2 t|}{[t b+(1-t) a]^{2}} \mathrm{~d} t, \\
= & -\frac{1}{b(b-a)}+\frac{1}{(b-a)^{4}}\left[[a(a+2 b)+b(b+2 a)] \ln \left(\frac{(a+b)^{2}}{4 a b}\right)\right. \\
& \left.-\frac{(a+b)^{2}(2 a-b)}{2 b}+b^{2}-3 a^{2}\right] .
\end{aligned}
$$

Now if we replace this values in (3.4), we get (3.2).
3.2. Fejér type inequalities The following result is a counterpart of the Fejér inequality for strongly reciprocally convex functions.

Theorem 3.5. Let $I \subset \mathbb{R} \backslash\{0\}$ be a real interval. If $f: I \rightarrow \mathbb{R}$ is a strongly reciprocally convex function with modulus $c, a, b \in I$ with $a<b$ and $f \in L[a, b]$ then

$$
\begin{align*}
f\left(\frac{2 a b}{a+b}\right) & \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x+c \int_{a}^{b}\left[\frac{1}{x^{2}}-\left(\frac{a+b}{2 a b}\right)^{2}\right] \frac{p(x)}{x^{2}} \mathrm{~d} x \\
& \leq \int_{a}^{b} \frac{f(x)}{x^{2}} p(x) \mathrm{d} x  \tag{3.5}\\
& \leq \frac{f(a)+f(b)}{2} \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x-c \int_{a}^{b}\left[\frac{1}{2}\left(\frac{a^{2}+b^{2}}{a^{2} b^{2}}\right)-\frac{1}{x^{2}}\right] \frac{p(x)}{x^{2}} \mathrm{~d} x
\end{align*}
$$

where $p:[a, b] \rightarrow[0, \infty)$ is an integrable function and satisfies

$$
\begin{equation*}
p\left(\frac{a b}{x}\right)=p\left(\frac{a b}{a+b-x}\right) . \tag{3.6}
\end{equation*}
$$

Proof. By Theorem 2.11 the function $g: I \rightarrow \mathbb{R}$, defined by $g(x):=f(x)-$ $\frac{c}{x^{2}}$ is harmonically convex, then in virtue of Theorem 1.3, we have that

$$
g\left(\frac{2 a b}{a+b}\right) \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x \leq \int_{a}^{b} \frac{g(x)}{x^{2}} p(x) \mathrm{d} x \leq \frac{g(a)+g(b)}{2} \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x .
$$

The above inequality is equivalent to

$$
\begin{aligned}
{\left[f\left(\frac{2 a b}{a+b}\right)-c\left(\frac{a+b}{2 a b}\right)^{2}\right] \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x } & \leq \int_{a}^{b} \frac{f(x)-\frac{c}{x^{2}}}{x^{2}} p(x) \mathrm{d} x \\
& \leq \frac{f(a)-\frac{c}{a^{2}}+f(b)-\frac{c}{b^{2}}}{2} \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x
\end{aligned}
$$

This last inequality can be simplified to

$$
\begin{aligned}
f\left(\frac{2 a b}{a+b}\right) & \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x-c\left(\frac{a+b}{2 a b}\right)^{2} \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x+c \int_{a}^{b} \frac{p(x)}{x^{4}} \mathrm{~d} x \\
\leq & \int_{a}^{b} \frac{f(x)}{x^{2}} \mathrm{~d} x \\
\leq & \frac{f(a)+f(b)}{2} \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x \\
& \quad-\frac{c}{2}\left(\frac{1}{a^{2}}+\frac{1}{b^{2}}\right) \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x+c \int_{a}^{b} \frac{p(x)}{x^{4}} \mathrm{~d} x
\end{aligned}
$$

which in turn is equivalent to the inequality

$$
\begin{aligned}
& f\left(\frac{2 a b}{a+b}\right) \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x+c \int_{a}^{b}\left[\frac{1}{x^{2}}-\left(\frac{a+b}{2 a b}\right)^{2}\right] \frac{p(x)}{x^{2}} \mathrm{~d} x \\
& \quad \leq \int_{a}^{b} \frac{f(x)}{x^{2}} p(x) \mathrm{d} x \\
& \quad \leq \frac{f(a)+f(b)}{2} \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x-c \int_{a}^{b}\left[\frac{1}{2}\left(\frac{a^{2}+b^{2}}{a^{2} b^{2}}\right)-\frac{1}{x^{2}}\right] \frac{p(x)}{x^{2}} \mathrm{~d} x
\end{aligned}
$$

Remarks 3.6. (a) Letting $c \rightarrow 0^{+}$, in inequality (3.5), we obtain (1.5) which is the Fejér type inequality for harmonically convex functions.
(b) Putting $p(x) \equiv 1$ into Theorem 3.5, we obtain the inequality (3.1).

Now, we establish a new Fejér-type inequality for strongly reciprocally convex functions.

Theorem 3.7. Suppose $f: I \subset \mathbb{R} \backslash\{0\} \rightarrow \mathbb{R}$ is a strongly reciprocally convex function with modulus $c$ on $I$. If $a, b \in I, a<b$, and $f \in L[a, b]$, then

$$
\begin{align*}
& f\left(\frac{2 a b}{a+b}\right) \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x+\frac{c}{2 a b} \int_{a}^{b} \frac{p(x)}{x^{4}}[2 a b-(a+b) x] \mathrm{d} x \\
& \quad \leq \int_{a}^{b} \frac{f(x)}{x^{2}} p(x) \mathrm{d} x  \tag{3.7}\\
& \quad \leq \frac{a[f(a)+f(b)]}{b-a} \int_{a}^{b}(b-x) \frac{p(x)}{x^{3}} \mathrm{~d} x-\frac{c}{a b} \int_{a}^{b}(b-x)(x-a) \frac{p(x)}{x^{4}} \mathrm{~d} x
\end{align*}
$$

where $p:[a, b] \rightarrow \mathbb{R}$ is a nonnegative integrable function that satisfies (3.6).

Proof. According to (3.6), for $x=t b+(1-t) a$, we have

$$
\begin{equation*}
p\left(\frac{a b}{t b+(1-t) a}\right)=p\left(\frac{a b}{t a+(1-t) b}\right) \tag{3.8}
\end{equation*}
$$

Since $f \in \operatorname{SRC}_{([a, b], c)}$, from the definition 2.6 , we obtain

$$
\begin{equation*}
f\left(\frac{2 x y}{x+y}\right) \leq \frac{f(y)+f(x)}{2}-\frac{c}{4}\left(\frac{1}{x}-\frac{1}{y}\right)^{2}, \quad x, y \in[a, b] \tag{3.9}
\end{equation*}
$$

Let $x=\frac{a b}{t b+(1-t) a}$ and $y=\frac{a b}{t a+(1-t) b}$ in (3.9), then

$$
\begin{aligned}
f\left(\frac{2 a b}{a+b}\right) \leq & \frac{f\left(\frac{a b}{t a+(1-t) b}\right)+f\left(\frac{a b}{t b+(1-t) a}\right)}{2} \\
& \quad-\frac{c}{4}\left(\frac{t b+(1-t) a}{a b}-\frac{t a+(1-t) b}{a b}\right)^{2}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f\left(\frac{2 a b}{a+b}\right) p( & \left.\frac{a b}{t b+(1-t) a}\right) \leq \frac{1}{2}\left[f\left(\frac{a b}{t a+(1-t) b}\right) p\left(\frac{a b}{t a+(1-t) b}\right)\right. \\
& \left.+f\left(\frac{a b}{t b+(1-t) a}\right) p\left(\frac{a b}{t b+(1-t) a}\right)\right] \\
& -\frac{c}{4}\left(\frac{t b+(1-t) a}{a b}-\frac{t a+(1-t) b}{a b}\right)^{2} p\left(\frac{a b}{t b+(1-t) a}\right)
\end{aligned}
$$

Integrating both sides of the above inequalities with respect to $t$ over $[0,1]$, we obtain

$$
\begin{aligned}
f\left(\frac{2 a b}{a+b}\right) & \int_{0}^{1} p\left(\frac{a b}{t b+(1-t) a}\right) \mathrm{d} t \\
& \leq \frac{1}{2} \int_{0}^{1} f\left(\frac{a b}{t a+(1-t) b}\right) p\left(\frac{a b}{t a+(1-t) b}\right) \mathrm{d} t \\
& +\frac{1}{2} \int_{0}^{1} f\left(\frac{a b}{t b+(1-t) a}\right) p\left(\frac{a b}{t b+(1-t) a}\right) \mathrm{d} t \\
& -\frac{c}{4} \int_{0}^{1}\left(\frac{t b+(1-t) a}{a b}-\frac{t a+(1-t) b}{a b}\right)^{2} p\left(\frac{a b}{t b+(1-t) a}\right) \mathrm{d} t
\end{aligned}
$$

By simple computation,

$$
\begin{aligned}
f\left(\frac{2 a b}{a+b}\right) & \frac{a b}{b-a} \int_{a}^{b} \frac{p(x)}{x^{2}} \mathrm{~d} x \\
& \leq \frac{1}{2} \frac{a b}{b-a} \int_{a}^{b} \frac{f(x)}{x^{2}} p(x) \mathrm{d} x+\frac{a b}{b-a} \int_{a}^{b} \frac{f(x)}{x^{2}} p(x) \mathrm{d} x \\
& -\frac{c}{4} \frac{2}{b-a} \int_{a}^{b} \frac{p(x)}{x^{4}}[2 a b-(a+b) x] \mathrm{d} x
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& f\left(\frac{a b}{t a+(1-t) b}\right) p\left(\frac{a b}{t a+(1-t) b}\right) \\
& \quad \leq\left[t f(b)+(1-t) f(a)-c t(1-t)\left(\frac{1}{a}-\frac{1}{b}\right)^{2}\right] p\left(\frac{a b}{t a+(1-t) b}\right)
\end{aligned}
$$

Again, integrating both sides of the above inequalities with respect to $t$ over $[0,1]$, we obtain

$$
\begin{aligned}
& \int_{0}^{1} f\left(\frac{a b}{t a+(1-t) b}\right) p\left(\frac{a b}{t a+(1-t) b}\right) \mathrm{d} t \\
& \quad \leq \int_{0}^{1}\left[t f(b)+(1-t) f(a)-c t(1-t)\left(\frac{1}{a}-\frac{1}{b}\right)^{2}\right] p\left(\frac{a b}{t a+(1-t) b}\right) \mathrm{d} t
\end{aligned}
$$

By simple computation,

$$
\begin{aligned}
& \int_{a}^{b} \frac{f(x)}{x^{2}} p(x) \mathrm{d} x \\
& \quad \leq \frac{a[f(a)+f(b)]}{b-a} \int_{a}^{b}(b-x) \frac{p(x)}{x^{3}} \mathrm{~d} x-\frac{c}{a b} \int_{a}^{b}(b-x)(x-a) \frac{p(x)}{x^{4}} \mathrm{~d} x
\end{aligned}
$$

This concludes the proof.
Remarks 3.8. (a) Letting $c \rightarrow 0^{+}$in the inequalities (3.7), we obtain the left-hand side of inequality of Fejér type inequalities for harmonically convex function (see [4]).
(b) Letting $p(x) \equiv 1$ in the inequalities (3.7) we obtain inequalities of Hermite-Hadamard type (see Theorem 3.1).

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